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Almost sure exponential stability of the Euler–Maruyama approximations for stochastic functional differential equations

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Abstract. By the continuous and discrete nonnegative semimartingale convergence theorems, this paper investigates conditions under which the Euler–Maruyama (EM) approximations of stochastic functional differential equations (SFDEs) can share the almost sure exponential stability of the exact solution. Moreover, for sufficiently small stepsize, the decay rate as measured by the Lyapunov exponent can be reproduced arbitrarily accurately.

Keywords. Stochastic functional differential equations (SFDEs), nonnegative semimartingale convergence theorem, almost sure stability, EM method.

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1 Introduction

Stochastic differential equations (SDEs) and their stability have been used with great success in a variety of applications areas, including biology, epidemiology, mechanics, neural networks, economics, finance and so on. Most SDEs arising in practice are nonlinear and cannot be solved explicitly, so stability analysis of numerical methods for SDEs has recently received a great deal of attention. Due to the stochastic nature, the stability concepts of numerical schemes for SDEs include, for example, moment stability, almost sure stability and stability in distribution. There is an extensive literature concerned with the moment stability, for example, [5, 6, 8–10, 24, 29] for SDEs and [3, 20] for stochastic differential equations (SDDEs). Regarding the almost sure stability of numerical methods for SDEs, it was shown, by using the Chebyshev inequality and the Borel–Cantelli lemma, that the moment exponential stability implies almost sure exponential stability under certain conditions (for example, see [9, 24]). Higham and his coauthors ([8, 9]) di-

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rectly studied the numerical sequence and obtained almost sure stability by the strong law of large numbers. Now there is little result to examine stability in distribution of numerical methods although this stability is also important. The only work can be seen in [4] and [30].

By the technique based on the continuous semimartingale convergence theorem (cf. [11,14]), Mao developed in his serial papers (see e.g. [15–18,32]) the stochastic versions of the LaSalle theorem, from which follows the almost sure asymptotic stability of SDEs, including SDDEs and SFDEs. On the other hand, by the discrete semimartingale convergence theorem (cf. [26,33]), [25–27] investigated the stability of stochastic difference equations and [31] examined almost sure stability of the stochastic theta methods of linear SDEs. Noting that there are similar expressions for the continuous and discrete semimartingale convergence theorems, [26] obtained the sufficient conditions for almost surely asymptotic stability of both exact and numerical solutions of linear stochastic differential equations. Then Wu et al. [35] examined conditions under which the numerical solutions of SDDEs may share the almost sure stability of exact solutions for nonlinear SDDEs.

However, so far, little is as yet known about the stability of numerical solutions for SFDEs although there are lots of such results for the exact solutions of SFDFs (for example, [14, 17, 23, 32]). The main aim of this paper is to extend the results in [26, 35] to SFDEs and examine conditions under which the numerical solutions of SFDEs may reproduce the almost sure stability of the exact solutions. Consider the n-dimensional SFDE

$$dx(t) = f(x_t, t)dt + g(x_t, t)dw(t), \quad t \geq 0, \tag{1.1}$$

with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, namely, ξ is a bounded, $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random process defined on $[-\tau, 0]$, where

$$x_t =: x_t(\theta) = \{x(t+\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n),$$

$f, g : C([-\tau, 0] \times \mathbb{R}_+; \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$ are Borel measurable, $w(t)$ is a scalar Brownian motion. For the purpose of stability, we assume $f(0, t) = g(0, t) = 0$. This shows that (1.1) admits a trivial solution. As a standing hypothesis, we shall impose the following local Lipschitz condition on coefficients f and g:

Assumption 1.1. For each $k = 1, 2, \ldots,$ there is $c_k > 0$ such that for any maps $\varphi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$ and all $t \geq 0$,

$$|f(\varphi, t) - f(\phi, t)| \vee |g(\varphi, t) - g(\phi, t)| \leq c_k \|\varphi - \phi\|,$$

where $\|\varphi - \phi\| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta) - \phi(\theta)|$.

Let the stepsize $\triangle$ be a fraction of the delay τ, namely, $\triangle = \tau/M$ for some integer M. Then the EM method (see [19,34]) applied to (1.1) produces

$$\begin{cases} x_k = \xi(k\triangle), & -M \le k \le 0, \\ x_{k+1} = x_k + f(y_{k\triangle}, k\triangle)\triangle + g(y_{k\triangle}, k\triangle)\triangle w_k, & k \ge 0, \end{cases} \tag{1.2}$$

where $\triangle w_k = w((k+1)\triangle) - w(k\triangle)$ is the Brownian motion increment and $y_{k\triangle}$ is a $C([-\tau,0];\mathbb{R}^n)$-valued random process defined by piecewise linear interpolation:

$$y_{k\triangle} =: y_{k\triangle}(\theta) = x_{k+i} + \frac{\theta - i\triangle}{\triangle}(x_{k+1+i} - x_{k+i}), \\ \text{for } i\triangle \le \theta \le (i+1)\triangle,\ i = -M, -M+1, \dots, -1. \tag{1.3}$$

In the next section, we give some necessary notations and the continuous and discrete semimartingale convergence theorems as lemmas for the use of this paper. Motivated by the existing stability results for the exact solution of the SFDE (1.1), by the Lyapunov technique and the discrete nonnegative semimartingale convergence theorem, Section 3 examines the almost sure exponential stability of the EM approximate sequence $\{x_k\}_{k\ge 0}$. Section 4 presents some conditions under which the EM approximation $\{x_k\}_{k\ge 0}$ may reproduce the almost sure exponential stability of the exact solution of (1.1). To illustrate the application of our results, the final section examines a linear integro-differential equation and gives its simulation.

2 Notations and lemmas

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. Let $\mathbb{R}_+ = [0,\infty)$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, then its norm $\|A\|$ is defined by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $\tau > 0$ and $C([-\tau,0],\mathbb{R}^n)$ denote the family of continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$. The inner product of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$ or $x^T y$. If $a, b \in \mathbb{R}$, let $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$. The symbol $\mathbb{N}$ represents the set of the integer numbers, namely, $\mathbb{N} = \{0,1,\dots\}$ and let $\mathbb{N}_{-M} = \{0,-1,-2,\dots,-M\}$ for some positive integer M.

Let $(\mathbf{\Omega}, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, namely, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be a scalar Brownian motion defined on this probability space. Denote by $C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ the family of all bounded, $\mathcal{F}_0$-measurable $C([-\tau,0];\mathbb{R}^n)$-valued stochastic processes on $[-\tau,0]$. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process, define $x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$ for $t \ge 0$. Moreover, for a function $V \in C^2(\mathbb{R}^n;\mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $C([-\tau,0];\mathbb{R}^n) \times \mathbb{R}_+$

to $\mathbb{R}$ by

$$\mathcal{L}V(\varphi,t) = V_x(\varphi(0))f(\varphi,t) + \frac{1}{2}\text{trace}[g^T(\varphi,t)V_{xx}(\varphi(0))g(\varphi,t)]. \quad (2.1)$$

Let us emphasize that $\mathcal{L}V$ is a functional defined on $C((-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+$ while V is a function on $\mathbb{R}^n$.

