GMV CONTROL OF NONLINEAR MULTIVARIABLE SYSTEMS

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Abstract

A *Generalized Minimum Variance* control law is derived for the control of *nonlinear, possibly time-varying multivariable systems.* The solution for the control law is original and was obtained in the time-domain using a simple operator representation of the process. The quadratic cost index involves both error and control signal costing terms. The controller obtained is simple to implement and includes an internal model of the process. In one form might be considered a nonlinear version of the Smith Predictor. However, unlike the Smith Predictor a stabilizing control law can be obtained even for some open-loop unstable processes.

1 Introduction

The control law introduced below for nonlinear multivariable systems is based on a rich heritage. Åström introduced the *Minimum Variance* (MV) controller assuming the linear plant was minimum phase and later derived the MV controller for processes that could be nonminimum phase Åström [1]. The latter was guaranteed to be stable on nonminimum phase processes, whereas the former was unstable.

Hastings [9] and later Clarke and Hastings [2], modified the first of these control laws by adding a control costing term. This was termed a *Generalized Minimum Variance* (GMV) control law and enabled nonminimum phase processes to be stabilized, although when the control weighting tended to zero the control law reverted to the initial algorithm of Åström, which was unstable. However, the control law had similar characteristics to LQG design in some cases and was much simpler to implement. This simplicity was exploited very successfully in the so-called generalized MV self-tuning controller introduced by Clarke and Gawthrop [3].

The control of nonlinear non-minimum phase linear systems using a GMV type algorithm was considered by Grimble [4]. The use of GMV control laws for linear systems designs was reviewed in Grimble [5]. The use of dynamic cost weightings in the GMV cost index Grimble [6] provided additional flexibility and the dynamic costing solution was exploited to obtain a Generalized H_∞ controller Grimble [7]. All of these results were applicable to linear discrete-time stochastic processes.

The aim in the following is to introduce a GMV controller for nonlinear multivariable, possibly time-varying, processes. The structure of the system was defined so that a simple controller structure and solution are obtained. When the system is linear the results revert to those for the GMV controller referred to above Grimble [8]. There is some loss of generality in assuming the reference and disturbance models are represented by linear subsystems. However, the plant model can be in a very general nonlinear operator form, which might involve state-space, transfer operators or even nonlinear function look up tables.

For linear systems stability is ensured when the combination of a control weighting function and an error weighted plant model is strictly minimum phase. For nonlinear systems a related operator equation must have a stable inverse. It is shown that if there exists say a PID controller that will stabilize the nonlinear system, without transport delay elements, then a set of cost weightings can easily be defined to guarantee the existence of this inverse and thereby ensure the stability of the closed loop.

If the plant is open-loop stable the solution can be realized in a particularly simple form which relates to the well known Smith Predictor for systems with significant transport delays. This has the advantage of providing some confidence in the practical utility of the solution and also introduces what might be termed an extension of these *Smith* controllers for nonlinear plants. A so-called *Nonlinear Smith Predictor* will therefore be introduced.

2 System Description

The system description is of restricted generality and is carefully chosen so that simple results are obtained. The plant itself is nonlinear and may be time-varying and have quite a general form. However, the reference and disturbance signals are assumed to have linear time-invariant model representations. This is not very restrictive, since in many applications the models for the disturbance and reference signals are only LTI approximations.

The system is shown in Figure 1 and includes the *nonlinear* plant model and the linear reference/disturbance models.

There is no loss of generality in assuming that the zero mean white noise sources $\{\omega(t)\}$ and $\{\xi(t)\}$ have identity covariance matrices. There is also no requirement to specify the distribution of the noise sources, since it will be shown that the special structure of the system leads to a prediction equation, which is dependent upon the *linear* disturbance and reference models.

