Special singularity function for continuous part of the spectral data in the associated eigenvalue problem for nonlinear equations

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The procedure for finding the solutions of the Vakhnenko-Parkes equation by means of the inverse scattering method is described. The continuous spectrum is taken into account in the associated eigenvalue problem. The suggested special form of the singularity function for continuous part of the spectral data gives rise to the multi-mode solutions. The sufficient conditions are proved in order that these solutions become real functions. The interaction of the N periodic waves is studied. The procedure is illustrated by considering a number of examples. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4726168]

I. INTRODUCTION

Various physical phenomena in engineering and physics may be described by nonlinear evolution equations. Looking for exact solutions to completely integrable equations is a difficult task. In recent years, a few methods for obtaining the exact solutions of nonlinear evolution equations have been suggested. One of the fundamental direct methods is undoubtedly the Hirota bilinear method[1,2] which possesses significant features that make it practical for the determination of multiple-soliton solutions. However, the direct methods can be applied only for finding the solitary wave solutions or the traveling-wave solutions. In this sense, the inverse scattering method is the most appropriate way of tackling the initial value problem although its employment is a fairly difficult procedure.[3–5]

In this paper, we will consider the nonlinear evolution equation

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{1}$$

This equation arises from the Vakhnenko equation[2,6–8]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0 \tag{2}$$

through the transformation[9,10]

$$
\begin{aligned}
&u(x, t) = U(X, T) = W_X(X, T), \\
&x = x_0 + T + W(X, T), \\
&t = X.
\end{aligned}
\tag{3}
$$

The corresponding governing equation for U, namely

$$UU_{XXT} - U_X U_{XT} + U^2 U_T = 0 \tag{4}$$

is given in Ref. 9.

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Equations (1), (2), and (4) arose as a result of describing high-frequency perturbations in a relaxing medium.[8] Following the papers,[11–13] hereafter (1) (or equivalently (4)) is referred to as the Vakhnenko-Parkes equation (VPE).

Recently the Hirota method[2, 9, 10] as well as the inverse scattering method[14] have been applied to obtain the exact *N*-soliton solutions of the VPE. In this paper, we use the inverse scattering transform method to study the periodic solutions of the VPE (1) associated with the continuous part of the spectral data.

In Sec. II, we formulate the spectral problem for the VPE by adapting the results given by Caudrey[15] and by Kaup.[16] In Sec. III, we find the solutions corresponding to the continuous part of the spectral data. In Sec. IV, we find the real N-mode solutions for $N = 1, 2, 3, 4$. Our results are summarized in Sec. V.

II. THE SPECTRAL PROBLEM FOR THE VPE

In order to use the inverse scattering method, one first has to formulate the associated eigenvalue problem. In Ref. 14, it is shown that the pair of equations

$$\psi_{XXX} + W_X \psi_X - \lambda \psi = 0, \tag{5}$$

$$3\psi_{XT} + (W_T + 1)\psi = 0 \tag{6}$$

is associated with the VPE (1) considered here. Note that the inverse scattering transform problem is related to a spectral equation of third order (5). The inverse problem for third-order spectral equations has been considered by Caudrey[15] and Kaup.[16] We adapt the results obtained by these authors to the present problem and describe a procedure for using the inverse scattering transform method to find the solutions of the VPE that are associated with the continuous part of the spectral data.

We use the general theory of the inverse scattering problem for *N* spectral equations which has been developed by Caudrey in Ref. 15. According to Ref. 15 the spectral Eq. (5) can be rewritten in the form

$$\frac{\partial}{\partial X} \boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X, \zeta)] \cdot \boldsymbol{\psi} \tag{7}$$

with

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -W_X & 0 \end{pmatrix}. \tag{8}$$

The matrix **A** has the eigenvalues $\lambda_j(\zeta)$ and left- and right-eigenvectors $\tilde{\boldsymbol{v}}_j(\zeta)$ and $\boldsymbol{v}_j(\zeta)$, respectively $(j = 1, 2, 3)$. In the case considered here we define

$$\lambda_j(\zeta) = \omega_j \zeta, \quad \lambda_j^3(\zeta) = \lambda,$$
$$\boldsymbol{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \end{pmatrix}, \quad \tilde{\boldsymbol{v}}_j(\zeta) = \begin{pmatrix} \lambda_j^2 & \lambda_j & 1 \end{pmatrix}, \tag{9}$$

where $\omega_j = e^{2\pi i(j-1)/3}$ are the cube roots of 1.

