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Solutions Associated with Discrete and Continuous Spectrums in the Inverse Scattering Method for the Vakhnenko-Parkes Equation

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In this paper the inverse scattering method is applied to the Vakhnenko-Parkes equation. We describe a procedure for using the inverse scattering transform to find the solutions that are associated with both the bound state spectrum and continuous spectrum of the spectral problem. The suggested special form of the singularity function gives rise to the multi-mode periodic solutions. Sufficient conditions are obtained in order that the solutions become real functions. The interaction of the solitons and multi-mode periodic waves is studied. The procedure is illustrated by considering a number of examples.

Subject Index: 010, 011

§1. Introduction

It is of significance to look for exact solutions of nonlinear evolution equations in many applications of physics and technology. Various effective approaches have been developed to construct exact wave solutions of completely integrable equations. One of the fundamental direct methods is undoubtedly the Hirota bilinear method,[1),2)] which possesses significant features that make it practical for the determination of multiple-soliton solutions. However, the direct methods can be applied only for finding solitary-wave solutions or traveling-wave solutions. In this sense, the inverse scattering method is the most appropriate way of tackling the initial value problem although its employment is a fairly difficult procedure.[3)–5)]

This paper deals with a nonlinear evolution equation

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{1·1}$$

This equation arises from the Vakhnenko equation[6)–8)]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0 \tag{1·2}$$

through the transformation[9),10)]

$$u(x,t) := U(X,T) = W_X(X,T), \quad x := x_0 + T + W(X,T), \quad t := X. \tag{1·3}$$

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The corresponding governing equation for U, namely

$$UU_{XXT} - U_X U_{XT} + U^2 U_T = 0 \tag{1·4}$$

is given in the paper.[9]

Equations (1·1), (1·2), and (1·4) arose as a result of describing high-frequency perturbations in a relaxing medium.[8] Following the papers,[11),12] hereafter the equation (1·1) is referred to as the Vakhnenko-Parkes equation (VPE).

Recently the Hirota method[2),9] as well as the inverse scattering method[13] have been applied to obtain the exact N-soliton solutions of the VPE. In this paper we use the inverse scattering transform method to study additionally the periodic solutions of the VPE (1·1) associated with the continuous part of the spectral data as well as to investigate the interaction of solitons with these periodic waves.

In §2 we formulate the spectral problem for the VPE by adapting the results given by Caudrey[14] and by Kaup.[15] In §3 we find the solutions corresponding to both the bound state and continuous part of the spectral data. In §4 we find the corresponding real M-soliton and periodic N-mode solutions and study their interactions. Our results are summarized in §5.

§2. The inverse spectral problem for the VPE

In order to use the inverse scattering method, one first has to formulate the associated eigenvalue problem. In the paper,[13] it is shown that the pair of equations

$$\psi_{XXX} + W_X \psi_X - \lambda\psi = 0, \tag{2·1}$$

$$3\psi_{XT} + (W_T + 1)\psi = 0 \tag{2·2}$$

is associated with the VPE (1·1) considered here through the quantity W. Indeed, the compatibility of Eqs. (2·1) and (2·2) yields the condition[13] $(W_{XXT} + (1 + W_T)W_X)_X = 0$ or $W_{XXT} + (1 + W_T)W_X + h(T) = 0$, where $h(T)$ is an arbitrary function of T. Now, according to (2·6) and (3·10), the inverse scattering method restricts the solutions to those that vanish as $|X| \to \infty$, so $h(T)$ is to be identically zero. Thus, the pair of equations (2·1) and (2·2) can be considered as the Lax pair for the VPE (1·1).

Note that the inverse scattering transform problem is related to a spectral equation of third order (2·1). The inverse problem for third-order spectral equations has been considered by Caudrey[14] and Kaup.[15] We adapt the results obtained by these authors to the present spectral problem and describe a procedure for using the inverse scattering transform method to find the solutions of the VPE.

We follow the general theory of the inverse scattering problem for N spectral equations which has been developed by Caudrey.[14] According to this paper[14] the spectral equation (2·1) can be rewritten in the form[13]

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\boldsymbol{A}(\zeta) + \boldsymbol{B}(X,\zeta)] \cdot \boldsymbol{\psi} \tag{2·3}$$

with

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -W_X & 0 \end{pmatrix}. \tag{2·4}$$

The matrix $\boldsymbol{A}$ has the eigenvalues $\lambda_j(\zeta)$ and left- and right-eigenvectors $\tilde{\boldsymbol{v}}_j(\zeta)$ and $\boldsymbol{v}_j(\zeta)$, respectively, $(j = 1, 2, 3)$. These quantities are defined through a spectral parameter λ as

$$\lambda_j(\zeta) = \omega_j\zeta, \quad \lambda_j^3(\zeta) = \lambda, \boldsymbol{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j(\zeta) \\ \lambda_j^2(\zeta) \end{pmatrix}, \quad \tilde{\boldsymbol{v}}_j(\zeta) = \begin{pmatrix} \lambda_j^2(\zeta) & \lambda_j(\zeta) & 1 \end{pmatrix}, \tag{2·5}$$

where $\omega_j = e^{2\pi \mathrm{i}(j-1)/3}$ are the cube roots of 1.

The solution of the linear equation (2·1), or equivalently Eq. (2·3), has been obtained by Caudrey[14] in terms of Jost functions $\boldsymbol{\phi}_j(X, \zeta)$ which have the asymptotic behaviour

$$\boldsymbol{\Phi}_j(X, \zeta) := \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X, \zeta) \to \boldsymbol{v}_j(\zeta) \quad \text{as} \quad X \to -\infty. \tag{2·6}$$

Here T is regarded as a parameter; the T-evolution of the scattering data will be taken into account later. The solution of the direct problem (2·3) is given by the equation system (4.5) in the paper.[14] Since there is a set of symmetry properties $\phi_1(X, \zeta/\omega_1) = \phi_2(X, \zeta/\omega_2) = \phi_3(X, \zeta/\omega_3)$ (see e.g. (6.14) and (6.15) in the paper[14]) for Jost functions $\boldsymbol{\phi}_j(X, \zeta)$, we need only consider the element $\phi_1(X, \zeta)$ (as well as $\Phi_1(X, \zeta)$). In the general case it is necessary to take into account both the bound state spectrum and the continuous spectrum. According to the relation (6.20) in the paper,[14] the solution of (2·3) is as follows:

$$\begin{aligned} \Phi_1(X, \zeta) = 1 - &\sum_{k=1}^{K}\sum_{j=2}^{3} \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Phi_1(X, \omega_j\zeta_1^{(k)}) \\ &+ \frac{1}{2\pi \mathrm{i}} \int \sum_{j=2}^{3} Q_{1j}(\zeta') \frac{\exp\{[\lambda_j(\zeta') - \lambda_1(\zeta')]X\}}{\zeta' - \zeta} \Phi_1^{\pm}(X, \omega_j\zeta')d\zeta'. \end{aligned} \tag{2·7}$$

Equation (2·7) contains the spectral data, namely, K poles with the quantities $\gamma_{1j}^{(k)}$ for the bound state spectrum as well as the functions $Q_{1j}(\zeta')$ given along all the boundaries of regular regions for the continuous spectrum. The boundaries between regions, where the Jost function $\boldsymbol{\phi}_1(X, \zeta)$ is regular, appear at $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ over all $j \neq 1$.[14] The singularities on boundaries of these regions within the complex ζ-plane are taken into account by the third term in the relation (2·7). The integral in (2·7) is along all the boundaries (see the dashed lines in Fig. 1). The direction of integration is taken so that the side chosen to be $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) < 0$ is shown by the arrows in Fig. 1. It is necessary to note that we should carry out the integration along the lines $\omega_2(\xi + \mathrm{i}\varepsilon)$ and $-\omega_3(\xi + \mathrm{i}\varepsilon)$ with $\varepsilon > 0$. In this case, as we will show in §4.1.1, the condition (2·6) is satisfied. Passing to the limit $\varepsilon \to 0$ we can obtain the

