Quantum tomographic cryptography with Bell diagonal states: non-equivalence of classical and quantum distillation protocols

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We present a generalized tomographic quantum key distribution protocol in which the two parties share a Bell diagonal mixed state of two qubits. We show that if an eavesdropper performs a coherent measurement on many quantum ancilla states simultaneously, classical methods of secure key distillation are less effective than quantum entanglement distillation protocols. We also show that certain Bell diagonal states are resistant to any attempt of incoherent eavesdropping.

INTRODUCTION

The security of quantum key distribution (QKD) [1, 2] is an important consequence of the application of the laws of physics to information and communication theory. A one-time pad provides perfect cryptographic security for sending messages between two parties but relies on being able to distribute a shared secret key [3, 4]. Classically, it is impossible to amplify a set of shared randomness, but quantum mechanics allows this to be done by the transmission of quantum states [5]. The full power of quantum cryptography rests on the ability to place upper bounds on the knowledge of a potential eavesdropper (Eve) about the distributed key shared by the legitimate parties (Alice and Bob). In this paper we present a generalization of the so-called tomographic quantum key distribution protocol [6]. We consider the situation where Alice and Bob use qubits in a maximally entangled state distributed by a central source. The qubits undergo a quantum channel that converts the state to a Bell diagonal mixed state.

We analyze the security of this protocol under two broad scenarios. In the first scenario, Alice and Bob agree on a cryptographic key if the correlations between their measurement results are stronger than any possible correlations between one of them and Eve, under the assumption that Eve has full control over the source of entangled qubits but she can only perform incoherent measurements. The tomographic element of the protocol allows Alice and Bob to compute the maximal strength of correlations between Eve and any one of them. The Csiszár-Körner [8] theorem then guarantees that if the correlations between Alice and Bob are stronger than those between Eve and either of them, a secure key can be established through *one-way* error correcting codes.

In the second scenario, we examine the situation when Eve's correlations are initially stronger than Alice and Bob's. It was shown that in some cases it is still possible to obtain a secure key [9]. The idea is that by means of two-way communication Alice and Bob can strengthen their correlations with respect to Eve's so that the CK theorem can be applied again. This procedure is called *advantage distillation* (AD).

There are two possible strategies for Eve within the second scenario: incoherent and coherent measurements. The first case was examined in [10] where it was shown that advantage distillation is possible as long as the two-qubit state shared by Alice and Bob is entangled. We re-derive this result using different reasoning than the one presented in [10].

In the second case, we show that the above result no longer holds in the case of coherent measurements by Eve. Indeed, if the qubits are affected by too many errors (caused by Eve's actions), advantage distillation fails despite Alice and Bob still sharing an entangled state. In such cases the only way for Alice and Bob to obtain a secure key is to revert to quantum entanglement distillation.

TOMOGRAPHIC QKD

In a tomographic QKD scheme, a central source distributes entangled qubits to Alice and Bob. They independently and randomly choose to measure three tomographically complete observables σ_x, σ_y and σ_z (Pauli matrices) on each qubit. At the end of the transmission, they publicly announce their choice of observables for each qubit pair. They then proceed to divide their measurement results according to those for which their measurement bases match, and those for which their measurement bases do not match. Exchanging a subset of their measurements allows Alice and Bob to tomographically reconstruct the density operator of the two-qubit state they share.

Ideally, in the absence of noise in the source or channels, they expect to receive the maximally entangled state

$$|\psi_{ideal}\rangle = \frac{1}{\sqrt{2}}\left(|z_0, z_0\rangle + |z_1, z_1\rangle\right), \qquad (1)$$

where $|z_k\rangle$ is the eigenstate of σ_z with the eigenvalue $(-1)^k$, and Alice (Bob) possesses the left (right) qubit. The results for matching bases can then be used to generate a cryptographic key as they are either perfectly correlated (for x and z bases) or anti-correlated (for y basis).

However, Alice and Bob cannot realistically expect to obtain the maximally entangled state Eq. (1) because either the source is not ideal, the channel conveying the qubits is noisy, or there is an eavesdropper tampering with the source. For security analysis, we assume that Eve has total control over the source and that all the errors are caused by her when she tries to extract information about the key.

To constrain Eve's information, Alice and Bob use part of their measurements to perform full tomography on the state distributed by the source. The protocol we consider here is such that Alice and Bob agree to communicate if and only if they see the Bell diagonal state

$$\varrho_{AB} = \sum_{a,b=0}^{1} p_{ab}|z_{ab}\rangle\langle z_{ab}|, \qquad (2)$$

where

$$|z_{ab}\rangle = \frac{1}{\sqrt{2}}\sum_{k=0}^{1}\omega^{kb}|z_k, z_{k+a}\rangle \qquad (3)$$

and $\sum_{a,b=0}^{1}p_{ab} = 1, \omega = -1$. Following the nomenclature of [11] we call a the amplitude bit and b the phase bit. Here, we assume that $p_{00} > \frac{1}{2}$ [7].