A stochastic sequence $\{\rho_k\}_{k\in\mathbb{N}}$ is said to be an *$\mathcal{F}_k$-martingale-difference* if $\mathbb{E}|\rho_k| < \infty$ and $\mathbb{E}(\rho_k|\mathcal{F}_{k-1}) = 0$ for all $k = 1, 2, \ldots$. (For more knowledge for the martingale-difference, please refer [11, 25, 26]). By the definitions of the martingale-difference and the martingale, it follows that

Lemma 2.1. *Suppose that $\{\pi_k\}_{k\geq 0}$ is any deterministic nonzero sequence. Then $X_k = \sum_{i=1}^{k}\pi_i\rho_i$ is a martingale and $X_1 = 0$ if and only if ρ_i is a martingale-difference, that is, the partial summation of martingale-difference leads at once to a martingale (and conversely).*

The following two lemmas will play important roles in this paper. The first one is the continuous semimartingale convergence theorem (cf. [11, 14]). The second one is the corresponding discrete version (cf. [26, 33]).

Lemma 2.2. *Let $A(t), U(t)$ be two $\mathcal{F}_t$-adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $X(t)$ is nonnegative and*

$$X(t) = \zeta + A(t) - U(t) + M(t) \quad \textit{for } t \geq 0.$$

If $\lim_{t\to\infty} A(t) < \infty$ a.s., then for almost all $\omega \in \mathbf{\Omega}$,

$$\lim_{t\to\infty} X(t) < \infty \quad \textit{and} \quad \lim_{t\to\infty} U(t) < \infty,$$

that is, both $X(t)$ and $U(t)$ converge to finite random variables.

Lemma 2.3. *Let $\{A_k\}_{k\in\mathbb{N}}$, $\{U_k\}_{k\in\mathbb{N}}$ be two sequences of nonnegative stochastic sequences such that both A_k and U_k are $\mathcal{F}_{k-1}$-measurable for $k = 1, 2, \ldots,$ and $A_0 = U_0 = 0$ a.s. Let M_k be a real-value local martingale with $M_0 = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $\{X_k\}$ is a nonnegative semimartingale with the Doob–Mayer decomposition*

$$X_k = \zeta + A_k - U_k + M_k.$$

If $\lim_{k\to\infty} A_k < \infty$ a.s., then for almost all $\omega \in \mathbf{\Omega}$,

$$\lim_{k\to\infty} X_k < \infty \quad \textit{and} \quad \lim_{k\to\infty} U_k < \infty,$$

that is, both X_k and U_k converge to finite random variables.

In the following sections we will employ these lemmas to establish the almost sure asymptotic stability theorems for both exact and numerical solutions to (1.1).

3 Stability of the exact solution and the EM approximation

Motivated by the continuous stability results in [17, 32], by the Lyapunov technique and the discrete nonnegative semimartingale convergence, this section will establish an almost sure stability theorem for the stochastic sequence $\{x_k\}_{k\geq 0}$. As comparison, this section also presents the corresponding continuous stability result. To be precise, let us give the definitions on the almost sure exponential stability of SFDEs and their numerical approximations.

Definition 3.1. The solution $x(t,\xi)$ to (1.1) is said to be *almost surely exponentially stable* if there exists a constant $\eta > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t,\xi)| \leq -\eta \quad \text{a.s.} \tag{3.1}$$

for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$.

Definition 3.2. The discrete sequence $\{x_k\}_{k\geq 0}$ defined by (1.2) is said to be *almost surely exponentially stable* if there exists a constant $\bar{\eta} > 0$ such that

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log |x_k| \leq -\bar{\eta} \quad \text{a.s.} \tag{3.2}$$

for any bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_{-M}}$.

Let us state a theorem, which does not only give the existence-and-uniqueness result of the solution but also provide us with a criterion on the almost sure exponential stability of the exact solution (please see [32] for the existence-and-existence result and [18] for the almost sure exponential stability).

Theorem 3.3. *Let Assumption* 1.1 *hold. Assume that there exist* $V \in C^2(\mathbb{R}^n;\mathbb{R}_+)$, *a number of positive constants* c, p, λ_1, λ_2 *and a probability measure* η *such that for any* $x \in \mathbb{R}^n$ *and* $(\varphi,t) \in C([-\tau,0];\mathbb{R}^n) \times \mathbb{R}_+$,

$$c|x|^p \leq V(x), \tag{3.3}$$

$$\mathcal{L}V(\varphi,t) \leq -\lambda_1 V(\varphi(0)) + \lambda_2 \int_{-\tau}^{0} V(\varphi(\theta))d\eta(\theta). \tag{3.4}$$

If $\lambda_1 > \lambda_2$, *then for any given initial data* $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, *there exists a unique global solution to* (1.1) *and this solution, denoted by* $x(t;\xi)$, *has the prop-*

erty that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t;\xi)|) \le -\frac{\gamma}{p} \quad a.s., \tag{3.5}$$

where $\gamma > 0$ *is the unique positive root of*

$$\lambda_1 - \gamma = \lambda_2 e^{\gamma\tau}. \tag{3.6}$$

We now establish the discrete counterpart of Theorem 3.3 for the stochastic sequence $\{x_k\}_{k\ge 0}$ defined by (1.2).

Theorem 3.4. *Fix* $\triangle > 0$. *Let* $c_\triangle$, p, $\lambda_{1\triangle}$, $\lambda_{2\triangle}$ *all be positive constants and* μ *be a probability measure. Assume that there exists a convex function* $V_\triangle : \mathbb{R}^n \to \mathbb{R}_+$ *and an* $\mathcal{F}_{k\triangle}$*-martingale-difference* ρ_k *such that*

$$c_\triangle |x|^p \le V_\triangle(x), \quad x \in \mathbb{R}^n, \tag{3.7}$$

and for the sequence $\{x_k\}_{k\ge 0}$ *defined by* (1.2),

$$V_\triangle(x_{k+1}) - V_\triangle(x_k) \le -\lambda_{1\triangle}\triangle V_\triangle(x_k) + \lambda_{2\triangle}\triangle \int_{-\tau}^0 V_\triangle(y_{k\triangle}(\theta))d\mu(\theta) + \rho_{k+1}, \tag{3.8}$$

where $y_{k\triangle}$ *is defined by* (1.3). *If* $\lambda_{1\triangle} > \lambda_{2\triangle}$ *and* $\lambda_{1\triangle}\triangle \le 1$, *then for any bounded initial sequence* $\{\xi(k\triangle)\}_{k\in\mathbb{N}_{-M}}$, *the EM approximate solution* (1.2) *obeys*

$$\limsup_{k\to\infty} \frac{1}{k\triangle}\log|x_k| \le -\frac{\gamma_\triangle}{p} \quad a.s., \tag{3.9}$$

where $\gamma_\triangle > 0$ *is the unique positive root of*

$$\lambda_{2\triangle}\triangle e^{(M+1)\gamma_\triangle\triangle} + (1-\lambda_{1\triangle}\triangle)e^{\gamma_\triangle\triangle} - 1 = 0, \tag{3.10}$$

namely, the stochastic sequence $\{x_k\}_{k\ge 0}$ *is almost surely exponentially stable.*