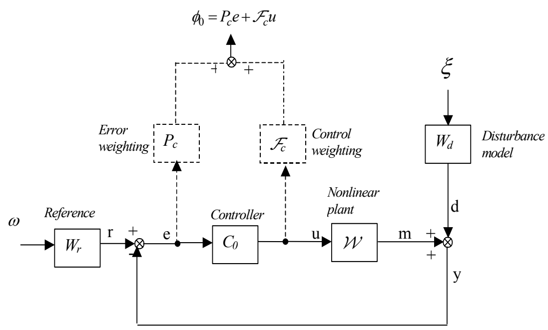

Figure 1: Single Degree of Freedom Closed Loop Feedback Control System for the Nonlinear Plant
(inferred output ϕ_0 is dependent on the weightings shown dotted)

3 System Models

The polynomial matrix system models, for the system shown in Figure 1, may be listed as follows:

Disturbance model: *(assumed linear)*

$$W_d(z^{-1}) = A_f^{-1}(z^{-1})C_d(z^{-1}) \tag{1}$$

Reference model: *(assumed linear)*

$$W_r(z^{-1}) = A_f^{-1}(z^{-1})E_r(z^{-1}) \tag{2}$$

where without loss of generality these models have the common denominator matrix $A_f(z^{-1})$. Note that the arguments of the polynomial matrices are often omitted for simplicity.

Nonlinear time-varying plant model:

$$(\mathcal{W}u)(t) = z^{-k}(\mathcal{W}_k u)(t) \tag{3}$$

where k denotes the magnitude of the common delay elements in the output signal paths. Most of the initial results do not need a more detailed breakdown of the plant model structure. However, in the later sections it will be assumed that any unstable modes of the plant are included in a stable/unstable linear time in variant block of polynomial matrix form: $\mathcal{W}_{2k} = A_2^{-1}B_{2k}$, where $(\mathcal{W}_k u)(t) = W_{2k}(\mathcal{W}_{1k}u)(t)$. Thence, the total plant model:

$$(\mathcal{W}u)(t) = z^{-k}W_{2k}(\mathcal{W}_{1k}u)(t) \tag{4}$$

and $B_2 = z^{-k}B_{2k}$.

3.1 Signals

The signals shown in Figure 1 may be listed as follows:

Error signal: $e(t) = r(t) - y(t)$ (5)

Plant output: $y(t) = d(t) + (\mathcal{W}u)(t)$ (6)

Reference: $r(t) = W_r\omega(t)$ (7)

Disturbance signal: $d(t) = W_d\xi(t)$ (8)

Combined signal: $f(t) = r(t) - d(t)$ (9)

The *power spectrum* for the combined reference and disturbance model can be computed, noting these are linear subsystems, using:

$$\Phi_{ff} = \Phi_{rr} + \Phi_{dd} = W_rW_r^* + W_dW_d^* \tag{10}$$

and the *generalized spectral-factor* Y_f may be computed using:

$$Y_fY_f^* = \Phi_{ff} \tag{11}$$

where the system models ensure Y_f is strictly minimum phase. Note that a measurement noise model has not been included to simplify the equations. This is appropriate so long as the control cost-function weighting, introduced in the next section, ensures controller roll-off at high frequencies.

4 Optimal Nonlinear Generalized Minimum Variance (NGMV) Problem and Solution

The optimal NGMV control problem involves the minimisation of the variance of the signal $\{\phi_0(t)\}$ in Figure 1. This signal involves an error signal *dynamic cost weighting* matrix: $P_c(z^{-1})$, represented by a linear polynomial matrix: $P_c = P_{cd}^{-1}P_{cn}$ and a possibly *nonlinear dynamic control costing* operator term: $(\mathcal{F}_c u)(t)$. The choice of dynamic weightings is critical to the design and typically P_c is low-pass and $\mathcal{F}_c$ is a high-pass transfer. The signal:

$$\{\phi_0(t)\} = P_ce(t) + (\mathcal{F}_cu)(t) \tag{12}$$

is to be minimized in a variance sense, so that the cost index to be minimised:

$$J = E\{\phi_0^T(t)\phi_0(t)\} = E\{trace\{\phi_0(t)\phi_0^T(t)\}\} \tag{13}$$

where $E\{\cdot\}$ denotes the expectation operator. Note that in some applications the signal $\phi_0(t)$ may represent an inferred output. That is, this signal represents the output from a subsystem that cannot be measured directly.

