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Convergence rate of numerical solutions to SFDEs with jumps

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ABSTRACT

In this paper, we are interested in numerical solutions of stochastic functional differential equations with jumps. Under a global Lipschitz condition, we show that the pth-moment convergence of Euler–Maruyama numerical solutions to stochastic functional differential equations with jumps has order $1/p$ for any $p \geq 2$. This is significantly different from the case of stochastic functional differential equations without jumps, where the order is $1/2$ for any $p \geq 2$. It is therefore best to use the mean-square convergence for stochastic functional differential equations with jumps. Moreover, under a local Lipschitz condition, we reveal that the order of mean-square convergence is close to $1/2$, provided that local Lipschitz constants, valid on balls of radius j, do not grow faster than $\log j$.

1. Introduction

Recently, the theory of Functional Differential Equations (FDEs) has received a great deal of attention. Hale and Lune [1] have studied deterministic FDEs and their stability. For Stochastic Functional Differential Equations (SFDEs), here we highlight the great contribution of Kolmanovskii and Nosov [2] and Mao [3]. Kolmanovskii and Nosov [2] not only established the theory of existence and uniqueness of SFDEs but also investigated the stability and asymptotic stability of the equations, while Mao [3] studied the exponential stability of the equations.

On the other hand, Stochastic Differential Equations (SDEs) with jumps have been widely used in many branches of science and industry, in particular, in economics, finance and engineering (see, e.g., [4–6] and the references therein). Since most SDEs with jumps cannot be solved explicitly, numerical methods have become essential. Under a local Lipschitz condition, Higham and Kloeden [7] showed strong convergence and nonlinear stability for Euler–Maruyama (*EM*) numerical solutions to SDEs with jumps, while, in [8], Higham and Kloeden further revealed strong convergence rate for Backward *EM* on SDEs with jumps, provided that the drift coefficient obeys a one-side Lipschitz condition and a polynomial growth condition.

Returning to SFDEs, under a local Lipschitz condition, Mao [9] showed strong convergence of *EM* numerical solutions, while revealing convergence rate under a global Lipschitz condition. To the best of our knowledge there has been no systematic work so far on numerical methods for SFDEs with jumps. The purpose of this paper is to take some steps in this direction, building extensively on the results of Mao [9] and Yuan and Mao [10] in the Brownian motion case. In reference to the existing results in the literature, our contributions are as follows:

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- Under a global Lipschitz condition, we show that the pth-moment convergence of *EM* numerical solutions to SFDEs with jumps has order $1/p$ for any $p \geq 2$. This is significantly different from the case of SFDEs without jumps, where the order is $1/2$ for any $p \geq 2$. In practice, it is therefore best to use the mean-square convergence for SFDEs with jumps.
- Under a local Lipschitz condition, Mao [9] showed strong convergence without rate of *EM* numerical solutions to SFDEs without jumps. However, in this work we shall reveal that the order of the mean-square convergence is close to 1/2, provided that local Lipschitz constants, valid on balls of radius j, do not grow faster than $\log j$. More precisely, the order of the mean-square convergence is $1/(2+\epsilon)$, provided that local Lipschitz constants do not grow faster than $(\log j)^{1/(1+\epsilon)}$.
- Some new techniques are developed to cope with the difficulties due to the jumps.

This paper is organized as follows: Section 2 gives some preliminary results, in particular, *EM* numerical solutions to SFDEs with jumps are set up. In Section 3, we discuss the pth-moment convergence of *EM* numerical solutions to SFDEs with jumps under a global Lipschitz condition. The rate of the mean-square convergence for *EM* numerical solutions to SFDEs with jumps under a local Lipschitz condition is provided in Section 4. Finally, in order to make the paper self-contained, an existence-and-uniqueness result of solutions to SFDEs with jumps is provided in the Appendix.

2. Preliminaries

Throughout this paper, we let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is continuous on the right and $\mathcal{F}_0$ contains all P-zero sets). Let $|\cdot|$ denote the Euclidean norm and the matrix trace norm. Let $\tau > 0$ and $D := D([-\tau, 0]; R^n)$ denote the family of all right-continuous functions with left-hand limits φ from $[-\tau, 0]$ to R^n, and $\hat{D} := \hat{D}([-\tau, 0]; R^n)$ denote the family of all left-continuous functions with right-hand limits φ from $[-\tau, 0]$ to R^n, we will always use $\|\varphi\| := \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$ to denote the norm in D and $\hat{D}$ potentially involved when no confusion possibly arises. $D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, D-valued random variables. For all $t \geq 0$, $x_t := \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is regarded as a D-valued stochastic process. Let $x(t^-) := \lim_{s\uparrow t} x(s)$ on $t \geq -\tau$ and $x_{t^-} := \{x(t+\theta)^- : -\tau \leq \theta \leq 0\}$. It is easy to see that $x(t^-)$ is a $\hat{D}$-valued stochastic process.

It should be pointed out that space D and $\hat{D}$ are not complete under the supremum norm $\|\cdot\|$. To make D a complete space, we need to define the following metric (see [11, Chapter 3]). Let Λ denote the class of strictly increasing, continuous mapping of $[-\tau, 0]$ onto itself and

$$\Lambda^*_\epsilon = \left\{\lambda \in \Lambda : \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \leq \epsilon\right\},$$

define

$$d(\xi, \zeta) = \inf\left\{\epsilon > 0 : \exists\lambda \in \Lambda^*_\epsilon \text{ such that } \sup_{t\in[-\tau,0]} |\xi(t) - \zeta(\lambda(t))| \leq \epsilon\right\}. \tag{2.1}$$

$d(\cdot, \cdot)$ is called a Skorohod metric, and by [11, Theorem 14.2, p115] we know that D is complete in the metric d. Since the supremum norm and the Skorohod metric are equivalent (see [11, Theorem 14.1, p114]), we shall use the supremum norm for studying the convergence, however, we use the Skorohod metric to investigate the existence and uniqueness of the equations in Appendix.

In this paper, we consider the following SFDE with jumps

$$dx(t) = f(x_t)dt + g(x_t)dB(t) + h(x_{t^-})dN(t), \quad 0 \leq t \leq T, \tag{2.2}$$

with the initial data $x_0 = \xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. Here, $f, h : \hat{D} \to R^n$, $g : \hat{D} \to R^{n\times m}$, $B(t)$ is an m-dimensional Brownian motion and $N(t)$ is a scalar Poisson process with intensity λ. We further assume that $B(t)$ and $N(t)$ are independent. It should be pointed out that the solution of Eq. (2.2) is in D.

For our purposes, we need the following assumptions which can also guarantee the existence and uniqueness of solution to (2.2) (see Appendix).

(**H1**) (Global Lipschitz condition) There exists a left-continuous nondecreasing function $\mu : [-\tau, 0] \to R_+$ such that for all $\varphi, \psi \in \hat{D}$

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq \int_{-\tau}^0 |\varphi(\theta) - \psi(\theta)|^2 d\mu(\theta). \tag{2.3}$$

Remark 2.1. For simplicity, we write $L := \mu(0) - \mu(-\tau)$, which is referred to as the global Lipschitz constant. Note from (2.3) that for all $\varphi, \psi \in \hat{D}$

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq L\|\varphi - \psi\|^2. \tag{2.4}$$

This further implies the linear growth condition, that is, for $\varphi \in \hat{D}$

$$|f(\varphi)|^2 \vee |g(\varphi)|^2 \vee |h(\varphi)|^2 \leq K(1 + \|\varphi\|^2), \tag{2.5}$$

where $K := 2(L \vee |f(0)|^2 \vee |g(0)|^2 \vee |h(0)|^2)$.