The above state can be obtained from the maximally entangled state Eq. (1) assuming that the travelling qubits undergo bit and phase flips. The so-called Werner state, i.e., the maximally entangled state with white noise, is a special case where $p_{01} = p_{10} = p_{11}$. Therefore, the protocol presented here is more general than the one studied in [6, 12] where only Werner states were considered.

As Alice and Bob perform their measurements in the three bases x, y and z, it is convenient to express the state ϱ_{AB} in the x and y bases. This can easily be done using the transformation rules on the Bell states,

$$|z_{ab}\rangle = \omega^{ab}|x_{ba}\rangle = (-\mathrm{i})^a\omega^{ab}|y_{a+b+1\,a}\rangle. \qquad (4)$$

Writing out (2) in the other bases, we have

$$\varrho_{AB} = \sum_{a,b=0}^{1} p_{ba}|x_{ab}\rangle\langle x_{ab}| = \sum_{a,b=0}^{1} p_{b\,a+1}|y_{ab}\rangle\langle y_{ab}|. \qquad (5)$$

We can then compute the probability of Alice and Bob obtaining correlated results conditional on a particular choice of basis:

$$
\begin{aligned}
\text{Prob(correlation}|x\text{ basis)} &= p_{00} + p_{11} \\
\text{Prob(correlation}|y\text{ basis)} &= p_{01} + p_{10} \\
\text{Prob(correlation}|z\text{ basis)} &= p_{00} + p_{11}, \qquad (6)
\end{aligned}
$$

and also the probability of getting anti-correlated results:

$$
\begin{aligned}
\text{Prob(anti-correlation}|x\text{ basis)} &= p_{01} + p_{10} \\
\text{Prob(anti-correlation}|y\text{ basis)} &= p_{00} + p_{11} \\
\text{Prob(anti-correlation}|z\text{ basis)} &= p_{01} + p_{10}. \qquad (7)
\end{aligned}
$$

Since $p_{00} > \frac{1}{2}$, Alice and Bob are more likely to obtain correlated results when they measure in the x and z bases, and anti-correlated results in the y basis; Alice and Bob will thus make use of correlation to generate their key when they measure in the x and z bases, and anti-correlation to generate their key when in the y basis.

EAVESDROPPING

In order to obtain as much information as possible about the key generated by Alice and Bob, Eve entangles their qubits with ancilla states $|e_{ab}\rangle$ in her possession. The best she can do is to prepare the following tripartite pure state

$$|\psi_{ABE}\rangle = \sum_{a,b=0}^{1} \sqrt{p_{ab}}|z_{ab}\rangle|e_{ab}\rangle, \qquad (8)$$

where $\langle e_{ab}|e_{cd}\rangle = \delta_{a,c}\delta_{b,d}$. Tracing out Eve gives the mixed state Eq. (2) that Alice and Bob measures, and this purification is the most general one as far as incoherent attacks are concerned.

Eve's purifications, when expressed in different bases, read

$$
\begin{aligned}
|\psi_{ABE}\rangle &= \frac{1}{\sqrt{2}}\sum_{k,a=0}^{1}|z_k, z_{k+a}\rangle\left(\sum_{b=0}^{1}\sqrt{p_{ab}}\omega^{kb}|e_{ab}\rangle\right) \\
&= \frac{1}{\sqrt{2}}\sum_{k,a=0}^{1}|x_k, x_{k+a}\rangle\left(\sum_{b=0}^{1}\sqrt{p_{ba}}\omega^{kb}\omega^{ab}|e_{ba}\rangle\right) \\
&= \frac{1}{\sqrt{2}}\sum_{k,a=0}^{1}|y_k, y_{k+a}\rangle \\
&\qquad \left(\sum_{b=0}^{1}\sqrt{p_{b\,a+b+1}}(-\mathrm{i})^b\omega^{kb}\omega^{b(a+b+1)}|e_{b\,a+b+1}\rangle\right),
\end{aligned}
$$
$$(9)$$