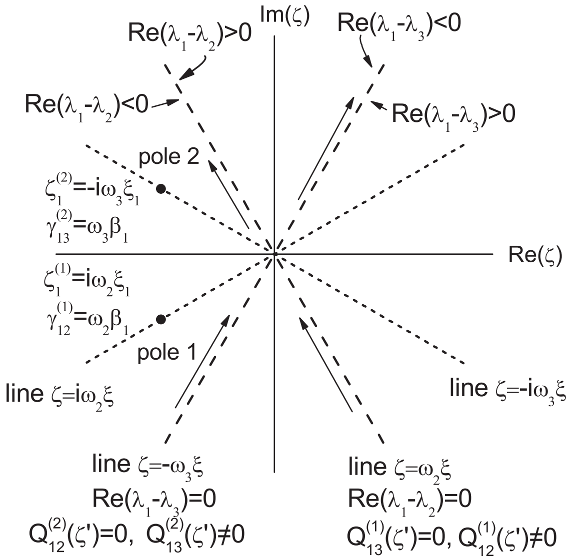

Fig. 1. The regular regions for Jost functions $\boldsymbol{\phi}_1(X,\zeta)$ in the complex ζ-plane. The dashed lines determine the boundaries between regular regions. These lines are lines where the singularity functions $Q_{1j}(\zeta')$ are given. The dotted lines are the lines where the poles appear.

solution which does not satisfy the condition (2·6) (see §4.1). However, for any finite $\varepsilon > 0$ the restricted region on X can be determined where the solution associated with a finite $\varepsilon > 0$ (for which the condition (2·6) is valid) and the solution associated with $\varepsilon = 0$ are sufficiently close to each other (see §4.1.1 for details). In this sense, taking the integration at $\varepsilon = 0$, we remain within the inverse scattering theory[14] whereas the condition (2·6) can be omitted. The solution obtained at $\varepsilon = 0$ can be extended to sufficiently large finite X. Thus, we will interpret the solution obtained at $\varepsilon = 0$ as the solution of Eq. (1·1) which is valid on arbitrary but finite X.

Provided $Q_{1j}(\zeta) \equiv 0$ in (2·7), the consideration of the bound state spectrum only gives rises to the purely soliton solutions. The procedure for finding the exact N-soliton solution of the VPE via the inverse scattering method is described in the paper.[13] In the next section we study the solutions of the VPE taking into account additionally the continuous part of the spectral data.

§3. The soliton and periodic solutions

Now additionally to the bound state spectrum we consider the continuous spectrum of the associated eigenvalue problem, i.e. assume that at least some of the functions $Q_{1j}(\zeta')$ are nonzero. At each fixed $j \neq 1$ the functions $Q_{1j}(\zeta')$ characterize the singularity of $\Phi_1(X,\zeta)$. This singularity can appear only on boundaries between the regular regions on the ζ-plane. The condition $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ determines these boundaries.[14] According to the paper,[14] we find that for $\Phi_1(X,\zeta)$ the complex ζ-plane is divided into four regions by two lines

$$\begin{aligned} &\text{(i)} \quad \zeta' = \omega_2\xi, \quad \text{with} \quad Q_{12}^{(1)}(\zeta') \neq 0, \quad Q_{13}^{(1)}(\zeta') \equiv 0, \\ &\text{(ii)} \quad \zeta' = -\omega_3\xi, \quad \text{with} \quad Q_{12}^{(2)}(\zeta') \equiv 0, \quad Q_{13}^{(2)}(\zeta') \neq 0, \end{aligned} \tag{3·1}$$

where ξ is real (see Fig. 1). Analysis shows that the direction of the integration in (2·7) is to be so that ξ sweeps from $-\infty$ to $+\infty$.

Let us consider the singularity functions $Q_{1j}(\zeta')$ on the boundaries, on which the Jost function $\boldsymbol{\phi}_1(X,\zeta)$ is singular, in the form ($n = 1, 2, ..., N$)

$$\left.\begin{aligned} Q_{12}^{(1)}(\zeta') &= -2\pi\mathrm{i}\sum_{n=1}^{N} q_{12}^{(2n-1)}\delta(\zeta' - \zeta'_{2n-1}) \\ Q_{13}^{(1)}(\zeta') &= -2\pi\mathrm{i}\sum_{n=1}^{N} q_{13}^{(2n-1)}\delta(\zeta' - \zeta'_{2n-1}) \equiv 0 \end{aligned}\right\} \quad \text{on the line} \quad \zeta' = \omega_2\xi,$$

$$\left.\begin{aligned} Q_{12}^{(2)}(\zeta') &= -2\pi\mathrm{i}\sum_{n=1}^{N} q_{12}^{(2n)}\delta(\zeta' - \zeta'_{2n}) \equiv 0 \\ Q_{13}^{(2)}(\zeta') &= -2\pi\mathrm{i}\sum_{n=1}^{N} q_{13}^{(2n)}\delta(\zeta' - \zeta'_{2n}) \end{aligned}\right\} \quad \text{on the line} \quad \zeta' = -\omega_3\xi. \tag{3·2}$$

For the singularity functions (3·2) and for M pairs of poles, the relationship (2·7) is reduced to the form

$$\begin{aligned} \Phi_1(X,\zeta) = 1 &- \sum_{k=1}^{2M}\sum_{j=2}^{3} \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Phi_1(X, \omega_j\zeta_1^{(k)}) \\ &- \sum_{l=1}^{2N}\sum_{j=2}^{3} q_{1j}^{(l)} \frac{\exp\{[\lambda_j(\zeta'_l) - \lambda_1(\zeta'_l)]X\}}{\zeta'_l - \zeta} \Phi_1(X, \omega_j\zeta'_l). \end{aligned} \tag{3·3}$$

In the paper,[13] it is proved that the poles appear in pairs only $\zeta_1^{(2m-1)} = \mathrm{i}\omega_2\xi_1$ and $\zeta_1^{(2m)} = -\mathrm{i}\omega_3\xi_1$, under the conditions $\gamma_{12}^{(2m-1)} = \omega_2\beta_m$, $\gamma_{13}^{(2m-1)} = 0$, $\gamma_{12}^{(2m)} = 0$, and $\gamma_{13}^{(2m)} = \omega_3\beta_m$, ($m = 1, 2, ..., M$). If we consider both the bound state spectrum and the continuous spectrum, the constants β_m are complex values in the general case. The restrictions on the constants β_m for real solutions U follow from a separate problem, which will be analyzed in the next section.

As follows from the relationship (3·3) and the formula

$$\phi_{1\,X}(X,\zeta) = \frac{\mathrm{i}}{\sqrt{3}}\left[\phi_{1\,X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) - \phi_{1\,X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)\right] \tag{3·4}$$

given in the paper,[13] for example, the singularities in the form (3·2) appear in pairs $\zeta'_{2n-1} = \omega_2\xi_n$ and $\zeta'_{2n} = -\omega_3\xi_n$. From (3·4), on considering the limits $\zeta \to \zeta'_l$ and $X \to -\infty$, it immediately follows that

$$q_{12}^{(2n-1)}\omega_2 = q_{13}^{(2n)} \quad \text{for} \quad n = 1, 2, ..., N. \tag{3·5}$$

Insofar as we have $2M$ poles and $2N$ coefficients $q_{12}^{(2n-1)}$ and $q_{13}^{(2n)}$ in the adopted specifications (3·2) of the singularity functions $Q_{1j}(\zeta')$, it is convenient to introduce the notation

$$\mu_{ji} = \begin{cases} \lambda_j(\zeta_1^{(i)}), \\ \lambda_j(\zeta'_{(i-K)}), \end{cases} \qquad p_{1j}^{(i)} = \begin{cases} \gamma_{1j}^{(i)} & \text{at} \quad i = 1, ..., K, \\ q_{1j}^{(i-K)} & \text{at} \quad i = K+1, ..., K+L, \end{cases} \tag{3·6}$$

where $K = 2M$ and $L = 2N$. Then the relationship (3·3) is rewritten as follows:

$$\Phi_1(X,\zeta) = 1 - \sum_{i=1}^{K+L}\sum_{j=2}^{3} p_{1j}^{(i)} \frac{\exp[(\mu_{ji} - \mu_{1i})X]}{\mu_{1i} - \zeta}\Phi_1(X, \mu_{ji}). \tag{3·7}$$

By defining

$$\Psi_i(X) = \sum_{j=2}^{3} p_{1j}^{(i)} \exp(\mu_{ji}X)\Phi_1(X, \mu_{ji}), \tag{3·8}$$

we may rewrite the relationship (3·7) as

$$\Phi_1(X,\zeta) = 1 - \sum_{i=1}^{K+L} \frac{\exp(-\mu_{1i}X)}{\mu_{1i} - \zeta}\Psi_i(X). \tag{3·9}$$

On the other hand, by expanding $\Phi_1(X,\zeta)$ as an asymptotic series in $\lambda_1^{-1}(\zeta)$, one can obtain (see Eq. (5.11) in the paper[13])

$$\Phi_1(X,\zeta) = 1 - \frac{1}{3\lambda_1(\zeta)}\left[W(X) - W(-\infty)\right] + O(\lambda_1^{-2}(\zeta)). \tag{3·10}$$