Proof. For any positive constant $\pi > 1$, by condition (3.8), we have

$$\begin{aligned}
&\pi^{(k+1)\triangle}V_\triangle(x_{k+1}) - \pi^{k\triangle}V_\triangle(x_k) \\
&\quad= \pi^{(k+1)\triangle}[V_\triangle(x_{k+1}) - V_\triangle(x_k)] + (\pi^{(k+1)\triangle} - \pi^{k\triangle})V_\triangle(x_k) \\
&\quad\le \pi^{(k+1)\triangle}\Big[-\lambda_{1\triangle}\triangle V_\triangle(x_k) + \lambda_{2\triangle}\triangle\int_{-\tau}^0 V_\triangle(y_{k\triangle}(\theta))d\mu(\theta) + \rho_{k+1}\Big] \\
&\qquad + (\pi^{(k+1)\triangle} - \pi^{k\triangle})V_\triangle(x_k) \\
&\quad= (\pi^\triangle - 1 - \lambda_{1\triangle}\triangle\pi^\triangle)\pi^{k\triangle}V_\triangle(x_k) \\
&\qquad + \lambda_{2\triangle}\triangle\pi^{(k+1)\triangle}\int_{-\tau}^0 V_\triangle(y_{k\triangle}(\theta))d\mu(\theta) + \pi^{(k+1)\triangle}\rho_{k+1},
\end{aligned} \tag{3.11}$$

which implies that

$$\pi^{k\triangle} V_{\triangle}(x_k) \le V_{\triangle}(\xi(0)) + [(\pi^{\triangle} - 1) - \lambda_{1\triangle}\triangle\pi^{\triangle}] \sum_{i=0}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) + \lambda_{2\triangle}\triangle \sum_{i=0}^{k-1} \pi^{(i+1)\triangle} \int_{-\tau}^{0} V_{\triangle}(y_{i\triangle}(\theta)) d\mu(\theta) + \sum_{i=0}^{k-1} \pi^{(i+1)\triangle} \rho_{i+1}. \tag{3.12}$$

Recall the elementary property of the convex function V: for any $x, y \in \mathbb{R}^n$ and $\varepsilon \in [0, 1]$,

$$V(\varepsilon x + (1-\varepsilon)y) \le \varepsilon V(x) + (1-\varepsilon)V(y). \tag{3.13}$$

This, together with the definition of $y_{k\triangle}$, yields

$$\begin{aligned}\int_{-\tau}^{0} V_{\triangle}(y_{i\triangle}(\theta)) d\mu(\theta) &= \sum_{m=-M}^{-1} \int_{m\triangle}^{(m+1)\triangle} V_{\triangle}\left(\frac{\theta - m\triangle}{\triangle} x_{i+m+1} + \frac{(m+1)\triangle - \theta}{\triangle} x_{i+m}\right) d\mu(\theta) \\ &\le \sum_{m=-M}^{-1} \int_{m\triangle}^{(m+1)\triangle} \left[\frac{\theta - m\triangle}{\triangle} V_{\triangle}(x_{i+m+1}) + \frac{(m+1)\triangle - \theta}{\triangle} V_{\triangle}(x_{i+m})\right] d\mu(\theta).\end{aligned}$$

By the Fubini theorem, we therefore have

$$\begin{aligned}\sum_{i=0}^{k-1} \pi^{(i+1)\triangle} \int_{-\tau}^{0} V_{\triangle}(y_{i\triangle}(\theta)) d\mu(\theta) &\le \sum_{m=-M}^{-1} \int_{m\triangle}^{(m+1)\triangle} \left[\frac{\theta - m\triangle}{\triangle} \sum_{i=0}^{k-1} \pi^{(i+1)\triangle} V_{\triangle}(x_{i+m+1}) + \frac{(m+1)\triangle - \theta}{\triangle} \sum_{i=0}^{k-1} \pi^{(i+1)\triangle} V_{\triangle}(x_{i+m})\right] d\mu(\theta).\end{aligned}$$

Noting that $M+m \geq 0$ for all $m=-M,-M+1,\ldots,-1$, we therefore have

$$
\begin{aligned}
&\sum_{i=0}^{k-1} \pi^{(i+1)\triangle} \int_{-\tau}^{0} V_{\triangle}(y_{i\triangle}(\theta))d\mu(\theta) \\
&\quad \leq \sum_{m=-M}^{-1} \int_{m\triangle}^{(m+1)\triangle} \Big[\frac{\theta - m\triangle}{\triangle} \sum_{i=0}^{k-1} \pi^{(i+1+M+1+m)\triangle} V_{\triangle}(x_{i+m+1}) \\
&\qquad\qquad + \frac{(m+1)\triangle - \theta}{\triangle} \sum_{i=0}^{k-1} \pi^{(i+1+M+m)\triangle} V_{\triangle}(x_{i+m}) \Big] d\mu(\theta) \\
&\quad \leq \sum_{m=-M}^{-1} \int_{m\triangle}^{(m+1)\triangle} \Big[\frac{\theta - m\triangle}{\triangle} \pi^{(M+1)\triangle} \sum_{i=-M}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) \\
&\qquad\qquad + \frac{(m+1)\triangle - \theta}{\triangle} \pi^{(M+1)\triangle} \sum_{i=-M}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) \Big] d\mu(\theta) \\
&\quad = \pi^{(M+1)\triangle} \sum_{i=-M}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) \sum_{m=-M}^{-1} \int_{m\triangle}^{(m+1)\triangle} d\mu(\theta) \\
&\quad = \pi^{(M+1)\triangle} \sum_{i=-M}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) \int_{-\tau}^{0} d\mu(\theta) \\
&\quad = \pi^{(M+1)\triangle} \sum_{i=-M}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) \\
&\quad = \pi^{(M+1)\triangle} \sum_{i=-M}^{-1} \pi^{i\triangle} V_{\triangle}(\xi(i\triangle)) + \pi^{(M+1)\triangle} \sum_{i=0}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i). \qquad (3.14)
\end{aligned}
$$