(H2) (Continuity of initial data) For $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ and some $p \geq 2$, there is a constant $\beta > 0$ such that

$$E\left(|\xi(s) - \xi(t)|^p\right) \leq \beta|t - s|, \quad t, s \in [-\tau, 0]. \tag{2.6}$$

For given $T \geq 0$ and $\tau > 0$, the time-step size $\triangle \in (0, 1)$ is defined by

$$\triangle := \frac{\tau}{N} = \frac{T}{M}$$

with some integers $N > \tau$ and $M > T$. Following [9], the *EM* method applied to (2.2) produces approximations $\bar{y}(k\triangle) \approx x(k\triangle)$ by setting $\bar{y}(k\triangle) := \xi(k\triangle)$, $-N \leq k \leq 0$, and

$$\bar{y}((k+1)\triangle) = \bar{y}(k\triangle) + f(\bar{y}_{k\triangle})\triangle + g(\bar{y}_{k\triangle})\triangle B_k + h(\bar{y}_{k\triangle})\triangle N_k, \tag{2.7}$$

where $\triangle B_k := B((k+1)\triangle) - B(k\triangle)$ is a Brownian increment, $\triangle N_k := N((k+1)\triangle) - N(k\triangle)$ is a Poisson increment, and $\bar{y}_{k\triangle} := \{\bar{y}_{k\triangle}(\theta) : -\tau \leq \theta \leq 0\}$ is a D-valued random variable defined by

$$\bar{y}_{k\triangle}(\theta) := \frac{(i+1)\triangle - \theta}{\triangle}\bar{y}((k+i)\triangle) + \frac{\theta - i\triangle}{\triangle}\bar{y}((k+i+1)\triangle) \tag{2.8}$$

for $i\triangle \leq \theta \leq (i+1)\triangle$, $i = -N, -(N-1), \ldots, -1$, where in order for $\bar{y}_{-\triangle}$ to be well defined, we set $\bar{y}(-(N+1)\triangle) = \xi(-N\triangle)$.

Given the discrete-time approximation $\{\bar{y}(k\triangle)\}_{k\geq 0}$, we define a continuous-time approximation $y(t)$ by setting $y(t) := \xi(t)$ for $-\tau \leq t \leq 0$, while for $t \in [0, T]$

$$y(t) = \xi(0) + \int_0^t f(\bar{y}_s)\mathrm{d}s + \int_0^t g(\bar{y}_s)\mathrm{d}B(s) + \int_0^t h(\bar{y}_{s-})\mathrm{d}N(s), \tag{2.9}$$

where

$$\bar{y}_{t-} := \lim_{s\uparrow t} \bar{y}_s, \qquad \bar{y}_t := \sum_{k=0}^{M-1} \bar{y}_{k\triangle} I_{[k\triangle,(k+1)\triangle)}(t).$$

It is easy to see that $y(k\triangle) = \bar{y}(k\triangle)$ for $k = -N, -N+1, \ldots, M$. That is, the discrete-time and continuous-time *EM* numerical solutions coincide at the gridpoints.

Remark 2.2. It is easy to observe from (2.8) that

$$\|\bar{y}_{k\triangle}\| = \max_{-N\leq i\leq 0} |\bar{y}((k+i)\triangle)|, \quad k = -1, 0, 1, \ldots, M-1, \tag{2.10}$$

which further yields

$$\|\bar{y}_{k\triangle}\| \leq \|y_{k\triangle}\|, \quad k = -1, 0, 1, \ldots, M-1,$$

by $y(k\triangle) = \bar{y}(k\triangle)$ and for any $t \in [0, T]$

$$\|\bar{y}_t\| = \|\bar{y}_{[\frac{t}{\triangle}]\triangle}\| \leq \|y_{[\frac{t}{\triangle}]\triangle}\| \leq \sup_{-\tau\leq s\leq t} |y(s)|, \tag{2.11}$$

where $[\frac{t}{\triangle}]$ is the integer part of $\frac{t}{\triangle}$.

3. Convergence rate under global Lipschitz condition

In this section, we shall investigate convergence rate of *EM* numerical scheme under global Lipschitz condition (2.3). Our results reveal a significant difference from these on the SFDEs without jumps.

Lemma 3.1. *Under condition (2.5), for $\|\xi\|^p < \infty$, $p \geq 2$, there exists a positive constant $H(p) := H(p, T, \xi, K)$ such that*

$$E\left(\sup_{-\tau\leq t\leq T} |x(t)|^p\right) \vee E\left(\sup_{-\tau\leq t\leq T} |y(t)|^p\right) \leq H(p). \tag{3.1}$$

Proof. Since the arguments of the moment bounds for the exact and continuous approximate solutions to (2.2) are very similar, here we only give an estimate for the continuous approximate solution $y(t)$. For every integer $R \geq 1$, define a stopping time

$$\theta_R := \inf\{t \geq 0 : \|y_t\| > R\}.$$

It is easy to see from (2.9) that for any $t \in [0, T]$

$$E\left(\sup_{0\le s\le t}|y(s\wedge\theta_R)|^p\right) \le 4^{p-1}\left[E\|\xi\|^p + E\left(\sup_{0\le s\le t}\left|\int_0^s f(\bar{y}_{r\wedge\theta_R})\mathrm{d}r\right|^p\right)\right.$$
$$\left. + E\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{y}_{r\wedge\theta_R})\mathrm{d}B(r)\right|^p\right) + E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})\mathrm{d}N(r)\right|^p\right)\right]. \tag{3.2}$$

By the Hölder inequality and (2.5)

$$\begin{aligned} E\left(\sup_{0\le s\le t}\left|\int_0^s f(\bar{y}_{r\wedge\theta_R})\mathrm{d}r\right|^p\right) &\le T^{p-1}\int_0^t E|f(\bar{y}_{r\wedge\theta_R})|^p\mathrm{d}r \\ &\le T^{p-1}\int_0^t E[K(1+\|\bar{y}_{r\wedge\theta_R}\|^2)]^{\frac{p}{2}}\mathrm{d}r \\ &\le 2^{\frac{p}{2}-1}T^pK^{\frac{p}{2}} + 2^{\frac{p}{2}-1}T^{p-1}K^{\frac{p}{2}}\int_0^t E\|\bar{y}_{r\wedge\theta_R}\|^p\mathrm{d}r. \end{aligned}$$