The solution of the linear Eq. (5), or equivalently Eq. (7), has been obtained by Caudrey[15] in terms of Jost functions $\boldsymbol{\phi}_j(X, \zeta)$ which have the asymptotic behaviour

$$\boldsymbol{\Phi}_j(X, \zeta) = \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X, \zeta) \to \boldsymbol{v}_j(\zeta) \tag{10}$$

as $X \to -\infty$.

Here *T* is regarded as a parameter; the *T*-evolution of the scattering data will be taken into account later. The solution of the direct problem (7) is given by the equation system (4.5) in Ref. 15. Since there is a set of symmetry properties $\phi_1(X, \zeta/\omega_1) = \phi_2(X, \zeta/\omega_2) = \phi_3(X, \zeta/\omega_3)$ (see (6.14) and (6.15) in Ref. 15, for example) for Jost functions $\boldsymbol{\phi}_j(X, \zeta)$, we need only consider the element
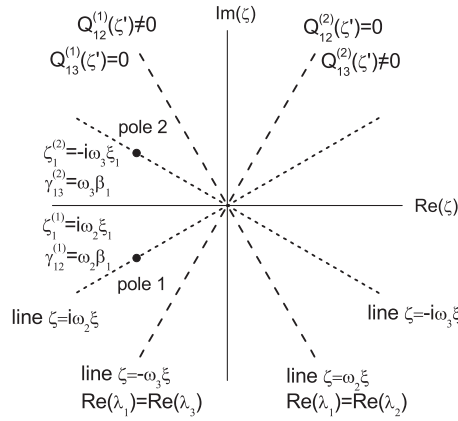
FIG. 1. The regular regions for Jost functions $\boldsymbol{\phi}_1(X, \zeta)$ in the complex ζ-plane. The dashed lines with singularity functions $Q_{1j}(\zeta')$ determine the boundaries between regular regions. The dotted lines are the lines where the poles appear.

$\phi_1(X, \zeta)$ (as well as $\Phi_1(X, \zeta)$). In the general case, it is necessary to take into account both the bound state spectrum and the continuous spectrum. According to the relation (6.20) in Ref. 15, the solution of (7) is as follows:

$$\Phi_1(X, \zeta) = 1 - \sum_{k=1}^{K} \sum_{j=2}^{3} \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Phi_1(X, \omega_j \zeta_1^{(k)})$$

$$+ \frac{1}{2\pi i} \int \sum_{j=2}^{3} Q_{1j}(\zeta') \frac{\exp\{[\lambda_j(\zeta') - \lambda_1(\zeta')]X\}}{\zeta' - \zeta} \Phi_1^{\pm}(X, \omega_j \zeta') d\zeta'. \qquad (11)$$

Equation (11) contains the spectral data, namely K poles with the quantities $\gamma_{1j}^{(k)}$ for the bound state spectrum as well as the functions $Q_{1j}(\zeta')$ given along all the boundaries of regular regions for the continuous spectrum. The boundaries between regions, where the Jost function $\boldsymbol{\phi}_1(X, \zeta)$ is regular, appear at $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ over all $j \neq 1$ (Ref. 15) (see Fig. 1). The singularities on boundaries of these regions within the complex ζ-plane are taken into account by the third term in the relation (11). The integral in (11) is along all the boundaries (see the dashed lines in Fig. 1).

   The bound state spectrum is associated with soliton solutions; in this case $Q_{1j}(\zeta) \equiv 0$ in (11). The procedure for finding the exact N-soliton solution of the VPE via the inverse scattering method is described in Ref. 14. In Sec. III, we study the solutions of the VPE which follow from the continuous part of the spectral data.

III. THE SOLUTIONS ASSOCIATED WITH THE CONTINUOUS PART OF THE SPECTRAL DATA

   Now we consider only the continuous spectrum of the associated eigenvalue problem, i.e., we assume that at least some of the functions $Q_{1j}(\zeta')$ are nonzero, while $\gamma_{1j}^{(k)} \equiv 0$ in Eq. (11). At each fixed $j \neq 1$ the functions $Q_{1j}(\zeta')$ characterize the singularity of $\Phi_1(X, \zeta)$. This singularity can appear only on boundaries between the regular regions on the ζ-plane. The condition $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ determines these boundaries.[15] According to Ref. 15, we find that for $\Phi_1(X, \zeta)$ the complex ζ-plane is divided into four regions by two lines:

$$\begin{aligned} &\text{(i) } \zeta = \omega_2 \xi, \quad \text{where} \quad Q_{12}^{(1)} \neq 0, \quad Q_{13}^{(1)} \equiv 0, \\ &\text{(ii) } \zeta = -\omega_3 \xi, \quad \text{where} \quad Q_{12}^{(2)} \equiv 0, \quad Q_{13}^{(2)} \neq 0, \end{aligned} \qquad (12)$$

where ξ is real (see Fig. 1). Analysis shows that the direction of the integration in (11) is such that ξ sweeps from $-\infty$ to $+\infty$.

Let us consider the singularity functions $Q_{1j}(\zeta')$ on the boundaries, on which the Jost function $\phi_1(X, \zeta)$ is singular, in the form ($n = 1, 2, ..., N$) on the line $\zeta' = \omega_2\xi$,

$$Q_{12}^{(1)}(\zeta') = -2\pi i \sum_{n=1}^{N} q_{12}^{(2n-1)}\delta(\zeta' - \zeta_1'),$$

$$Q_{13}^{(1)}(\zeta') = -2\pi i \sum_{n=1}^{N} q_{13}^{(2n-1)}\delta(\zeta' - \zeta_1') \equiv 0,$$

(13)

and on the line $\zeta' = -\omega_3\xi$,

$$Q_{12}^{(2)}(\zeta') = -2\pi i \sum_{n=1}^{N} q_{12}^{(2n)}\delta(\zeta' - \zeta_2') \equiv 0,$$

$$Q_{13}^{(2)}(\zeta') = -2\pi i \sum_{n=1}^{N} q_{13}^{(2n)}\delta(\zeta' - \zeta_2').$$

(14)

For the singularity functions (13) and (14), the relationship (11) is reduced to the form ($M = 2N$),

$$\Phi_1(X, \zeta) = 1 - \sum_{m=1}^{M} \sum_{j=2}^{3} q_{1j}^{(m)} \frac{\exp\{[\lambda_j(\zeta_m') - \lambda_1(\zeta_m')]X\}}{\zeta_m' - \zeta}$$

$$\times \Phi_1(X, \omega_j\zeta_m').$$

(15)

As follows from the relationship (15) and the formula

$$\phi_{1X}(X, \zeta) = \frac{i}{\sqrt{3}} [\phi_{1X}(X, -\omega_2\zeta)\phi_1(X, -\omega_3\zeta)$$

$$-\phi_{1X}(X, -\omega_3\zeta)\phi_1(X, -\omega_2\zeta)]$$

(16)

given in Ref. 14, for example, the singularities in the form (13) and (14) appear in pairs $\zeta_{2n-1}' = \omega_2\xi_n$, $\zeta_{2n}' = -\omega_3\xi_n$. From (16), considering the limits $\zeta \to \zeta_m'$ and $X \to -\infty$, it also follows immediately that

$$q_{12}^{(2n-1)}\omega_2 = q_{13}^{(2n)}, \quad \text{with} \quad n = 1, 2, ..., N.$$

(17)

We call attention to the fact that, at the special choice of the singularity function $Q_{1j}(\zeta')$ for continuous part of the spectral data as in (13) and (14), the second term on the right-hand side of the relation (15) is similar in mathematical structure to the second term in relation (5.5) from Ref. 14. Indeed, the formal substitutions $\xi_m = i\xi_m$, $q_{1j}^{(m)} = \gamma_{1j}^{(m)}$ transform the second term in (15) into the second term in (5.5) from Ref. 14. Since there is this transformation, we can apply the procedure developed for solving the N-soliton interaction to obtain the solutions connected with the continuous part of the spectral data for the associated eigenvalue problem.[14, 15] According to Ref. 14 (see Eqs. (5.11)– (5.15) therein), we can find $\Phi_1(X, \zeta)$ and can connect $\Phi_1(X, \zeta)$ with the solution $W(X)$, by expanding $\Phi_1(X, \zeta)$ as an asymptotic series in $\lambda_1^{-1}(\zeta)$ (see Eq. (5.11) in Ref. 14) as follows:

$$\Phi_1(X, \zeta) = 1 - \frac{1}{3\lambda_1(\zeta)} [W(X) - W(-\infty)] + O(\lambda_1^{-2}(\zeta)).$$