We can express Eq. (9) more conveniently as

$$
\begin{aligned}
|\psi_{ABE}\rangle &= \sum_{k,a=0}^{1} \sqrt{\frac{p_a}{2}} |z_k, z_{k+a}\rangle |f_{ka}^z\rangle \\
&= \sum_{k,a=0}^{1} \sqrt{\frac{q_a}{2}} |x_k, x_{k+a}\rangle |f_{ka}^x\rangle \\
&= \sum_{k,a=0}^{1} \sqrt{\frac{r_a}{2}} |y_k, y_{k+a}\rangle |f_{ka}^y\rangle,
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
p_a &= \sum_{b=0}^{1} p_{ab} \\
q_a &= \sum_{b=0}^{1} p_{ba} \\
r_a &= \sum_{b=0}^{1} p_{b\ a+b+1}
\end{aligned} \tag{11}
$$

and the normalized kets

$$
\begin{aligned}
|f_{ka}^z\rangle &= \frac{1}{\sqrt{p_a}} \sum_{b=0}^{1} \sqrt{p_{ab}}\, \omega^{kb} |e_{ab}\rangle \\
|f_{ka}^x\rangle &= \frac{1}{\sqrt{q_a}} \sum_{b=0}^{1} \sqrt{p_{ba}}\, \omega^{kb} \omega^{ab} |e_{ba}\rangle \\
|f_{ka}^y\rangle &= \frac{1}{\sqrt{r_a}} \sum_{b=0}^{1} \sqrt{p_{b\ a+b+1}}(-\mathrm{i})^b \omega^{kb} \omega^{b(a+b+1)} |e_{b\ a+b+1}\rangle
\end{aligned} \tag{12}
$$

are such that their inner products are given by

$$
\begin{aligned}
\langle f_{0a}^z | f_{1a}^z \rangle &= \frac{p_{a0} - p_{a1}}{p_{a0} + p_{a1}} \equiv \lambda_a^z \\
\langle f_{0a}^x | f_{1a}^x \rangle &= \frac{p_{0a} - p_{1a}}{p_{0a} + p_{1a}} \equiv \lambda_a^x \\
\langle f_{0a}^y | f_{1a}^y \rangle &= \frac{p_{0\ a+1} - p_{1a}}{p_{0\ a+1} + p_{1a}} \equiv \lambda_a^y.
\end{aligned} \tag{13}
$$

The ancillas with different a's are orthogonal.

Eve's eavesdropping strategy proceeds as follows. After Alice and Bob announce their measurement bases, Eve knows on which pairs of qubits they measured the same observables and that her ancilla is a mixture of four possible states. Formally this can be viewed as a transmission of information from Alice and Bob to Eve encoded in the quantum state of Eve's ancilla. To find the optimal eavesdropping strategy, she has to maximize this information transfer by a choice of a suitable generalized measurement known as a Positive Operator Value Measure (POVM). For example, if Alice and Bob measured in the x basis, Eve will obtain the following mixed state of her ancilla,

$$
\varrho_E^x = \sum_{k,a=0}^{1} \frac{q_a}{2} |f_{ka}^x\rangle\langle f_{ka}^x|. \tag{14}
$$

This is equivalent to Alice and Bob "communicating" to Eve that they measured $\{00, 01, 10, 11\}$ by sending her the quantum states $\{|f_{00}^x\rangle, |f_{01}^x\rangle, |f_{11}^x\rangle, |f_{10}^x\rangle\}$ with prior probabilities $\{\frac{q_0}{2}, \frac{q_1}{2}, \frac{q_1}{2}, \frac{q_0}{2}\}$ respectively. Eve has to find the optimal measurement that will extract from the transmission as much information as possible, called the *accessible information*. Note that this is not equivalent to finding a measurement that minimizes the error of distinguishing between these states [15].

INCOHERENT ATTACK

We first assume that Eve carries out an *incoherent* attack in which she performs measurements on her ancillas one at a time. In contrast, in a coherent attack, she would measure joint observables of more than one ancilla, or construct her initial state Eq. (8) so that more than one pair of qubits were entangled with each ancilla.

The ancilla states for each basis can be divided into two groups. The first group corresponds to $a = 0$ and refers to the case when Alice and Bob obtain correlated results. The second group corresponds to the case $a = 1$ and refers to the case when Alice and Bob obtain anti-correlated results.

For example, if Alice and Bob both measure in the y basis, Eve will have the state

$$
\varrho_E^y = \sum_{k,a=0}^{1} \frac{r_a}{2} |f_{ka}^y\rangle\langle f_{ka}^y|. \tag{15}
$$

The first group $a = 0$ occurs with probability r_0 and the second group $a = 1$ occurs with probability r_1. Similarly, if they measure in the x (z) basis, the first group occurs with probability q_0 (p_0) while the second group occurs with probability q_1 (p_1). The ancillas in the first group $|f_{k0}^m\rangle$ ($m = x, y, z$) are orthogonal to those in the second group $|f_{k1}^m\rangle$.

For the purpose of applying the Csiszár-Körner theorem, we need only to compute the mutual information between Eve and Bob and compare this with the mutual information between Alice and Bob; Eve would have to optimize her measurements on her ancilla so that it maximizes the information she gains about Bob's measurement results.