This, together with (2.11), immediately reveals that

$$E\left(\sup_{0\le s\le t}\left|\int_0^s f(\bar{y}_{r\wedge\theta_R})\mathrm{d}r\right|^p\right) \le c_1T + c_1\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)|^p\right)\mathrm{d}s, \tag{3.3}$$

where $c_1 = 2^{\frac{p}{2}-1}T^{p-1}K^{\frac{p}{2}}$. Now, using the Burkholder–Davis–Gundy inequality [3, Theorem 7.3, p40] and the Hölder inequality, we deduce that there exists a positive constant c_p such that

$$\begin{aligned} E\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{y}_{r\wedge\theta_R})\mathrm{d}B(r)\right|^p\right) &\le c_pE\left(\int_0^t |g(\bar{y}_{r\wedge\theta_R})|^2\mathrm{d}r\right)^{p/2} \\ &\le c_pT^{\frac{p-2}{2}}\int_0^t E|g(\bar{y}_{r\wedge\theta_R})|^p\mathrm{d}r. \end{aligned}$$

In the same way as (3.3) was done, it then follows easily that

$$E\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{y}_{r\wedge\theta_R})\mathrm{d}B(r)\right|^p\right) \le c_2T + c_2\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)|^p\right)\mathrm{d}s,$$

where $c_2 = 2^{\frac{p}{2}-1}T^{\frac{p-2}{2}}K^{\frac{p}{2}}c_p$. Moreover, observing that $\tilde{N}(t) = N(t) - \lambda t, t \geq 0$ is a martingale measure, using the Burkholder–Davis–Gundy inequality [12, Theorem 48, p193], Hölder inequality and (2.5), we obtain for some positive constant $\bar{c}_p$,

$$\begin{aligned} E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})\mathrm{d}N(r)\right|^p\right) &\le E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})\mathrm{d}\tilde{N}(r) + \lambda\int_0^s h(\bar{y}_{r\wedge\theta_R})\mathrm{d}r\right|^p\right) \\ &\le 2^p\left[E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})\mathrm{d}\tilde{N}(r)\right|^p + \lambda^p\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{r\wedge\theta_R})\mathrm{d}r\right|^p\right)\right] \\ &\le 2^p\left[\bar{c}_p\lambda^{p/2}E\left(\int_0^t |h(\bar{y}_{r\wedge\theta_R})|^2\mathrm{d}r\right)^{p/2} + \lambda^pT^{p-1}\int_0^t E|h(\bar{y}_{r\wedge\theta_R})|^p\mathrm{d}r\right] \\ &\le c_3T + c_3\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)|^p\right)\mathrm{d}s, \end{aligned}$$

where $c_3 = 2^{\frac{3p}{2}-1}K^{\frac{p}{2}}\left[\bar{c}_p\lambda^{p/2}T^{\frac{p-2}{2}} + \lambda^pT^{p-1}\right]$. Hence, in (3.2)

$$E\left(\sup_{0\le s\le t}|y(s\wedge\theta_R)|^p\right) \le 4^{p-1}\left[E\|\xi\|^p + (c_1+c_2+c_3)T + (c_1+c_2+c_3)\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)|^p\right)\mathrm{d}s\right].$$

Note that

$$E\left(\sup_{-\tau\le s\le t}|y(s\wedge\theta_R)|^p\right)\le E\|\xi\|^p+E\left(\sup_{0\le s\le t}|y(s\wedge\theta_R)|^p\right).$$

Applying the Gronwall inequality and letting $R\to\infty$, we then obtain

$$E\left(\sup_{-\tau\le t\le T}|y(t)|^p\right)\le H(p).$$

Since T is any fixed positive number, the required assertion follows. □

In order to obtain our main results, we need to estimate the pth moment of $y(s+\theta)-\bar{y}_s(\theta)$.

Lemma 3.2. *Let conditions (2.5) and (2.6) hold. Then, for $p\ge 2$ and $s\in[0,T]$*

$$E|y(s+\theta)-\bar{y}_s(\theta)|^p\le\gamma\triangle,\quad -\tau\le\theta\le 0, \tag{3.4}$$

where γ is a positive constant independent of $\triangle$.

Proof. Fix $s\in[0,T]$ and $\theta\in[-\tau,0]$. Let $k_s\in\{0,1,2,\cdots,M-1\}$, $k_\theta\in\{-N,-N+1,\ldots,-1\}$ be the integers for which $s\in[k_s\triangle,(k_s+1)\triangle)$, $\theta\in[k_\theta\triangle,(k_\theta+1)\triangle)$, respectively. For convenience, we write $v=s+\theta$ and $k_v=k_s+k_\theta$. Clearly, $0\le s-k_s\triangle<\triangle$ and $0\le\theta-k_\theta\triangle\le\triangle$, so

$$0\le v-k_v\triangle<2\triangle.$$

Recalling the definition of $\bar{y}_s$, $s\in[0,T]$, we then obtain from (2.8) that

$$\bar{y}_s(\theta)=\bar{y}_{k_s\triangle}(\theta)=\bar{y}(k_v\triangle)+\frac{\theta-k_\theta\triangle}{\triangle}[\bar{y}((k_v+1)\triangle)-\bar{y}(k_v\triangle)],$$

which implies

$$E|y(s+\theta)-\bar{y}_s(\theta)|^p\le 2^{p-1}E|\bar{y}((k_v+1)\triangle)-\bar{y}(k_v\triangle)|^p+2^{p-1}E|y(v)-\bar{y}(k_v\triangle)|^p. \tag{3.5}$$

For $k_v\le -1$, it thus follows from (2.6) that

$$E|\bar{y}((k_v+1)\triangle)-\bar{y}(k_v\triangle)|^p\le\beta\triangle. \tag{3.6}$$

Note that for some $\bar{H}:=\bar{H}(m,p)$

$$E|B(t)|^p\le\bar{H}t^{\frac{p}{2}},\quad t\ge 0, \tag{3.7}$$

and, by the characteristic functions' argument, for $\triangle\in(0,1)$

$$E|\triangle N_k|^p\le C\triangle, \tag{3.8}$$

where C is a positive constant which is independent of $\triangle$. For $k_v\ge 0$, using (2.7) and noting $g(\bar{y}_{k_v\triangle})$ and B_{k_v}, $h(\bar{y}_{k_v\triangle})$ and N_{k_v} are independent, respectively, we compute

$$E|\bar{y}((k_v+1)\triangle)-\bar{y}(k_v\triangle)|^p\le 3^{p-1}\left[E|f(\bar{y}_{k_v\triangle})|^p\triangle^p+E|g(\bar{y}_{k_v\triangle})|^pE|\triangle B_{k_v}|^p+E|h(\bar{y}_{k_v\triangle})|^pE|\triangle N_{k_v}|^p\right].$$

Taking (2.5) into consideration and applying Lemma 3.1, we then obtain that for $\triangle\in(0,1)$

$$E|\bar{y}((k_v+1)\triangle)-\bar{y}(k_v\triangle)|^p\le 3^{p-1}2^{\frac{p}{2}-1}K^{\frac{p}{2}}(1+H(p))(1+\bar{H}+C)\triangle. \tag{3.9}$$

Hence, in (3.5)

$$E|y(s+\theta)-\bar{y}_s(\theta)|^p\le 2^{p-1}\beta+3^{p-1}2^{\frac{3p}{2}-2}K^{\frac{p}{2}}(1+H(p))(1+\bar{H}+C)\triangle+2^{p-1}E|y(v)-\bar{y}(k_v\triangle)|^p. \tag{3.10}$$

In what follows, we divide the following five cases to estimate the second term on the right-hand side of (3.10).