(18)

On the other hand, by defining

$$\Psi_m(X) = \sum_{j=2}^{3} q_{1j}^{(m)} \exp\{\lambda_j(\zeta_m')X\}\Phi_1(X, \omega_j\zeta_m'),$$

(19)

we may rewrite the relationship (15) as

$$\Phi_1(X, \zeta) = 1 - \sum_{m=1}^{M} \frac{\exp\{-\lambda_1(\zeta_m')X\}}{\zeta_m' - \zeta}\Psi_m(X).$$

(20)

From (18) and (20), the following key relationship can be derived (see also Eq. (6.38) in Ref. 15):

$$W(X) - W(-\infty) = -3 \sum_{m=1}^{M} \exp\{-\lambda_1(\zeta'_m)X\}\Psi_m(X)$$

$$= 3\frac{\partial}{\partial X} \ln(\det M(X)). \tag{21}$$

Here $M(X)$ is the $2N \times 2N$ matrix given by elements

$$M_{ml}(X) = \delta_{ml} - \sum_{j=2}^{3} q_{1j}^{(m)} \frac{\exp\{[\lambda_j(\zeta'_m) - \lambda_1(\zeta'_l)]X\}}{\zeta'_l - \omega_j\zeta'_m}. \tag{22}$$

Now let us consider the T-evolution of the spectral data. By analyzing the solution of Eq. (6) when $X \to -\infty$, we find that $\phi_i(X, T, \zeta) = \exp[-(3\lambda_i(\zeta))^{-1}T]\phi_i(X, 0, \zeta)$. Hence the T-evolution of the scattering data is given by the relationships (with $m = 1, 2, ..., M$)

$$q_{ij}^{(m)}(T) = q_{ij}^{(m)}(0) \exp\{[-(3\lambda_j(\zeta'_m)^{-1} + (3\lambda_1(\zeta'_m))^{-1}]T\},$$

$$\lambda_j(T) = \lambda_j(0). \tag{23}$$

Consequently, the final result for the solution of the VPE, when we consider the spectral data from the continuous spectrum, as well as taking into account their T-evolution, is as follows:

$$W(X, T) = 3\frac{\partial}{\partial X} \ln(\det M(X, T)) + \text{const.} \tag{24}$$

The $2N \times 2N$ matrix $M(X, T)$ is defined as follows:

$$M_{ml}(X, T) = \delta_{ml} - \sum_{j=2}^{3} q_{1j}^{(m)}(0)\frac{\exp\{[-(3\lambda_j(\zeta'_m)^{-1} + (3\lambda_1(\zeta'_m))^{-1}]T + [\lambda_j(\zeta'_m) - \lambda_1(\zeta'_l)]X\}}{\zeta'_l - \omega_j\zeta'_m}, \tag{25}$$

with the relations ($n = 1, 2, ..., N$)

$$\begin{array}{llll} \lambda_1(\zeta'_{2n-1}) = \omega_2\xi_{2n-1}, & \lambda_2(\zeta'_{2n-1}) = \omega_3\xi_{2n-1}, & q_{12}^{(2n-1)} = \omega_2\beta_{2n-1}, & q_{13}^{(2n-1)} = 0, \\ \lambda_1(\zeta'_{2n}) = -\omega_3\xi_{2n-1}, & \lambda_3(\zeta'_{2n}) = -\omega_2\xi_{2n-1}, & q_{12}^{(2n)} = 0, & q_{13}^{(2n)} = \omega_3\beta_{2n-1}. \end{array} \tag{26}$$

As will be clear from the examples in Sec. IV, the solution (24) and (25) include N frequencies from the continuous part of the spectral data. For this reason, the solution (24) and (25) will be referred to as the N-mode solution of the VPE. Evidently, these discrete modes emanate from the special choice (13) and (14) of the singularity functions $Q_{1j}(\zeta')$ for continuous part of the spectral data.

For the solution (24) and (25), there are N arbitrary constants ξ_n and N arbitrary constants β_n. The constants ξ_n are real, while the constants β_n, in the general case, are complex. The solution (24) obtained through the matrix (25) is, in general, a complex function. Consequently, there is a problem in selecting the real solutions from the complex solutions. It turns out that we can obtain the real solutions by means of restriction of arbitrariness in the choice of the constants β_i. For the N-mode solution, we have succeeded in finding these restrictions.

