Stable Local Bases for Multivariate Spline Spaces

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Abstract. We present an algorithm for constructing stable local bases for the spaces $\mathcal{S}_d^r(\triangle)$ of multivariate polynomial splines of smoothness $r \geq 1$ and degree $d \geq r2^n + 1$ on an arbitrary triangulation $\triangle$ of a bounded polyhedral domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

§1. Introduction

Let $\triangle$ be a triangulation of a bounded polyhedral domain $\Omega \subset \mathbb{R}^n$, *i.e.*, $\triangle$ is a finite set of non-degenerate n-simplices such that

1) $\Omega = \bigcup_{T \in \triangle} T$;
2) the interiors of the simplices in $\triangle$ are pairwise disjoint; and
3) each facet of a simplex $T \in \triangle$ either lies on the boundary of Ω or is a common face of exactly two simplices in $\triangle$.

Given $1 \leq r \leq d$, we consider the spline space

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \ s|_T \in \Pi_d^n \ \text{ for all } n\text{-simplices } T \in \triangle\},$$

where Π_d^n is the linear space of all n-variate polynomials of total degree at most d. It is well-known that $\dim \Pi_d^n = \binom{n+d}{n}$.

The application of splines in numerical computations requires efficient algorithms for constructing locally supported bases for the space $\mathcal{S}_d^r(\triangle)$ or its subspaces (such as finite element spaces). Moreover, if a *local* basis $\{s_1, \ldots, s_m\}$ for $\mathcal{S}_d^r(\triangle)$ is in addition *stable*, *i.e.*, for all $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$,

$$K_1 \|\alpha\|_{\ell_p} \leq \Big\| \sum_{k=1}^m \alpha_k s_k \Big\|_{L_p(\Omega)} \leq K_2 \|\alpha\|_{\ell_p},$$

then a *nested* sequence of spaces

$$S_d^r(\triangle_1) \subset S_d^r(\triangle_2) \subset \cdots \subset S_d^r(\triangle_q) \subset \cdots, \tag{1.1}$$

may be used for designing multilevel methods of approximation on a bounded domain $\Omega \subset \mathbb{R}^n$, see *e.g.* [27] and references therein. In particular, the sequence (1.1)

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constitutes a *multiresolution analysis* on Ω if the maximal diameter of the triangles in $\triangle_q$ tends to zero as $q \to \infty$, and if the constants $0 < K_1, K_2 < \infty$ are independent of q. Note that the bases for the *full space* $\mathcal{S}_d^r(\triangle)$ are particularly interesting since $S_d^r(\triangle_q) \subset S_d^r(\triangle_{q+1})$ if $\triangle_{q+1}$ is a *refinement* of $\triangle_q$. (This is not the case for the finite element subspaces of $\mathcal{S}_d^r(\triangle)$ when $r \geq 1$; see [14,25,27].)

The famous B-splines constitute a stable locally supported basis for the space $\mathcal{S}_d^r(\triangle)$ in the one-dimensional case $n = 1$ for all $d \geq r + 1$. Moreover, the dual basis is also local and therefore provides a quasi-interpolant possessing optimal approximation order. There are well known constructions of local bases for $\mathcal{S}_d^r(\triangle)$ in the bivariate case $n = 2$ for all $d \geq 3r + 2$, see [1,21,22,26]. Stable local bases were constructed in [7,23] for some superspline subspaces, and in [17,19] for the full bivariate spline spaces $\mathcal{S}_d^r(\triangle)$, $d \geq 3r + 2$. In the trivariate case $n = 3$ local bases are known for all $d \geq 8r + 1$ [2]. It was conjected in [2] that in general locally supported bases for $\mathcal{S}_d^r(\triangle)$ exist if $d \geq r(2^n - 1) + n$.

The main objective of this paper is to construct stable locally supported bases for $\mathcal{S}_d^r(\triangle)$ and its superspline subspaces for all $n \geq 2$ and $r \geq 1$ provided $d \geq r2^n+1$.

We make use of the *nodal approach* originated in the finite element method, see *e.g.* [12], and extended to the problems of spline spaces on general triangulations in [26] and more recently in [8–11,15,16,17]. We show that in the multivariate case the *nodal smoothness conditions* can be better localized than usual Bernstein-Bézier smoothness conditions [5,20]. The key point for our analysis is that certain matrices associated with the smoothness conditions have a block diagonal structure, which in the same time makes it possible to handle them efficiently in numerical computations, see Sections 5 and 6. In particular, the dimension of any given spline space $\mathcal{S}_d^r(\triangle)$, $d \geq r2^n + 1$ can be efficiently computed by a formula obtained in Section 5.

The paper is organized as follows. In Section 2 we give some definitions and preliminary lemmas. The nodal functionals that we use are described in Section 3. Section 4 is devoted to a detailed analysis of nodal smoothness conditions. In Section 5 we construct local bases for $\mathcal{S}_d^r(\triangle)$, $d \geq r2^n + 1$. In Section 6 we show how to achieve stability of these bases. Finally, in Section 7 we extend the results to the superspline subspaces of $\mathcal{S}_d^r(\triangle)$.

§2. Preliminaries

2.1. Bases and minimal determining sets

It is obvious that the linear space $\mathcal{S}_d^r(\triangle)$ has finite dimension. In this subsection we consider an abstract finite-dimensional linear space $\mathcal{S}$, although in all our applications we have $\mathcal{S} \subset \mathcal{S}_d^r(\triangle)$.

Let $\mathcal{S}^*$ denote, as usual, the dual space of linear functionals on $\mathcal{S}$. Given a basis $\{s_j\}_{j\in J}$ for $\mathcal{S}$, its dual basis is a basis $\{\lambda_j\}_{j\in J}$ for $\mathcal{S}^*$ such that

$$\lambda_i s_j = \delta_{i,j}, \qquad \text{all } i, j \in J. \tag{2.1}$$

It is easy to see that the dual basis $\{\lambda_j\}_{j\in J}$ is uniquely determined by $\{s_j\}_{j\in J}$, and vice versa, a basis $\{\lambda_j\}_{j\in J}$ for $\mathcal{S}^*$ uniquely determines a basis $\{s_j\}_{j\in J}$ for $\mathcal{S}$ satisfying (2.1).

In order to construct a basis $\{s_j\}_{j\in J}$ for a spline space $\mathcal{S}$ it is often useful to find first a basis $\{\lambda_j\}_{j\in J}$ for $\mathcal{S}^*$ and then determine $\{s_j\}_{j\in J}$ from the duality condition (2.1). Usually, the required basis for $\mathcal{S}^*$ can be selected by an algorithm from a larger set $\Lambda \subset \mathcal{S}^*$ that spans $\mathcal{S}^*$. A common example of such a set Λ is the set of linear functionals picking off a coefficient of the Bernstein-Bézier representation of splines $s \in \mathcal{S}$, see *e.g.* [2]. Keeping in mind the tradition upheld in the literature on bivariate and multivariate splines, we will use the following terminology.

Definition 2.1. Any finite spanning set for $\mathcal{S}^*$ is called a determining set for $\mathcal{S}$. Any basis for $\mathcal{S}^*$ is called a minimal determining set for $\mathcal{S}$.

A standard argument in linear algebra shows that a set $\Lambda \subset \mathcal{S}^*$ is a determining set for $\mathcal{S}$ if and only if $\lambda s = 0$ for all $\lambda \in \Lambda$ implies $s = 0$ whenever $s \in \mathcal{S}$. Moreover, a determining set Λ is a minimal determining set for $\mathcal{S}$ if and only if no proper subset of Λ is a determining set. Since every linear functional on $\mathcal{S}$ is well-defined on any subspace $\tilde{\mathcal{S}}$ of $\mathcal{S}$, it is easy to see that a determining set for $\mathcal{S}$ is also a determining set for $\tilde{\mathcal{S}}$.

Suppose Λ is a determining set for $\mathcal{S}$. If Λ is not a minimal determining set for $\mathcal{S}$, then Λ is linearly dependent. It is particularly useful to know a complete system of linear relations for Λ.

Definition 2.2. Let $\Lambda = \{\lambda_j\}_{j\in J} \subset \mathcal{S}^*$ be a determining set for $\mathcal{S}$. Suppose that the functionals λ_j satisfy linear conditions

$$\sum_{j\in J} c_{i,j}\lambda_j = 0, \qquad i \in I, \tag{2.2}$$

where $c_{i,j}$ are some real coefficients. We say that (2.2) is a complete system of linear relations for Λ over $\mathcal{S}$ if for any $a = (a_j)_{j\in J}$, with $a_j \in \mathbb{R}$, $j \in J$, such that

$$\sum_{j\in J} c_{i,j}a_j = 0, \qquad i \in I, \tag{2.3}$$

there exists an element $s \in \mathcal{S}$ such that $\lambda_j s = a_j$ for all $j \in J$.

Note that the element $s \in \mathcal{S}$ as above is necessarily *unique*. Indeed, if there are $s_1, s_2 \in \mathcal{S}$ such that $\lambda_j s_1 = \lambda_j s_2 = a_j$ for all $j \in J$, then $\lambda_j(s_1 - s_2) = 0$, $j \in J$, which implies $s_1 = s_2$ since Λ is a determining set for $\mathcal{S}$.

Let $C := (c_{i,j})_{i\in I,\, j\in J}$. Then (2.3) means that the vector a lies in the null space $N(C) := \{a : \; Ca^T = 0\}$ of the matrix C. Thus, there is a 1-1 correspondence between elements $s \in \mathcal{S}$ and vectors $a \in N(C)$, where $a = (a_j)_{j\in J}$, $a_j = \lambda_j s$. In particular, the dimension of $\mathcal{S}$ can be computed as follows.

Lemma 2.3. *We have*

$$\dim \mathcal{S} = \dim N(C) = \#\Lambda - \operatorname{rank} C. \tag{2.4}$$

Moreover, given a determining set Λ for $\mathcal{S}$ and a complete system of linear relations for Λ over $\mathcal{S}$ with matrix C, it is straightforward to construct a basis for $\mathcal{S}$; see also [6].

Algorithm 2.4. *Suppose $\Lambda = \{\lambda_j\}_{j\in J} \subset \mathcal{S}^*$ is a determining set for $\mathcal{S}$, and (2.2) is a complete system of linear relations for Λ over $\mathcal{S}$. Let $a^{[k]} = (a_j^{[k]})_{j\in J}$, $k = 1,\ldots,m$, form a basis for the null space $N(C)$ of C. For each $k = 1,\ldots,m$, construct the unique element $\tilde{s}_k \in \mathcal{S}$ satisfying $\lambda_j \tilde{s}_k = a_j^{[k]}$ for all $j \in J$. Then $\{\tilde{s}_1,\ldots,\tilde{s}_m\}$ is a basis for $\mathcal{S}$.*

It is not difficult to determine corresponding minimal determining set, *i.e.*, the basis $\{\tilde{\lambda}_1,\ldots,\tilde{\lambda}_m\}$ for $\mathcal{S}^*$ dual to $\{\tilde{s}_1,\ldots,\tilde{s}_m\}$. Let

$$A := [a_j^{[k]}]_{j\in J,\, k=1,\ldots,m}.$$

Since the columns $a^{[k]}$ of this matrix are linearly independent, A has full column rank. Hence, there exists a left inverse of A, *i.e.*, a matrix

$$B = [b_{k,j}]_{k=1,\ldots,m,\, j\in J}$$

satisfying $BA = I_m$, where I_m is the $m \times m$ identity matrix. Note that B is not unique in general.