Case 1: $k_v\ge 0$ and $0\le v-k_v\triangle<\triangle$. By (2.9)

$$\begin{aligned}E|y(v)-\bar{y}(k_v\triangle)|^p&=E|f(\bar{y}_{k_v\triangle})(v-k_v\triangle)+g(\bar{y}_{k_v\triangle})(B(v)-B(k_v\triangle))+h(\bar{y}_{k_v\triangle})(N(v)-N(k_v\triangle))|^p\\&\le 3^{p-1}E|f(\bar{y}_{k_v\triangle})|^p(v-k_v\triangle)^p+3^{p-1}E|g(\bar{y}_{k_v\triangle})|^pE|B(v)-B(k_v\triangle)|^p\\&\quad+3^{p-1}E|h(\bar{y}_{k_v\triangle})|^pE|N(v)-N(k_v\triangle)|^p.\end{aligned}$$

Then, in the same way as (3.9) was done, we have for $\triangle\in(0,1)$

$$E|y(v)-\bar{y}(k_v\triangle)|^p\le 3^{p-1}2^{\frac{p}{2}-1}K^{\frac{p}{2}}(1+H(p))(1+\bar{H}+C)\triangle.$$

Case 2: $k_v \geq 0$ and $\triangle \leq v - k_v\triangle < 2\triangle$. It then follows easily that

$$\begin{aligned} E|y(v) - \bar{y}(k_v\triangle)|^p &= E|y(v) - \bar{y}((k_v+1)\triangle) + \bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p \\ &\leq 2^{p-1}E|y(v) - \bar{y}((k_v+1)\triangle)|^p + 2^{p-1}E|\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p. \end{aligned}$$

This, together with (3.9) and Case 1, leads to

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq 3^{p-1}2^{\frac{3p}{2}-1}K^{\frac{p}{2}}(1+H(p))(1+\bar{H}+C)\triangle.$$

Case 3: $k_v = -1$ and $0 \leq v - k_v\triangle \leq \triangle$. In this case, $-\triangle \leq v \leq 0$. We then have from (2.6) that

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq \beta\triangle.$$

Case 4: $k_v = -1$ and $\triangle \leq v - k_v\triangle < 2\triangle$. In such a case, $0 \leq v < \triangle$. Case 1 and Case 2 can be used to estimate the term

$$\begin{aligned} E|y(v) - \bar{y}(k_v\triangle)|^p &\leq 2^{p-1}E|y(v) - \xi(0)|^p + 2^{p-1}E|\xi(0) - \bar{y}(k_v\triangle)|^p \\ &\leq [2^{p-1}\beta + 3^{p-1}2^{\frac{3p}{2}-2}K^{\frac{p}{2}}(1+H(p))(1+\bar{H}+C)]\triangle. \end{aligned}$$

Case 5: $k_v \leq -2$. In this case, $v < 0$. So, by (2.6)

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq 2\beta\triangle.$$

Combining Case 1 to Case 5, we therefore complete the proof. □

The following theorem will tell us the error of the pth moment between the true solution and numerical solution under a global Lipschitz condition.

Theorem 3.1. *Under conditions (2.4) and (2.6), for $p \geq 2$*

$$E\left(\sup_{0\leq t\leq T}|x(t) - y(t)|^p\right) \leq \delta_1 L^{\frac{p}{2}} e^{\delta_2 L^{\frac{p}{2}}}\triangle, \tag{3.11}$$

where δ_1, δ_2 are constants, independent of $\triangle$.

Proof. It is easy to see from (2.2) and (2.9) that for any $t_1 \in [0, T]$

$$\begin{aligned} E\left(\sup_{0\leq t\leq t_1}|x(t) - y(t)|^p\right) \leq\ & 3^{p-1}E\left(\sup_{0\leq t\leq t_1}\left|\int_0^t f(x_s) - f(\bar{y}_s)\mathrm{d}s\right|^p\right) + 3^{p-1}E\left(\sup_{0\leq t\leq t_1}\left|\int_0^t g(x_s) - g(\bar{y}_s)\mathrm{d}B(s)\right|^p\right) \\ & + 3^{p-1}E\left(\sup_{0\leq t\leq t_1}\left|\int_0^t h(x_{s^-}) - h(\bar{y}_{s^-})\mathrm{d}N(s)\right|^p\right) \\ :=\ & I_1 + I_2 + I_3. \end{aligned} \tag{3.12}$$

In what follows, we estimate the three terms, respectively. By the Hölder inequality, (2.4) and Lemma 3.2,

$$\begin{aligned} I_1 &\leq 3^{p-1}T^{p-1}\int_0^{t_1} E|f(x_s) - f(\bar{y}_s)|^p\mathrm{d}s \\ &\leq 6^{p-1}T^{p-1}\int_0^{t_1} E|f(x_s) - f(y_s)|^p\mathrm{d}s + 6^{p-1}T^{p-1}\int_0^{t_1} E|f(y_s) - f(\bar{y}_s)|^p\mathrm{d}s \\ &\leq 6^{p-1}T^{p-1}\int_0^{t_1} E\left(\int_{-\tau}^0 |x(s+\theta) - y(s+\theta)|^2\mathrm{d}\mu(\theta)\right)^{\frac{p}{2}}\mathrm{d}s \\ &\quad + 6^{p-1}T^{p-1}\int_0^{t_1} E\left(\int_{-\tau}^0 |y(s+\theta) - \bar{y}_s(\theta)|^2\mathrm{d}\mu(\theta)\right)^{\frac{p}{2}}\mathrm{d}s \\ &\leq 6^{p-1}T^{p-1}L^{\frac{p}{2}}\int_0^{t_1} E\left(\sup_{0\leq r\leq s}|x(r) - y(r)|^p\right)\mathrm{d}s + 6^{p-1}T^{p-1}L^{\frac{p-2}{2}}\int_0^{t_1}\int_{-\tau}^0 E|y(s+\theta) - \bar{y}_s(\theta)|^p\mathrm{d}\mu(\theta)\mathrm{d}s \\ &\leq 6^{p-1}T^pL^{\frac{p}{2}}\gamma\triangle + 6^{p-1}T^{p-1}L^{\frac{p}{2}}\int_0^{t_1} E\left(\sup_{0\leq r\leq s}|x(r) - y(r)|^p\right)\mathrm{d}s. \end{aligned}$$

Now, the Burkholder–Davis–Gundy inequality [3, Theorem 7.3, p40], (2.4) and Lemma 3.2 also give that for some positive constant C_p

$$
\begin{aligned}
I_2 &\le 3^{p-1}C_pE\left(\int_0^{t_1}|g(x_s)-g(\bar{y}_s)|^2\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le 3^{p-1}T^{\frac{p-2}{2}}C_p\int_0^{t_1}E|g(x_s)-g(\bar{y}_s)|^p\mathrm{d}s\\
&\le 6^{p-1}T^{\frac{p-2}{2}}C_p\left[L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)\mathrm{d}s+L^{\frac{p-2}{2}}\int_0^{t_1}\int_{-\tau}^0E|y(s+\theta)-\bar{y}_s(\theta)|^p\mathrm{d}\mu(\theta)\mathrm{d}s\right]\\
&\le 6^{p-1}\gamma T^{\frac{p}{2}}C_pL^{\frac{p}{2}}\triangle+6^{p-1}T^{\frac{p-2}{2}}C_pL^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)\mathrm{d}s.
\end{aligned}
\tag{3.13}
$$