IV. REAL PERIODIC SOLUTIONS OF THE VPE

This research has culminated in finding the real N-mode solution. For convenience, we consider the solutions for $N = 1, 2, 3, 4$. For $N \geq 5$, all formulas can be easily obtained beginning with these examples.

1. In order to obtain the one-mode solution of the VPE (1), we need first to calculate the 2×2 matrix $M(X, T)$ according to (25). For the matrix elements $M_{kl}(X, T)$, we have

$$
\begin{aligned}
M_{11} &= 1 - \frac{i\omega_2 \beta_1}{\sqrt{3}\xi_1} \exp[-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T], \\
M_{12} &= -\frac{\omega_3 \beta_1}{2\xi_1} \exp[2\omega_3\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T], \\
M_{21} &= \frac{\omega_2 \beta_1}{2\xi_1} \exp[-2\omega_2\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T], \\
M_{22} &= 1 - \frac{i\omega_3 \beta_1}{\sqrt{3}\xi_1} \exp[-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T],
\end{aligned}
\tag{27}
$$

so that its determinant is

$$
\det M = \left[1 + c_1 \exp(-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T)\right]^2,
\tag{28}
$$

where $c_1 = -\dfrac{i\beta_1}{2\sqrt{3}\xi_1}$.

As has already been noted, the singularity functions in the form (13) and (14) give rise to a single frequency for the continuous part of the spectral data. Hence, once the expression (28) has been substituted into the key formula (24), (24) must provide us with the one-mode solution.

The condition that W_X is real requires a restriction on the constant β_1 (if the constant ξ_1 is arbitrary, but real). We have succeeded in obtaining this restriction (see Appendix), namely that the constant c_1, which in general is the complex-valued one $c_1 = |c_1|\exp(i\chi_1)$, should possess unit modulus $|c_1| = 1$, while the arbitrary real constant χ_1 defines an initial shift of solution $X_1 = \chi_1/(\sqrt{3}\xi_1)$ so that

$$
\det M = \left[1 + \exp\left(-i\sqrt{3}\xi_1(X - X_1) + \frac{T}{i\sqrt{3}\xi_1}\right)\right]^2.
\tag{29}
$$

The final result for one mode of the continuous spectrum is the solution (24) with (29), namely,

$$
W = -3\sqrt{3}\,\xi_1 \tan\left(\frac{\sqrt{3}}{2}\xi_1(X - X_1) + \frac{T}{2\sqrt{3}\xi_1}\right) + \text{const.}
\tag{30}
$$

The corresponding solution for $U = W_X$ (with U governed by (4)) was obtained recently by other methods, for example, by the sine-cosine method,[17] the (G'/G)-expansion method,[13] and the extended tanh-function method.[17–19] However, only the approach developed here and the solution in the form (24) and (25) enable us to study the interaction the periodic N-mode waves.

2. Let us consider the two-mode solution of the VPE. In this case $M(X, T)$ is a 4×4 matrix. We will not give the explicit form of this matrix here, but we find its determinant

$$
\det M(X, T) = (1 + q_1 + q_2 + b_{12}q_1 q_2)^2,
\tag{31}
$$

where

$$
q_i = c_i \exp[-i\sqrt{3}\xi_i X + (i\sqrt{3}\xi_i)^{-1}T], \quad c_i = -\frac{i\beta_i}{2\sqrt{3}\xi_i},
$$

$$
b_{12} = \left(\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1}\right)^2 \frac{\xi_1^2 + \xi_2^2 - \xi_1\xi_2}{\xi_1^2 + \xi_2^2 + \xi_1\xi_2}, \quad b_{12} \geq 0.
\tag{32}
$$

In Appendix, the restrictions on the constants $c_i = |c_i|\exp(i\chi_i)$ for real solutions are found. The real constants χ_i define the initial shifts of solutions $X_i = \chi_i/(\sqrt{3}\xi_i)$. The analysis in considerable detail shows (see Appendix) that the relations $|c_1| = |c_2| = 1/\sqrt{b_{12}}$ are sufficient conditions in order that W may become real. Consequently, the real solution describing the interaction of two periodic waves for the VPE is defined by the key relationship (24), where

$$
\det M(X, T) = \left(1 + \frac{1}{\sqrt{b_{12}}}q_1 + \frac{1}{\sqrt{b_{12}}}q_2 + q_1 q_2\right)^2
\tag{33}
$$

and b_{12} is as in (32), while q_i should contain the phase shifts $X_i = \chi_i/(\sqrt{3}\xi_i)$ as in (34), namely

$$q_i = \exp\left[-i\sqrt{3}\xi_i(X - X_i) + (i\sqrt{3}\xi_i)^{-1}T\right]. \tag{34}$$