Let us now present the POVM measurement that maximizes the information transferred by Bob to Eve. In the first step, Eve sorts the mixture of the ancillas into two sub-ensembles according to the index a. This can easily be done using a projective measurement. This sorting is an auxiliary step as, at this stage, she does not gain any

more information about the result of Bob's measurement. After that, depending on the outcome of the projection ($a = 0$ or $a = 1$), Eve has an equiprobable mixture of two non-orthogonal ancilla states each corresponding to Alice and Bob's result. For example, if the chosen measurement basis was the z basis, Eve will receive the mixed state

$$\varrho_E^z = \sum_{k,a=0}^{1} \frac{p_a}{2}|f_{ka}^z\rangle\langle f_{ka}^z|. \qquad (16)$$

Projecting into either the $a = 0$ or $a = 1$ orthogonal subspaces (depending on Alice and Bob's measurement outcomes), she will obtain one of the two equiprobable mixtures of non-orthogonal ancilla states

$$\varrho_0^z = \frac{1}{2}|f_{00}^z\rangle\langle f_{00}^z| + \frac{1}{2}|f_{10}^z\rangle\langle f_{10}^z| \quad \text{if Alice and Bob obtained}$$

$$\text{correlated results, } a = 0;$$

$$\varrho_1^z = \frac{1}{2}|f_{01}^z\rangle\langle f_{01}^z| + \frac{1}{2}|f_{11}^z\rangle\langle f_{11}^z| \quad \text{if Alice and Bob obtained}$$

$$\text{anti-correlated results, } a = 1.$$

$$(17)$$

Next, she applies the measurement that maximizes the accessible information encoded in the mixture of the two ancilla states given by the outcome of her projective measurement. In the case of two equally likely states, this optimum measurement is given by the so-called *square-root measurement* [13, 14]. The outcome probabilities of the square-root measurement are

$$\eta_a^x = \frac{1}{2}\left(1 + \sqrt{1 - (\lambda_a^x)^2}\right)$$

$$\eta_a^y = \frac{1}{2}\left(1 + \sqrt{1 - (\lambda_a^y)^2}\right)$$

$$\eta_a^z = \frac{1}{2}\left(1 + \sqrt{1 - (\lambda_a^z)^2}\right), \qquad (18)$$

where η_a^m is the probability of correctly inferring a given ancilla state in the m basis ($m = x, y, z$). The index a refers to the correlation/anti-correlation subspace in which the ancilla lies.

It is straightforward to compute the mutual information between Bob and Eve:

$$I_{BE} = \frac{1}{3}I_{BE}^x + \frac{1}{3}I_{BE}^y + \frac{1}{3}I_{BE}^z, \qquad (19)$$

where I_{BE}^m is the mutual information when Alice and Bob measure in the same basis m. We have

$$I_{BE}^x = q_0\left(1 - H(\eta_0^x)\right) + q_1\left(1 - H(\eta_1^x)\right)$$

$$I_{BE}^y = r_0\left(1 - H(\eta_0^y)\right) + r_1\left(1 - H(\eta_1^y)\right)$$

$$I_{BE}^z = p_0\left(1 - H(\eta_0^z)\right) + p_1\left(1 - H(\eta_1^z)\right). \qquad (20)$$

Here, $H(\eta_a^m) = -\eta_a^m \log_2 \eta_a^m - (1 - \eta_a^m)\log_2(1 - \eta_a^m)$ is the *binary entropy* of the respective probability distributions. Also, the mutual information between Alice and Bob is given by

$$I_{AB} = 1 - \frac{1}{3}\left(H(p_0) + H(q_0) + H(r_0)\right). \qquad (21)$$

We are interested in the conditions for which our protocol is secure against Eve's incoherent eavesdropping attack. Now, even if Eve obtains some information about the transmitted key through her incoherent measurement, Alice and Bob can still obtain a secure key with a few additional steps. According to the Csiszár-Körner (CK) theorem, a secure key can be generated from a raw key sequence by means of a suitably chosen error-correcting code and classical one-way communication between Alice and Bob if the mutual information between Alice and Bob exceeds that between Eve and either one of them (the CK regime). For the protocol considered, the mutual information between Alice and Eve, and Bob and Eve, are the same so that security is assured as long as

$$I_{AB} > I_{BE}. \qquad (22)$$

QUANTUM ENTANGLEMENT DISTILLATION

If there is too much noise in the two-qubit state, the CK theorem is not immediately applicable. Instead, Alice and Bob need to either select a subsequence of their bit values in a systematic way or pre-process their two-qubit state before measuring, so that the CK theorem is applicable once more. One method of doing this is *quantum entanglement distillation* (QED), a quantum procedure by which many weakly entangled qubit pairs are distilled into a smaller number of more strongly entangled qubit pairs by means of local operations and classical communication.