Substituting (3.14) into (3.12) yields

$$
\begin{aligned}
\pi^{k\triangle} V_{\triangle}(x_k) \leq{}& V_{\triangle}(\xi_0) + \lambda_{2\triangle} \triangle \pi^{(M+1)\triangle} \sum_{i=-M}^{-1} \pi^{i\triangle} V_{\triangle}(\xi(i\triangle)) \\
&+ [\lambda_{2\triangle} \triangle \pi^{(M+1)\triangle} + (1 - \lambda_{1\triangle}\triangle)\pi^{\triangle} - 1] \sum_{i=0}^{k-1} \pi^{i\triangle} V_{\triangle}(x_i) \\
&+ \sum_{i=0}^{k-1} \pi^{(i+1)\triangle} \rho_{i+1}. \qquad (3.15)
\end{aligned}
$$

Let us introduce the function

$$\kappa(\pi) = \lambda_{2\Delta}\Delta\pi^{(M+1)\Delta} + (1-\lambda_{1\Delta}\Delta)\pi^{\Delta} - 1. \tag{3.16}$$

Noting that $\lambda_{1\Delta} > \lambda_{2\Delta}$ and $\lambda_{1\Delta}\Delta \le 1$, we therefore have

$$\kappa(1) = -(\lambda_{1\Delta} - \lambda_{2\Delta})\Delta < 0 \quad \text{and} \quad \kappa'(\pi) > 0 \ \text{ for all } \pi > 1,$$

which implies that there exists a unique $\pi^*_\Delta > 1$ such that $\kappa(\pi^*_\Delta) = 0$. Choosing $\pi = \pi^*_\Delta$, equation (3.15) may be rewritten as

$$\pi^{*\,k\Delta}_\Delta V_\Delta(x_k) \le X_k,$$

where

$$X_k = V_\Delta(\xi_0) + \lambda_{2\Delta}\Delta\pi^{*\,(M+1)\Delta}_\Delta \sum_{i=-M}^{-1} \pi^{*\,i\Delta}_\Delta V_\Delta(\xi(i\Delta)) \\ + \sum_{i=0}^{k-1} \pi^{*\,(i+1)\Delta}_\Delta \rho_{i+1}.$$

We define $M_k = \sum_{i=0}^{k} \pi^{*\,(i+1)\Delta}_\Delta \rho_{i+1}$. Noting that ρ_i is an $\mathcal{F}_{i\Delta}$-martingale-difference, Lemma 2.1 shows that M_k is a martingale with $M_0 = 0$. Noting that the initial sequence $\{\xi(k\Delta)\}_{k\in\mathbb{N}_{-M}}$ is bounded, by Lemma 2.3, $\lim_{k\to\infty} X_k < \infty$ a.s. We therefore have

$$\limsup_{k\to\infty} \pi^{*\,k\Delta}_\Delta V_\Delta(x_k) \le \lim_{k\to\infty} X_k < \infty \qquad \text{a.s.}$$

Recall that $\pi^*_\Delta > 1$. Choosing the constant $\gamma_\Delta > 0$ such that $\pi^*_\Delta = e^{\gamma_\Delta}$, we therefore obtain

$$\limsup_{k\to\infty} e^{\gamma_\Delta k\Delta} V_\Delta(x_k) < \infty \quad \text{a.s.}$$

This, together with condition (3.7), gives the desired assertion. □

In Theorem 3.4, the role of condition (3.8) is similar to (3.4) in Theorem 3.3. Clearly, it is not convenient to check conditions (3.4) and (3.8) since they are not related to coefficients f and g explicitly. To specify conditions (3.4) and (3.8), in the next section, we impose some conditions on functionals f and g to guarantee Theorems 3.3 and 3.4. These conditions also show that the EM approximate sequence $\{x_k\}_{k\ge 0}$ may accurately reproduce the almost sure exponential stability of the exact solution of (1.1).

4 Reproduction of stability of the numerical solution for the exact solution

In this section, by Theorems 3.3 and 3.4, we will present some conditions under which the EM method (1.2) may reproduce the almost sure exponential stability of the exact solution of (1.1). By using Theorem 3.3, we firstly give a theorem for the almost sure exponential stability of the exact solution.

Theorem 4.1. *Let Assumption* 1.1 *hold. Assume that there are three nonnegative constants $\zeta_1, \zeta_2, \zeta_3$ and two probability measures η and μ such that*

$$2\varphi(0)^T f(\varphi,t) \le -\zeta_1|\varphi(0)|^2 + \zeta_2 \int_{-\tau}^{0} |\varphi(\theta)|^2 d\eta(\theta), \tag{4.1a}$$

$$|g(\varphi,t)|^2 \le \zeta_3 \int_{-\tau}^{0} |\varphi(\theta)|^2 d\mu(\theta) \tag{4.1b}$$

for all $\varphi \in C([-\tau,0];\mathbb{R}^n)$ and $t \ge 0$. If

$$\zeta_1 > \zeta_2 + \zeta_3, \tag{4.2}$$

then for any given initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, there exists a unique global solution $x(t;\xi)$ to (1.1) *and this solution has the property that*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t;\xi)|) \le -\frac{\gamma}{2} \quad a.s., \tag{4.3}$$

where $\gamma > 0$ is the unique positive root of

$$\zeta_1 - \gamma = (\zeta_2 + \zeta_3)e^{\gamma\tau}. \tag{4.4}$$

Proof. Choose $V(x) = |x|^2$. To apply Theorem 3.3, it is important to test condition (3.4). Applying (2.1) to $V(x) = |x|^2$ and using conditions (4.1a) and (4.1b) yield

$$\begin{aligned}\mathcal{L}V(\varphi,t) &= 2\varphi(0)^T f(\varphi,t) + |g(\varphi,t)|^2 \\ &\le -\zeta_1|\varphi(0)|^2 + \zeta_2\int_{-\tau}^{0}|\varphi(\theta)|^2 d\eta(\theta) + \zeta_3\int_{-\tau}^{0}|\varphi(\theta)|^2 d\mu(\theta).\end{aligned}$$

Define the probability measure ν such that

$$d\nu = \frac{\zeta_2 d\eta + \zeta_3 d\mu}{\zeta_2 + \zeta_3}.$$

We therefore have

$$\mathcal{L}V(\varphi,t) \le -\zeta_1|\varphi(0)|^2 + (\zeta_2+\zeta_3)\int_{-\tau}^{0}|\varphi(\theta)|^2 d\nu(\theta),$$

which implies that condition (3.4) holds. Applying Theorem 3.3 gives the desired assertions. □

Let us now discuss the reproduction of the EM approximate solution (1.2) for the almost sure exponential stability of the exact solution (1.1).