Lemma 2.5. *The dual basis $\{\tilde{\lambda}_1,\ldots,\tilde{\lambda}_m\}$ can be computed by*

$$\tilde{\lambda}_k = \sum_{j\in J} b_{k,j}\lambda_j, \qquad k = 1,\ldots,m.$$

Proof: It is straightforward to check that the duality condition (2.1) is satisfied. □

2.2. Geometry of a triangulation in $\mathbb{R}^n$

Recall that an ℓ-simplex τ $(0 \le \ell \le n)$ is the convex hull $\langle v_0,\ldots,v_\ell\rangle$ of $\ell+1$ points $v_0,\ldots,v_\ell \in \mathbb{R}^n$ called vertices of τ. The simplex τ is non-degenerate if its ℓ-dimensional volume is non-zero and degenerate otherwise. The dimension of a non-degenerate ℓ-simplex is ℓ. By the interior of an ℓ-simplex we mean its ℓ-dimensional interior. The convex hull of a subset of $\{v_0,\ldots,v_\ell\}$ containing $m+1 \le \ell+1$ elements is an m-face of τ. Thus, an m-face is itself an m-simplex. An $(\ell-1)$-face of τ is also called a facet of τ, and any 1-face of τ is also called an edge of τ. Note that the only ℓ-face of τ is τ itself, and the vertices of τ are its 0-faces. (We identify a vertex v and its convex hull $\{v\}$.)

Denote by $\mathcal{T}_\ell$ the set of all ℓ-faces of the simplices in $\triangle$ ($\ell = 0, \dots, n-1$) and set

$$\mathcal{T} := \bigcup_{\ell=0}^{n} \mathcal{T}_\ell,$$

where $\mathcal{T}_n := \triangle$. We will also use notation $\mathcal{V} := \mathcal{T}_0$, $\mathcal{E} := \mathcal{T}_1$ and $\mathcal{F} := \mathcal{T}_{n-1}$ for the sets of all vertices, edges and facets of $\triangle$, respectively. The star of a simplex $\tau \in \mathcal{T}$, denoted by $\mathrm{star}(\tau)$, is the union of all n-simplices $T \in \triangle$ containing τ, *i.e.*,

$$\mathrm{star}(\tau) = \bigcup_{\substack{T \in \triangle \\ \tau \subset T}} T.$$

In particular, $\mathrm{star}(T) = T$ for each $T \in \triangle$.

Furthermore, given $\tau \in \mathcal{T}_\ell$, $\ell \leq n-1$, we denote by (τ) the linear manifold in $\mathbb{R}^n$ parallel to the affine span $\mathrm{aff}(\tau)$ of τ and by $(\tau)^\perp$ the orthogonal complement of (τ) in $\mathbb{R}^n$. Note that $\dim(\tau)^\perp = n - \ell$. In particular, $(v)^\perp = \mathbb{R}^n$ for all $v \in \mathcal{V}$.

Let $\tau = \langle v_0, \dots, v_\ell \rangle \in \mathcal{T}_\ell$, $\ell \leq n-1$, and let $w \in \mathcal{V}$ be such that $\tau' = \langle \tau, w \rangle := \langle v_0, \dots, v_\ell, w \rangle$ is in $\mathcal{T}_{\ell+1}$. Since $\dim(\tau)^\perp = n - \ell$ and $\dim(\tau') = \ell + 1$, the linear manifold $(\tau)^\perp \cap (\tau')$ has dimension 1. Moreover, since $\mathrm{aff}(\tau)$ has codimension 1 as an affine subspace of $\mathrm{aff}(\tau')$, it defines two half-spaces of $\mathrm{aff}(\tau')$, and there is a unique unit vector in $(\tau)^\perp \cap (\tau')$ pointing into the half-space of $\mathrm{aff}(\tau')$ containing w. We denote this unit vector by

$$\sigma_{\tau,w}.$$

If v is a vertex in $\mathcal{V}$, then $\sigma_{v,w}$ is obviously the unit vector in the direction of the edge $\langle v, w \rangle$. If $w_1, \dots, w_m \in \mathcal{V}$ and $\tilde{\tau} = \langle \tau, w_1, \dots, w_m \rangle$ is in $\mathcal{T}_{\ell+m}$, $\ell + m \leq n$, then we set

$$\sigma(\tau, \tilde{\tau}) := (\sigma_{\tau,w_1}, \dots, \sigma_{\tau,w_m}).$$

2.3. Nodal functionals

Given $\sigma = (\sigma_1, \dots, \sigma_m)$ a linearly independent sequence of *unit* vectors in $\mathbb{R}^n$, and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$, let D_σ^α denote the partial derivative

$$D_\sigma^\alpha := D_{\sigma_1}^{\alpha_1} \cdots D_{\sigma_m}^{\alpha_m},$$

where D_{σ_i} is the derivative in the direction σ_i,

$$D_{\sigma_i} f(x) := \lim_{t \to +0} t^{-1} \{ f(x + \sigma_i t) - f(x) \},$$

for a differentiable f. By a nodal functional we mean any linear functional on $\mathcal{S}_d^r(\triangle)$ of the form $\eta = \delta_x D_\sigma^\alpha$, where x is a point in Ω, and δ_x is the point-evaluation functional,

$$\delta_x f := f(x).$$

We denote by

$$q(\eta) = |\alpha| := \sum_{i=1}^{m} \alpha_i \le r \tag{2.5}$$

the order of η. Given $s \in \mathcal{S}_d^r(\triangle)$, the partial derivative $D_\sigma^\alpha s$ is continuous everywhere in Ω if $|\alpha| \le r$, and piecewise continuous if $|\alpha| > r$. In this last case we have to choose an n-simplex $T \in \triangle$, with $x \in T$, and apply our functional to $s|_T$. The following situation is of special interest since, for it, a *natural* choice for T exists. Assume that for some $\tau \in \mathcal{T}$ we have $x \in \tau$ and $x + \varepsilon\sigma_i \in \tau$, $i = 1, \ldots, m$, if $\varepsilon > 0$ is small enough. Then $\delta_x D_\sigma^\alpha s|_T$ is the same for all $T \in \triangle$ such that $\tau \subset T$. We will choose T in this way whenever the above situation occurs.

We will often use the following simple lemma.

Lemma 2.6. *Let L be a linear manifold in $\mathbb{R}^n$, $\dim L = m \le n$, and let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be a basis of L, where $\sigma_1, \ldots, \sigma_m \in L$ are unit vectors. Suppose that all components of $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m)$ are also some unit vectors in L. Then for any $\alpha \in \mathbb{Z}^m$ there exist real coefficients c_β such that*

$$D_{\tilde{\sigma}}^\alpha = \sum_{\substack{\beta \in \mathbb{Z}^m \\ |\beta| = |\alpha|}} c_\beta D_\sigma^\beta.$$

Proof: Since σ is a basis for L, there are real coefficients a_{ij} such that

$$\tilde{\sigma}_i = \sum_{j=1}^{m} a_{ij}\sigma_j \qquad i = 1, \ldots, m.$$

Therefore,

$$D_{\tilde{\sigma}_i} = \sum_{j=1}^{m} a_{ij} D_{\sigma_j} \qquad i = 1, \ldots, m,$$

and

$$D_{\tilde{\sigma}}^\alpha = \Big(\sum_{j=1}^{m} a_{1j} D_{\sigma_j}\Big)^{\alpha_1} \cdots \Big(\sum_{j=1}^{m} a_{mj} D_{\sigma_j}\Big)^{\alpha_m},$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$. $\square$

2.4. Polynomial unisolvent sets

Let τ be a non-degenerate ℓ-simplex in $\mathbb{R}^n$. We set

$$\Pi_m^\ell(\tau) := \{p|_\tau : \ p \in \Pi_m^n\}, \qquad m = -1, 0, 1, 2, \ldots,$$

where Π_m^n is the space of all n-variate polynomials of total degree at most m, $m = 0, 1, 2, \ldots$, and $\Pi_{-1}^n := \{0\}$. By a change of variables, the elements of $\Pi_m^\ell(\tau)$

may be considered as ℓ-variate polynomials of total degree at most m defined on τ. In particular, $\dim \Pi_m^\ell(\tau) = \dim \Pi_m^\ell = \binom{\ell+m}{\ell}$, $m = 0, 1, 2, \ldots$, $\dim \Pi_{-1}^\ell(\tau) = 0$. A finite set $\Xi \subset \tau$ is said to be Π_m^ℓ-unisolvent if for any real a_ξ, $\xi \in \Xi$, there exists a unique $p \in \Pi_m^\ell(\tau)$ such that $p(\xi) = a_\xi$ for all $\xi \in \Xi$. Obviously, the number of elements in any Π_m^ℓ-unisolvent set is equal to the dimension of Π_m^ℓ.

As a well known example of a Π_m^ℓ-unisolvent set we mention the set of $\binom{\ell+m}{\ell}$ uniformly distributed points in the ℓ-simplex $\tau = \langle v_0, \ldots, v_\ell \rangle$,

$$\tilde{\Xi}_m(\tau) := \{\xi : \ \xi = \frac{i_0 v_0 + \cdots + i_\ell v_\ell}{m}, \text{ where } i_0 + \cdots + i_\ell = m\}. \tag{2.6}$$

Moreover, its subsets

$$\tilde{\Xi}_m^k(\tau) := \{\xi \in \tilde{\Xi}_m(\tau) : \ i_j > k, \ j = 0, \ldots, \ell\}, \qquad 0 \le k \le \frac{m-\ell}{\ell+1}, \tag{2.7}$$

are examples of $\Pi_{m-(k+1)(\ell+1)}^\ell$-unisolvent sets in the *interior* of τ.

The following technical lemma will be very useful later.

Lemma 2.7. *Let $p \in \Pi_m^\ell(\tau)$ and $0 \le k \le \frac{m-\ell}{\ell+1}$. Suppose that*

1) *for each facet τ' of τ,*

$$\delta_x D_{\sigma(\tau',\tau)}^{k'} p = 0, \qquad \text{all } x \in \tau', \ \ k' = 0, \ldots, k,$$

2) *for some $\Pi_{m-(k+1)(\ell+1)}^\ell$-unisolvent set Ξ in the interior of τ,*

$$\delta_\xi p = 0, \qquad \text{all } \xi \in \Xi.$$

Then $p = 0$.