3. For $N = 3$ in relationship obtained from (25)

$$\det M(X, T) = (1 + c_1q_1 + c_2q_2 + c_3q_3 + c_1c_2b_{12}q_1q_2$$
$$+ c_1c_3b_{13}q_1q_3 + c_2c_3b_{23}q_2q_3$$
$$+ c_1c_2c_3b_{12}b_{13}b_{23}q_1q_2q_3)^2 \tag{35}$$

with q_i, c_i as in (32) and

$$b_{ij} = \left(\frac{\xi_j - \xi_i}{\xi_j + \xi_i}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}, \quad b_{ij} = b_{ji}, \tag{36}$$

we write $c_i = |c_i|\exp(i\chi_i)$, then the arguments χ_i determine the initial phase shifts of mode $X_i = \chi_i/(\sqrt{3}\xi_i)$. As is proved in Appendix, the conditions on the constants c_1 (or the same on β_i) are

$$|c_1| = 1/\sqrt{b_{12}b_{13}}, \quad |c_2| = 1/\sqrt{b_{12}b_{23}}, \quad |c_3| = 1/\sqrt{b_{13}b_{23}}. \tag{37}$$

Hence, the three-mode solution is the relation (24) with

$$\det M = \left[1 + \frac{1}{\sqrt{b_{12}b_{13}}}(q_1 + q_2q_3) + \frac{1}{\sqrt{b_{12}b_{23}}}(q_2 + q_1q_3)\right.$$
$$\left. + \frac{1}{\sqrt{b_{13}b_{23}}}(q_3 + q_1q_2) + q_1q_2q_3\right]^2. \tag{38}$$

Here the phase shifts X_i are taken into account in q_i by way of (34).

4. For $N = 4$ the restrictions are as follows (see Appendix):

$$|c_i| = \prod_{j \neq i}^4 b_{ij}^{-\frac{1}{2}}, \quad b_{ij} = b_{ji}, \quad i = 1, 2, 3, 4. \tag{39}$$

The determinant for a real solution (24) is as follows:

$$\det M = \left[1 + \frac{1}{\sqrt{b_{12}b_{13}b_{14}}}(q_1 + q_2q_3q_4)\right.$$
$$+ \frac{1}{\sqrt{b_{12}b_{23}b_{24}}}(q_2 + q_1q_3q_4)$$
$$+ \frac{1}{\sqrt{b_{13}b_{23}b_{34}}}(q_3 + q_1q_2q_4)$$
$$+ \frac{1}{\sqrt{b_{14}b_{24}b_{34}}}(q_4 + q_1q_2q_3)$$
$$+ \frac{1}{\sqrt{b_{13}b_{14}b_{23}b_{24}}}(q_1q_2 + q_3q_4)$$
$$+ \frac{1}{\sqrt{b_{12}b_{14}b_{23}b_{34}}}(q_1q_3 + q_2q_4)$$
$$\left. + \frac{1}{\sqrt{b_{12}b_{13}b_{24}b_{34}}}(q_1q_4 + q_2q_3) + q_1q_2q_3q_4\right]^2. \tag{40}$$

As before, the q_i and b_{ij} are defined by (34) and (36), respectively.

V. CONCLUSION

We have adapted and applied the inverse scattering method to the Vakhnenko-Parkes equation in order to find the solutions that are associated with the continuous spectrum of the spectral problem. The special form of the singularity function for continuous part of the spectral data enabled us to obtain the multi-mode solutions. The sufficient conditions have been proved in order that the solutions become real functions. We have described how to define the interaction of the multi-mode periodic waves. The procedure has been illustrated by considering a number of examples.