In the same way as (3.13) was done, together with the Burkholder–Davis–Gundy inequality [12, Theorem 48, p193], we can deduce from (2.4) that for some positive constant $\bar{C}_p$

$$
\begin{aligned}
I_3 &\le 6^{p-1}E\left(\sup_{0\le t\le t_1}\left|\int_0^t h(x_{s-})-h(\bar{y}_{s-})\mathrm{d}\tilde{N}(s)\right|^p+\lambda^p\sup_{0\le t\le t_1}\left|\int_0^t h(x_s)-h(\bar{y}_s)\mathrm{d}s\right|^p\right)\\
&\le 6^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})\int_0^{t_1}E|h(x_s)-h(\bar{y}_s)|^p\mathrm{d}s\\
&\le 12^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)\mathrm{d}s\\
&\quad+12^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})L^{\frac{p-2}{2}}\int_0^{t_1}\int_{-\tau}^0E|y(s+\theta)-\bar{y}_s(\theta)|^p\mathrm{d}\mu(\theta)\mathrm{d}s\\
&\le 12^{p-1}\gamma(\bar{C}_pT^{\frac{p}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^p)L^{\frac{p}{2}}\triangle+12^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)\mathrm{d}s.
\end{aligned}
$$

Therefore

$$
E\left(\sup_{0\le t\le t_1}|x(t)-y(t)|^p\right)\le\delta_1L^{\frac{p}{2}}\triangle+\delta_2L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)\mathrm{d}s,
$$

where $\delta_1 = 6^{p-1}\gamma T^{\frac{p}{2}}\left(T^{\frac{p}{2}}+C_p+2^{p-1}\bar{C}_p\lambda^{\frac{p}{2}}+\lambda^pT^{\frac{p}{2}}\right)$ and $\delta_2 = 6T^{\frac{p-2}{2}}\left(T^{\frac{p}{2}}+C_p+2^{p-1}\bar{C}_p\lambda^{\frac{p}{2}}+\lambda^pT^{\frac{p}{2}}\right)$. The desired assertion thus follows from the Gronwall inequality. □

Remark 3.1. The result of Theorem 3.1 tells us that

$$
E\left(\sup_{0\le t\le T}|x(t)-y(t)|^2\right)\le\delta_3Le^{\delta_4L}\triangle,
\tag{3.14}
$$

where δ_3, δ_4 are constants which are independent of $\triangle$ under global Lipschitz condition (2.4). This means that the order of the mean-square convergence is 1/2, while Eq. (3.11) tells us that the order of the pth-moment convergence is $1/p$ ($p\ge 2$). In other words, the lower moment has a better convergence rate for SFDEs with jumps, whence it is best in practice to use the mean-square convergence. This is significantly different from the result on SFDEs without jumps. Letting $h\equiv 0$ in (2.2), i.e. there are no jumps, we already known that for $p\ge 2$ (see [10])

$$
E\left(\sup_{0\le t\le T}|x(t)-y(t)|^p\right)\le\hat{C}_1\,\triangle^{p/2},
$$

where $\hat{C}_1$ is a constant independent of $\triangle$. This means that the order of the pth-moment convergence is 1/2 for all $p\ge 2$. Why is there a significant difference? Actually, it is due to the following fact: all moments of the Poisson increments $N((k+1)\triangle)-N(k\triangle)$ have the same order of $\triangle$ (see (3.8)), while the moments of increments $\triangle B_k = B((k+1)\triangle)-B(k\triangle)$ have different orders, namely $E|\triangle B_k|^{2n}=O(\triangle^n)$ and $E|\triangle B_k|^{2n+1}=0$.

The differences are already relevant to simple simulations. Consider the case of constant coefficients as example and compare the exact values of the moments. The moments are

$$
E|\triangle B_k|^2=\triangle,\qquad E|\triangle N_k|^2=\triangle+\triangle^2,\qquad E|\triangle B_k+\triangle N_k|^2=2\triangle+\triangle^2
$$

and

$$
E|\triangle B_k|^4=3\,\triangle^2,\qquad E|\triangle N_k|^4=\triangle+7\,\triangle^2+6\,\triangle^3+\triangle^4,\qquad E|\triangle B_k+\triangle N_k|^4=\triangle+16\,\triangle^2+12\,\triangle^3+\triangle^4\,.
$$

Thus in the case of Brownian motion without additional jumps for $\triangle > \frac{1}{3}$ the second moment yields the smaller error, for $\triangle = \frac{1}{3}$ the errors coincide and for $\triangle < \frac{1}{3}$ the fourth moment yields the smaller error. Furthermore the fourth moment is $O(\triangle^2)$. In the pure jump case the fourth moment is always larger than the second and in the mixed case the moments coincide for $\triangle \approx 0.0634$, for larger $\triangle$ the second moment is smaller than the fourth and for $\triangle < 0.0634$ the fourth is smaller than the second moment. Therefore in the mixed case, if one just goes by magnitude, one should stick to the first moment at least for all $\triangle \geq 0.0634$. For smaller $\triangle$ one could consider the fourth moment, but the cost of calculating the additional power have to be measured against the minor gain, which is only an improvement of the slope ($\triangle$ instead of $2\triangle$) but not of the order (i.e. both moments are $O(\triangle)$).

We have done a simple simulation to visualize the errors of the above toy example. For this we simulated 1000 sample paths of Brownian motion resp. of the Poisson process (with intensity 1) up to time 1 on a 0.0001 grid. Next we defined the approximations for $\triangle = 0.1$ and 0.01 and calculated the empirical moments of the errors at times 0.0999, 0.1999, ... The result of the simulation can be seen in Fig. 1. One difference between the theoretical discussion above and the actual simulation, is that the second and fourth empirical moments of the error in the Poisson case do usually coincide. This happens, since in a simulation one considers only finitely many paths and thus for $\triangle$ small enough, each of these paths has only at most one jump during a time step of size $\triangle$. Therefore the error is zero or one and these are invariant under taking the second or fourth power.

Remark 3.2. As we stated in the Introduction section, there has been no systematic work so far on numerical schemes for SFDEs with jumps (pure jumps). As sequels to this work, we shall report two extensions in future work:

(i) Strong convergence of *EM* numerical schemes of SFDE with pure jumps

$$\mathrm{d}x(t) = f(x_t)\mathrm{d}t + h(x_{t-})\mathrm{d}N(t), \quad 0 \leq t \leq T, \tag{3.15}$$

under condition

$$|f(\varphi) - f(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq L\|\varphi - \psi\|^2 \tag{3.16}$$

for $L > 0$. Since all moments of the Poisson increments $N((k+1)\triangle) - N(k\triangle)$ have the same order of $\triangle(\in (0, 1))$, the challenge is to estimate

$$E\left(\sup_{1\leq k\leq M-1} \|\bar{y}_{k\triangle} - \bar{y}_{(k-1)\triangle}\|^2\right)$$

under condition (3.16), where $\bar{y}_{k\triangle}$ is determined similarly by (2.7).