From (3·8) and (3·9), the following key relationship may be found (see also Eq. (6.38) in the paper[14]):

$$W(X) - W(-\infty) = -3\sum_{k=1}^{K+L} \exp(-\mu_{1k}X)\Psi_k(X) = 3\frac{\partial}{\partial X}\ln(\det M). \tag{3·11}$$

Here the matrix $M(X)$ is defined as follows:

$$M_{il}(X) = \delta_{il} - \sum_{j=2}^{3} p_{1j}^{(i)} \frac{\exp[(\mu_{ji} - \mu_{1l})X]}{\mu_{ji} - \mu_{1l}}. \tag{3·12}$$

Now let us consider the T-evolution of the spectral data. By analyzing the solution of Eq. (2·2) when $X \to -\infty$, we find that $\phi_j(X,T,\zeta) = \exp[-(3\lambda_j(\zeta))^{-1}T] \times \phi_j(X,0,\zeta)$. Hence, the T-evolution of the scattering data is given by the relationships (with $i = 1,\ 2, ..., K+L$)

$$\lambda_j(T) = \lambda_j(0), \qquad p_{1j}^{(i)}(T) = p_{1j}^{(i)}(0)\exp\{[-(3\mu_{ji})^{-1} + (3\mu_{1i})^{-1}]T\}. \tag{3·13}$$

Consequently, the final result for the solution of the VPE, when we consider the spectral data from both the bound state spectrum and the continuous spectrum, as well as taking into account their T-evolution, is as follows:

$$U(X,T) = W_X(X,T) = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right). \tag{3·14}$$

Here $M(X,T)$ is the $(K+L)\times(K+L)$ matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^{3} p_{1j}^{(k)} \frac{\exp\{(\mu_{jk}-\mu_{1l})X + [-(3\mu_{jk})^{-1} + (3\mu_{1k})^{-1}]T\}}{\mu_{jk}-\mu_{1l}}, \tag{3·15}$$

where, for $i \le M$,

$$\begin{aligned}
&\mu_{1(2i-1)} = \lambda_1(\zeta_1^{(2i-1)}) = \mathrm{i}\omega_2\xi_i, && \mu_{2(2i-1)} = \lambda_2(\zeta_1^{(2i-1)}) = \mathrm{i}\omega_3\xi_i,\\
&p_{12}^{(2i-1)} = \gamma_{12}^{(2i-1)} = \omega_2\beta_i, && p_{13}^{(2i-1)} = \gamma_{13}^{(2i-1)} = 0,\\
&\mu_{1(2i)} = \lambda_1(\zeta_1^{(2i)}) = -\mathrm{i}\omega_3\xi_i, && \mu_{3(2i)} = \lambda_3(\zeta_1^{(2i)}) = -\mathrm{i}\omega_2\xi_i,\\
&p_{12}^{(2i)} = \gamma_{12}^{(2i)} = 0, && p_{13}^{(2i)} = \gamma_{13}^{(2i)} = \omega_3\beta_i,
\end{aligned} \tag{3·16}$$

and for $M < i \le M+N$,

$$\begin{aligned}
&\mu_{1(2i-1)} = \lambda_1(\zeta'_{2(i-M)-1}) = \omega_2\xi_i, && \mu_{2(2i-1)} = \lambda_2(\zeta'_{2(i-M)-1}) = \omega_3\xi_i,\\
&p_{12}^{(2i-1)} = q_{12}^{(2(i-M)-1)} = \omega_2\beta_i, && p_{13}^{(2i-1)} = q_{13}^{(2(i-M)-1)} = 0,\\
&\mu_{1(2i)} = \lambda_1(\zeta'_{2(i-M)}) = -\omega_3\xi_i, && \mu_{3(2i)} = \lambda_3(\zeta'_{2(i-M)}) = -\omega_2\xi_i,\\
&p_{12}^{(2i)} = q_{12}^{(2(i-M))} = 0, && p_{13}^{(2i)} = q_{13}^{(2(i-M))} = \omega_3\beta_i.
\end{aligned} \tag{3·17}$$

For the solution (3·14), (3·15) there are $(M+N)$ arbitrary constants ξ_i and $(M+N)$ arbitrary constants β_i. The constants ξ_i are real, while the constants β_i, in the general case, are complex.

As will be clear from the examples in the next section, the solution (3·14), (3·15) includes N discrete frequencies from the continuous part of the spectral data. For this reason, the solution (3·14), (3·15), without solitons (i.e. with $M=0$), will be referred to as N-mode solution of the VPE. Evidently these discrete modes emanate from the special choice (3·2) of the singularity functions $Q_{1j}(\zeta')$.

The solution obtained through the matrix (3·15) is in general a complex function. Consequently, there is a problem in selecting the real solutions from the complex solutions. It turns out that we can obtain the real solutions by means of restriction of arbitrariness in the choice of the constants β_i. We have succeeded in finding these restrictions.

§4. Real solutions for the VPE

The goal of this research is to find real functions for $U = W_X$. We analyze a number of examples, as well as the general case, for the interaction of the solitons and multi-mode waves. To obtain the solutions of the VPE, one has to calculate the determinant of matrix (3·15). Firstly, we present four results of such a calculation for

$M+N \leq 4$. For the sake of convenience we will use the auxiliary function $F(X,T)$ given by the definition $F(X,T)=\sqrt{\det M(X,T)}$. In particular, from (3·15),

1) for $M+N=1$ we have

$$F = 1 + c_1 q_1; \tag{4·1}$$

2) for $M+N=2$ we have

$$F = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2; \tag{4·2}$$

3) for $M+N=3$ we have

$$\begin{aligned} F = 1 &+ c_1 q_1 + c_2 q_2 + c_3 q_3 + b_{12} c_1 c_2 q_1 q_2 + b_{13} c_1 c_3 q_1 q_3 + b_{23} c_2 c_3 q_2 q_3 \\ &+ b_{12} b_{13} b_{23} c_1 c_2 c_3 q_1 q_2 q_3; \end{aligned} \tag{4·3}$$

4) for $M+N=4$ we have

$$\begin{aligned} F = 1 &+ c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 + b_{12} c_1 c_2 q_1 q_2 + b_{13} c_1 c_3 q_1 q_3 \\ &+ b_{14} c_1 c_4 q_1 q_4 + b_{23} c_2 c_3 q_2 q_3 + b_{24} c_2 c_4 q_2 q_4 + b_{34} c_3 c_4 q_3 q_4 \\ &+ b_{12} b_{13} b_{23} c_1 c_2 c_3 q_1 q_2 q_3 + b_{12} b_{14} b_{24} c_1 c_2 c_4 q_1 q_2 q_4 \\ &+ b_{13} b_{14} b_{34} c_1 c_3 c_4 q_1 q_3 q_4 + b_{23} b_{24} b_{34} c_2 c_3 c_4 q_2 q_3 q_4 \\ &+ b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} c_1 c_2 c_3 c_4 q_1 q_2 q_3 q_4. \end{aligned} \tag{4·4}$$

For $M+N>4$, the explicit expression for the function $F(X,T)$ can be obtained in a similar manner. It is reasonable to present the quantities c_i, q_i, and b_{ij} involved in the above formulas (4·1)–(4·4) separately for three distinct cases:

1. The purely solitonic case $(i,j) \leq M$ assumes

$$\begin{aligned} &q_i = \exp(2\theta_i) \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i}, \\ &b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i \xi_j}{\xi_i^2 + \xi_j^2 + \xi_i \xi_j}, \quad b_{ij} \geq 0; \end{aligned} \tag{4·5}$$

2. The case of purely multi-mode waves $M < (i,j) \leq M+N$ assumes

$$\begin{aligned} &q_i = \exp(2\theta_i) \quad 2\theta_i = -\mathrm{i}\sqrt{3}\xi_i X + (\mathrm{i}\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{\mathrm{i}\beta_i}{2\sqrt{3}\xi_i}, \\ &b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i \xi_j}{\xi_i^2 + \xi_j^2 + \xi_i \xi_j}, \quad b_{ij} \geq 0; \end{aligned} \tag{4·6}$$