APPENDIX: THE CONDITIONS ON THE CONSTANTS c_i FOR REAL SOLUTIONS

We use the case $N = 4$ as an example to prove the restrictions on the constants, at which the solution $W(X, T)$ is real. We will consider the auxiliary function $f = \sqrt{\det M(X, T)}$ for convenience, namely

$$
\begin{aligned}
f = {} & 1 + c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 + c_1 c_2 b_{12} q_1 q_2 \\
& + c_1 c_3 b_{13} q_1 q_3 + c_1 c_4 b_{14} q_1 q_4 + c_2 c_3 b_{23} q_2 q_3 \\
& + c_2 c_4 b_{24} q_2 q_4 + c_3 c_4 b_{34} q_3 q_4 + c_1 c_2 c_3 b_{12} b_{13} b_{23} q_1 q_2 q_3 \\
& + c_1 c_2 c_4 b_{12} b_{14} b_{24} q_1 q_2 q_4 + c_1 c_3 c_4 b_{13} b_{14} b_{34} q_1 q_3 q_4 \\
& + c_2 c_3 c_4 b_{23} b_{24} b_{34} q_2 q_3 q_4 \\
& + c_1 c_2 c_3 c_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} q_1 q_2 q_3 q_4.
\end{aligned}
\tag{A1}
$$

We here redefine the values $c_i = |c_i|$, since the arguments χ_i can always be introduced in the variables $q_i = \exp(i\theta_i)$ with $\theta = -\sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T$.

The solution (21) then has a form

$$
W(X, T) = 6 \frac{\partial}{\partial X} \ln(f) + \text{const.}
\tag{A2}
$$

The function f is complex-valued, i.e.,

$$
\begin{aligned}
& f = f_{Re} + i f_{Im} = |f| \exp(i\chi_f), \quad f_{Re} = \text{Re}(f), \\
& f_{Im} = \text{Im}(f), \quad \tan(\chi_f) = f_{Im}/f_{Re},
\end{aligned}
\tag{A3}
$$

hence,

$$
W(X, T)/6 = \frac{\partial}{\partial X} \ln(|f|) + i \frac{\partial \chi_f}{\partial X} + \text{const.}
\tag{A4}
$$

If we succeed in making $\partial^2 \chi_f / \partial X^2 \equiv 0$ by the choice of the constants c_i, then the solution $W(X, T)$ will be a real function.

Let us write f_{Im} and f_{Re} in explicit forms

$$
\begin{aligned}
f_{Im} = {}& c_1 \sin(\theta_1) + c_2 \sin(\theta_2) + c_3 \sin(\theta_3) + c_4 \sin(\theta_4) \\
&+ c_1 c_2 b_{12} \sin(\theta_1 + \theta_2) + c_1 c_3 b_{13} \sin(\theta_1 + \theta_3) \\
&+ c_1 c_4 b_{14} \sin(\theta_1 + \theta_4) + c_2 c_3 b_{23} \sin(\theta_2 + \theta_3) \\
&+ c_2 c_4 b_{24} \sin(\theta_2 + \theta_4) + c_3 c_4 b_{34} \sin(\theta_3 + \theta_4) \\
&+ c_1 c_2 c_3 b_{12} b_{13} b_{23} \sin(\theta_1 + \theta_2 + \theta_3) \\
&+ c_1 c_2 c_4 b_{12} b_{14} b_{24} \sin(\theta_1 + \theta_2 + \theta_4) \\
&+ c_1 c_3 c_4 b_{13} b_{14} b_{34} \sin(\theta_1 + \theta_3 + \theta_4) \\
&+ c_2 c_3 c_4 b_{23} b_{24} b_{34} \sin(\theta_2 + \theta_3 + \theta_4) \\
&+ c_1 c_2 c_3 c_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4),
\end{aligned} \tag{A5}
$$

$$
\begin{aligned}
f_{Re} = {}& 1 + c_1 \cos(\theta_1) + c_2 \cos(\theta_2) + c_3 \cos(\theta_3) + c_4 \cos(\theta_4) \\
&+ c_1 c_2 b_{12} \cos(\theta_1 + \theta_2) + c_1 c_3 b_{13} \cos(\theta_1 + \theta_3) \\
&+ c_1 c_4 b_{14} \cos(\theta_1 + \theta_4) + c_2 c_3 b_{23} \cos(\theta_2 + \theta_3) \\
&+ c_2 c_4 b_{24} \cos(\theta_2 + \theta_4) + c_3 c_4 b_{34} \cos(\theta_3 + \theta_4) \\
&+ c_1 c_2 c_3 b_{12} b_{13} b_{23} \cos(\theta_1 + \theta_2 + \theta_3) \\
&+ c_1 c_2 c_4 b_{12} b_{14} b_{24} \cos(\theta_1 + \theta_2 + \theta_4) \\
&+ c_1 c_3 c_4 b_{13} b_{14} b_{34} \cos(\theta_1 + \theta_3 + \theta_4) \\
&+ c_2 c_3 c_4 b_{23} b_{24} b_{34} \cos(\theta_2 + \theta_3 + \theta_4) \\
&+ c_1 c_2 c_3 c_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4).
\end{aligned} \tag{A6}
$$