Alice and Bob's two-qubit state Eq. (2) can be distilled successfully using local operations and classical communication as long as they satisfy the Peres–Horodecki partial transposition criterion [17]: A two-qubit state ϱ is quantum distillable if and only if it is a *non-positive partial transposed* (NPPT) state. A state ϱ is NPPT if $\varrho^{T_B} \not\geq 0$ so that it has at least one negative eigenvalue. Here, ϱ^{T_B} denotes the transposition with respect to Bob's basis only. The partial transpose of each of our Bell states gives,

$$|z_{kl}\rangle\langle z_{kl}| \longrightarrow \frac{1}{2} - |z_{k+1\ l+1}\rangle\langle z_{k+1\ l+1}|. \qquad (23)$$

Applying the Peres–Horodecki criterion to our Bell diagonal mixture, we find that the state Eq. (2) is quantum distillable provided that

$$\max_{ab} p_{ab} > \frac{1}{2}. \qquad (24)$$

ADVANTAGE DISTILLATION

Instead of manipulating their qubits in QED, Alice and Bob can process the raw key sequence they have established in the protocol in order to obtain a more secure key sequence. One such procedure is known as *advantage distillation* (AD).

In the AD protocol, Alice and Bob divide their raw key sequence into blocks of length L. For each block, Alice generates a random bit and adds this, modulo 2, to each bit of the block. She then sends this processed block to Bob via a public channel. After receiving the block, Bob subtracts his corresponding block from it (modulo 2). If all the bit values are the same, it is a deemed a good block. Otherwise it is a bad block. Bob then informs Alice whether the block he received was good or bad. If it is a good block, Alice will record the random bit she initially generated into her distilled bit sequence while Bob enters into his distilled sequence the common bit value he found after subtraction. If it is a bad block, they will both reject the bits and it plays no further part in the distillation procedure.

Now for a good block, two cases can occur:

(I) Alice and Bob's distilled bits are the same;

(II) Alice and Bob's distilled bits are different.

Case (I) occurs when Alice and Bob started out with an identical raw block (i.e. their length L blocks are perfectly correlated). On the other hand, Case (II) occurs when Alice and Bob start out with raw L-blocks that are anti-correlated with each other.

Now, for large L, there will be approximately $\frac{L}{3}$ bits in the good block that result from Alice and Bob's z basis measurement. For these, p_0 is the probability that Alice and Bob obtain correlated results while p_1 is the probability that they obtain anti-correlated results. Similarly, $\frac{L}{3}$ bits will result from x (y) basis measurement — q_0 (r_0) is the probability that Alice and Bob obtain correlated results while q_1 (r_1) is the probability that they obtain anti-correlated results. Thus for a good block, Case (I) occurs with probability $\frac{p_0^{L/3} q_0^{L/3} r_1^{L/3}}{p_0^{L/3} q_0^{L/3} r_1^{L/3} + p_1^{L/3} q_1^{L/3} r_0^{L/3}}$ while Case (II) occurs with probability $\frac{p_1^{L/3} q_1^{L/3} r_0^{L/3}}{p_0^{L/3} q_0^{L/3} r_1^{L/3} + p_1^{L/3} q_1^{L/3} r_0^{L/3}}$ (remember that for the y basis, Alice and Bob generate their raw key from anti-correlation, which corresponds to probability r_1). The error rate for Alice and Bob (the proportion of Case (II) blocks) is given by

$$E_{AB} = \frac{p_1^{L/3} q_1^{L/3} r_0^{L/3}}{p_0^{L/3} q_0^{L/3} r_1^{L/3} + p_1^{L/3} q_1^{L/3} r_0^{L/3}}, \qquad (25)$$

which for $L \gg 1$, $p_1 < p_0$ and $q_1 < q_0$ (since $p_{00} > \frac{1}{2}$) is

approximately

$$E_{AB} \approx \left(\frac{p_1 q_1 r_0}{p_0 q_0 r_1} \right)^{L/3}. \qquad (26)$$

Eve is able to intercept the processed blocks that Alice sends to Bob via the classical channel. From their public communication, she will also be able to know which of the blocks are accepted or rejected. For the good blocks, she has to deduce the distilled bit for each block. To do this, she can either resort to incoherent or coherent measurements on her ancillas.