3. The case of a combination of solitons $(i,i') \leq M$ and multi-mode waves $M < (j,j') \leq M+N$ assumes

$$\begin{aligned} &q_i = \exp(2\theta_i) \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i}, \\ &q_j = \exp(2\theta_j) \quad 2\theta_j = -\mathrm{i}\sqrt{3}\xi_j X + (\mathrm{i}\sqrt{3}\xi_j)^{-1} T, \quad c_j = \frac{\mathrm{i}\beta_j}{2\sqrt{3}\xi_j}, \end{aligned}$$

$$b_{ii'} = \left(\frac{\xi_i - \xi_{i'}}{\xi_i + \xi_{i'}}\right)^2 \frac{\xi_i^2 + \xi_{i'}^2 - \xi_i\xi_{i'}}{\xi_i^2 + \xi_{i'}^2 + \xi_i\xi_{i'}}, \quad 0 \le b_{ii'} \le 1,$$

$$b_{jj'} = \left(\frac{\xi_j - \xi_{j'}}{\xi_j + \xi_{j'}}\right)^2 \frac{\xi_j^2 + \xi_{j'}^2 - \xi_j\xi_{j'}}{\xi_j^2 + \xi_{j'}^2 + \xi_j\xi_{j'}}, \quad 0 \le b_{jj'} \le 1,$$

$$b_{ij} = \left(\frac{\xi_i + \mathrm{i}\xi_j}{\xi_i - \mathrm{i}\xi_j}\right)^2 \frac{\xi_i^2 - \xi_j^2 + \mathrm{i}\xi_i\xi_j}{\xi_i^2 - \xi_j^2 - \mathrm{i}\xi_i\xi_j}, \quad |b_{ij}| \equiv 1. \tag{4·7}$$

With the above found representation of the auxiliary function $F(X,T)$ and taking into account the key relationship (3·11), we can write the explicit solution to the basic nonlinear evolution equation (1·1) in the following concise form:

$$W(X,T) = 6\frac{\partial}{\partial X}\ln(F(X,T)) + \text{const.} \tag{4·8}$$

The function F is complex-valued in the general case because the values of β_i (and hence of c_i) are complex constants.

Since we are interested only in the real solution W_X under real constants ξ_i, we need the restrictions on the constants c_i in (4·1)–(4·4).

4.1. *The solutions associated with the continuous spectrum*

We study the multi-mode solutions for $M = 0$ and $N = 1, 2, 3, 4$, while for $N \ge 5$ all formulas can easily be obtained by means of a generalization of these examples.

4.1.1. *The one-mode solution*

In order to obtain the one-mode solution of the VPE (1·1) we need first to calculate the 2×2 matrix $M(X,T)$ according to (3·15) with $M = 0$ and $N = 1$. For the matrix elements $M_{kl}(X,T)$ we have

$$M_{11}(X,T) = 1 - \frac{\mathrm{i}\omega_2\beta_1}{\sqrt{3}\xi_1}\exp[-\mathrm{i}\sqrt{3}\xi_1 X + (\mathrm{i}\sqrt{3}\xi_1)^{-1}T],$$

$$M_{12}(X,T) = -\frac{\omega_3\beta_1}{2\xi_1}\exp[2\omega_3\xi_1 X + (\mathrm{i}\sqrt{3}\xi_1)^{-1}T],$$

$$M_{21}(X,T) = \frac{\omega_2\beta_1}{2\xi_1}\exp[-2\omega_2\xi_1 X + (\mathrm{i}\sqrt{3}\xi_1)^{-1}T],$$

$$M_{22}(X,T) = 1 - \frac{\mathrm{i}\omega_3\beta_1}{\sqrt{3}\xi_1}\exp[-\mathrm{i}\sqrt{3}\xi_1 X + (\mathrm{i}\sqrt{3}\xi_1)^{-1}T], \tag{4·9}$$

so that the respective determinant is

$$\det M(X,T) = \left[1 + c_1\exp(-\mathrm{i}\sqrt{3}\xi_1 X + (\mathrm{i}\sqrt{3}\xi_1)^{-1}T)\right]^2, \quad c_1 = \frac{\mathrm{i}\beta_1}{2\sqrt{3}\xi_1}. \tag{4·10}$$

As has been noted already, the singularity functions in the form (3·2) with $N = 1$ give rise to a single frequency for the continuous part of the spectral data. Hence, the expression (4·10), having been substituted into the concise formula (4·8), must provide us with the one-mode solution.

The condition that W_X is real requires a restriction on the constant β_1 (if the constant ξ_1 is arbitrary but real). We have succeeded in obtaining this restriction (see Appendix A), namely that the constant c_1, which in general is the complex-valued one $c_1 = |c_1| \exp(\mathrm{i}\chi_1)$, should possess the unity modulus $|c_1| = 1$, while the arbitrary real constant χ_1 defines an initial shift of solution $X_1 = \chi_1/(\sqrt{3}\xi_1)$ so that

$$\det M(X,T) = \left[1 + \exp\left(-\mathrm{i}\sqrt{3}\xi_1(X - X_1) + \frac{T}{\mathrm{i}\sqrt{3}\xi_1}\right)\right]^2. \qquad (4{\cdot}11)$$

The final result for one mode of the continuous spectrum is the solution (4·8) with (4·11), namely,

$$W(X,T) = -3\sqrt{3}\,\xi_1 \tan\left(\frac{\sqrt{3}}{2}\xi_1(X - X_1) + \frac{T}{2\sqrt{3}\xi_1}\right) + \text{const.} \qquad (4{\cdot}12)$$

The corresponding solution for $U = W_X$ (with U governed by (1·4)) was obtained recently by other methods, for example, by the sine-cosine method,[16] the (G'/G)-expansion method,[12] and the extended tanh-function method.[16),17] However, only the approach developed here and the solution in the form (3·15) enable us to study the interaction of solitons and periodic waves.

We obtain the periodic solutions even for $N = 1$. Let us call attention once again to the condition (2·6) which in the final result is shown below to restrict the region X for periodic solutions. At first glance it would seem that there is a contradiction between the condition (2·6) and the periodic solution. Indeed, on the one hand, the condition (2·6) demands that the solution $W(X,T)$ should vanish as $X \to -\infty$; on the other hand, the periodic solution obtained here does not satisfy the condition (2·6). Nevertheless, consideration of the details enables us to find a reasonable explanation. So, in the paper[14] at the derivation of the relation (2·7) (see also (4.5) in the paper[14]), the integral in (2·7) appears as a result of the integration on two sides of the boundaries between regular regions. For an understanding of this fact, the relationship (3.14) from the paper[14] plays an important role. Hence, the integration in (2·7) (also as in (4.5) in the paper[14]) should be carried out over the lines $\omega_2(\xi + \mathrm{i}\varepsilon)$ and $-\omega_3(\xi + \mathrm{i}\varepsilon)$ as ξ sweeps from $-\infty$ to $+\infty$, where $\varepsilon > 0$. As a result, in the relationship (4·11) we should exchange ξ_1 by $(\xi_1 + \mathrm{i}\varepsilon)$ and that enables us to define the solution in the form

$$W(X,T) = -\mathrm{i}6\sqrt{3}\,(\xi_1 + \mathrm{i}\varepsilon)\frac{\exp(\sqrt{3}\varepsilon X)\exp\left(-\mathrm{i}\sqrt{3}\xi_1(X - X_1) + \frac{T}{\mathrm{i}\sqrt{3}\xi_1}\right)}{1 + \exp(\sqrt{3}\varepsilon X)\exp\left(-\mathrm{i}\sqrt{3}\xi_1(X - X_1) + \frac{T}{\mathrm{i}\sqrt{3}\xi_1}\right)}, \qquad (4{\cdot}13)$$

which tends to constants as $|X| \to \infty$ at arbitrary $\varepsilon > 0$. Thus, on the one hand, the condition (2·6) is satisfied, and, on the other hand, at small $\varepsilon > 0$ we have a sufficiently large region over X where the solution associated with a finite $\varepsilon > 0$ and the periodic solution associated with $\varepsilon = 0$ are sufficiently close to each other. The region X with periodic solutions can be extended to sufficiently large, but finite, $|X|$. For any sequence $\varepsilon_n \to 0$ we remain within the inverse scattering theory[14] where the condition (2·6) is not violated. Consequently, periodic solution obtained at $\varepsilon = 0$ is to be interpreted as the solution of Eq. (1·1) which is valid on arbitrary but finite $|X|$.

4.1.2. *The two-mode solution*

Let us consider a two-mode solution of the VPE for which $M = 0$ and $N = 2$. In this case $M(X,T)$ is a 4×4 matrix. According to (4·2) we find that

$$\sqrt{\det M(X,T)} = F(X,T) = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2, \tag{4·14}$$

where q_i, c_i, and b_{12} are defined by (4·6).