Proof: Let $\tau_1, \ldots, \tau_{\ell+1}$ be all facets of τ. For each τ_i, let p_i be a linear n-variate polynomial such that $p_i|_{\tau_i} = 0$ and $p_i|_\tau \ne 0$. It follows from 1) that

$$p = \tilde{p} \prod_{i=1}^{\ell+1} (p_i|_\tau)^{k+1},$$

where $\tilde{p}$ is a polynomial in $\Pi_{m-(k+1)(\ell+1)}^\ell(\tau)$. Since p_i, $i = 1, \ldots, \ell+1$, do not vanish in the interior of τ, 2) implies that $\tilde{p}(\xi) = 0$ for all $\xi \in \Xi$. Therefore, $\tilde{p} = 0$, and hence $p = 0$. $\square$

§3. A nodal determining set for $\mathcal{S}_d^r(\triangle)$

Suppose $r \geq 1$ and $d \geq r2^n + 1$. We now associate with each $\tau \in \mathcal{T}$ a set $\mathcal{N}_\tau$ of nodal functionals on $\mathcal{S}_d^r(\triangle)$. First, let v be a vertex in $\mathcal{V} = \mathcal{T}_0$. For each n-simplex $T \in \triangle$ containing v we define

$$\mathcal{N}_{v,q}(T) := \{\delta_v D^\alpha_{\sigma(v,T)} :\ \alpha \in \mathbb{Z}^n_+,\ |\alpha| = q\}, \qquad 0 \leq q \leq r2^{n-1},$$

$$\mathcal{N}_v(T) := \bigcup_{q=0}^{r2^{n-1}} \mathcal{N}_{v,q}(T).$$

Moreover, we set

$$\mathcal{N}_{v,q} := \bigcup_{\substack{T\in\triangle \\ v\in T}} \mathcal{N}_{v,q}(T), \qquad \mathcal{N}_v := \bigcup_{q=0}^{r2^{n-1}} \mathcal{N}_{v,q} = \bigcup_{\substack{T\in\triangle \\ v\in T}} \mathcal{N}_v(T).$$

Suppose now $\tau \in \mathcal{T}_\ell$ for some $\ell \in \{1, \ldots, n-1\}$. For each $0 \leq q \leq r2^{n-\ell-1}$, let $\Xi_{\tau,q}$ be a $\Pi^\ell_{\mu_{\ell,q}}$-unisolvent set in the *interior* of τ, where

$$\mu_{\ell,q} := d - q - (r2^{n-\ell} - q + 1)(\ell + 1). \tag{3.1}$$

Given any n-simplex $T \in \triangle$ containing τ, we define for each $\xi \in \Xi_{\tau,q}$,

$$\mathcal{N}_{\tau,q,\xi}(T) := \{\delta_\xi D^\alpha_{\sigma(\tau,T)} :\ \alpha \in \mathbb{Z}^{n-\ell}_+,\ |\alpha| = q\}.$$

Moreover, we set

$$\mathcal{N}_\tau(T) := \bigcup_{q=0}^{r2^{n-\ell-1}} \bigcup_{\xi\in\Xi_{\tau,q}} \mathcal{N}_{\tau,q,\xi}(T), \qquad \mathcal{N}_{\tau,q,\xi} := \bigcup_{\substack{T\in\triangle \\ \tau\subset T}} \mathcal{N}_{\tau,q,\xi}(T),$$

$$\mathcal{N}_{\tau,q} := \bigcup_{\xi\in\Xi_{\tau,q}} \mathcal{N}_{\tau,q,\xi}, \qquad \mathcal{N}_\tau := \bigcup_{q=0}^{r2^{n-\ell-1}} \mathcal{N}_{\tau,q} = \bigcup_{\substack{T\in\triangle \\ \tau\subset T}} \mathcal{N}_\tau(T).$$

Finally, for each $T \in \triangle = \mathcal{T}_n$ we define

$$\mathcal{N}_T := \{\delta_\xi :\ \xi \in \Xi_T\},$$

where Ξ_T is a $\Pi^n_{d-(r+1)(n+1)}$-unisolvent set in the interior of T.

Note that in general the sets $\mathcal{N}_{\tau,q,\xi}(T)$ are not mutually disjoint for different T containing τ. For example, let $\tau = \langle v_0, \ldots, v_{n-2}\rangle \in \mathcal{T}_{n-2}$, and suppose that both $T = \langle \tau, u, w\rangle$ and $\widetilde{T} = \langle \tau, u, \tilde{w}\rangle$ are in $\triangle$. Then the nodal functional $\delta_\xi D^{r+1}_{\sigma_{\tau,u}}$ belongs to $\mathcal{N}_{\tau,r+1,\xi}(T) \cap \mathcal{N}_{\tau,r+1,\xi}(\widetilde{T})$. On the other hand, if an n-simplex $T \in \triangle$ is fixed, then the sets $\mathcal{N}_{\tau,q,\xi}(T)$ are mutually disjoint for all τ, q, ξ.

Theorem 3.1. *The set*

$$\mathcal{N} := \bigcup_{\tau \in \mathcal{T}} \mathcal{N}_\tau$$

is a determining set for $\mathcal{S}_d^r(\triangle)$.

Proof: Let $s \in \mathcal{S}_d^r(\triangle)$ satisfy $\eta s = 0$ for all $\eta \in \mathcal{N}$. We have to show that $s = 0$. To this end we choose an arbitrary $T \in \triangle$ and show that $s|_T = 0$. For each vertex v of T, the set

$$\mathcal{N}_v(T) = \{\delta_v D^\alpha_{\sigma(v,T)} : \ \alpha \in \mathbb{Z}_+^n, \ |\alpha| \le r2^{n-1}\}$$

is included in $\mathcal{N}$. Since $\sigma(v,T)$ is a basis of $\mathbb{R}^n$, we have by Lemma 2.6,

$$\delta_v D^\alpha_\sigma s|_T = 0, \qquad \text{all } \alpha \in \mathbb{Z}_+^n, \ |\alpha| \le r2^{n-1},$$

for any sequence σ of unit vectors.

For $\ell = 0, \ldots, n-1$, we now show by induction that for each ℓ-face τ of T, if the components of σ are some unit vectors in $(\tau)^\perp$, then

$$\delta_x D^\alpha_\sigma s|_T = 0, \qquad \text{all } x \in \tau, \ \alpha \in \mathbb{Z}_+^{n-\ell}, \ |\alpha| \le r2^{n-\ell-1}. \tag{3.2}$$

The validity of (3.2) for $\ell = 0$ is shown above. Suppose $1 \le \ell \le n-1$. Let $\alpha \in \mathbb{Z}_+^{n-\ell}$, $|\alpha| = q$, with $1 \le q \le r2^{n-\ell-1}$. In view of Lemma 2.6, it suffices to prove (3.2) for $\sigma = \sigma(\tau, T)$. We have $p := D^\alpha_{\sigma(\tau,T)} s|_T \in \Pi^n_{d-q}$ and $p|_\tau \in \Pi^\ell_{d-q}(\tau)$. By the induction hypothesis, for each facet τ' of τ,

$$\delta_x D^{q'}_{\sigma(\tau',\tau)} p|_\tau = 0, \qquad \text{all } x \in \tau', \ q' = 0, \ldots, r2^{n-\ell} - q.$$

Since the nodal functionals $\delta_\xi D^\alpha_{\sigma(\tau,T)}$, $\xi \in \Xi_{\tau,q}$, are included in $\mathcal{N}_\tau(T) \subset \mathcal{N}$, we have in addition

$$\delta_\xi p|_\tau = 0, \qquad \text{all } \xi \in \Xi_{\tau,q}.$$

Since $\Xi_{\tau,q}$ is $\Pi^\ell_{\mu_{\ell,q}}$-unisolvent, Lemma 2.7 implies that $p|_\tau = 0$, which confirms (3.2).

In particular, (3.2) holds for each facet F of T, *i.e.*,

$$\delta_x D^q_{\sigma(F,T)} s|_T = 0, \qquad \text{all } x \in F, \ q = 0, \ldots, r.$$

Since $\mathcal{N}_T$ is included in $\mathcal{N}$, we have in addition

$$\delta_\xi s|_T = 0, \qquad \text{all } \xi \in \Xi_T.$$

Since Ξ_T is $\Pi^n_{d-(r+1)(n+1)}$-unisolvent, Lemma 2.7 implies that $s|_T = 0$, which completes the proof. $\square$

Theorem 3.2. *For each $T \in \triangle$, let*

$$\mathcal{N}(T) := \mathcal{N}_T \cup \bigcup_{\ell=0}^{n-1} \bigcup_{\tau \in \mathcal{T}_\ell(T)} \mathcal{N}_\tau(T),$$

where $\mathcal{T}_\ell(T)$ denotes the set of all ℓ-faces of T. Then $\mathcal{N}(T)$ is a minimal determining set for Π_d^n.

Proof: It is easy to see that the set of nodal functionals $\mathcal{N}(T)$ is the same, whatever the triangulation $\triangle$ containing T may be. If we take $\triangle = \{T\}$, then obviously $\mathcal{S}_d^r(\triangle) = \Pi_d^n$ and $\mathcal{N} = \mathcal{N}(T)$. Therefore, $\mathcal{N}(T)$ is a determining set for Π_d^n by Theorem 3.1. It thus remains to show that $\#\mathcal{N}(T) = \dim \Pi_d^n = \binom{n+d}{n}$. We have

$$\#\mathcal{N}(T) = \#\mathcal{N}_T + \sum_{v \in \mathcal{T}_0(T)} \#\mathcal{N}_v(T) + \sum_{\ell=1}^{n-1} \sum_{\tau \in \mathcal{T}_\ell(T)} \#\mathcal{N}_\tau(T).$$

It is easy to see that

$$\begin{aligned}
\#\mathcal{N}_T &= \binom{n+d-(r+1)(n+1)}{n}, \\
\#\mathcal{N}_v(T) &= \sum_{q=0}^{r2^{n-1}} \binom{n-1+q}{n-1} = \binom{n+r2^{n-1}}{n}, \quad v \in \mathcal{T}_0(T), \\
\#\mathcal{N}_\tau(T) &= \sum_{q=0}^{r2^{n-\ell-1}} \binom{\ell+\mu_{\ell,q}}{\ell} \binom{n-\ell-1+q}{n-\ell-1}, \quad \tau \in \mathcal{T}_\ell(T), \quad 1 \le \ell \le n-1,
\end{aligned}$$

where $\mu_{\ell,q}$ is defined in (3.1).

We now consider the set

$$Z := \{\alpha \in \mathbb{Z}_+^{n+1} : \ |\alpha| = d\}.$$

Obviously, $\#Z = \binom{n+d}{n}$. Therefore, the theorem will be established if we show that

$$\#Z = \#\mathcal{N}(T). \tag{3.3}$$

For any nonempty subset I of $\{1, \ldots, n+1\}$, let

$$\begin{aligned}
Z_I &:= \{\alpha \in Z : \sum_{i \in I} \alpha_i \ge d - r2^{n-\ell-1}\}, \qquad \text{if } \ell := \#I - 1 < n, \\
Z_{\{1,\ldots,n+1\}} &:= Z,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{Z}_{\{i\}} &:= Z_{\{i\}}, \qquad i = 1, \ldots, n+1, \\
\tilde{Z}_I &:= Z_I \setminus \bigcup_{i \in I} Z_{I \setminus \{i\}}, \qquad \#I \ge 2.
\end{aligned}$$

Taking into account the assumption $d \geq r2^n + 1$, it is not difficult to see that Z is a disjoint union of the sets $\tilde{Z}_I$. Hence,

$$\#Z = \sum_{\ell=0}^{n} \sum_{\#I=\ell+1} \#\tilde{Z}_I.$$

We have

$$\begin{aligned}\tilde{Z}_{\{1,\dots,n+1\}} &= \{\alpha \in Z : \sum_{\substack{i=1\\ i\neq j}}^{n+1} \alpha_i < d-r, \quad j=1,\dots,n+1\} \\ &= \{\alpha \in \mathbb{Z}_+^{n+1} : \ |\alpha| = d, \quad \alpha_j \geq r+1, \quad j=1,\dots,n+1\},\end{aligned}$$

and it follows that

$$\#\tilde{Z}_{\{1,\dots,n+1\}} = \binom{n+d-(r+1)(n+1)}{n} = \#\mathcal{N}_T.$$

Furthermore, for each $i=1,\dots,n+1$, we have

$$\tilde{Z}_{\{i\}} = \{\alpha \in \mathbb{Z}_+^{n+1} : \ |\alpha| = d, \quad \alpha_i \geq d - r2^{n-1}\},$$

so that $\#\tilde{Z}_{\{i\}} = \binom{n+r2^{n-1}}{n}$, and hence

$$\sum_{i=1}^{n+1} \#\tilde{Z}_{\{i\}} = (n+1)\binom{n+r2^{n-1}}{n} = \sum_{v\in\mathcal{T}_0(T)} \#\mathcal{N}_v(T).$$