Incoherent Attack on Advantage Distillation

In the incoherent attack, Eve performs a square root measurement to distinguish her ancillas one by one and, from her results, tries to deduce what Alice and Bob measured for each entry in an L-block. She then subtracts Alice's transmitted block from her own corresponding block, as Bob does. Typically, Eve's block will be inhomogeneous after subtraction so she decides by majority voting which bit value to assign to a particular block – she bets on the value which occurs most frequently in her block, and if there are the same number of 0s as 1s, she picks one of them at random.

Consider Case (I) blocks, i.e. Alice and Bob start out with correlated raw blocks. From her square-root measurement, Eve guesses each entry in the block correctly with the following probabilities:

- η_0^x if Alice and Bob measured in the x basis;

- η_1^y if they measured in the y basis;

- η_0^z if they measured in the z-basis.

She guesses an entry incorrectly with probabilities

- $1 - \eta_0^x$ if Alice and Bob measured in the x basis;

- $1 - \eta_1^y$ if they measured in the y basis;

- $1 - \eta_0^z$ if they measured in the z-basis.

Because Eve applies majority voting, she makes errors whenever she guesses more than half of the entries in a block wrongly. If the same number of 0s and 1s appear in her guesses, she picks one of them at random and makes errors half of the time. We can thus compute Eve's error

rate:

$$E_{BE}^{(I)} = \sum_{\sum_i e_i > \frac{L}{2}} \binom{\frac{L}{3}}{e_x} (1-\eta_0^x)^{e_x} (\eta_0^x)^{\frac{L}{3}-e_x}$$

$$\times \binom{\frac{L}{3}}{e_y} (1-\eta_1^y)^{e_y} (\eta_1^y)^{\frac{L}{3}-e_y}$$

$$\times \binom{\frac{L}{3}}{e_z} (1-\eta_0^z)^{e_z} (\eta_0^z)^{\frac{L}{3}-e_z}$$

$$+\frac{1}{2} \sum_{\sum_i e_i = \frac{L}{2}} \binom{\frac{L}{3}}{e_x} (1-\eta_0^x)^{e_x} (\eta_0^x)^{\frac{L}{3}-e_x}$$

$$\times \binom{\frac{L}{3}}{e_y} (1-\eta_1^y)^{e_y} (\eta_1^y)^{\frac{L}{3}-e_y}$$

$$\times \binom{\frac{L}{3}}{e_z} (1-\eta_0^z)^{e_z} (\eta_0^z)^{\frac{L}{3}-e_z}, \quad (27)$$

where e_i is the number of errors made in the ith basis. The second summation arises from the situation when Eve has to assign 0 or 1 at random to the block because the number of 0s and 1s in the block are equal.

For $L \gg 1$, we can lower bound the summations in Eq. (27) by approximating them with the main contributing terms, i.e., terms for which the binomial factor $\binom{\frac{L}{3}}{e_m}$, $(m = x, y, z)$ has its peak:

$$E_{BE}^{(I)} \sim \binom{\frac{L}{3}}{\frac{L}{6}} (1-\eta_0^x)^{\frac{L}{6}} (\eta_0^x)^{\frac{L}{6}}$$

$$\times \binom{\frac{L}{3}}{\frac{L}{6}} (1-\eta_1^y)^{\frac{L}{6}} (\eta_1^y)^{\frac{L}{6}}$$

$$\times \binom{\frac{L}{3}}{\frac{L}{6}} (1-\eta_0^z)^{\frac{L}{6}} (\eta_0^z)^{\frac{L}{6}}. \quad (28)$$

By applying Stirling's approximation we have

$$E_{BE}^{(I)} \sim 2^L \left(\eta_0^x \eta_1^y \eta_0^z (1-\eta_0^x)(1-\eta_1^y)(1-\eta_0^z) \right)^{\frac{L}{6}}. \quad (29)$$

Similarly for Case (II) blocks in which Alice and Bob start out with anti-correlated raw blocks, we can obtain the error rate for Eve:

$$E_{BE}^{(II)} \sim 2^L \left(\eta_1^x \eta_0^y \eta_1^z (1-\eta_1^x)(1-\eta_0^y)(1-\eta_1^z) \right)^{\frac{L}{6}}. \quad (30)$$

Finally, the total error rate for Eve is given by

$$E_{BE} \sim \frac{p_0^{\frac{L}{3}} q_0^{\frac{L}{3}} r_1^{\frac{L}{3}}}{p_0^{\frac{L}{3}} q_0^{\frac{L}{3}} r_1^{\frac{L}{3}} + p_1^{\frac{L}{3}} q_1^{\frac{L}{3}} r_0^{\frac{L}{3}}} E_{BE}^{(I)}$$

$$+ \frac{p_1^{\frac{L}{3}} q_1^{\frac{L}{3}} r_0^{\frac{L}{3}}}{p_0^{\frac{L}{3}} q_0^{\frac{L}{3}} r_1^{\frac{L}{3}} + p_1^{\frac{L}{3}} q_1^{\frac{L}{3}} r_0^{\frac{L}{3}}} E_{BE}^{(II)} \quad (31)$$