Since the solution W_X should be real and the constants ξ_i are arbitrary, but real, there are restrictions on the constants $c_i = |c_i|\exp(\mathrm{i}\chi_i)$. The real constants χ_i define the initial shifts of solutions $X_i = \chi_i/(\sqrt{3}\xi_i)$. The analysis in considerable detail shows (see Appendix A) that the relations $|c_1| = |c_2| = 1/\sqrt{b_{12}}$ are the sufficient conditions in order that W_X be real. Thus, the interaction of two periodic waves for the VPE is described by the relationship (4·8) with

$$F(X,T) = 1 + \frac{1}{\sqrt{b_{12}}}q_1 + \frac{1}{\sqrt{b_{12}}}q_2 + q_1q_2, \tag{4·15}$$

where b_{12} is as in (4·6), and the quantities q_i now contain the phaseshifts $X_i = \chi_i/(\sqrt{3}\xi_i)$ as follows:

$$q_i = \exp(\mathrm{i}2\theta_i), \quad 2\theta_i = -\sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T. \tag{4·16}$$

In explicit form, the two-mode solution is as follows:

$$W(X,T) = -3\sqrt{3}\,\frac{(\xi_1+\xi_2)\sin(\theta_1+\theta_2) + b_{12}^{-1/2}(\xi_1-\xi_2)\sin(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2) + b_{12}^{-1/2}\cos(\theta_1-\theta_2)}. \tag{4·17}$$

4.1.3. *The three-mode solution*

For $N = 3$ and $M = 0$ in the relationship

$$\begin{aligned} F(X,T) = 1 &+ c_1q_1 + c_2q_2 + c_3q_3 + c_1c_2b_{12}q_1q_2 + c_1c_3b_{13}q_1q_3 \\ &+ c_2c_3b_{23}q_2q_3 + c_1c_2c_3b_{12}b_{13}b_{23}q_1q_2q_3 \end{aligned} \tag{4·18}$$

obtained from (3·15) (see, also (4·3)) with q_i, c_i, and b_{12} as in (4·6), we write $c_i = |c_i|\exp(\mathrm{i}\chi_i)$. Then the arguments χ_i determine the initial phaseshifts of modes $X_i = \chi_i/(\sqrt{3}\xi_i)$. As is proved in Appendix A, the conditions on the constants c_i (or the same on β_i) are

$$|c_1| = 1/\sqrt{b_{12}b_{13}}, \qquad |c_2| = 1/\sqrt{b_{12}b_{23}}, \qquad |c_3| = 1/\sqrt{b_{13}b_{23}}. \tag{4·19}$$

Hence, the three-mode solution is the relation (4·8) with

$$\begin{aligned} F(X,T) = 1 &+ \frac{1}{\sqrt{b_{12}b_{13}}}(q_1 + q_2q_3) + \frac{1}{\sqrt{b_{12}b_{23}}}(q_2 + q_1q_3) \\ &+ \frac{1}{\sqrt{b_{13}b_{23}}}(q_3 + q_1q_2) + q_1q_2q_3. \end{aligned} \tag{4·20}$$

Here the phaseshifts X_i are taken into account in q_i by way of (4·16).

4.1.4. *The four-mode solution*

For $N = 4$ and $M = 0$ the restrictions have the form (see Appendix A)

$$|c_i| = \prod_{\substack{j=1 \\ j\neq i}}^{4} b_{ij}^{-\frac{1}{2}}, \quad 0 \le b_{ij} = b_{ji} \le 1, \quad i = 1, 2, 3, 4. \tag{4·21}$$

The function F for a real solution (4·8) is as follows:

$$\begin{aligned} F(X,T) = 1 &+ \frac{1}{\sqrt{b_{12}b_{13}b_{14}}}(q_1 + q_2q_3q_4) + \frac{1}{\sqrt{b_{12}b_{23}b_{24}}}(q_2 + q_1q_3q_4) \\ &+ \frac{1}{\sqrt{b_{13}b_{23}b_{34}}}(q_3 + q_1q_2q_4) + \frac{1}{\sqrt{b_{14}b_{24}b_{34}}}(q_4 + q_1q_2q_3) \\ &+ \frac{1}{\sqrt{b_{13}b_{14}b_{23}b_{24}}}(q_1q_2 + q_3q_4) + \frac{1}{\sqrt{b_{12}b_{14}b_{23}b_{34}}}(q_1q_3 + q_2q_4) \\ &+ \frac{1}{\sqrt{b_{12}b_{13}b_{24}b_{34}}}(q_1q_4 + q_2q_3) + q_1q_2q_3q_4. \end{aligned} \tag{4·22}$$

As before, the b_{ij} and q_i are defined by (4·6) and (4·16), respectively.

4.2. *The solutions associated with the bound state spectrum*

The features of the solutions associated with bound state spectrum can be shown by considering the two-soliton solution for which $M = 2$, $N = 0$. The solution (4·8) can be obtained through (4·2), namely

$$F(X,T) = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2 \tag{4·23}$$

with (4·5), namely

$$\begin{aligned} &q_i = \exp(2\theta_i) \quad 2\theta_i = \sqrt{3}\xi_iX - (\sqrt{3}\xi_i)^{-1}T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i}, \\ &b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}, \quad b_{ij} \ge 0. \end{aligned} \tag{4·24}$$

In Appendix B it is proved that the constants c_i can be only real ones. Moreover, the signs of $\alpha_i = c_i/|c_i|$ can independently take the values ± 1, i.e. we have four variants, namely $\alpha_1 = \alpha_2 = 1$, $\alpha_1 = \alpha_2 = -1$, $\alpha_1 = -\alpha_2 = 1$ and $\alpha_1 = -\alpha_2 = -1$. Note that in the paper[2)] only the first two variants are observed. The standard soliton solution for which $\alpha_1 = \alpha_2 = 1$ and the singular soliton solutions for which $\alpha_1 = \alpha_2 = -1$, $\alpha_1 = -\alpha_2 = 1$ and $\alpha_1 = -\alpha_2 = -1$, are obtained by means of the relation (4·8)

$$U(X,T) = W(X,T)_X = 6\frac{\partial^2}{\partial X^2}\ln(F) = 6\frac{\partial^2}{\partial X^2}\ln(G_i), \tag{4·25}$$

where the G_i are defined by (B·6)–(B·9).

The forms (B·3), (B·6)–(B·9) for F are more preferable, since we see that the solution is dependent on two combinations of the spectral parameters $\xi_1+\xi_2$ and $\xi_1-\xi_2$, but not three values ξ_1, ξ_2, and $\xi_1+\xi_2$ as it may appear from the relation (4·23).

For $N \ge 3$ we give the conditions without proof. All the constants c_i are to be real and the signs of $\alpha_i = c_i/|c_i|$ can equal to ± 1 independently of each other.

4.3. *Real soliton and multi-mode solutions of the VPE*

In this subsection we will consider the general case, when both the bound state spectrum and the continuous spectrum are taken into account in the associated spectral problem. We will find the conditions on c_i for real solutions of the VPE. To obtain the solution, we need to know the function F (see (4·1)–(4·4)).

Let the indexes i and i' be related to the values involved in the bound state spectrum for which $(i,\, i') \leq M$, while the indexes j and j' are related to the values involved in the continuous part of the spectral data for which $M < (j, j') \leq M + N$.

4.3.1. *The interaction of a soliton with one-mode wave*

The interaction of a standard soliton with periodic one-mode wave can be described by means of the relations (4·2)

$$F(X,T) = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2 \tag{4·26}$$

with q_i and b_{12} as in (4·7), namely

$$\begin{aligned} q_1 &= \exp(\sqrt{3}\xi_1 X - (\sqrt{3}\xi_1)^{-1} T), \quad c_1 = \frac{\beta_1}{2\sqrt{3}\xi_1}, \\ q_2 &= \exp(-\mathrm{i}\sqrt{3}\xi_2 X + (\mathrm{i}\sqrt{3}\xi_2)^{-1} T), \quad c_2 = \frac{\mathrm{i}\beta_2}{2\sqrt{3}\xi_2}, \\ b_{12} &= \left(\frac{\xi_1 + \mathrm{i}\xi_2}{\xi_1 - \mathrm{i}\xi_2}\right)^2 \frac{\xi_1^2 - \xi_2^2 + \mathrm{i}\xi_1\xi_2}{\xi_1^2 - \xi_2^2 - \mathrm{i}\xi_1\xi_2}, \quad |b_{12}| \equiv 1. \end{aligned} \tag{4·27}$$

First, we emphasize that the soliton and one-mode wave (4·12) propagate in opposite directions. The soliton propagates in the positive direction of the X-axis, while the one-mode wave (4·12) propagates in the negative direction of the X-axis.