(ii) Invariant measure for *EM* numerical solutions to Eq. (3.15). Based on the strong convergence established in problem (i), the key ingredient is to show the Markovian property of $\bar{y}_{k\triangle}$ to show problem (ii).

4. Convergence rate under local Lipschitz condition

In this section, we shall discuss convergence rate of *EM* numerical solutions to (2.2) under the following local Lipschitz condition.

(**H3**) (Local Lipschitz condition) For each integer $j \geq 1$, there is a left-continuous nondecreasing function $\mu_j : [-\tau, 0] \to R_+$ such that

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq \int_{-\tau}^{0} |\varphi(\theta) - \psi(\theta)|^2 \mathrm{d}\mu_j(\theta), \tag{4.1}$$

for those $\varphi, \psi \in \hat{D}$ with $\|\varphi\| \vee \|\psi\| \leq j$.

(**H4**) (Linear growth condition) Assume that there is a constant $h > 0$ such that for $\varphi \in \hat{D}$

$$|f(\varphi)|^2 \vee |g(\varphi)|^2 \vee |h(\varphi)|^2 \leq h(1 + \|\varphi\|^2). \tag{4.2}$$

Remark 4.1. Under conditions (4.1) and (4.2), for any initial data $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, (2.2) admits a unique solution $x(t), t \in [0, T]$, by using the standard truncation procedure (see [3, Theorem 3.4, p56]). Moreover, (4.1) implies for those $\varphi, \psi \in \hat{D}$ with $\|\varphi\| \vee \|\psi\| \leq j$

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq L_j\|\varphi - \psi\|^2, \tag{4.3}$$

where $L_j := \mu_j(0) - \mu_j(-\tau)$.

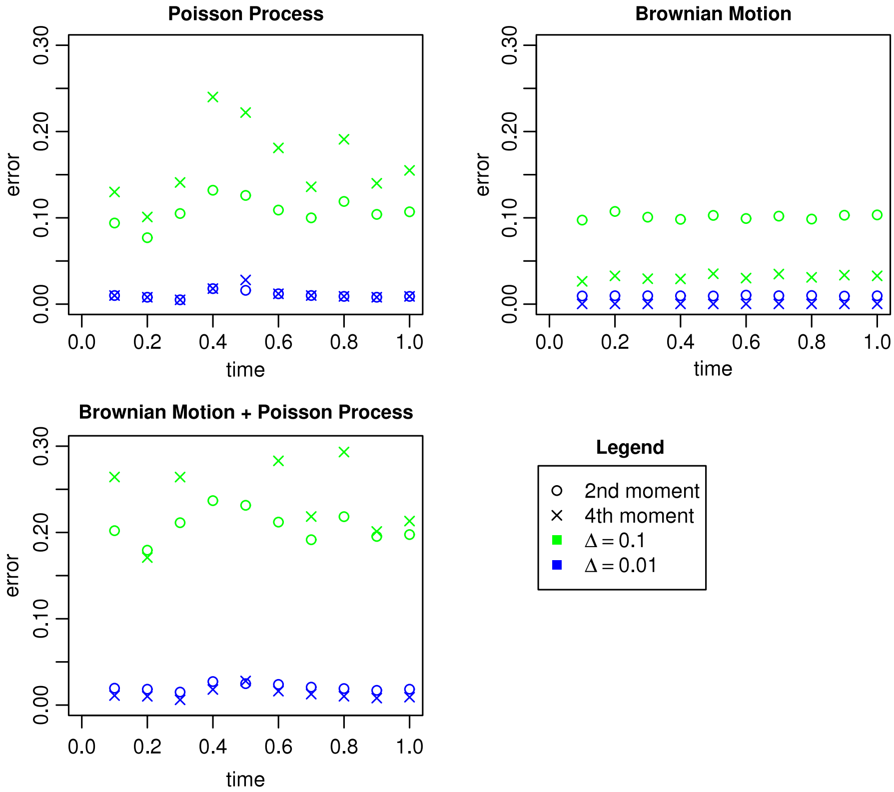

Fig. 1. Simulation of the example—empirical errors.

Theorem 4.1. *Let conditions (2.6), (4.1) and (4.2) hold. If there exist positive constants* α *and* $\tilde{\varepsilon} \in (0, 1)$ *such that local Lipschitz constants obey*

$$L_j^{1+\tilde{\varepsilon}} \leq \alpha \log j, \tag{4.4}$$

then

$$E\left(\sup_{0\leq t\leq T} |x(t) - y(t)|^2\right) = O(\triangle^{\frac{2}{2+\epsilon}}), \tag{4.5}$$

where $\epsilon \in (0, \tilde{\varepsilon})$ *is an arbitrarily fixed small positive number.*

Proof. Let $j \geq 1$ be an integer, and let $S_j = \{x \in R^n : |x| \leq j\}$. Define the projection $\pi_j : R^n \to S_j$ by

$$\pi_j(x) = \frac{j \wedge |x|}{|x|} x,$$

where we set $\pi_j(0) = 0$ as usual. It is easy to see that for all $x, y \in R^n$

$$|\pi_j(x) - \pi_j(y)| \leq |x - y|.$$

Define the operator $\bar{\pi}_j : \hat{D} \to \hat{D}$ by

$$\bar{\pi}_j(\varphi) = \{\pi_j(\varphi(\theta)) : -\tau \leq \theta \leq 0\}.$$

Clearly,

$$\|\bar{\pi}_j(\varphi)\| \leq j, \quad \forall \varphi \in \hat{D}.$$

Define the truncation functions $f_j : \hat{D} \to R^n, g_j : \hat{D} \to R^{n\times m}$ and $h_j : \hat{D} \to R^n$ by

$$f_j(\varphi) = f(\bar{\pi}_j(\varphi)), \qquad g_j(\varphi) = g(\bar{\pi}_j(\varphi)), \qquad h_j(\varphi) = h(\bar{\pi}_j(\varphi)), \tag{4.6}$$

respectively. Then, by (4.1), for any $\varphi, \psi \in \hat{D}$

$$\begin{aligned}
&|f_j(\varphi) - f_j(\psi)|^2 \vee |g_j(\varphi) - g_j(\psi)|^2 \vee |h_j(\varphi) - h_j(\psi)|^2 \\
&\quad \le |f(\bar{\pi}_j(\varphi)) - f(\bar{\pi}_j(\psi))|^2 \vee |g(\bar{\pi}_j(\varphi)) - g(\bar{\pi}_j(\psi))|^2 \vee |h(\bar{\pi}_j(\varphi)) - h(\bar{\pi}_j(\psi))|^2 \\
&\quad \le \int_{-\tau}^{0} |\pi_j(\varphi(\theta)) - \pi_j(\psi(\theta))|^2 \mathrm{d}\mu_j(\theta) \\
&\quad \le \int_{-\tau}^{0} |\varphi(\theta) - \psi(\theta)|^2 \mathrm{d}\mu_j(\theta).
\end{aligned} \tag{4.7}$$