Let now $I \subset \{1,\dots,n+1\}$, $\ell := \#I - 1 < n$. Then

$$\begin{aligned}\tilde{Z}_I &= \{\alpha \in Z : \sum_{i\in I} \alpha_i \geq d - r2^{n-\ell-1}, \quad \sum_{i\in I\setminus\{j\}} \alpha_i < d - r2^{n-\ell}, \quad j \in I\} \\ &= \bigcup_{q=0}^{r2^{n-\ell-1}} \{\alpha \in Z : \sum_{i\in I} \alpha_i = d-q, \quad \alpha_j \geq r2^{n-\ell} - q + 1, \quad j \in I\}.\end{aligned}$$

A standard combinatorial argument shows that the cardinality of the set

$$\{\alpha \in Z : \sum_{i\in I} \alpha_i = d-q, \quad \alpha_j \geq r2^{n-\ell} - q + 1, \quad j \in I\}$$

is $\binom{\ell+\mu_{\ell,q}}{\ell}\binom{n-\ell-1+q}{n-\ell-1}$. Since the number of subsets I of $\{1,\dots,n+1\}$ consisting of $\ell+1$ elements is equal to $\binom{n+1}{\ell+1} = \#\mathcal{T}_\ell(T)$, we conclude that

$$\sum_{\#I=\ell+1} \#\tilde{Z}_I = \sum_{\tau\in\mathcal{T}_\ell(T)} \#\mathcal{N}_\tau(T), \qquad \ell = 1,\dots,n-1.$$

Thus, (3.3) holds, and the proof is complete. $\square$

Theorem 3.2 shows that the set $\mathcal{N}(T)$ defines a *Hermite interpolation operator* $\mathcal{H}_T : C^{r2^{n-1}}(T) \to \Pi_d^n$ as follows. Given $f \in C^{r2^{n-1}}(T)$, let $\mathcal{H}_T f$ be the unique polynomial in Π_d^n satisfying

$$\eta \mathcal{H}_T f = \eta f, \qquad \text{all } \eta \in \mathcal{N}(T). \tag{3.4}$$

Obviously, this is a standard finite-element interpolation scheme, see *e.g.* [24,30].

The following estimation of the norm of $\mathcal{H}_T f$ in the case of uniformly distributed points easily follows from the general results given in [13]; see also the proof of Lemma 3.9 in [16].

Lemma 3.3. *Choose*

$$\begin{aligned} \Xi_{\tau,q} &= \tilde{\Xi}_{d-q}^{r2^{n-\ell}-q}, \qquad \text{all } \tau \in \mathcal{T}_\ell,\ 1 \le \ell \le n-1,\ 0 \le q \le r2^{n-\ell-1}, \\ \Xi_T &= \tilde{\Xi}_d^r, \qquad \text{all } T \in \mathcal{T}_n, \end{aligned} \tag{3.5}$$

where $\tilde{\Xi}_m^k$ are defined in (2.7). Then

$$\|\mathcal{H}_T f\|_{L_\infty(T)} \le K \max_{\eta \in \mathcal{N}(T)} h_T^{q(\eta)} |\eta f|, \tag{3.6}$$

where h_T is the diameter of T, $q(\eta)$ is the order of the nodal functional η, and K is a constant depending only on n, r and d.

§4. Smoothness conditions

As shown in the previous section, $\mathcal{N} \subset \mathcal{S}_d^r(\triangle)^*$ is a determining set for $\mathcal{S}_d^r(\triangle)$. Therefore, $\mathcal{N}$ is a spanning set for $\mathcal{S}_d^r(\triangle)^*$. However, as we will see, there are some linear dependencies between the elements of $\mathcal{N}$, called *nodal smoothness conditions*. Our next task is to describe these conditions.

Let $\tau \in \mathcal{T}_\ell$ for some $0 \le \ell \le n-1$, and let $F = \langle \tau, u_1, \ldots, u_{n-\ell-1} \rangle \in \mathcal{T}_{n-1}$ be an *interior* facet of $\triangle$ attached to τ. Then there are exactly two n-simplices $T_1, T_2 \in \triangle$ sharing the facet F. Let $T_1 = \langle F, u_{n-\ell} \rangle$, $T_2 = \langle F, w \rangle$. Since the components of

$$\sigma(\tau, T_1) = (\sigma_{\tau,u_1}, \ldots, \sigma_{\tau,u_{n-\ell}})$$

form a basis for $(\tau)^\perp$, and since $\sigma_{\tau,w}$ also lies in $(\tau)^\perp$, there exists $\mu \in \mathbb{R}^{n-\ell}$, $\mu = (\mu_1, \ldots, \mu_{n-\ell})$, such that

$$\sigma_{\tau,w} = \sum_{i=1}^{n-\ell} \mu_i \sigma_{\tau,u_i}.$$

Lemma 4.1. *If $s \in \mathcal{S}_d^r(\triangle)$, then for all $\xi \in \tau$, $\alpha \in \mathbb{Z}_+^{n-\ell-1}$ and $0 \le r' \le r$,*

$$\delta_\xi D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} s = \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta|=r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_\xi D^\alpha_{\sigma(\tau,F)} D^\beta_{\sigma(\tau,T_1)} s, \tag{4.1}$$

where $\binom{|\beta|}{\beta} := \frac{|\beta|!}{\beta_1! \cdots \beta_{n-\ell}!}$, $\mu^\beta := \mu_1^{\beta_1} \cdots \mu_{n-\ell}^{\beta_{n-\ell}}$.

Proof: Let $p_1 := s|_{T_1}$, $p_2 := s|_{T_2}$ and $\sigma_i := \sigma_{\tau,u_i}$, $i = 1, \ldots, n-\ell$. We have

$$\begin{aligned}
\delta_\xi D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} p_1 &= \delta_\xi D^\alpha_{\sigma(\tau,F)} \Big(\sum_{i=1}^{n-\ell} \mu_i D_{\sigma_i} \Big)^{r'} p_1 \\
&= \delta_\xi D^\alpha_{\sigma(\tau,F)} \Big(\sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta|=r'}} \binom{|\beta|}{\beta} \mu^\beta D^{\beta_1}_{\sigma_1} \cdots D^{\beta_{n-\ell}}_{\sigma_{n-\ell}} \Big) p_1 \\
&= \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta|=r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_\xi D^\alpha_{\sigma(\tau,F)} D^\beta_{\sigma(\tau,T_1)} p_1.
\end{aligned}$$

Since $s \in C^r(T_1 \cup T_2)$ and $r' \le r$,

$$D^{r'}_{\sigma_{\tau,w}} p_1(x) = D^{r'}_{\sigma_{\tau,w}} p_2(x), \qquad \text{all } x \in F = T_1 \cap T_2.$$

Therefore,

$$\delta_\xi D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} p_1 = \delta_\xi D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} p_2,$$

for all $\xi \in F$, in particular for $\xi \in \tau$. Thus,

$$\delta_\xi D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} p_2 = \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta|=r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_\xi D^\alpha_{\sigma(\tau,F)} D^\beta_{\sigma(\tau,T_1)} p_1. \tag{4.2}$$

Finally, we note that

$$D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} = D^\gamma_{\sigma(\tau,T_2)}, \qquad D^\alpha_{\sigma(\tau,F)} D^\beta_{\sigma(\tau,T_1)} = D^{\tilde\gamma}_{\sigma(\tau,T_1)}, \tag{4.3}$$

where $\gamma = (\alpha_1, \ldots, \alpha_{n-\ell-1}, r')$, $\tilde\gamma = (\alpha_1 + \beta_1, \ldots, \alpha_{n-\ell-1} + \beta_{n-\ell-1}, \beta_{n-\ell})$, and the observation that by definition

$$\delta_\xi D^\gamma_{\sigma(\tau,T_2)} s = \delta_\xi D^\gamma_{\sigma(\tau,T_2)} p_2, \qquad \delta_\xi D^{\tilde\gamma}_{\sigma(\tau,T_1)} s = \delta_\xi D^{\tilde\gamma}_{\sigma(\tau,T_1)} p_1$$

(see Section 2.3) completes the proof. $\square$

Remark 4.2. Lemma 4.1 shows that the condition (4.2) holds for all $\xi \in \tau$, $\alpha \in \mathbb{Z}_+^{n-\ell}$ and $0 \le r' \le r$ if the two polynomials p_1 and p_2 defined on T_1 and T_2, respectively, join together with C^r-smoothness across $F = T_1 \cap T_2$. It is not difficult to see that the converse is also true. Note that for $\tau \in \mathcal{T}_0$, Lemma 4.1 as well as its converse were given (in a slightly different form) in Theorem 4.1.2 of [11], and (in the bivariate case) in [16].

We now concentrate on the conditions (4.1) that involve the nodal functionals in the set $\mathcal{N}$ defined in Section 3. Namely, Lemma 4.1 implies that the following linear relations between the elements of $\mathcal{N}$ hold:

1) given $v \in \mathcal{T}_0$ and $0 \le q \le r2^{n-1}$, the system $\mathcal{R}_{v,q}$ of linear conditions

$$\delta_v D^\alpha_{\sigma(v,F)} D^{r'}_{\sigma_{v,w}} = \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_v D^\alpha_{\sigma(v,F)} D^\beta_{\sigma(v,T_1)}, \tag{4.4}$$

for all $0 \le r' \le \min\{r,q\}$, all $\alpha \in \mathbb{Z}_+^{n-1}$, with $|\alpha| = q - r'$, and all interior facets $F \in \mathcal{T}_{n-1}$ such that $v \in F$,

2) given $\tau \in \mathcal{T}_\ell$ (where $1 \le \ell \le n-2$), $0 \le q \le r2^{n-\ell-1}$, and $\xi \in \Xi_{\tau,q}$, the system $\mathcal{R}_{\tau,q,\xi}$ of linear conditions

$$\delta_\xi D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} = \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_\xi D^\alpha_{\sigma(\tau,F)} D^\beta_{\sigma(\tau,T_1)}, \tag{4.5}$$

for all $0 \le r' \le \min\{r,q\}$, all $\alpha \in \mathbb{Z}_+^{n-\ell-1}$, with $|\alpha| = q - r'$, and all interior facets $F \in \mathcal{T}_{n-1}$ such that $\tau \subset F$, and

3) given an interior facet $F \in \mathcal{T}_{n-1}$, $0 \le q \le r$, and $\xi \in \Xi_{F,q}$, the linear condition $\mathcal{R}_{F,q,\xi}$,

$$\delta_\xi D^q_{\sigma_{F,w}} = (-1)^q \delta_\xi D^q_{\sigma(F,T_1)}. \tag{4.6}$$

(Here and above w, T_1 and μ_i correspond to a particular F and are defined as in Lemma 4.1.)