If the smallest delay in each output channel of the plant is of magnitude k steps this implies the control at time t affects the output at least k steps later. For this reason the control costing can be defined to have the form:

$$(\mathcal{F}_cu)(t) = z^{-k}(\mathcal{F}_{ck}u)(t) \tag{14}$$

Typically this will be a linear operator but it may also be chosen to be nonlinear to cancel the plant input nonlinearities in appropriate cases. The control weighting operator $\mathcal{F}_{ck}$ is assumed to be full rank and invertible.

Theorem: NGMV Optimal Controller

The NGMV optimal controller to minimize the variance of the weighted error and control signals may be computed from the following equations. The assumption is made that the nonlinear possibly time-varying operator $\left(P_c \mathcal{W}_k' - \mathcal{F}_{ck}\right)$ has a stable causal inverse, due to the choice of weighting operators P_c and $\mathcal{F}_c$.

Diophantine equation: The smallest degree solution (G_0, F_0), with respect to F_0, must be computed from the polynomial matrix equation:

$$A_{pf} P_{cd} F_0 + z^{-k} G_0 = P_{cf} D_f \qquad (15)$$

where the left coprime polynomial matrices A_{pf} and P_{cf} satisfy:

$$A_{pf}^{-1} P_{cf} = P_{cn} A_f^{-1} \qquad (16)$$

and the spectral factor Y_f is written in the polynomial matrix form: $Y_f = A_f^{-1} D_f$.

Optimal control: The optimal control may be computed as:

$$u(t) = \left(F_0 Y_f^{-1} \mathcal{W}_k' - \mathcal{F}_{ck}\right)^{-1} \left(\left(A_{pf} P_{cd}\right)^{-1} G_0 Y_f^{-1} e\right)(t) \qquad (17)$$

5 Concluding Remarks

A relatively simple controller shown in Figure 2 for nonlinear multivariable and possibly time-varying systems was introduced. The closed loop stability of the system was shown to depend upon the existence of a stable inverse for a particular loop operator. This operator depended upon the cost weighting definitions. It was shown that a possible starting point for weighting selection was through the relationship to a PID controller. That is, if it is assumed that PID controller exists, to stabilize the delay free plant model $\mathcal{W}_k'$, then this guarantees that existence of at least one set of control weightings that will ensure closed-loop stability.

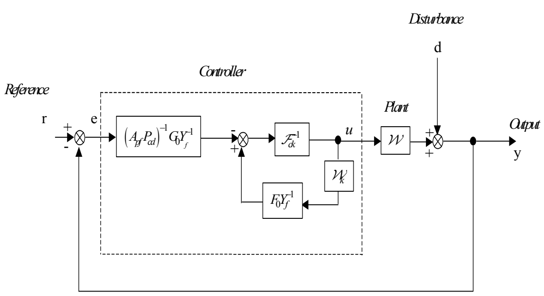

Figure 2: Control Signal Generation and Controller Modules

The assumptions made in the definition of the system reference and disturbance models and the specification of the cost index, were all aimed at leading to a simple controller solution. However, note that the plant description can be very general. The structure of the system was chosen so that the main polynomial matrix equations to be solved are all linear. The controller is therefore simple to compute and implement.

A major advantage of the NGMV solution is that the only knowledge of the nonlinear plant model $\mathcal{W}_k'$ that is required is the ability to compute an output $\mathrm{m}_k(t) = (\mathcal{W}_k' u)(t)$ for a given control input sequence $\{u(t)\}$. Such a model could be in Fortran or C code, or might even be a neural network. The remaining computations concern the linear disturbance and reference signal models and knowledge of the transport delay element of length k. These are representative linear approximations and experience suggests they will be adequate so long as they capture the dominant frequency response behaviour. It follows that such a controller can be calculated without the usual model information required in traditional model base control law design.

The relationship to the Smith Predictor was discussed for two reasons. Firstly the extension of *Smith's* ideas to the nonlinear problem is interesting and provides a practical method of implementing these controllers, when the plant is open-loop stable. Secondly the physical structure is useful to provide an intuitive understanding of the operation and properties of the proposed Nonlinear GMV controller. The *Nonlinear Smith Predictor*, is particularly valuable, since it relates to a well known technique and thereby provides some confidence in the nonlinear version.

6 References

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