Since the coefficient in front of $E_{BE}^{(I)}$ goes to 1 while the coefficient in front of $E_{BE}^{(II)}$ goes to 0, we are left with

$$E_{BE} \approx 2^L \left(\eta_0^x \eta_1^y \eta_0^z (1-\eta_0^x)(1-\eta_1^y)(1-\eta_0^z) \right)^{\frac{L}{6}}. \quad (32)$$

By comparing the error rates [9], we can obtain the condition for AD to be successful under an incoherent attack:

$$\lim_{L \to \infty} \frac{E_{AB}}{E_{BE}} < 1 \quad (33)$$

which reduces to

$$\frac{p_1}{p_0} \frac{q_1}{q_0} \frac{r_0}{r_1} < 8\sqrt{\eta_0^x \eta_1^y \eta_0^z (1-\eta_0^x)(1-\eta_1^y)(1-\eta_0^z)}. \quad (34)$$

For the special case of Werner states ($p_{01} = p_{10} = p_{11} = \frac{1-p_{00}}{3}$, so that $p_0 = q_0 = r_1$, $p_1 = q_1 = r_0$ and $\eta_0^x = \eta_1^y = \eta_0^z$), we find that Eq. (34) reduces to

$$\frac{p_1}{p_0} < 2\sqrt{\eta_0^z (1-\eta_0^z)}. \quad (35)$$

A similar result was obtained by Bruß et al. [12].

Coherent Attack on Advantage Distillation

We consider a particularly simple scheme of coherent attack that is similar to that presented in [16]. Eve's strategy is as follows.

For each good block, Eve has a corresponding set of ancilla states and rather than measuring her ancillas one-by-one (an incoherent attack), she performs a joint measurement on *all* L of them to acquire knowledge about the value that Alice assigned to the block. By also making use of the classical information that is exchanged between Alice and Bob during the distillation process, Eve can learn a lot more than if she were to measure her ancillas one by one.

Consider first a Case (I) block. As an example, suppose that Alice and Bob start out with the same block 01001 for $L = 5$, and Alice's random bit is 1. After addition (modulo 2), she sends the processed block 10110 to Bob via the public channel which Eve is able to intercept. Eve can also project her block of ancilla states into the orthogonal subspace corresponding to Alice and Bob having a correlated or anti-correlated block. Doing this, she can know that Alice and Bob started out with the same raw blocks (i.e. Case (I) blocks). From this, Eve can then deduce the following possibilities:

1. If Alice's random bit is '0', Alice and Bob must have started out with raw blocks 10110. If Alice and Bob had measured in the bases x, y, y, z, x for the respective entries in the block, the ancilla state that she holds will be $|f_{11}^x\rangle|f_{01}^y\rangle|f_{10}^y\rangle|f_{11}^z\rangle|f_{00}^x\rangle$.

2. If Alice's random bit is '1', Alice and Bob must have started out with raw blocks 01001. If x, y, y, z, x is the order of basis measurements for the entries, the ancilla state that she holds will then be $|f_{00}^{(x)}\rangle|f_{10}^y\rangle|f_{01}^y\rangle|f_{00}^z\rangle|f_{11}^x\rangle$.

The mutual inner product between the two ancilla states is $(\lambda_0^x)^{n_x}(\lambda_1^y)^{n_y}(\lambda_0^z)^{n_z}$, where n_a is the number of times the basis a was measured. The optimal measurement to distinguish these two states is again the square root measurement. In general, for each Case (I) block of length L, Eve needs to distinguish just 2 possible L-ancilla states with mutual inner product $(\lambda_0^x)^{n_x}(\lambda_1^y)^{n_y}(\lambda_0^z)^{n_z}$.

Now, for large L, we have $n_x, n_y, n_z \approx \frac{L}{3}$. Eve's probability of correctly inferring a particular L-ancilla state is given by

$$\frac{1}{2}\left(1 + \sqrt{1 - (\lambda_0^x\lambda_1^y\lambda_0^z)^{\frac{2L}{3}}}\right) \approx 1 - \frac{1}{4}(\lambda_0^x\lambda_1^y\lambda_0^z)^{\frac{2L}{3}}. \quad (36)$$

Her error rate for Case (I) blocks is thus

$$E_{BE}^{(I)} \approx \frac{1}{4}(\lambda_0^x\lambda_1^y\lambda_0^z)^{\frac{2L}{3}}. \quad (37)$$