Theorem 4.2. *Let conditions* (4.1a), (4.1b) *and* (4.2) *hold. Assume also that f satisfies the linear growth condition, namely, there exist a constant $K > 0$ and a probability measure $\bar{\eta}$ such that*

$$|f(\varphi,t)| \le K\int_{-\tau}^{0}|\varphi(\theta)|d\bar{\eta}(\theta). \tag{4.5}$$

Let $\gamma > 0$ be the number determined by (4.4) *and $\varepsilon \in (0,\gamma/2)$ be arbitrary. Then there exists a $\triangle^* > 0$ such that if $\triangle < \triangle^*$, then for any given bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_M}$, the EM approximate solution* (1.2) *obeys*

$$\limsup_{k\to\infty}\frac{1}{k\triangle}\log(|x_k|) \le -\frac{\gamma}{2}+\varepsilon \quad a.s., \tag{4.6}$$

namely, the random sequence $\{x_k\}_{k\ge 0}$ defined by (1.2) *is almost surely exponentially stable.*

In order to prove this theorem we need to prepare a lemma which shows that under Assumption 1.1, conditions (4.1b) and (4.5) may guarantee the pth moment of x_k is bounded (see Lemma 3.2 in [19] or Theorem 2.1 in [34]).

Lemma 4.3. *Under conditions* (4.1b) *and* (4.5), *for any $p > 0$ and any given bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_M}$, the random sequence $\{x_k\}_{k\ge 0}$ defined by* (1.2) *holds the property that $\mathbb{E}(\sup_{k\ge -M}|x_k|^p) < \infty$.*

Proof of Theorem 4.2. Choose $V_\triangle(x) = |x|^2$. It is obvious that V is a convex function. To use Theorem 3.4, it is key to test condition (3.8). Note that

$$\begin{aligned}|x_{k+1}|^2 &= \langle x_k + f(y_{k\triangle},k\triangle)\triangle + g(y_{k\triangle},k\triangle)\triangle w_k, x_k + f(y_{k\triangle},k\triangle)\triangle \\ &\qquad + g(y_{k\triangle},k\triangle)\triangle w_k\rangle \\ &= |x_k|^2 + 2x_k^T f(y_{k\triangle},k\triangle)\triangle + |f(y_{k\triangle},k\triangle)\triangle|^2 \\ &\qquad + |g(y_{k\triangle},k\triangle)\triangle w_k|^2 \\ &\qquad + 2\langle x_k + f(y_{k\triangle},k\triangle)\triangle, g(y_{k\triangle},k\triangle)\triangle w_k\rangle. \end{aligned} \tag{4.7}$$

By the definition of $y_{k\triangle}$, $y_{k\triangle}(\theta)$ is a continuous function for all $\theta \in [-\tau, 0]$ with $y_{k\triangle}(0) = x_k$. By condition (4.1a), we have

$$2x_k^T f(y_{k\triangle}, k\triangle) \leq -\zeta_1 |x_k|^2 + \zeta_2 \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\eta(\theta). \tag{4.8}$$

By condition (4.5), applying the Hölder inequality gives

$$|f(y_{k\triangle}, k\triangle)|^2 \leq K^2 \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\bar{\eta}(\theta). \tag{4.9}$$

By condition (4.1b),

$$\begin{aligned} &|g(y_{k\triangle}, k\triangle)\triangle w_k|^2 \\ &\quad\leq |g(y_{k\triangle}, k\triangle)|^2 |\triangle w_k|^2 \\ &\quad= |g(y_{k\triangle}, k\triangle)|^2 \triangle + |g(y_{k\triangle}, k\triangle)|^2 (|\triangle w_k|^2 - \triangle) \\ &\quad\leq \zeta_3 \triangle \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\mu(\theta) + |g(y_{k\triangle}, k\triangle)|^2 (|\triangle w_k|^2 - \triangle). \end{aligned} \tag{4.10}$$

Substituting (4.8)–(4.10) into (4.7) yields

$$\begin{aligned} |x_{k+1}|^2 - |x_k|^2 \leq -\zeta_1 \triangle |x_k|^2 + \zeta_2 \triangle \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\eta(\theta) \\ + K^2 \triangle^2 \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\bar{\eta}(\theta) \\ + \zeta_3 \triangle \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\mu(\theta) + \bar{\rho}_{k+1}, \end{aligned}$$

where

$$\begin{aligned} \bar{\rho}_{k+1} = 2\langle x_k + f(y_{k\triangle}, k\triangle)\triangle, g(y_{k\triangle}, k\triangle)\triangle w_k\rangle \\ + |g(y_{k\triangle}, k\triangle)|^2 (|\triangle w_k|^2 - \triangle). \end{aligned} \tag{4.11}$$

Define the probability measure $\bar{\nu}_\triangle$ for any given $\triangle > 0$ such that

$$d\bar{\nu}_\triangle = \frac{\zeta_2 d\eta + \zeta_3 d\mu + K^2 \triangle d\bar{\eta}}{\zeta_2 + \zeta_3 + K^2 \triangle}.$$

We therefore have

$$|x_{k+1}|^2 - |x_k|^2 \leq -\zeta_1 \triangle |x_k|^2 + (\zeta_2 + \zeta_3 + K^2 \triangle) \triangle \int_{-\tau}^{0} |y_{k\triangle}(\theta)|^2 d\bar{\nu}_\triangle(\theta) + \bar{\rho}_{k+1}, \tag{4.12}$$

We now prove that $\bar{\rho}_k$ is an $\mathcal{F}_{k\triangle}$-martingale-difference. It is obvious that

$$\begin{aligned}\mathbb{E}(\bar{\rho}_{k+1}|\mathcal{F}_{k\triangle}) &= \mathbb{E}[(2\langle x_k + f(y_{k\triangle}, k\triangle)\triangle, g(y_{k\triangle}, k\triangle)\triangle w_k\rangle \\ &\qquad + |g(y_{k\triangle}, k\triangle)|^2(|\triangle w_k|^2 - \triangle))|\mathcal{F}_{k\triangle}] \\ &= \langle x_k + f(y_{k\triangle}, k\triangle)\triangle, g(y_{k\triangle}, k\triangle)\mathbb{E}(\triangle w_k|\mathcal{F}_{k\triangle})\rangle \\ &\qquad + |g(y_{k\triangle}, k\triangle)|^2\mathbb{E}((|\triangle w_k|^2 - \triangle)|\mathcal{F}_{k\triangle}). \end{aligned} \tag{4.13}$$