Here we restrict ourselves to the simplest case $b_{12}c_1c_2 = 1$ that describes the interaction of a standard soliton with a one-mode wave. As follows immediately from Appendix C, for real solutions (4·8),

$$W(X,T) = 6\frac{\partial}{\partial X}\ln(F(X,T)) + \text{const},$$

where $F(X,T)$ is

$$F(X,T) = 1 + \frac{1}{\sqrt{b_{12}}} q_1 + \frac{1}{\sqrt{b_{12}}} q_2 + q_1 q_2. \tag{4·28}$$

There is an exceptional case at $\xi_1 = \xi_2$. Then we have $b_{12} = 1$, and $F = (1+q_1)(1+q_2)$. Consequently, the solution (4·8) is reduced to the relation

$$\begin{aligned} W = W_1 + W_2 = 3\sqrt{3}\xi_1 \tanh&\left(\frac{\sqrt{3}}{2}\xi_1(X - X_1) - \frac{T}{2\sqrt{3}\xi_1}\right) \\ &-3\sqrt{3}\,\xi_1 \tan\left(\frac{\sqrt{3}}{2}\xi_1(X - X_0) + \frac{T}{2\sqrt{3}\xi_1}\right) + \text{const}. \end{aligned} \tag{4·29}$$

Here W_1 is the one-soliton solution and W_2 is the solution (4·12) associated with one mode in the continuous part of the spectral data. The relationship $W = W_1 + W_2$

is easily verified also by direct substitution into Eq. (1·1). The two waves W_1 and W_2 propagate in different directions with the same speed without change of wave profile and phaseshift. In other words, only in the case $\xi_1 = \xi_2$ is there a simple superposition of the solutions W_1 and W_2. It is obvious that interactions of the two solitons with a one-mode wave and/or of the two-mode solution with one soliton do not satisfy this form of the interaction.

4.3.2. *Real solutions for M solitons and the N-mode wave*

The interaction of M solitons and the N-mode wave (4·8) can be obtained by means of the function $F(X,T)$ with restrictions (C·7) given in Appendix C, namely

$$c_i = \pm 1 \Bigg/ \sqrt{\prod_{\substack{j=1 \\ j\neq i}}^{M+N} b_{ij}}\ , \quad b_{ij} = b_{ji}, \quad i = 1, ..., M+N, \tag{4·30}$$

and with the retention of the phaseshifts X_i in the quantities q_i (C·2). The signs for c_i in (4·30) can be chosen independently of each other. If the index i in (4·30) is connected with the continuous part of the spectral data ($M < i \leq M+N$), then the solutions generated by 'plus' and 'minus' signs in (4·30) are different only in the phaseshifts. However, for the index i from the bound state spectrum ($i \leq M$), the solutions have different forms of function dependencies. Here it is relevant to remember that there are standard soliton solutions and singular soliton solutions generated by different signs in the constants c_i (4·30).

The solution will contain $(M+N)$ real constants ξ_i for determining the values b_{ij} and $(M+N)$ real constants X_i to define the phaseshifts.

§5. Conclusion

The procedure for finding the solutions of the Vakhnenko-Parkes equation by means of the inverse scattering method is described. Both the bound state spectrum and the continuous spectrum are taken into account in the associated eigenvalue problem. The special form of the singularity functions enables us to obtain the multi-mode solutions. Sufficient conditions have been proved in order that the solutions become real functions. Finally we studied the interaction of solitons and the multi-mode wave.

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Appendix A

—— *The Conditions on the Constants c_i for Multi-Mode Waves* ——

In this appendix we will prove the conditions on the constants $c_i = |c_i| \exp(\mathrm{i}\chi_i)$

for solutions associated with the continuous part of the spectral data only. We use the case $N = 4$ as an example to prove the restrictions on the constants, at which the solution $W_X(X,T)$ is real. The auxiliary function $F(X,T) = \sqrt{\det M(X,T)}$ for finding the solution is (4·4), namely

$$\begin{aligned} F(X,T) = 1 &+ c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 + c_1 c_2 b_{12} q_1 q_2 + c_1 c_3 b_{13} q_1 q_3 \\ &+ c_1 c_4 b_{14} q_1 q_4 + c_2 c_3 b_{23} q_2 q_3 + c_2 c_4 b_{24} q_2 q_4 + c_3 c_4 b_{34} q_3 q_4 \\ &+ c_1 c_2 c_3 b_{12} b_{13} b_{23} q_1 q_2 q_3 + c_1 c_2 c_4 b_{12} b_{14} b_{24} q_1 q_2 q_4 \\ &+ c_1 c_3 c_4 b_{13} b_{14} b_{34} q_1 q_3 q_4 + c_2 c_3 c_4 b_{23} b_{24} b_{34} q_2 q_3 q_4 \\ &+ c_1 c_2 c_3 c_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} q_1 q_2 q_3 q_4. \end{aligned} \qquad (A·1)$$

Here we redefine the values c_i in such a way that $c_i = |c_i|$, since the arguments χ_i can always be introduced into the variables $q_i = \exp(\mathrm{i}2\theta_i)$ with $2\theta_i = -\sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T$ and $X_i = \chi_i/(\sqrt{3}\xi_i)$ serving as the shifts of solutions. The solution then has the form (4·8)

$$W(X,T) = 6\frac{\partial}{\partial X}\ln(F(X,T)) + \text{const.} \qquad (A·2)$$

The function F is complex-valued, i.e.

$$\begin{aligned} &F = F_{Re} + \mathrm{i}F_{Im} = |F|\exp(\mathrm{i}\chi_F), \quad F_{Re} = \mathrm{Re}(F), \quad F_{Im} = \mathrm{Im}(F), \\ &\tan(\chi_F) = F_{Im}/F_{Re}, \end{aligned} \qquad (A·3)$$

hence,

$$W(X,T)/6 = \frac{\partial}{\partial X}\ln(|F|) + \mathrm{i}\frac{\partial \chi_F}{\partial X} + \text{const.} \qquad (A·4)$$

If we succeed in making $\partial^2\chi_F/\partial X^2 \equiv 0$ by the choice of the constants c_i, then the solution $W_X(X,T)$ will be a real function.

Let us write F_{Im} and F_{Re} in explicit forms, namely

$$\begin{aligned} F_{Im} = &\; c_1\sin(2\theta_1) + c_2\sin(2\theta_2) + c_3\sin(2\theta_3) + c_4\sin(2\theta_4) \\ &+ c_1 c_2 b_{12}\sin[2(\theta_1+\theta_2)] + c_1 c_3 b_{13}\sin[2(\theta_1+\theta_3)] \\ &+ c_1 c_4 b_{14}\sin[2(\theta_1+\theta_4)] + c_2 c_3 b_{23}\sin[2(\theta_2+\theta_3)] \\ &+ c_2 c_4 b_{24}\sin[2(\theta_2+\theta_4)] + c_3 c_4 b_{34}\sin[2(\theta_3+\theta_4)] \\ &+ c_1 c_2 c_3 b_{12} b_{13} b_{23}\sin[2(\theta_1+\theta_2+\theta_3)] \\ &+ c_1 c_2 c_4 b_{12} b_{14} b_{24}\sin[2(\theta_1+\theta_2+\theta_4)] \\ &+ c_1 c_3 c_4 b_{13} b_{14} b_{34}\sin[2(\theta_1+\theta_3+\theta_4)] \\ &+ c_2 c_3 c_4 b_{23} b_{24} b_{34}\sin[2(\theta_2+\theta_3+\theta_4)] \\ &+ c_1 c_2 c_3 c_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34}\sin[2(\theta_1+\theta_2+\theta_3+\theta_4)], \end{aligned} \qquad (A·5)$$

$$\begin{aligned} F_{Re} = 1 &+ c_1\cos(2\theta_1) + c_2\cos(2\theta_2) + c_3\cos(2\theta_3) + c_4\cos(2\theta_4) \\ &+ c_1 c_2 b_{12}\cos[2(\theta_1+\theta_2)] + c_1 c_3 b_{13}\cos[2(\theta_1+\theta_3)] \\ &+ c_1 c_4 b_{14}\cos[2(\theta_1+\theta_4)] + c_2 c_3 b_{23}\cos[2(\theta_2+\theta_3)] \end{aligned}$$

$$
\begin{aligned}
&+c_2c_4b_{24}\cos[2(\theta_2+\theta_4)]+c_3c_4b_{34}\cos[2(\theta_3+\theta_4)]\\
&+c_1c_2c_3b_{12}b_{13}b_{23}\cos[2(\theta_1+\theta_2+\theta_3)]\\
&+c_1c_2c_4b_{12}b_{14}b_{24}\cos[2(\theta_1+\theta_2+\theta_4)]\\
&+c_1c_3c_4b_{13}b_{14}b_{34}\cos[2(\theta_1+\theta_3+\theta_4)]\\
&+c_2c_3c_4b_{23}b_{24}b_{34}\cos[2(\theta_2+\theta_3+\theta_4)]\\
&+c_1c_2c_3c_4b_{12}b_{13}b_{14}b_{23}b_{24}b_{34}\cos[2(\theta_1+\theta_2+\theta_3+\theta_4)].
\end{aligned}
\tag{A·6}
$$