That is, f_j, g_j and h_j satisfy the global Lipschitz condition. For $t \in [0, T]$, let $x^j(t)$ be the solution to the following SFDE with jumps

$$\mathrm{d}x^j(t) = f_j(x_t^j)\mathrm{d}t + g_j(x_t^j)\mathrm{d}B(t) + h_j(x_{t-}^j)\mathrm{d}N(t)$$

with the initial data $x_0^j = \xi$ and $y^j(t)$ be the corresponding continuous-time *EM* solution with the step size $\triangle$. By Theorem 3.1 for any sufficiently small $\epsilon \in (0, \tilde{\varepsilon})$

$$E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right) \le \delta_1 L_j^{1+\epsilon/2} e^{\delta_2 L_j^{1+\epsilon/2}} \triangle.$$

Furthermore, by (4.4) (here we assume $L_j \ge 1$ without any loss of generality),

$$E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right) \le e^{(\delta_1+\delta_2)L_j^{1+\epsilon/2}} \triangle \le j^{\alpha(\delta_1+\delta_2)} \triangle. \tag{4.8}$$

Set

$$\hat{x}(T) = \sup_{0\le t\le T} |x(t)| \quad \text{and} \quad \hat{y}(T) = \sup_{0\le t\le T} |y(t)|.$$

For any integer $j \ge 1$, define stopping time

$$\tau_j = T \wedge \inf\{t \in [0, T] : \|x_t^j\| \vee \|y_t^j\| > j\}.$$

It is easy to see that $\|x_s^j\| \le j$ for any $0 \le s < \tau_j$. Then, combining (4.6) gives that for any $0 \le s < \tau_j$

$$f_j(x_s^j) = f\left(\frac{\|x_s^j\| \wedge j}{\|x_s^j\|} x_s^j\right) = f\left(\frac{\|x_s^j\| \wedge (j+1)}{\|x_s^j\|} x_s^j\right) = f_{j+1}(x_s^j) = f(x_s^j).$$

Similarly,

$$g_j(x_s^j) = g_{j+1}(x_s^j) = g(x_s^j), \qquad h_j(x_s^j) = h_{j+1}(x_s^j) = h(x_s^j).$$

While on $0 \le t < \tau_j$

$$\begin{aligned}
x^j(t) &= \xi(0) + \int_0^t f_j(x_s^j)\mathrm{d}s + \int_0^t g_j(x_s^j)\mathrm{d}B(s) + \int_0^t h_j(x_{s-}^j)\mathrm{d}N(s) \\
&= \xi(0) + \int_0^t f_{j+1}(x_s^j)\mathrm{d}s + \int_0^t g_{j+1}(x_s^j)\mathrm{d}B(s) + \int_0^t h_{j+1}(x_{s-}^j)\mathrm{d}N(s) \\
&= \xi(0) + \int_0^t f(x_s^j)\mathrm{d}s + \int_0^t g(x_s^j)\mathrm{d}B(s) + \int_0^t h(x_{s-}^j)\mathrm{d}N(s).
\end{aligned}$$

Consequently, we must have that

$$x(t) = x^j(t) = x^{j+1}(t)$$

on $0 \le t < \tau_j$. Likewise, we can also derive that

$$y(t) = y^j(t) = y^{j+1}(t)$$

for $0 \le t < \tau_j$. These imply that τ_j is nondecreasing and, by Lemma 3.1, $\lim_{j\to\infty} \tau_j = T$ a.s. Let $\tau_0 = 0$ and compute for $t \in [0, T]$

$$\begin{aligned} |x(t) - y(t)|^2 &= \sum_{j=1}^{\infty} |x(t) - y(t)|^2 I_{[\tau_{j-1} \le t < \tau_j)} \\ &= \sum_{j=1}^{\infty} |x^j(t) - y^j(t)|^2 I_{[\tau_{j-1} \le t < \tau_j)} \\ &\le \sum_{j=1}^{\infty} |x^j(t) - y^j(t)|^2 I_{\{j-1 \le \hat{x}(T) \vee \hat{y}(T)\}}. \end{aligned}$$

Therefore, by the Hölder inequality

$$\begin{aligned} E\left(\sup_{0\le t\le T} |x(t) - y(t)|^2\right) &\le \sum_{j=1}^{\infty} \left(E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right)\right)^{\frac{2}{2+\epsilon}} \left(EI_{\{j-1 \le \hat{x}(T) \vee \hat{y}(T)\}}\right)^{\frac{\epsilon}{2+\epsilon}} \\ &\le \sum_{j=1}^{\infty} \left(E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right)\right)^{\frac{2}{2+\epsilon}} [P(j-1 \le \hat{x}(T) \vee \hat{y}(T))]^{\frac{\epsilon}{2+\epsilon}}. \end{aligned} \tag{4.9}$$

On the other hand, for any $q \ge 2$, we obtain from Lemma 3.1

$$P(j-1 \le \hat{x}(T) \vee \hat{y}(T)) \le \frac{E|\hat{x}(T)|^q + E|\hat{y}(T)|^q}{(\frac{j}{2})^q} \le \frac{2H(q)}{(\frac{j}{2})^q} \tag{4.10}$$

with $j \ge 2$. Substituting (4.8) and (4.10) into (4.9), one has

$$E\left(\sup_{0\le t\le T} |x(t) - y(t)|^2\right) \le \left(1 + 2^{\frac{q\epsilon}{2+\epsilon}} (2H(q))^{\frac{\epsilon}{2+\epsilon}} \sum_{j=2}^{\infty} j^{\frac{2\alpha(\delta_1+\delta_2)-q\epsilon}{2+\epsilon}}\right) \triangle^{\frac{2}{2+\epsilon}}. \tag{4.11}$$

For any fixed $\epsilon > 0$ letting q be sufficiently large for

$$q \ge \frac{\alpha(\delta_1 + \delta_2) + 2(2+\epsilon)}{\epsilon},$$

we see that the right-hand side of (4.11) is convergent, whence the desired assertion (4.5) follows.

Remark 4.2. Under a local Lipschitz condition, Mao [9] showed strong convergence of the numerical solutions to SFDEs without jumps, and convergence rate was revealed under a global Lipschitz condition. In the present paper, under a local Lipschitz condition, we reveal the convergence rate for the numerical solutions to SFDEs with jumps. The convergence rate for jump processes (2.2) we revealed here is $1/(2+\epsilon)$ (close to $1/2$) under the logarithm growth condition (4.4). This is different from the case for diffusion processes (without jumps) which was studied in [10], where it was shown that the rate of convergence is still $1/2$ under the logarithm growth condition. The reason for such a difference has already been pointed out in Remark 3.1.

Remark 4.3. Logarithm growth condition (4.4) holds under global Lipschitz condition (2.3). Taking $L := \mu(0) - \mu(-\tau)$ in Remark 2.1 and $\tilde{\varepsilon} \in (0, 1)$ we can choose constants $\alpha, j > 0$ such that

$$L^{1+\tilde{\varepsilon}} \le \alpha \log j.$$

On the other hand, the logarithm growth condition (4.4) on the coefficients has been used in Fang et al. [13, Theorem 1.8, (1.17)] to study the global flow for SDEs under a local Lipschitz condition. To the best of our knowledge, (for local Lipschitz case) convergence rate of *EM* schemes for SFDEs (with jumps), even for SDEs, is still open for the general local Lipschitz condition instead of (4.4).

Appendix. An existence-and-uniqueness theorem

To make our paper self-contained, in this section we shall discuss the existence and uniqueness of solutions to (2.2) under assumption (H1).