Remark 4.3. In view of (4.3) it is easy to see that the smoothness conditions in $\mathcal{R}_{v,q}$, $\mathcal{R}_{\tau,q,\xi}$ or $\mathcal{R}_{F,q,\xi}$ involve only the nodal functionals in $\mathcal{N}_{v,q}$, $\mathcal{N}_{\tau,q,\xi}$ or $\mathcal{N}_{F,q,\xi}$, respectively. (See the definition of the sets of nodal functionals $\mathcal{N}_{v,q}$ and $\mathcal{N}_{\tau,q,\xi}$ in Section 3.)

Let

$$\begin{aligned} \mathcal{R}_v &:= \bigcup_{q=0}^{r2^{n-1}} \mathcal{R}_{v,q}, \qquad v \in \mathcal{T}_0, \\ \mathcal{R}_\tau &:= \bigcup_{q=0}^{r2^{n-\ell-1}} \mathcal{R}_{\tau,q}, \qquad \mathcal{R}_{\tau,q} := \bigcup_{\xi \in \Xi_{\tau,q}} \mathcal{R}_{\tau,q,\xi}, \qquad \tau \in \mathcal{T}_\ell, \quad 1 \le \ell \le n-1. \end{aligned} \tag{4.7}$$

Theorem 4.4. *The set*

$$\mathcal{R} := \bigcup_{\tau \in \mathcal{T} \setminus \mathcal{T}_n} \mathcal{R}_\tau \tag{4.8}$$

is a complete system of linear relations for $\mathcal{N}$ over $\mathcal{S}_d^r(\triangle)$.

Proof: By Theorem 3.1, $\mathcal{N}$ is a determining set for $\mathcal{S}_d^r(\triangle)$. Suppose the system $\mathcal{R}$ is written as

$$\sum_{j \in J} c_{i,j} \eta_j = 0, \qquad i \in I,$$

where I, J are some index sets, $\{\eta_j\}_{j \in J} = \mathcal{N}$, and $c_{i,j}$ real coefficients. Let a_j, $j \in J$, be any real numbers satisfying

$$\sum_{j \in J} c_{i,j} a_j = 0, \qquad i \in I.$$

According to Definition 2.2, we have to show that there exists a spline $s \in \mathcal{S}_d^r(\triangle)$ such that $\eta_j s = a_j$ for all $j \in J$. We first construct the polynomial pieces of s, $p_T = s|_T$, $T \in \triangle$, as follows. By Theorem 3.2, $\mathcal{N}(T)$ is a minimal determining set for Π_d^n. We define p_T to be the unique polynomial in Π_d^n such that

$$\eta_j p_T = a_j, \qquad \text{all } \eta_j \in \mathcal{N}(T).$$

We thus have to prove that p_T, $T \in \triangle$, join together with C^r-smoothness. To this end it suffices to consider two n-simplices $T_1, T_2 \in \triangle$ sharing a facet $F \in \mathcal{T}_{n-1}$ and show that the two polynomials $p_1 := p_{T_1}$ and $p_2 := p_{T_2}$ join with C^r-smoothness across F. This, in turn, will follow if we show that

$$\delta_x D^{r'}_{\sigma_{F,w}}(p_2 - p_1) = 0, \qquad \text{all } x \in F, \quad r' = 0, \dots, r. \tag{4.9}$$

where w is the vertex of T_2 not lying in F. (That is, $T_2 = \langle F, w \rangle$.)

We first prove by induction on ℓ that for each ℓ-face τ of F, $\ell = 0, \dots, n-2$, and for all $r' = 0, \dots, r$, and $\alpha \in \mathbb{Z}^{n-\ell-1}$, with $|\alpha| \le r2^{n-\ell-1} - r'$,

$$\delta_x D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}}(p_2 - p_1) = 0, \qquad \text{all } x \in \tau. \tag{4.10}$$

Let $\ell = 0$, and let v be a vertex of F. Given $r' = 0, \dots, r$ and $\alpha \in \mathbb{Z}^{n-1}$, with $|\alpha| \le r2^{n-1} - r'$, the functional $\eta_{j_0} := \delta_v D^\alpha_{\sigma(v,F)} D^{r'}_{\sigma_{v,w}}$ is in $\mathcal{N}(T_2)$. Hence, $\eta_{j_0} p_2 = a_{j_0}$. Let us compute $\eta_{j_0} p_1$. We set $\eta_{j_\beta} := \delta_v D^\alpha_{\sigma(v,F)} D^\beta_{\sigma(v,T_1)} \in \mathcal{N}(T_1)$, $|\beta| = r'$. By (4.4), the equation

$$\eta_{j_0} - \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} = 0$$

belongs to $\mathcal{R}$. Therefore,

$$a_{j_0} - \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta a_{j_\beta} = 0.$$

On the other hand, since $\eta_{j_\beta} \in \mathcal{N}(T_1)$, we have $\eta_{j_\beta} p_1 = a_{j_\beta}$, and it follows that

$$\eta_{j_0} p_1 = \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} p_1 = \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta a_{j_\beta} = a_{j_0}.$$

Thus, $\eta_{j_0}(p_2 - p_1) = 0$, which confirms (4.10) for $\ell = 0$.

Suppose $1 \le \ell \le n-2$, and let τ be an ℓ-face of F. Given $r' = 0, \ldots, r$ and $\alpha \in \mathbb{Z}^{n-\ell-1}$, with $|\alpha| \le r2^{n-\ell-1} - r'$, consider

$$p := D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} (p_2 - p_1)|_\tau \in \Pi^\ell_{d-q}(\tau),$$

where $q := |\alpha| + r'$. Let us show that for each facet τ' of τ,

$$\delta_x D^{q'}_{\sigma(\tau',\tau)} p = 0, \qquad \text{all } x \in \tau',\ q' = 0, \ldots, r2^{n-\ell} - q. \tag{4.11}$$

Since the components of $\sigma(\tau',\tau)$ and $\sigma(\tau,F)$ form a basis for $(\tau')^\perp \cap (F)$, we have by Lemma 2.6, that

$$D^{q'}_{\sigma(\tau',\tau)} D^\alpha_{\sigma(\tau,F)} = \sum_{\substack{\gamma \in \mathbb{Z}^{n-\ell} \\ |\gamma| = |\alpha| + q'}} c_\gamma D^\gamma_{\sigma(\tau',F)}.$$

Moreover, since $\sigma_{\tau,w} \in (\tau)^\perp \subset (\tau')^\perp$,

$$D^{r'}_{\sigma_{\tau,w}} = \sum_{\tilde{r}=0}^{r'} \sum_{\substack{\gamma \in \mathbb{Z}^{n-\ell} \\ |\gamma| = r' - \tilde{r}}} \tilde{c}_{\gamma,\tilde{r}} D^\gamma_{\sigma(\tau',F)} D^{\tilde{r}}_{\sigma_{\tau',w}}.$$

Therefore, we have for $x \in \tau'$,

$$\begin{aligned}
\delta_x D^{q'}_{\sigma(\tau',\tau)} p &= \delta_x D^{q'}_{\sigma(\tau',\tau)} D^\alpha_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} (p_2 - p_1) \\
&= \sum_{\tilde{r}=0}^{r'} \sum_{\substack{\gamma \in \mathbb{Z}^{n-\ell} \\ |\gamma| = |\alpha| + q'}} \sum_{\substack{\tilde{\gamma} \in \mathbb{Z}^{n-\ell} \\ |\tilde{\gamma}| = r' - \tilde{r}}} c_\gamma \tilde{c}_{\tilde{\gamma},\tilde{r}} \delta_x D^{\gamma+\tilde{\gamma}}_{\sigma(\tau',F)} D^{\tilde{r}}_{\sigma_{\tau',w}} (p_2 - p_1).
\end{aligned}$$

By the induction hypothesis, every term in this last sum is zero (since $\tilde{r} \le r$ and $|\gamma| + |\tilde{\gamma}| + \tilde{r} = |\alpha| + q' + r' = q + q' \le r2^{n-\ell}$), and (4.11) follows. We show now that

$$\delta_\xi p = 0, \qquad \text{all } \xi \in \Xi_{\tau,q}, \tag{4.12}$$

where $\Xi_{\tau,q}$ is a $\Pi^{\ell}_{\mu_{\ell,q}}$-unisolvent set in the interior of τ as defined in Section 3. Let $\xi \in \Xi_{\tau,q}$ be given. Similar to the proof in case $\ell = 0$, we set $\eta_{j_0} := \delta_\xi D^{\alpha}_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} \in \mathcal{N}(T_2)$, $\eta_{j_\beta} := \delta_\xi D^{\alpha}_{\sigma(\tau,F)} D^{\beta}_{\sigma(\tau,T_1)} \in \mathcal{N}(T_1)$, $|\beta| = r'$. By (4.5), the equation

$$\eta_{j_0} - \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} = 0$$

belongs to $\mathcal{R}$. Hence, we get

$$\begin{aligned} \eta_{j_0} p_1 &= \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} p_1 = \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta a_{j_\beta} \\ &= a_{j_0} = \eta_{j_0} p_2, \end{aligned}$$

and (4.12) is proved. In view of (4.11) and (4.12), we conclude by Lemma 2.7 that $p = 0$, which establishes (4.10).

To prove (4.9) for any given $r' = 0, \ldots, r$, we set

$$p := D^{r'}_{\sigma_{F,w}} (p_2 - p_1)|_F \in \Pi^{n-1}_{d-r'}.$$

Analysis similar to the above shows that by (4.10) it follows that for each facet τ of F,

$$\delta_x D^q_{\sigma(\tau,F)} p = 0, \qquad \text{all } x \in \tau,\ q = 0, \ldots, 2r - r'.$$

Furthermore, given $\xi \in \Xi_{F,r'}$, the nodal functionals $\eta_{j_1} := \delta_\xi D^{r'}_{\sigma(F,T_1)}$ and $\eta_{j_2} := \delta_\xi D^{r'}_{\sigma_{F,w}}$ are in $\mathcal{N}(T_1)$ and $\mathcal{N}(T_2)$, respectively. By (4.6),

$$\delta_\xi D^{r'}_{\sigma_{F,w}} = (-1)^{r'} \delta_\xi D^{r'}_{\sigma(F,T_1)},$$

and hence

$$\delta_\xi p = \eta_{j_2} p_2 - (-1)^{r'} \eta_{j_1} p_1 = a_{j_2} - (-1)^{r'} a_{j_1} = 0.$$

Thus, Lemma 2.7 implies that $p = 0$, which establishes (4.9) and completes the proof of the theorem. □

§5. Construction of a local basis for $\mathcal{S}^r_d(\triangle)$

Let $d \geq r2^n + 1$. Since $\mathcal{N}$ is a determining set for $\mathcal{S}^r_d(\triangle)$ by Theorem 3.1, and $\mathcal{R}$ is a complete system of linear relations for $\mathcal{N}$ over $\mathcal{S}^r_d(\triangle)$ by Theorem 4.4, Algorithm 2.4 can be applied to construct a basis $\{\tilde{s}_1, \ldots, \tilde{s}_m\}$ for $\mathcal{S}^r_d(\triangle)$. To this end we only need to choose a basis $\{a^{[1]}, \ldots, a^{[m]}\}$ for the null space $N(C)$ of the corresponding matrix C. In this section we will show how to choose the basis for $N(C)$ so that the resulting basis for $\mathcal{S}^r_d(\triangle)$ is *local* as defined below.

Let v be a vertex of $\triangle$. We set $\operatorname{star}^1(v) := \operatorname{star}(v)$, and define $\operatorname{star}^\gamma(v)$, $\gamma \geq 2$, recursively as the union of the stars of the vertices in $\mathcal{T}_0 \cap \operatorname{star}^{\gamma-1}(v)$.