Let us try to present F_{Im} and F_{Re} in the forms

$$F_{Im}=2G\sin(\theta_1+\theta_2+\theta_3+\theta_4) \tag{A·7}$$

and

$$F_{Re}=2G\cos(\theta_1+\theta_2+\theta_3+\theta_4), \tag{A·8}$$

where G is the same in both above formulas (A·7) and (A·8). This can be done, if the following conditions are satisfied:

$$
\begin{aligned}
&c_1=c_2c_3c_4b_{23}b_{24}b_{34},\quad c_2=c_1c_3c_4b_{13}b_{14}b_{34},\quad c_3=c_1c_2c_4b_{12}b_{14}b_{24},\\
&c_4=c_1c_2c_3b_{12}b_{13}b_{23},\quad c_1c_2b_{12}=c_3c_4b_{34},\quad c_1c_3b_{13}=c_2c_4b_{24},\\
&c_1c_4b_{14}=c_2c_3b_{23},\quad c_1c_2c_3c_4b_{12}b_{13}b_{14}b_{23}b_{24}b_{34}=1.
\end{aligned}
\tag{A·9}
$$

It turns out that all these relations are valid when

$$c_1=\frac{1}{\sqrt{b_{12}b_{13}b_{14}}},\quad c_2=\frac{1}{\sqrt{b_{12}b_{23}b_{24}}},\quad c_3=\frac{1}{\sqrt{b_{13}b_{23}b_{34}}},\quad c_4=\frac{1}{\sqrt{b_{14}b_{24}b_{34}}}. \tag{A·10}$$

At these conditions (A·10) the expression for G reads as follows:

$$
\begin{aligned}
G=\;&\cos(\theta_1+\theta_2+\theta_3+\theta_4)+\frac{1}{\sqrt{b_{12}b_{13}b_{14}}}\cos(\theta_1-\theta_2-\theta_3-\theta_4)\\
&+\frac{1}{\sqrt{b_{12}b_{23}b_{24}}}\cos(\theta_2-\theta_1-\theta_3-\theta_4)+\frac{1}{\sqrt{b_{13}b_{23}b_{34}}}\cos(\theta_3-\theta_1-\theta_2-\theta_4)\\
&+\frac{1}{\sqrt{b_{14}b_{24}b_{34}}}\cos(\theta_4-\theta_1-\theta_2-\theta_3)+\frac{1}{\sqrt{b_{13}b_{14}b_{23}b_{24}}}\cos(\theta_1+\theta_2-\theta_3-\theta_4)\\
&+\frac{1}{\sqrt{b_{12}b_{14}b_{23}b_{34}}}\cos(\theta_1+\theta_3-\theta_2-\theta_4)+\frac{1}{\sqrt{b_{12}b_{13}b_{24}b_{34}}}\cos(\theta_1+\theta_4-\theta_2-\theta_3).
\end{aligned}
\tag{A·11}
$$

Now it is readily seen from (A·3) that

$$\chi_F=\theta_1+\theta_2+\theta_3+\theta_4 \tag{A·12}$$

and as consequence we have

$$\frac{\partial^2\chi_F}{\partial X^2}=\frac{\partial^2\chi_F}{\partial X\partial T}=0. \tag{A·13}$$

Hence, as follows from (A·4), the four-mode solution of the VPE can be reduced to real form with four real constants X_i and four real constants ξ_i (see (4·22)).

Without proof here we give the following conditions on the constants c_i that ensure the real N-mode solution of the VPE:

$$|c_i| = \prod_{\substack{j=1 \\ j\neq i}}^{N} b_{ij}^{-\frac{1}{2}}, \quad b_{ij} = b_{ji}, \quad i = 1, ..., N, \tag{A·14}$$

where the N constants ξ_i determine the values b_{ij} and the N constants X_i define the phaseshifts for each mode. Note that these relations (A·14) are sufficient conditions, but not necessary ones.

Appendix B
—— *The Conditions on the Constants c_i under the Interaction of Two Solitons* ——

Here we consider the conditions on signs for the constants c_i under the interaction of two solitons ($M = 2$, $N = 0$). We start with the relationship (4·2) and (4·5)

$$F = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2. \tag{B·1}$$

Let us present the constants c_i in the form

$$c_i = \alpha_i |c_i| \exp(\mathrm{i}\chi_i) = b_{12}^{-1/2} \exp\left(-\sqrt{3}\xi_i X_i + \mathrm{i}\sigma_i\right), \quad \sigma_i = \chi_i + \pi(1-\alpha_i)/2. \tag{B·2}$$

All new constants χ_i and $X_i = -\ln(|c_i\sqrt{b_{12}}|)/(\sqrt{3}\xi_i)$ are real. We assume that $-\pi/2 < \chi_i \le \pi/2$, then the values α_i retain the signs of the constants $\mathrm{Re}(c_i)$, i.e. $\alpha_i = \mathrm{Re}(c_i)/|\mathrm{Re}(c_i)|$. It is convenient for analyzing to rewrite (B·1) (the same as (4·2)) in the form

$$F = 2\exp\left(\theta_1 + \theta_2 + \frac{\mathrm{i}}{2}(\sigma_1 + \sigma_2)\right) G \tag{B·3}$$

with

$$G = \cosh\left(\theta_1 + \theta_2 + \frac{\mathrm{i}}{2}(\sigma_1 + \sigma_2)\right) + b_{12}^{-1/2} \cosh\left(\theta_1 - \theta_2 + \frac{\mathrm{i}}{2}(\sigma_1 - \sigma_2)\right),$$
$$2\theta_i = \sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1} T. \tag{B·4}$$

It is easily seen that only G defines the solution, since $\frac{\partial^2}{\partial X^2}\ln(F) = \frac{\partial^2}{\partial X^2}\ln(G)$, while the conditions that the function G is real are as follows:

$$\chi_i = 0, \quad \sigma_i + \sigma_2 = 2\pi k_1, \quad \sigma_i - \sigma_2 = 2\pi k_2 \tag{B·5}$$

with $k_i = 0,\ 1$. These restrictions (B·5) lead to the requirements $\alpha_1 = \pm 1$, $\alpha_2 = \pm 1$, independently of each other, and $\chi_i = 0$. Then the function F has the following forms:

1) for $\alpha_1 = \alpha_2 = 1$,

$$F = 2\exp(\theta_1 + \theta_2)\, G_1, \quad G_1 = \cosh(\theta_1 + \theta_2) + b_{12}^{-1/2}\cosh(\theta_1 - \theta_2)\,; \tag{B·6}$$

2) for $\alpha_1 = \alpha_2 = -1$,

$$F = 2\exp(\theta_1 + \theta_2)\, G_2, \quad G_2 = \cosh(\theta_1 + \theta_2) - b_{12}^{-1/2}\cosh(\theta_1 - \theta_2)\,; \tag{B·7}$$

3) for $\alpha_1 = -\alpha_2 = 1$,

$$F = 2\exp(\theta_1 + \theta_2)\, G_3, \quad G_3 = -\sinh(\theta_1 + \theta_2) + b_{12}^{-1/2}\sinh(\theta_1 - \theta_2)\,; \tag{B·8}$$

4) for $\alpha_1 = -\alpha_2 = -1$,

$$F = 2\exp(\theta_1 + \theta_2)\, G_4, \quad G_4 = -\sinh(\theta_1 + \theta_2) - b_{12}^{-1/2}\sinh(\theta_1 - \theta_2)\,. \tag{B·9}$$

Hence, the standard soliton solution that follows from (B·6) and the singular soliton solutions that follow from (B·7)–(B·9) are the real functions

$$U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(G_i). \tag{B·10}$$

Now we rewrite the restrictions in somewhat different form. By retaining the values of the phaseshifts X_i in the quantities q_i, we require

$$c_1 = \pm\sqrt{b_{12}}, \quad c_2 = \pm\sqrt{b_{12}}, \tag{B·11}$$

where the signs are independent of each other. Note that for this case there are two arbitrary real constants ξ_i, and two arbitrary real constants X_i ($i = 1, 2$).

The notation in (B·6)–(B·9) shows that the solution is defined by two combinations of the spectral parameters, namely $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$, but not three values ξ_1, ξ_2, $\xi_1 + \xi_2$ as it may appear from (B·1).