Theorem A.1. *Under conditions* (2.3)*, there exists a unique solution* $x(t), t \in [0, T]$*, to* (2.2) *for any initial value* $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$.

Proof. Since our proof is an application of the proof for the case without jumps in [3, Theorem 2.2, p150], here we give only a sketch for the proof of jump case.

Uniqueness. Let $x(t)$ and $\bar{x}(t)$ be two solutions to (2.2) on $[0, T]$. Noting from (2.2) that

$$x(t) - \bar{x}(t) = \int_0^t [f(x_s) - f(\bar{x}_s)]\mathrm{d}s + \int_0^t [g(x_s) - g(\bar{x}_s)]\mathrm{d}B(s) + \int_0^t [h(x_{s-}) - h(\bar{x}_{s-})]\mathrm{d}N(s)$$

and $\tilde{N}(t) = N(t) - \lambda t$ is a martingale measure for $t \in [0, T]$, along with (2.4) we have

$$E\left(\sup_{0\le t\le T} |x(t) - \bar{x}(t)|^2\right) \le 3L(T + 4 + 8\lambda + 2\lambda^2 T) \int_0^T E\left(\sup_{0\le r\le s} |x(r) - \bar{x}(r)|^2\right) \mathrm{d}s.$$

By the Gronwall inequality

$$E\left(\sup_{0\le t\le T} |x(t) - \bar{x}(t)|^2\right) = 0,$$

which implies that $x(t) = \bar{x}(t)$ for $t \in [0, T]$ almost surely. The uniqueness has been proved.

Existence. Define $x_0^0 = \xi$ and $x^0(t) = \xi(0)$ for $0 \le t \le T$. For each $n = 1, 2, \ldots$, set $x_0^n = \xi$ and define, by the Picard iterations,

$$x^n(t) = \xi(0) + \int_0^t f(x_s^{n-1})\mathrm{d}s + \int_0^t g(x_s^{n-1})\mathrm{d}B(s) + \int_0^t h(x_{s-}^{n-1})\mathrm{d}N(s) \tag{A.1}$$

for $t \in [0, T]$. It also follows from (A.1) that for any integer $k \ge 1$

$$E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le \bar{c}_1 + \bar{c}_2 \int_0^T E\left(\sup_{0\le r\le s} |x^{n-1}(r)|^2\right) \mathrm{d}s,$$

where $\bar{c}_1 = 4[E\|\xi\|^2 + K(T^2 + 4T) + (8\lambda + 2\lambda^2 T)T] + \bar{c}_2 TE\|\xi\|^2$ and $\bar{c}_2 = 4K[T + 4 + 8\lambda + 2\lambda^2 T]$. This further implies that

$$\max_{1\le n\le k} E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le \bar{c}_1 + \bar{c}_2 \int_0^T \max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^{n-1}(s)|^2\right) \mathrm{d}t.$$

Observing

$$\begin{aligned}
\max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^{n-1}(s)|^2\right) &= \max\left\{E|\xi(0)|^2, E\left(\sup_{0\le s\le t} |x^1(s)|^2\right), \ldots, E\left(\sup_{0\le s\le t} |x^{k-1}(s)|^2\right)\right\} \\
&\le \max\left\{E\|\xi\|^2, E\left(\sup_{0\le s\le t} |x^1(s)|^2\right), \ldots, E\left(\sup_{0\le s\le t} |x^{k-1}(s)|^2\right), E\left(\sup_{0\le s\le t} |x^k(s)|^2\right)\right\} \\
&\le E\|\xi\|^2 + \max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^n(s)|^2\right),
\end{aligned}$$

we hence deduce that

$$\max_{1\le n\le k} E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le \bar{c}_1 + \bar{c}_2 TE\|\xi\|^2 + \bar{c}_2 \int_0^T \max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^n(s)|^2\right) \mathrm{d}t.$$

The Gronwall inequality implies

$$\max_{1\le n\le k} E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le (\bar{c}_1 + \bar{c}_2 TE\|\xi\|^2)e^{\bar{c}_2 T}.$$

Since k is arbitrary, we must have for $n \ge 1$

$$E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le (\bar{c}_1 + \bar{c}_2 TE\|\xi\|^2)e^{\bar{c}_2 T}.$$

Next, by (A.1)

$$\begin{aligned}
E\left(\sup_{0\le t\le T} |x^1(t) - x^0(t)|^2\right) &\le 3K(T + 4 + 8\lambda + 2\lambda^2 T) \int_0^T (1 + E\|x_s^0\|^2)\mathrm{d}s \\
&\le 3KT(T + 4 + 8\lambda + 2\lambda^2 T)(1 + E\|\xi\|^2) := \bar{C}.
\end{aligned} \tag{A.2}$$

We now claim that for $n \geq 0$

$$E\left(\sup_{0\leq s\leq t} |x^{n+1}(s) - x^n(s)|^2\right) \leq \frac{\bar{C}M^n t^n}{n!}, \quad 0 \leq t \leq T, \tag{A.3}$$

where $M = 3K(T + 4 + 8\lambda + 2\lambda^2 T)$. We shall show this by induction. In view of (A.2) we see that (A.3) holds whenever $n = 0$. Now, assume that (A.3) holds for some $n \geq 0$. Then,

$$\begin{aligned} E\left(\sup_{0\leq s\leq t} |x^{n+2}(s) - x^{n+1}(s)|^2\right) &\leq M \int_0^t E\|x_s^{n+1} - x_s^n\|^2 \mathrm{d}s \\ &\leq M \int_0^t E\left(\sup_{0\leq r\leq s} |x^{n+1}(r) - x^n(r)|^2\right) \mathrm{d}s \\ &\leq M \int_0^t \frac{\bar{C}M^n s^n}{n!} \mathrm{d}s = \frac{\bar{C}M^{n+1} t^{n+1}}{(n+1)!}. \end{aligned}$$

Following the proof of [3, Theorem 3.1, p55], we can show that for almost all $\omega \in \Omega$ there exists a positive integer $n_0 = n_0(\omega)$ such that

$$\sup_{0\leq s\leq T} |x^{n+1}(s) - x^n(s)| \leq \frac{1}{2^n} \quad \text{whenever } n \geq n_0(\omega). \tag{A.4}$$

This implies that $\{x^n(\cdot)\}_{n\geq 1}$ is a Cauchy sequence under $\sup|\cdot|$. However, since our space $D([0, T]; R^n)$ is not a complete space under $\sup|\cdot|$, we do not know whether $\{x^n(\cdot)\}_{n\geq 1}$ has a limit in $D([0, T]; R^n)$. We shall use the Skorohod metric $d(\cdot, \cdot)$. Taking $\lambda(t) = t$ in (2.1), we can see that $\{x^n(\cdot)\}_{n\geq 1}$ is still a Cauchy sequence under d. By [11, Theorem 14.2, p115] we know that $D([0, T]; R^n)$ is complete in the metric d. Therefore there exists unique $x(t)$, $t \in [0, T] \in D([0, T]; R^n)$ such that $d(x^n(\cdot), x(\cdot)) \to 0$ as $n \to \infty$. Taking the limit in (A.1), we then can show that $x(t)$ is the solution of (2.2). □

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