Definition 5.1. Let $\mathcal{S}$ be a linear subspace of $\mathcal{S}_d^r(\triangle)$. A basis $\{s_1, \ldots, s_m\}$ for $\mathcal{S}$ is called local (or γ-local) if there is an integer γ such that for each $k = 1, \ldots, m$, supp $s_k \subset \text{star}^\gamma(v_k)$, for some vertex v_k of $\triangle$, and the dual functionals $\lambda_1, \ldots, \lambda_m$, defined by (2.1), can be localized in the same sets $\text{star}^\gamma(v_1), \ldots, \text{star}^\gamma(v_k)$, *i.e.*, for each $k = 1, \ldots, m$, $\lambda_k s = 0$ for all $s \in \mathcal{S}$ satisfying $s|_{\text{star}^\gamma(v_k)} = 0$.

We say that an *algorithm produces local bases* if there exists an absolute (integer) constant γ such that any basis constructed by that algorithm is at most γ-local.

The key observation for our construction is that the matrix C of the system $\mathcal{R}$ has a *block diagonal structure*. More precisely, by Remark 4.3 we have

$$\begin{aligned} C &= [\tilde{C}\ O], \\ \tilde{C} &= \operatorname{diag}(C_\tau)_{\tau \in \mathcal{T} \setminus \mathcal{T}_n}, \end{aligned} \tag{5.1}$$

where C_τ is the matrix of the system $\mathcal{R}_\tau$ defined in (4.7), and O is the zero matrix corresponding to the nodal functionals in $\mathcal{N}_T$, $T \in \mathcal{T}_n$, not involved in any smoothness conditions. Moreover, each matrix C_τ itself is block diagonal. Namely,

$$C_\tau = \operatorname{diag}(C_{\tau,q})_{q=0,\ldots,r2^{n-\ell-1}}, \qquad \tau \in \mathcal{T}_\ell, \quad 0 \le \ell \le n-1, \tag{5.2}$$

where $C_{\tau,q}$ is the matrix of the system $\mathcal{R}_{\tau,q}$ defined in (4.4)–(4.7). If $1 \le \ell \le n-1$, then the matrix $C_{\tau,q}$ is again block diagonal,

$$C_{\tau,q} = \operatorname{diag}(C_{\tau,q,\xi})_{\xi \in \Xi_{\tau,q}},$$

with $C_{\tau,q,\xi}$ being the matrix of the system $\mathcal{R}_{\tau,q,\xi}$. By Lemma 2.3, we have

$$\begin{aligned} \dim \mathcal{S}_d^r(\triangle) &= \#\mathcal{N} - \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_n} \operatorname{rank} C_\tau \\ &= \#\mathcal{N} - \sum_{v \in \mathcal{T}_0} \sum_{q=0}^{r2^{n-1}} \operatorname{rank} C_{v,q} - \sum_{\ell=1}^{n-1} \sum_{\tau \in \mathcal{T}_\ell} \sum_{q=0}^{r2^{n-\ell-1}} \sum_{\xi \in \Xi_{\tau,q}} \operatorname{rank} C_{\tau,q,\xi} \end{aligned} \tag{5.3}$$

Remark 5.2. The formula (5.3) leads to an efficient computation of the dimension of the space $\mathcal{S}_d^r(\triangle)$ by applying to the *small* matrices $C_{v,q}$ and $C_{\tau,q,\xi}$ the standard numerical algorithms of rank determination (see *e.g.* [29]).

In view of (5.1) and (5.2), $N(\tilde{C})$ is an (outer) direct sum of $N(C_{\tau,q})$, $q = 0, \ldots, r2^{n-\ell-1}$, $\tau \in \mathcal{T}_\ell$, $0 \le \ell \le n-1$. Hence, if we know bases for all $N(C_{\tau,q})$, then we can combine them into a basis for $N(\tilde{C})$ that trivially extends to a basis for $N(C)$. Let $\mathcal{N}_{\tau,q} = \{\eta_j^{[\tau,q]}\}_{j \in J_{\tau,q}}$ and $C_{\tau,q} = (c_{i,j}^{[\tau,q]})_{i \in I_{\tau,q}, j \in J_{\tau,q}}$, so that $\mathcal{R}_{\tau,q}$ has the form

$$\sum_{j \in J_{\tau,q}} c_{i,j}^{[\tau,q]} \eta_j^{[\tau,q]} = 0, \qquad i \in I_{\tau,q}.$$

For each $\tau \in \mathcal{T}_\ell$, $0 \le \ell \le n-1$, and $q = 0, \ldots, r2^{n-\ell-1}$, suppose

$$a^{[\tau,q,k]} = (a_j^{[\tau,q,k]})_{j \in J_{\tau,q}}, \qquad k = 1, \ldots, m_{\tau,q}, \tag{5.4}$$

form a basis for $N(C_{\tau,q})$. In addition, for each $T \in \mathcal{T}_n$, let $a^{[T,0,k]} = (a_j^{[T,0,k]})_{j \in J_{T,0}}$, $k = 1, \ldots, m_T$, be any basis of $\mathbb{R}^{m_T}$, where $m_T = \#J_{T,0} = \#\mathcal{N}_T = \#\Xi_T$. We define $\tilde{a}^{[\tau,q,k]} = (\tilde{a}_j^{[\tau,q,k]})_{j \in J}$, with $J = \cup_{\tau,q} J_{\tau,q}$, by

$$\tilde{a}_j^{[\tau,q,k]} := \begin{cases} a_j^{[\tau,q,k]}, & \text{if } j \in J_{\tau,q}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the vectors $\tilde{a}^{[\tau,q,k]}$, $k = 1, \ldots, m_{\tau,q}$, $q = 0, \ldots, q_\ell$, $\tau \in \mathcal{T}_\ell$, $0 \le \ell \le n$, where

$$q_\ell = \begin{cases} r2^{n-\ell-1}, & \text{if } 0 \le \ell \le n-1, \\ 0, & \text{if } \ell = n, \end{cases} \tag{5.5}$$

obviously form a basis for $N(C)$. The corresponding basis

$$\tilde{s}^{[\tau,q,k]}, \qquad k = 1, \ldots, m_{\tau,q}, \quad q = 0, \ldots, q_\ell, \quad \tau \in \mathcal{T}_\ell, \quad 0 \le \ell \le n, \tag{5.6}$$

for $\mathcal{S}_d^r(\triangle)$ produced by Algorithm 2.4 satisfies

$$\begin{aligned} \eta_j^{[\tau,q]} \tilde{s}^{[\tau,q,k]} &= a_j^{[\tau,q,k]}, \qquad j \in J_{\tau,q}, \\ \eta \tilde{s}^{[\tau,q,k]} &= 0, \qquad \text{all } \eta \in \mathcal{N} \setminus \mathcal{N}_{\tau,q}. \end{aligned} \tag{5.7}$$

Denote by

$$\tilde{\lambda}^{[\tau,q,k]}, \qquad k = 1, \ldots, m_{\tau,q}, \quad q = 0, \ldots, q_\ell, \quad \tau \in \mathcal{T}_\ell, \quad 0 \le \ell \le n, \tag{5.8}$$

the dual basis for $\mathcal{S}_d^r(\triangle)^*$ determined by the duality condition

$$\tilde{\lambda}^{[\tau,q,k]} \tilde{s}^{[\tau',q',k']} = \begin{cases} 1, & \text{if } \tau = \tau', \ q = q' \text{ and } k = k', \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5.3. *The basis (5.6) for $\mathcal{S}_d^r(\triangle)$, where $d \ge r2^n + 1$, is local. Moreover,*

$$\operatorname{supp} \tilde{s}^{[\tau,q,k]} \subset \operatorname{star}(\tau), \tag{5.9}$$

and the dual basis (5.8) satisfies

$$\tilde{\lambda}^{[\tau,q,k]} s = 0 \quad \text{for all } s \in \mathcal{S}_d^r(\triangle) \text{ such that } s|_{\operatorname{star}(\tau)} = 0. \tag{5.10}$$

Proof: By (5.7) we have $\eta \tilde{s}^{[\tau,q,k]} = 0$ for all $\eta \in \mathcal{N} \setminus \mathcal{N}_{\tau,q}$. Since $\mathcal{N}_{\tau,q} \cap \mathcal{N}(T) \ne \emptyset$ only if $\tau \subset T$, (5.9) follows from the fact that $\mathcal{N}(T)$ is a determining set for Π_d^n, see Theorem 3.2. To show (5.10), we consider the matrix A with columns

$$\tilde{a}^{[\tau,q,k]}, \qquad k = 1, \ldots, m_{\tau,q}, \quad q = 0, \ldots, q_\ell, \quad \tau \in \mathcal{T}_\ell, \quad 0 \le \ell \le n.$$

This matrix is block diagonal,

$$\begin{aligned} A &= \operatorname{diag}(A_\tau)_{\tau \in \mathcal{T}}, \\ A_\tau &= \operatorname{diag}(A_{\tau,q})_{q=0,\ldots,q_\ell}, \qquad \tau \in \mathcal{T}_\ell, \quad 0 \le \ell \le n, \end{aligned}$$

where $A_{\tau,q} := (a_j^{[\tau,q,k]})_{j \in J_{\tau,q}, k=1,\ldots,m_{\tau,q}}$. Let $B_{\tau,q}$ be a left inverse of $A_{\tau,q}$. Then $B := \operatorname{diag}(B_\tau)_{\tau \in \mathcal{T}}$, with $B_\tau = \operatorname{diag}(B_{\tau,q})_{q=0,\ldots,q_\ell}$, $\tau \in \mathcal{T}_\ell$, $0 \le \ell \le n$, is a left inverse of A. Hence, by Lemma 2.5, $\tilde{\lambda}^{[\tau,q,k]}$ is a linear combination of $\eta_j^{[\tau,q]}$, $j \in J_{\tau,q}$. This implies (5.10) since for every $\eta \in \mathcal{N}_{\tau,q}$ we obviously have $\eta s = 0$ if $s|_{\operatorname{star}(\tau)} = 0$. $\square$

Remark 5.4. A similar analysis of the space $\mathcal{S}_d^r(\triangle)$, $d \geq r2^n+1$, was done in [2] by using Bernstein-Bézier smoothness conditions [5]. However, the existence of a local basis for $\mathcal{S}_d^r(\triangle)$ was shown in [2] only for $n \leq 3$. The main advantage of the nodal techniques used here is that the matrix $\tilde{C}$ in (5.1) is block diagonal, while the matrix of Bernstein-Bézier smoothness conditions is block triangular (see [6]).

§6. A stable local basis for $\mathcal{S}_d^r(\triangle)$

In this section we show that if the sets $\Xi_{\tau,q}$ and Ξ_T as well as the bases (5.4) for $N(C_{\tau,q})$ are properly chosen, then an appropriately renormalized version of the local basis for $\mathcal{S}_d^r(\triangle)$ constructed above is in addition stable.

Let us denote by $\omega_\triangle$ the shape regularity constant of the triangulation $\triangle$,

$$\omega_\triangle := \max_{T\in\triangle} \frac{h_T}{\rho_T},$$

where h_T and ρ_T are the diameter of T and the diameter of its inscribed sphere, respectively. Given $M = \bigcup_{T\in\tilde{\triangle}} T$, where $\tilde{\triangle} \subset \triangle$, we denote by $|M|$ the *n-dimensional volume* of M.