Now we select a factor $\sin(\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4))$ from f_{Im} and a factor $\cos(\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4))$ from f_{Re}. This can be done if the following conditions are satisfied:

$$
\begin{aligned}
&c_1 = c_2 c_3 c_4 b_{23} b_{24} b_{34}, \quad c_2 = c_1 c_3 c_4 b_{13} b_{14} b_{34}, \\
&c_3 = c_1 c_2 c_4 b_{12} b_{14} b_{24}, \quad c_4 = c_1 c_2 c_3 b_{12} b_{13} b_{23}, \\
&c_1 c_2 b_{12} = c_3 c_4 b_{34}, \quad c_1 c_3 b_{13} = c_2 c_4 b_{24}, \quad c_1 c_4 b_{14} = c_2 c_3 b_{23}, \\
&c_1 c_2 c_3 c_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} = 1.
\end{aligned} \tag{A7}
$$

It turns out that all these relations are valid, when

$$
\begin{aligned}
c_1 = \frac{1}{\sqrt{b_{12} b_{13} b_{14}}}, \quad & c_2 = \frac{1}{\sqrt{b_{12} b_{23} b_{24}}}, \\
c_3 = \frac{1}{\sqrt{b_{13} b_{23} b_{34}}}, \quad & c_4 = \frac{1}{\sqrt{b_{14} b_{24} b_{34}}}.
\end{aligned} \tag{A8}
$$

The conditions (A8) enable us to reduce both f_{Im} and f_{Re} to the forms

$$
\begin{aligned}
f_{Im} = 2g \sin(\tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4)), \\
f_{Re} = 2g \cos(\tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4)),
\end{aligned} \tag{A9}
$$

where

$$
\begin{aligned}
g = {} & \frac{1}{\sqrt{b_{12}b_{13}b_{14}}} \cos\left(\tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 - \theta_4)\right) \\
& + \frac{1}{\sqrt{b_{12}b_{23}b_{24}}} \cos\left(\tfrac{1}{2}(\theta_2 - \theta_1 - \theta_3 - \theta_4)\right) \\
& + \frac{1}{\sqrt{b_{13}b_{23}b_{34}}} \cos\left(\tfrac{1}{2}(\theta_3 - \theta_1 - \theta_2 - \theta_4)\right) \\
& + \frac{1}{\sqrt{b_{14}b_{24}b_{34}}} \cos\left(\tfrac{1}{2}(\theta_4 - \theta_1 - \theta_2 - \theta_3)\right) \\
& + \frac{1}{\sqrt{b_{13}b_{14}b_{23}b_{24}}} \cos\left(\tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4)\right) \\
& + \frac{1}{\sqrt{b_{12}b_{14}b_{23}b_{34}}} \cos\left(\tfrac{1}{2}(\theta_1 + \theta_3 - \theta_2 - \theta_4)\right) \\
& + \frac{1}{\sqrt{b_{12}b_{13}b_{24}b_{34}}} \cos\left(\tfrac{1}{2}(\theta_1 + \theta_4 - \theta_2 - \theta_3)\right).
\end{aligned}
\tag{A10}
$$

Now it is readily seen from (A3) that

$$
\chi_f = \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4)
\tag{A11}
$$

and as a consequence we have

$$
\frac{\partial^2 \chi_f}{\partial X^2} = \frac{\partial^2 \chi_f}{\partial X \partial T} = 0.
\tag{A12}
$$

Hence, as follows from (A4), the four-mode solution of the VPE can be reduced to real form with four real constants X_i and four real constants ξ_i (see (41)).

Without proof here we give the following conditions on the constants c_i that ensure the real N-mode solution of the VPE:

$$
|c_i| = \prod_{j \neq i}^{N} b_{ij}^{-\frac{1}{2}}, \quad b_{ij} = b_{ji}, \quad i = 1, ..., N,
\tag{A13}
$$

whereas the N constants ξ_i determine the values b_{ij} and the N constants X_i through β_i define the phase shifts of the separate modes. Note that these relations are sufficient conditions, but not necessary ones.

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