Under the linear growth conditions (4.1b) and (4.5), for any $p > 0$, Lemma 4.3 shows

$$\mathbb{E}\Big(\sup_{k\geq -M} |x_k|^p\Big) < \infty,$$

which implies that

$$\sup_{k\geq -M} |x_k| < \infty \quad \text{a.s.} \tag{4.14}$$

By the definition of $y_{k\triangle}$, equation (4.5) shows that

$$\begin{aligned}|x_k + f(y_{k\triangle}, k\triangle)| &\leq |x_k| + |f(y_{k\triangle}, k\triangle)| \\ &\leq |x_k| + K\int_{-\tau}^{0} |y_{k\triangle}(\theta)| d\bar{\eta}(\theta) \\ &\leq |x_k| + K\sum_{i=-M}^{-1}\int_{i\triangle}^{(i+1)\triangle}\Bigg|\frac{(i+1)\triangle - \theta}{\triangle}x_{k+i} \\ &\qquad\qquad + \frac{\theta - i\triangle}{\triangle}x_{k+i+1}\Bigg| d\bar{\eta}(\theta) \\ &\leq |x_k| + K\sum_{i=-M}^{-1}\int_{i\triangle}^{(i+1)\triangle}\Bigg[\frac{(i+1)\triangle - \theta}{\triangle}|x_{k+i}| \\ &\qquad\qquad + \frac{\theta - i\triangle}{\triangle}|x_{k+i+1}|\Bigg] d\bar{\eta}(\theta) \\ &\leq |x_k| + K\sup_{k\geq -M}|x_k|\sum_{i=-M}^{-1}\int_{i\triangle}^{(i+1)\triangle} d\bar{\eta}(\theta) \\ &\leq \sup_{k\geq -M}|x_k| + K\sup_{k\geq -M}|x_k|\int_{-\tau}^{0} d\bar{\eta}(\theta) \\ &\leq (1+K)\sup_{k\geq -M}|x_k| < \infty. \end{aligned} \tag{4.15}$$

Similarly, (4.1b) gives that

$$
\begin{aligned}
|g(y_{k\triangle}, k\triangle)|^2 &\le \sum_{i=-M}^{-1} \int_{i\triangle}^{(i+1)\triangle} \left[\frac{(i+1)\triangle - \theta}{\triangle} |x_{k+i}|^2 + \frac{\theta - i\triangle}{\triangle} |x_{k+i+1}|^2 \right] \\
&\le \sup_{k \ge -M} |x_k|^2 < \infty.
\end{aligned}
\tag{4.16}
$$

Noting that $\triangle w_k$ is a Brownian motion increment independent of the filtration $\{\mathcal{F}_{k\triangle}\}_{k\ge 0}$, we have $\mathbb{E}(\triangle w_k|\mathcal{F}_{k\triangle}) = \mathbb{E}\triangle w_k = 0$ and $\mathbb{E}((|\triangle w|^2 - \triangle)|\mathcal{F}_{k\triangle}) = \mathbb{E}(|\triangle w|^2 - \triangle) = 0$. These, together with (4.13), (4.15) and (4.16), yield

$$
\mathbb{E}(\bar{\rho}_{k+1}|\mathcal{F}_{k\triangle}) = 0. \tag{4.17}
$$

Noting that $\triangle w_k \sim N(0, \triangle)$ is independent of $\mathcal{F}_{k\triangle}$, it is easy to compute that

$$
\mathbb{E}(|\triangle w_k|\mathcal{F}_{k\triangle}) = \mathbb{E}|\triangle w_k| = \sqrt{\frac{2\triangle}{\pi}}
$$

and

$$
\mathbb{E}(||\triangle w_k|^2 - \triangle)||\mathcal{F}_{k\triangle}) \le \mathbb{E}|\triangle w_k|^2 + \triangle = 2\triangle.
$$

By the definition (4.11) of $\bar{\rho}_{k+1}$, we therefore have

$$
\begin{aligned}
\mathbb{E}|\bar{\rho}_{k+1}| &\le \mathbb{E}|\langle x_k + f(y_{k\triangle}, k\triangle)\triangle, g(y_{k\triangle}, k\triangle)\triangle w_k \rangle \\
&\quad + |g(y_{k\triangle}, k\triangle)|^2 (|\triangle w_k|^2 - \triangle)| \\
&\le \mathbb{E}[|x_k + f(y_{k\triangle}, k\triangle)\triangle||g(y_{k\triangle}, k\triangle)||\triangle w_k|] \\
&\quad + \mathbb{E}[|g(y_{k\triangle}, k\triangle)|^2 ||\triangle w_k|^2 - \triangle|] \\
&\le \mathbb{E}[|x_k + f(y_{k\triangle}, k\triangle)\triangle||g(y_{k\triangle}, k\triangle)|\mathbb{E}(|\triangle w_k||\mathcal{F}_{k\triangle})] \\
&\quad + \mathbb{E}[|g(y_{k\triangle}, k\triangle)|^2 \mathbb{E}(||\triangle w_k|^2 - \triangle||\mathcal{F}_{k\triangle})] \\
&\le \sqrt{\frac{2\triangle}{\pi}} \mathbb{E}[|x_k + f(y_{k\triangle}, k\triangle)\triangle||g(y_{k\triangle}, k\triangle)|] + 2\triangle \mathbb{E}[|g(y_{k\triangle}, k\triangle)|]^2 \\
&\le \sqrt{\frac{\triangle}{2\pi}} \mathbb{E}[|x_k + f(y_{k\triangle}, k\triangle)\triangle|]^2 + \left(2\triangle + \sqrt{\frac{\triangle}{2\pi}}\right) \mathbb{E}[|g(y_{k\triangle}, k\triangle)|]^2 \\
&\le \sqrt{\frac{2\triangle}{\pi}} \mathbb{E}|x_k|^2 + \triangle^2 \sqrt{\frac{2\triangle}{\pi}} \mathbb{E}|f(y_{k\triangle}, k\triangle)|^2 \\
&\quad + \left(2\triangle + \sqrt{\frac{\triangle}{2\pi}}\right) \mathbb{E}|g(y_{k\triangle}, k\triangle)|^2.
\end{aligned}
\tag{4.18}
$$

By condition (4.5), applying the Fubini Theorem and the Hölder inequality give

$$\mathbb{E}|f(y_{k\triangle},k\triangle)|^2 \le K^2 \int_{-\tau}^{0} \mathbb{E}|y_{k\triangle}(\theta)|^2 d\bar{\eta}(\theta).$$

Note that $y_{k\triangle}$ is the linear interpolation of $x_{k-M}, x_{k-M+1}, \ldots, x_0$, so (1.3) can be rewritten as

$$y_{k\triangle}(\theta) = \frac{(i+1)\triangle - \theta}{\triangle} x_{k+i} + \frac{\theta - i\triangle}{\triangle} x_{k+i+1}$$

for $\theta \in [i\triangle, (i+1)\triangle]$ ($i = -M, -M+1, \ldots, -1$), which yields

$$\begin{aligned}\mathbb{E}|y_{k\triangle}(\theta)|^2 &\le \frac{(i+1)\triangle - \theta}{\triangle}\mathbb{E}|x_{k+i}|^2 + \frac{\theta - i\triangle}{\triangle}\mathbb{E}|x_{k+i+1}|^2 \\ &\le \mathbb{E}|x_{k+i}|^2 \vee \mathbb{E}|x_{k+i+1}|^2 \\ &\le \mathbb{E}\Big(\sup_{i\in\mathbb{N}_{-M}} |x_{k+i}|^2\Big).\end{aligned}$$

This shows that

$$\mathbb{E}|f(y_{k\triangle},k\triangle)|^2 \le K^2 \mathbb{E}\Big(\sup_{i\in\mathbb{N}_{-M}} |x_{k+i}|^2\Big). \tag{4.19}$$