Definition 6.1. *Let $\mathcal{S}$ be a linear subspace of $\mathcal{S}_d^r(\triangle)$. We say that a basis $\{\tilde{s}_1,\ldots,\tilde{s}_m\}$ for $\mathcal{S}$ is L_p-*stable *if there exist constants K_1, K_2 depending only on n, r, d and $\omega_\triangle$, such that for any $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m$,*

$$K_1\|\alpha\|_{\ell_p} \leq \Big\|\sum_{k=1}^m \alpha_k \tilde{s}_k\Big\|_{L_p(\Omega)} \leq K_2\|\alpha\|_{\ell_p}.$$

To establish stability of a local basis it seems most convenient to use the following general lemma; see also [23].

Lemma 6.2. *Let $\{s_1,\ldots,s_m\}$ be a γ-local basis for $\mathcal{S}$, and let $\{\lambda_1,\ldots,\lambda_m\} \subset \mathcal{S}^*$ be its dual basis. Suppose that*

$$\|s_k\|_{L_\infty(\Omega)} \leq C_1, \qquad k=1,\ldots,m, \tag{6.1}$$

and

$$|\lambda_k s| \leq C_2\|s\|_{L_\infty(\mathrm{star}^\gamma(v_k))}, \qquad \text{all } s\in\mathcal{S}, \quad k=1,\ldots,m, \tag{6.2}$$

where $\mathrm{supp}\, s_k \subset \mathrm{star}^\gamma(v_k)$ *as in Definition 5.1. Then for any $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m$,*

$$K_1C_2^{-1}\|\alpha\|_{\ell_p} \leq \Big\|\sum_{k=1}^m \alpha_k \frac{s_k}{|\mathrm{supp}\, s_k|^{1/p}}\Big\|_{L_p(\Omega)} \leq K_2C_1\|\alpha\|_{\ell_p}, \quad 1\leq p\leq\infty, \tag{6.3}$$

where K_1, K_2 are some constants depending only on n, r, d, γ and $\omega_\triangle$.

Proof: Let $s = \sum_{k=1}^m \alpha_k \frac{s_k}{|\operatorname{supp} s_k|^{1/p}}$. We first prove the upper bound in (6.3). Given an n-simplex $T \in \triangle$, we have by (6.1)

$$\|s|_T\|_{L_p(T)} \le C_1(\#\Sigma_T)^{1-1/p} \begin{cases} \Big(\sum_{k\in\Sigma_T} |\alpha_k|^p\Big)^{1/p}, & \text{if } 1 \le p < \infty, \\ \max_{k\in\Sigma_T} |\alpha_k|, & \text{if } p = \infty, \end{cases}$$

where

$$\Sigma_T := \{k : \ T \subset \operatorname{supp}\, s_k\}. \tag{6.4}$$

As in the bivariate case (see Lemmas 3.1 and 3.2 in [23]), it is not difficult to show that

$$\#\{T \in \triangle : \ T \subset \operatorname{star}^\gamma(v_k)\} \le \tilde K_1, \tag{6.5}$$

and

$$\max\left\{ \frac{|\operatorname{star}^\gamma(v_k)|}{|T|} : \ T \subset \operatorname{star}^\gamma(v_k)\right\} \le \tilde K_2, \tag{6.6}$$

where $\tilde K_1, \tilde K_2$ are some constants depending only on n, γ and $\omega_\triangle$. Hence, for $1 \le p < \infty$ we have

$$\|s\|^p_{L_p(\Omega)} = \sum_{T\in\triangle} \|s|_T\|^p_{L_p(T)} \le \tilde K_1 C_1^p (\#\Sigma_T)^{p-1} \|\alpha\|^p_{\ell_p},$$

which shows that the upper bound will be established for all $1 \le p \le \infty$ if we prove that $\#\Sigma_T$ is bounded by a constant depending only on n, r, d, γ and $\omega_\triangle$. To this end we note that since the basis $\{s_1, \ldots, s_m\}$ is γ-local, $\operatorname{supp} s_k \subset \operatorname{star}^{2\gamma}(v)$, for all $k \in \Sigma_T$, where v is any vertex of T. Therefore, the set $\{s_k : \ k \in \Sigma_T\}$ is linearly independent on $\operatorname{star}^{2\gamma}(v)$, and its cardinality $\#\Sigma_T$ does not exceed the dimension of the space of all piecewise polynomials of degree d on $\operatorname{star}^{2\gamma}(v)$, *i.e.*, $\#\Sigma_T \le N\binom{n+d}{n}$, where N is the number of n-simplices of $\triangle$ lying in $\operatorname{star}^{2\gamma}(v)$. By (6.5), N is bounded by a constant depending only on n, γ and $\omega_\triangle$, and the assertion follows.

To establish the lower bound in (6.3), we obtain by (6.2),

$$|\alpha_k| = |\operatorname{supp} s_k|^{1/p} |\lambda_k s| \le C_2 |\operatorname{supp} s_k|^{1/p} \|s\|_{L_\infty(\operatorname{star}^\gamma(v_k))}, \qquad k = 1, \ldots, m.$$

Since $\|s\|_{L_\infty(\operatorname{star}^\gamma(v_k))} \le \|s\|_{L_\infty(\Omega)}$, this completes the proof in the case $p = \infty$. Suppose $1 \le p < \infty$. By a Nikolskii-type inequality, see *e.g.* [27, p. 56], for some n-simplex $T_k \subset \operatorname{star}^\gamma(v_k)$,

$$\|s\|_{L_\infty(\operatorname{star}^\gamma(v_k))} = \|s|_{T_k}\|_{L_\infty(T_k)} \le \tilde K_3 |T_k|^{-1/p} \|s|_{T_k}\|_{L_p(T_k)},$$

where $\tilde K_3$ is a constant depending only on n and d. Since $\operatorname{supp} s_k \subset \operatorname{star}^\gamma(v_k)$, we have by (6.6),

$$\frac{|\operatorname{supp} s_k|}{|T_k|} \le \tilde K_2.$$

Therefore,

$$\sum_{k=1}^{m} |\alpha_k|^p \le \tilde{K}_2(\tilde{K}_3 C_2)^p \sum_{k=1}^{m} \int_{T_k} |s|^p.$$

We now have to bound the number of appearances of a given n-simplex T_k on the right-hand side of the above inequality. If $T_{k_1} = T_{k_2}$, then $\mathrm{star}^\gamma(v_{k_1}) \cap \mathrm{star}^\gamma(v_{k_2}) \ne \emptyset$. Hence, $\mathrm{supp}\, s_{k_2} \subset \mathrm{star}^{3\gamma}(v_{k_1})$. Thus, for all k such that $T_k = T_{k_1}$,

$$\mathrm{supp}\, s_k \subset \mathrm{star}^{3\gamma}(v_{k_1}).$$

The set $\{s_k : T_k = T_{k_1}\}$ is linearly independent on $\mathrm{star}^{3\gamma}(v_{k_1})$, and it can be shown as above that its cardinality is bounded by a constant $\tilde{K}_4$ depending only on n, γ and $\omega_\triangle$. Therefore,

$$\sum_{k=1}^{m} \int_{T_k} |s|^p \le \tilde{K}_4 \int_\Omega |s|^p,$$

which completes the proof. $\square$

We are ready to formulate our main result about stability of the local basis constructed in Section 5. For each $\tau \in \mathcal{T}$, denote by h_τ the *diameter* of the set $\mathrm{star}(\tau)$. (This is compatible with the above notation h_T for $T \in \mathcal{T}_n = \triangle$ since $\mathrm{star}(T) = T$.)

Theorem 6.3. *Suppose that*

1) *every $\Xi_{\tau,q}$, $q = 0, \dots, q_\ell$, $\tau \in \mathcal{T}_\ell$, $1 \le \ell \le n$ (where $\Xi_{T,0} := \Xi_T$ if $T \in \mathcal{T}_n$), is chosen to be the set of uniformly distributed points in the interior of τ, as defined in (3.5); and*
2) *for each $q = 0, \dots, q_\ell$ and $\tau \in \mathcal{T}_\ell$, $0 \le \ell \le n$, the vectors*

$$a^{[\tau,q,k]} = (a_j^{[\tau,q,k]})_{j \in J_{\tau,q}}, \qquad k = 1, \dots, m_{\tau,q}, \tag{6.7}$$

form an orthonormal basis for $N(C_{\tau,q})$.

Let $\tilde{s}^{[\tau,q,k]}$ be the local basis functions for $\mathcal{S}_d^r(\triangle)$, $d \ge r2^n + 1$, constructed as in Section 5. Then for every $1 \le p \le \infty$, the splines

$$h_\tau^{-q} |\mathrm{star}(\tau)|^{-\frac{1}{p}} \tilde{s}^{[\tau,q,k]}, \qquad k = 1, \dots, m_{\tau,q}, \quad q = 0, \dots, q_\ell, \quad \tau \in \mathcal{T}_\ell, \quad 0 \le \ell \le n,$$

form an L_p-stable local basis for $\mathcal{S}_d^r(\triangle)$.

Proof: As shown in Section 5, the splines $\tilde{s}^{[\tau,q,k]}$ are 1-local, and $\mathrm{supp}\, \tilde{s}^{[\tau,q,k]} \subset \mathrm{star}(\tau)$. By (6.6),

$$|\mathrm{supp}\, \tilde{s}^{[\tau,q,k]}| \le |\mathrm{star}(\tau)| \le \tilde{K}_2 |\mathrm{supp}\, \tilde{s}^{[\tau,q,k]}|,$$

where $\tilde{K}_2$ depends only on n and $\omega_\triangle$. Hence, in view of Lemma 6.2, the theorem will be established once we prove that

$$\|\tilde{s}^{[\tau,q,k]}\|_{L_\infty(\Omega)} \le C_1 h_\tau^q, \tag{6.8}$$

and
$$|\tilde\lambda^{[\tau,q,k]}s| \le C_2 h_\tau^{-q}\|s\|_{L_\infty(\mathrm{star}(\tau))}, \qquad \text{all } s\in \mathcal{S}_d^r(\triangle), \tag{6.9}$$
where the constants C_1, C_2 depend only on n, r, d and $\omega_\triangle$.

We first show (6.8). Since $\operatorname{supp}\tilde s^{[\tau,q,k]}\subset \mathrm{star}(\tau)$, we have $\|\tilde s^{[\tau,q,k]}\|_{L_\infty(\Omega)} = \|\tilde s^{[\tau,q,k]}\|_{L_\infty(\mathrm{star}(\tau))}$. Let T be an n-simplex in $\mathrm{star}(\tau)$, and let $\mathcal{H}_T$ be the Hermite interpolation operator defined in (3.4). Since $\tilde s^{[\tau,q,k]}|_T = \mathcal{H}_T\tilde s^{[\tau,q,k]}|_T$, we have by Lemma 3.3,
$$\|\tilde s^{[\tau,q,k]}|_T\|_{L_\infty(T)} \le \tilde K_5 \max_{\eta\in\mathcal{N}(T)} h_T^{q(\eta)}|\eta\tilde s^{[\tau,q,k]}|,$$
where $\tilde K_5$ depends only on n, r and d. Now, by (5.7), $\eta\tilde s^{[\tau,q,k]} = 0$ for all $\eta\in\mathcal{N}(T)\setminus\mathcal{N}_{\tau,q}$, and
$$\eta_j^{[\tau,q]}\tilde s^{[\tau,q,k]} = a_j^{[\tau,q,k]}, \qquad j\in J_{\tau,q}.$$
Since the vectors $a^{[\tau,q,k]}$, $k=1,\dots,m_{\tau,q}$, are orthonormal, we have $|a_j^{[\tau,q,k]}|\le 1$. Taking into account that $q(\eta)=q$ for all $\eta\in\mathcal{N}_{\tau,q}$, we arrive at the estimate
$$\|\tilde s^{[\tau,q,k]}|_T\|_{L_\infty(T)} \le \tilde K_5 h_T^q \le \tilde K_5 h_\tau^q,$$
and (6.8) is proved.