The foregoing proof points to a way for finding the restrictions for any M with $N = 0$. Here it should be underlined that only at real c_i with any sign of $\alpha_i = c_i/|c_i|$, the soliton (or singular soliton) solutions are determined by a real function. The conditions on the constants c_i are as follows:

$$c_i = \pm 1 \Bigg/ \sqrt{\prod_{\substack{j=1 \\ j\neq i}}^{M} b_{12}}\,, \quad i = 1, \ldots, M, \tag{B·12}$$

with the retention of the phaseshifts X_i in the quantities q_i. The signs for c_i are independent of each other. The solution will contain the M real constants ξ_i for determining the values b_{ij} and the M real constants X_i to define the phaseshifts.

Appendix C

—— The Restrictions on the Constants c_i in the General Case ——

In this appendix we will obtain the restrictions on the constants c_i for real solutions, in the general case, taking into account the spectral data from both the bound state spectrum and the continuous spectrum. All features are inherent in the case $M+N=4$ considered here as an example. To find the solution by means of the inverse scattering method, one needs to know the function (4·4)

$$\begin{aligned} F = {} & 1 + c_1q_1 + c_2q_2 + c_3q_3 + c_4q_4 + b_{12}c_1c_2q_1q_2 + b_{13}c_1c_3q_1q_3 \\ & + b_{14}c_1c_4q_1q_4 + b_{23}c_2c_3q_2q_3 + b_{24}c_2c_4q_2q_4 + b_{34}c_3c_4q_3q_4 \\ & + b_{12}b_{13}b_{23}c_1c_2c_3q_1q_2q_3 + b_{12}b_{14}b_{24}c_1c_2c_4q_1q_2q_4 \\ & + b_{13}b_{14}b_{34}c_1c_3c_4q_1q_3q_4 + b_{23}b_{24}b_{34}c_2c_3c_4q_2q_3q_4 \\ & + b_{12}b_{13}b_{14}b_{23}b_{24}b_{34}c_1c_2c_3c_4q_1q_2q_3q_4. \end{aligned} \tag{C·1}$$

For convenience we rewrite the variables q_i in the somewhat different form

$$\begin{aligned} & q_i = \exp(2\theta_i), \quad q_j = \exp(\mathrm{i}2\theta_j), \quad 2\theta_i = \sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T, \\ & 2\theta_j = -\sqrt{3}\xi_j(X - X_j) - (\sqrt{3}\xi_j)^{-1}T. \end{aligned} \tag{C·2}$$

The phaseshifts X_i are the arbitrary real constants. The values b_{ij} in (C·1) are as in (4·7)

$$\begin{aligned} b_{ii'} &= \left(\frac{\xi_i - \xi_{i'}}{\xi_i + \xi_{i'}}\right)^2 \frac{\xi_i^2 + \xi_{i'}^2 - \xi_i\xi_{i'}}{\xi_i^2 + \xi_{i'}^2 + \xi_i\xi_{i'}}, \quad 0 \le b_{ii'} \le 1, \\ b_{jj'} &= \left(\frac{\xi_j - \xi_{j'}}{\xi_j + \xi_{j'}}\right)^2 \frac{\xi_j^2 + \xi_{j'}^2 - \xi_j\xi_{j'}}{\xi_j^2 + \xi_{j'}^2 + \xi_j\xi_{j'}}, \quad 0 \le b_{jj'} \le 1, \\ b_{ij} &= \left(\frac{\xi_i + \mathrm{i}\xi_j}{\xi_i - \mathrm{i}\xi_j}\right)^2 \frac{\xi_i^2 - \xi_j^2 + \mathrm{i}\xi_i\xi_j}{\xi_i^2 - \xi_j^2 - \mathrm{i}\xi_i\xi_j}, \quad |b_{ij}| \equiv 1, \end{aligned} \tag{C·3}$$

where $(i,\, i') \le M$, and $M < (j,\, j') \le M+N$. Note that $b_{ii'}$ and $b_{jj'}$ are real values, and $b_{ij}^* = 1/b_{ij}$.

Without loss of generality, we will consider one set of values M and N, for example $M=1$ and $N=3$. Now we will show that the restrictions (A·10)

$$\begin{aligned} & c_1 = \pm 1/\sqrt{b_{12}b_{13}b_{14}}, \quad c_2 = \pm 1/\sqrt{b_{12}b_{23}b_{24}}, \\ & c_3 = \pm 1/\sqrt{b_{13}b_{23}b_{34}}, \quad c_4 = \pm 1/\sqrt{b_{14}b_{24}b_{34}} \end{aligned} \tag{C·4}$$

(with b_{ij} determined by (C·3)) are sufficient in order to obtain the real solutions.

For definiteness, we assume that $\sqrt{b_{ij}}$ is a root of an equation $x^2 = b_{ij}$ with $-\pi/2 < \arg\sqrt{b_{ij}} \le \pi/2$. Let us rewrite the relations (C·4) in the form $c_i = \alpha_i/\prod_{\substack{j=1\\ j\ne i}}^{4}\sqrt{b_{ij}}$, where $\alpha_i = \pm 1$. It is evident that we can always attain $\alpha_2 = \alpha_3 = \alpha_4 = 1$ by choosing the phaseshifts X_2, X_3, and X_4, while we need to consider the two cases $\alpha_1 = \pm 1$. By defining $\sigma = (1-\alpha_1)/2$, we can rewrite the auxiliary function F from (C·1) in the form

$$F(X,T) = 2G\mathrm{e}^{\mathrm{i}\pi\sigma}(b_{12}b_{13}b_{14})^{-1/4}\exp(\theta_1 + \mathrm{i}\pi\sigma/2 + \mathrm{i}\theta_2 + \mathrm{i}\theta_3 + \mathrm{i}\theta_4),$$

$$G\mathrm{e}^{\mathrm{i}\pi\sigma} = \Big[(b_{12}b_{13}b_{14})^{1/4}\cos(-\mathrm{i}\theta_1 + \pi\sigma/2 + \theta_2 + \theta_3 + \theta_4)$$

$$
\begin{aligned}
&+(b_{12}b_{13}b_{14})^{-1/4}\cos(-\mathrm{i}\theta_1+\pi\sigma/2-\theta_2-\theta_3-\theta_4)\Big] \\
&+(b_{23}b_{24})^{-1/2}\Big[(b_{13}b_{14}/b_{12})^{1/4}\cos(\mathrm{i}\theta_1-\pi\sigma/2+\theta_2-\theta_3-\theta_4) \\
&+(b_{13}b_{14}/b_{12})^{-1/4}\cos(-\mathrm{i}\theta_1+\pi\sigma/2+\theta_2-\theta_3-\theta_4)\Big] \\
&+(b_{23}b_{34})^{-1/2}\Big[(b_{12}b_{14}/b_{13})^{1/4}\cos(\mathrm{i}\theta_1-\pi\sigma/2+\theta_3-\theta_2-\theta_4) \\
&+(b_{12}b_{14}/b_{13})^{-1/4}\cos(-\mathrm{i}\theta_1+\pi\sigma/2+\theta_3-\theta_2-\theta_4)\Big] \\
&+(b_{24}b_{34})^{-1/2}\Big[(b_{12}b_{13}/b_{14})^{1/4}\cos(\mathrm{i}\theta_1-\pi\sigma/2+\theta_4-\theta_2-\theta_3) \\
&+(b_{12}b_{13}/b_{14})^{-1/4}\cos(-\mathrm{i}\theta_1+\pi\sigma/2+\theta_4-\theta_2-\theta_3)\Big]. \qquad \text{(C·5)}
\end{aligned}
$$

Since b_{23}, b_{24}, and b_{34} are real, and $b_{1j}^* = 1/b_{1j}$ for $j = 2, 3, 4$, it is evident that $G^* = G$, i.e. the variable G in the solution is a real-valued function. Hence, the solution of the VPE

$$
U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(F) = 6\frac{\partial^2}{\partial X^2}\ln(G) \qquad \text{(C·6)}
$$

represents a real quantity.

Using this example, one can prove without difficulty that the procedure considered above can be extended to any M and N with restrictions (see also (A·14), (B·12), and (C·4))

$$
c_i = \pm 1 \Bigg/ \sqrt{\prod_{\substack{j=1\\ j\neq i}}^{M+N} b_{ij}}\,, \quad b_{ij} = b_{ji}, \quad i = 1, ..., M+N, \qquad \text{(C·7)}
$$

while the quantities q_i retain the phaseshifts X_i (see (C·2)). The signs in (C·7) can be chosen independently of each other. For interaction of M solitons and the N-mode wave there are $(M+N)$ real constants ξ_i and $(M+N)$ real constants X_i.

Note that the restrictions (C·7) are sufficient conditions in order that the solution of the VPE becomes real.

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