Similarly, condition (4.1b) gives

$$\mathbb{E}|g(y_{k\triangle},k\triangle)|^2 \le \zeta_3 \mathbb{E}\Big(\sup_{i\in\mathbb{N}_{-M}} |x_{k+i}|^2\Big). \tag{4.20}$$

Substituting (4.19) and (4.20) into (4.18) yields

$$\mathbb{E}|\bar{\rho}_{k+1}| \le \left[(1+K^2\triangle^2)\sqrt{\frac{2\triangle}{\pi}} + \zeta_3\Big(2\triangle + \sqrt{\frac{\triangle}{2\pi}}\Big)\right]\mathbb{E}\Big(\sup_{i\in\mathbb{N}_{-M}} |x_{k+i}|^2\Big).$$

Lemma 4.3 shows that $\mathbb{E}|\bar{\rho}_{k+1}| < \infty$. This, together with (4.17), implies that $\bar{\rho}_k$ is an $\mathcal{F}_{k\triangle}$-martingale-difference. Define $\triangle_1^* = [(\zeta_1 - \zeta_2 - \zeta_3)/K^2] \wedge (1/\zeta_1)$. It is clear that for any $\triangle < \triangle_1^*$, $\zeta_1 > \zeta_2 + \zeta_3 + K^2\triangle$ and $1 - \zeta_1\triangle > 0$. These show that (4.12) satisfies condition (3.8) with $\lambda_{1\triangle} = \zeta_1$ and $\lambda_{2\triangle} = \zeta_2 + \zeta_3 + K^2\triangle$. Applying Theorem 3.4 yields that

$$\limsup_{k\to\infty} \frac{\log|x_k|}{k\triangle} \le -\frac{\gamma_\triangle}{2} \quad \text{a.s.,} \tag{4.21}$$

where $\gamma_\triangle > 0$ is the unique positive root of

$$(\zeta_2 + \zeta_3 + K^2\triangle)\triangle e^{(M+1)\gamma_\triangle\triangle} + (1 - \zeta_1\triangle)e^{\gamma_\triangle\triangle} - 1 = 0. \tag{4.22}$$

Noting that $M\triangle = \tau$, equation (4.22) may be rewritten as

$$(\zeta_2 + \zeta_3 + K^2\triangle)e^{\gamma_\triangle \tau} + \frac{1 - e^{-\gamma_\triangle \triangle}}{\triangle} - \zeta_1 = 0. \tag{4.23}$$

For any $z \in \mathbb{R}_+$, define

$$h_\triangle(z) = (\zeta_2 + \zeta_3 + K^2\triangle)e^{z\tau} + \frac{1 - e^{-z\triangle}}{\triangle} - \zeta_1.$$

Noting that $\lim_{\triangle\to 0}(1 - e^{-z\triangle})/\triangle = z$, for any $z \in \mathbb{R}_+$, we have

$$\limsup_{\triangle\to 0} h_\triangle(z) = (\zeta_2 + \zeta_3)e^{z\tau} + z - \zeta_1. \tag{4.24}$$

By the definition of γ, (4.23) and (4.24) show

$$\limsup_{\triangle\to 0} \gamma_\triangle = \gamma,$$

which implies that for any positive $\varepsilon \in (0, \gamma/2)$, there exists a $\triangle_2^* > 0$ such that for any $\triangle < \triangle_2^*$, we have

$$\gamma_\triangle > \gamma - 2\varepsilon.$$

This, together with (4.21), yields that for any $\triangle < \triangle^* =: \triangle_1^* \wedge \triangle_2^*$, the desired assertion (4.6) holds. □

Theorem 4.2 shows that if the coefficient f obeys the linear growth condition, in addition to the conditions imposed in Theorem 4.1, then the EM approximate solution (1.2) may reproduce the almost sure exponential stability of exact solutions of (1.1). Moreover, for sufficiently small stepsize $\triangle$, the decay rate as measured by the Lyapunov exponent can be reproduced arbitrarily accurately.

If there is no linear growth condition (4.5) on f, for SDDEs, [35] shows that the backward EM approximations may reproduce the almost sure exponential stability. But for SFDEs, we need to give conditions which can guarantee that the backward EM method is well defined. We also need some auxiliary results, for example, establishing the result similar to Lemma 4.3 for the backward EM. Hence we hope to be able to discuss this method elsewhere.

As a special case of SFDEs, let us use Theorem 4.2 to investigate reproduction of the EM method of SDDEs for the almost sure exponential stability of the exact solutions. Let us consider the SDDE

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dw(t) \tag{4.25}$$

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with the initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, whose EM approximation is defined by

$$\begin{cases} x_k = \xi(k\triangle) & k = -m, -m+1, \dots, 0, \\ x_{k+1} = x_k + f(x_k, x_{k-m}, k\triangle)\triangle \\ \qquad\qquad + g(x_k, x_{k-m}, k\triangle)\triangle w_k, & k = 0, 1, 2, \dots. \end{cases} \tag{4.26}$$

Define u and v to be the Dirac measures in 0 and $-\tau$, respectively. Choose

$$\eta = v, \quad \mu = \bar{\eta} = \frac{1}{2}(u + v).$$

Then conditions (4.1a), (4.1b) and (4.5) may be specialized as

$$2x^T f(x, y, t) \le -\zeta_1 |x|^2 + \zeta_2 |y|^2, \tag{4.27}$$

$$|g(x, y, t)|^2 \le \frac{\zeta_3}{2}(|x|^2 + |y|^2), \tag{4.28}$$

$$|f(x, y, t)| \le \frac{K}{2}(|x| + |y|) \tag{4.29}$$

for any $x, y \in \mathbb{R}^n$ and all $t \ge 0$. Applying Theorems 4.1 and 4.2 gives the following theorem (also see Theorems 1 and 2 in [35]):

Theorem 4.4. *Under Assumption* 1.1, *assume that there exist* $\zeta_1, \zeta_2, \zeta_3$ *such that conditions* (4.27) *and* (4.28) *hold and condition* (4.2) *is satisfied. Let* γ *be determined by* (4.4). *Then for any initial data* $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, *equation* (4.25) *almost surely admits a global solution and this solution has property* (4.3). *Moreover, if condition* (4.29) *is also satisfied, for any* $\varepsilon \in (0, \gamma/2)$, *there exists a* $\triangle^* > 0$ *such that for any* $\triangle < \triangle^*$, *the EM approximation* (4.26) *has property* (4.6).