By our hypotheses, the columns of the matrix
$$A_{\tau,q} = [a_j^{[\tau,q,k]}]_{j\in J_{\tau,q},\,k=1,\dots,m_{\tau,q}} \tag{6.10}$$
are orthonormal. Hence, $A_{\tau,q}^T$ is a left inverse of $A_{\tau,q}$. By Lemma 2.5 and the proof of Theorem 5.3, it follows that the dual functional $\tilde\lambda^{[\tau,q,k]}$ can be computed as
$$\tilde\lambda^{[\tau,q,k]} = \sum_{j\in J_{\tau,q}} a_j^{[\tau,q,k]}\eta_j^{[\tau,q]}.$$
Therefore, for any $s\in\mathcal{S}_d^r(\triangle)$,
$$|\tilde\lambda^{[\tau,q,k]}s| = \Big|\sum_{j\in J_{\tau,q}} a_j^{[\tau,q,k]}\eta_j^{[\tau,q]}s\Big| \le \#J_{\tau,q}\max_{j\in J_{\tau,q}}|\eta_j^{[\tau,q]}s|.$$
Given $j\in J_{\tau,q}$, let T be an n-simplex such that $\tau\subset T$ and $\eta_j^{[\tau,q]}\in\mathcal{N}(T)$. Since $\eta_j^{[\tau,q]}$ is a nodal functional of order q, we have by Markov inequality (see, *e.g.* [13]),
$$|\eta_j^{[\tau,q]}s| = |\eta_j^{[\tau,q]}s|_T| \le \tilde K_6\rho_T^{-q}\|s|_T\|_{L_\infty(T)} \le \tilde K_6\omega_\triangle^q h_T^{-q}\|s\|_{L_\infty(\mathrm{star}(\tau))},$$
where $\tilde K_6$ is a constant depending only on n and d. Since $\#J_{\tau,q} = \#\mathcal{N}_{\tau,q}$ is bounded above by a constant depending only on n, r, d and $\omega_\triangle$, the estimate (6.9) follows, and the proof is complete. □

It is easy to see that Theorem 6.3 remains valid for any $\Xi_{\tau,q}$ such that the Hermite interpolation operator defined by (3.4) satisfies (3.6), and for any choice of the bases (6.7) for $N(C_{\tau,q})$ such that the *condition number* of the matrix (6.10) is bounded by a constant K depending only on n, r, d and $\omega_\triangle$; compare [6]. However, there is a good reason to prefer, at least in practice, an *orthonormal basis* for $N(C_{\tau,q})$, as explaned in the following remark.

Remark 6.4. There is a numerically efficient way to compute an orthonormal basis $a^{[\tau,q,k]} = (a_j^{[\tau,q,k]})_{j\in J_{\tau,q}}$, $k = 1,\ldots,m_{\tau,q}$, for each $N(C_{\tau,q})$, as required in the above theorem. Namely, construct by an appropriate algorithm a *singular value decomposition* $C_{\tau,q} = Q_L X Q_R^T$ of the matrix $C_{\tau,q}$, where Q_L, Q_R are orthogonal matrices, and $X = [D\ O]$, $D = \mathrm{diag}(\sigma_1,\ldots,\sigma_p)$, with $\sigma_1 \geq \cdots \geq \sigma_p \geq 0$ being the singular values of $C_{\tau,q}$, see *e.g.* [29]. Obviously, $m_{\tau,q}$ is equal to the number of zero columns in X (including the columns corresponding to zero singular values). Hence, the columns of the matrix $[O\ I_{m_{\tau,q}}]^T$ constitute an orthonormal basis for $N(X)$. Since $C_{\tau,q}Q_R = Q_L X$, the columns of $A_{\tau,q} = Q_R\,[O\ I_{m_{\tau,q}}]^T$ form the desired orthonormal basis for $N(C_{\tau,q})$. Thus, the matrix $A_{\tau,q}$ consists of the last $m_{\tau,q}$ columns of Q_R.

§7. Superspline spaces

In this section we construct stable local bases for the superspline subspaces of $\mathcal{S}_d^r(\triangle)$.

Definition 7.1. Let $\rho = (\rho_\tau)_{\tau\in\mathcal{T}\setminus(\mathcal{T}_{n-1}\cup\mathcal{T}_n)}$ be a sequence of integers satisfying

$$r \leq \rho_\tau \leq 2^{n-\ell-1}, \qquad \tau\in\mathcal{T}_\ell, \quad 0\leq\ell\leq n-2. \tag{7.1}$$

The linear space of splines

$$\begin{aligned}\mathcal{S}_d^{r,\rho}(\triangle) := \{s\in\mathcal{S}_d^r(\triangle) : s \text{ is } \rho_\tau\text{-times differentiable across } \tau,\\ \text{for all } \tau\in\mathcal{T}\setminus(\mathcal{T}_{n-1}\cup\mathcal{T}_n)\}\end{aligned} \tag{7.2}$$

is called a superspline space.

In the limiting case $\rho_\tau = 2^{n-\ell-1}$, $\tau\in\mathcal{T}\setminus(\mathcal{T}_{n-1}\cup\mathcal{T}_n)$, the superspline spaces were introduced and studied in [8–11], see also [3,4]. In particular, local bases for $\mathcal{S}_d^{r,\rho}(\triangle)$, where $\rho_\tau = 2^{n-\ell-1}$, were constructed in [11] and [4]. For general ρ_τ, but only in the bivariate case $n = 2$, the superspline spaces were explored in [22,28] and, more recently, in [18,19].

As we will see, our method of construction of a stable local basis can be applied to the spaces (7.2). We first have to extend the system $\mathcal{R}$ of smoothness conditions defined in (4.4)–(4.8) to a larger system $\hat{\mathcal{R}}$, by allowing a larger range of r' in (4.4) and (4.5). Namely, we include in the extended systems $\hat{\mathcal{R}}_{v,q}$ and $\hat{\mathcal{R}}_{\tau,q,\xi}$ all conditions (4.4) and (4.5), respectively, where $0\leq r'\leq\min\{\rho_\tau, q\}$. The systems $\mathcal{R}_{F,q,\xi}$ are not enlarged, *i.e.*, we set $\hat{\mathcal{R}}_{F,q,\xi} = \mathcal{R}_{F,q,\xi}$.

By the method of proof of Theorem 4.4 it is not difficult to establish the following analogue of it.

Theorem 7.2. *The set $\hat{\mathcal{R}}$ is a complete system of linear relations for $\mathcal{N}$ over $\mathcal{S}_d^{r,\rho}(\triangle)$.*

It is easy to see that the matrix $\hat{C}$ of the system $\hat{\mathcal{R}}$ possesses a block diagonal structure similar to the structure of the matrix C considered in Section 5. Therefore, all results about the dimension and the local bases carry over to the superspline spaces. Thus, we have

$$\dim \mathcal{S}_d^{r,\rho}(\triangle) = \#\mathcal{N} - \sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_n} \operatorname{rank} \hat{C}_\tau$$
$$= \#\mathcal{N} - \sum_{v\in\mathcal{T}_0} \sum_{q=0}^{r2^{n-1}} \operatorname{rank} \hat{C}_{v,q} - \sum_{\ell=1}^{n-1} \sum_{\tau\in\mathcal{T}_\ell} \sum_{q=0}^{r2^{n-\ell-1}} \sum_{\xi\in\Xi_{\tau,q}} \operatorname{rank} \hat{C}_{\tau,q,\xi}, \tag{7.3}$$

where $\hat{C}_\tau$, $\hat{C}_{v,q}$ and $\hat{C}_{\tau,q,\xi}$ are the appropriate blocks of $\hat{C}$. Define the splines

$$\hat{s}^{[\tau,q,k]}, \qquad k = 1,\ldots,\hat{m}_{\tau,q}, \quad q = 0,\ldots,q_\ell, \quad \tau\in\mathcal{T}_\ell, \quad 0\le\ell\le n, \tag{7.4}$$

by the condition

$$\begin{aligned} \eta_j^{[\tau,q]}\hat{s}^{[\tau,q,k]} &= \hat{a}_j^{[\tau,q,k]}, \qquad j\in J_{\tau,q}, \\ \eta\hat{s}^{[\tau,q,k]} &= 0, \qquad \text{all } \eta\in\mathcal{N}\setminus\mathcal{N}_{\tau,q}, \end{aligned} \tag{7.5}$$

where

$$\hat{a}^{[\tau,q,k]} = (\hat{a}_j^{[\tau,q,k]})_{j\in J_{\tau,q}}, \qquad k=1,\ldots,\hat{m}_{\tau,q}, \tag{7.6}$$

is a basis for $N(\hat{C}_{\tau,q})$.

Theorem 7.3. *The splines (7.4) form a local basis for $\mathcal{S}_d^{r,\rho}(\triangle)$, where ρ satisfies (7.1), and $d\ge r2^n+1$. Moreover,*

$$\operatorname{supp} \hat{s}^{[\tau,q,k]} \subset \operatorname{star}(\tau), \tag{7.7}$$

and the dual basis (5.8) satisfies

$$\hat{\lambda}^{[\tau,q,k]}s = 0 \quad \textit{for all } s\in\mathcal{S}_d^r(\triangle) \textit{ such that } s|_{\operatorname{star}(\tau)} = 0. \tag{7.8}$$

Since (7.4) is a local basis for $\mathcal{S}_d^{r,\rho}(\triangle)$, Lemma 6.2 can be applied, and the same argument as in the proof of Theorem 6.3 shows that the following result holds.

Theorem 7.4. *Suppose that*

1) *every $\Xi_{\tau,q}$, $q=0,\ldots,q_\ell$, $\tau\in\mathcal{T}_\ell$, $1\le\ell\le n$ (where $\Xi_{T,0} := \Xi_T$ if $T\in\mathcal{T}_n$), is chosen to be the set of uniformly distributed points in the interior of τ, as defined in (3.5), and*
2) *for each $q=0,\ldots,q_\ell$ and $\tau\in\mathcal{T}_\ell$, $0\le\ell\le n$, vectors $\hat{a}^{[\tau,q,k]} = (\hat{a}_j^{[\tau,q,k]})_{j\in J_{\tau,q}}$, $k=1,\ldots,m_{\tau,q}$, form an orthonormal basis for $N(\hat{C}_{\tau,q})$.*

Let $\hat{s}^{[\tau,q,k]}$ be the local basis functions (7.4) for $\mathcal{S}_d^{r,\rho}(\triangle)$, where ρ satisfies (7.1), and $d\ge r2^n+1$. Then for every $1\le p\le\infty$, the splines

$$h_\tau^{-q}|\operatorname{star}(\tau)|^{-\frac{1}{p}}\hat{s}^{[\tau,q,k]}, \qquad k=1,\ldots,m_{\tau,q}, \quad q=0,\ldots,q_\ell, \quad \tau\in\mathcal{T}_\ell, \quad 0\le\ell\le n,$$

form an L_p-stable local basis for $\mathcal{S}_d^{r,\rho}(\triangle)$.

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