5 The linear cases and simulation

To illustrate the application of our theory, let us consider the following n-dimensional linear stochastic integro-differential equation

$$\begin{aligned} dx(t) = &\left[-Ax(t) + B\int_{-\tau}^{0} \kappa_1(\theta)x(t+\theta)d\theta\right]dt \\ &+ C\left[\int_{-\tau}^{0} \kappa_2(\theta)x(t+\theta)d\theta\right]dw(t) \end{aligned} \tag{5.1}$$

for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$, where $A, B, C \in \mathbb{R}^{n\times n}$, κ_1 and κ_2 represent the kernel functions with bounded variation on $[-\tau, 0]$. Define

$$u_1 = \int_{-\tau}^{0} \kappa_1(\theta)d\theta, \quad u_2 = \int_{-\tau}^{0} \kappa_2(\theta)d\theta.$$

It is obvious that $u_1, u_2 < \infty$. For any $\theta \in [-\tau, 0]$, define

$$\eta_1(\theta) = \frac{1}{u_1}\int_{-\tau}^{\theta} \kappa_1(s)ds, \quad \eta_2(\theta) = \frac{1}{u_2}\int_{-\tau}^{\theta} \kappa_2(s)ds.$$

Clearly, η_1 and η_2 are probability measures and (5.1) may be rewritten as

$$\begin{aligned} dx(t) = \Big[-Ax(t) + Bu_1\int_{-\tau}^{0} x(t+\theta)d\eta_1(\theta)\Big]dt \\ + Cu_2\int_{-\tau}^{0} x(t+\theta)d\eta_2(\theta)dw(t). \end{aligned} \tag{5.2}$$

For any $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$, define

$$f(\varphi, t) = -A\varphi(0) + Bu_1\int_{-\tau}^{0} \varphi(\theta)d\eta_1(\theta),$$

$$g(\varphi, t) = Cu_2\int_{-\tau}^{0} \varphi(\theta)d\eta_2(\theta).$$

It is easy to compute

$$\begin{aligned} 2\varphi^T(0)f(\varphi, t) &\le -2\varphi^T(0)A\varphi(0) + 2u_1\varphi^T(0)B\int_{-\tau}^{0} \varphi(\theta)d\eta_1(\theta) \\ &\le -\lambda_{\min}(A + A^T)|\varphi(0)|^2 \\ &\quad + u_1\|B\|\Big(|\varphi(0)|^2 + \int_{-\tau}^{0} |\varphi(\theta)|^2 d\eta_1(\theta)\Big) \\ &= -\Big(\lambda_{\min}(A + A^T) - u_1\|B\|\Big)|\varphi(0)|^2 \\ &\quad + u_1\|B\|\int_{-\tau}^{0} |\varphi(\theta)|^2 d\eta_1(\theta), \end{aligned}$$

which implies that condition (4.1a) holds with

$$\zeta_1 = \lambda_{\min}(A + A^T) - u_1\|B\|, \quad \zeta_2 = u_1\|B\|,$$

where $\lambda_{\min}(A + A^T)$ denotes the smallest eigenvalue of $A + A^T$. Applying the Hölder inequality gives that

$$|g(\varphi, t)|^2 \le u_2^2\|C\|^2\int_{-\tau}^{0} |\varphi(\theta)|^2 d\eta_2(\theta),$$

which implies that condition (4.1b) is satisfied with $\zeta_3 = u_2^2\|C\|^2$. Condition (4.2) may be specialized as

$$\lambda_{\min}(A + A^T) > 2u_1\|B\| + u_2^2\|C\|^2. \tag{5.3}$$

Let ν_1 be a Dirac measure in the origin. By the definition of f, we have

$$|f(\varphi,t)| \le \|A\||\varphi(0)| + u_1\|B\| \int_{-\tau}^{0} |\varphi(\theta)|d\eta_1(\theta)$$
$$= \|A\| \int_{-\tau}^{0} |\varphi(\theta)|d\nu_1(\theta) + u_1\|B\| \int_{-\tau}^{0} |\varphi(\theta)|d\eta_1(\theta).$$

Define the probability measure $\bar{\eta}$ such that

$$d\bar{\eta} = \frac{\|A\|d\nu_1 + u_1\|B\|d\eta_1}{\|A\| + u_1\|B\|}.$$

We therefore have

$$|f(\varphi,t)| \le (\|A\| + u_1\|B\|) \int_{-\tau}^{0} |\varphi(\theta)|d\bar{\eta}(\theta),$$

which implies that f satisfies the linear growth condition (4.5).

The EM method (1.2) applied to (5.1) has the form

$$\begin{cases} x_k = \xi(k\triangle), & k \in \mathbb{N}_{-M}, \\ x_{k+1} = (1 - A\triangle)x_k + B\triangle \displaystyle\sum_{i=-M}^{-1} \kappa_1(i\triangle)x_{k+i} & \\ \qquad + C\triangle \displaystyle\sum_{i=-M}^{-1} \kappa_2(i\triangle)x_{k+i}\triangle w_k, & k \ge 0. \end{cases} \tag{5.4}$$

Applying Theorems 4.1 and 4.2 gives the following corollary:

Corollary 5.1. *Let condition* (5.3) *hold and* $\gamma > 0$ *be the unique positive root of*

$$\lambda_{\min}(A + A^T) - u_1\|B\| - \gamma = (u_1\|B\| + u_2^2\|C\|^2)e^{\gamma\tau}. \tag{5.5}$$

For any initial data $\xi \in C([-\tau,0];\mathbb{R})$ *and* $\varepsilon \in (0,\gamma/2)$*, there exists* $\triangle^* > 0$ *such that for any* $\triangle < \triangle^*$*, the EM approximation* (5.4) *of* (5.1) *has property* (4.6).

Consider the scalar case of (5.1)

$$dx(t) = \left[-3x(t) + \int_{-\tau}^{0} x(t+\theta)d\theta\right]dt + \sigma \int_{-\tau}^{0} x(t+\theta)d\theta dw(t) \tag{5.6}$$

with initial data $\xi(\theta) = \theta + 1$ for any $\theta \in [-1,0]$. Then condition (5.3) may be rewritten as $\sigma \in (-2,2)$ under which (5.6) is almost surely exponentially stable. Choosing $\sigma = 1$, we give the following simulation (see Figure 1).

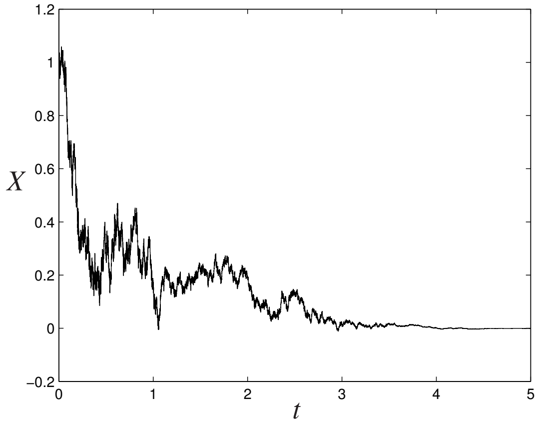

Figure 1. The figure shows a stochastic trajectory generated by the Euler–Maruyama scheme for time step $\triangle t = 10^{-5}$ for the system (5.6).

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