Similarly when we consider Case (II) blocks, Eve's error rate is

$$E_{BE}^{(II)} \approx \frac{1}{4}(\lambda_1^x\lambda_0^y\lambda_1^z)^{\frac{2L}{3}}. \quad (38)$$

Eve's *total* error rate is thus

$$
\begin{aligned}
E_{BE} &= \frac{p_0^{\frac{L}{3}}q_0^{\frac{L}{3}}r_1^{\frac{L}{3}}}{p_0^{\frac{L}{3}}q_0^{\frac{L}{3}}r_1^{\frac{L}{3}} + p_1^{\frac{L}{3}}q_1^{\frac{L}{3}}r_0^{\frac{L}{3}}}E_{BE}^{(I)} \\
&+ \frac{p_1^{\frac{L}{3}}q_1^{\frac{L}{3}}r_0^{\frac{L}{3}}}{p_0^{\frac{L}{3}}q_0^{\frac{L}{3}}r_1^{\frac{L}{3}} + p_1^{\frac{L}{3}}q_1^{\frac{L}{3}}r_0^{\frac{L}{3}}}E_{BE}^{(II)} \\
&\approx \frac{1}{4}(\lambda_0^x\lambda_1^y\lambda_0^z)^{\frac{2L}{3}}
\end{aligned} \quad (39)
$$

since once again, the coefficient in front of $E_{BE}^{(I)}$ goes to 1 while the coefficient in front of $E_{BE}^{(II)}$ goes to 0.

Finally by comparing error rates (Eq. (33)), we obtain the condition for AD to be possible under a coherent attack by Eve:

$$\frac{p_1}{p_0}\frac{q_1}{q_0}\frac{r_0}{r_1} < (\lambda_0^x\lambda_1^y\lambda_0^z)^2. \quad (40)$$

DISCUSSION

We now analyze the above results. A Bell diagonal density matrix is characterized by four real parameters and a normalization condition so we can parameterize such a state by the probability p_{00} (the amount of the state $|z_{00}\rangle$ in the Bell mixture) and two angles θ, ϕ characterizing the remaining three probabilities p_{01}, p_{10}, p_{11}:

$$
\begin{aligned}
p_{01} &= (1 - p_{00})\cos^2\theta\cos^2\phi \\
p_{10} &= (1 - p_{00})\sin^2\theta\cos^2\phi \\
p_{11} &= (1 - p_{00})\sin^2\phi.
\end{aligned} \quad (41)
$$

This means that for a fixed p_{00}, all the quantities such as $I_{AB}, I_{BE}, E_{AB}, E_{BE}$ for incoherent and coherent attacks are two-argument functions.

First, for each p_{00} we can plot a region characterizing all the Bell diagonal states which lead to secure *raw* keys. As long as p_{00} is greater than around 0.765 all corresponding states are secure. Below this, fewer and fewer states are secure (white regions in Fig. 1) until, for $p_{00} = \frac{1}{2}$, the Bell diagonal mixture becomes separable and no secret bits can be obtained. Even then we can still identify certain states that are resistant against incoherent eavesdropping as long as p_{00} is greater than half. These are states of the form $p_{00}|z_{00}\rangle\langle z_{00}| + p_{01}|z_{01}\rangle\langle z_{01}|$, $p_{00}|z_{00}\rangle\langle z_{00}| + p_{10}|z_{10}\rangle\langle z_{10}|$ and $p_{00}|z_{00}\rangle\langle z_{00}| + p_{11}|z_{11}\rangle\langle z_{11}|$. It is interesting to note that this threshold of 0.765, below which it is no longer possible to generate secure keys for every state, is the same threshold as that for the Werner state — this means that the Werner state will be the first state to become insecure as the p_{00} threshold is exceeded.

Second, using Eq. (34) we verified the results presented in [10], namely that QED is equivalent to AD if Eve can only perform incoherent attacks. In other words, as long as p_{00} is greater than $\frac{1}{2}$, Alice and Bob do not need QED because AD works equally well and does not require collective operations on qubits, which are difficult to realize experimentally.

However, if Eve is capable of carrying out a coherent attack, QED is much more powerful than AD (Fig. 2). We see that as $p_{00} \to \frac{1}{2}$, more states fall into the black regions where AD fails and only QED is possible. As before, the same states that are resistant to incoherent attack in the CK regime are resistant to the above coherent attack on AD.

CONCLUSION

We have generalized the tomographic QKD scheme to Bell diagonal states and analyzed its resistance to various eavesdropping attacks, both in the CK regime and when Alice and Bob perform advantage distillation. We have shown the inequivalence of advantage distillation and entanglement distillation in the presence of coherent measurement by a potential eavesdropper. It still remains to be seen whether Eve can further increase her information gain by entangling more than one pair of Alice and Bob's qubits with her ancilla.

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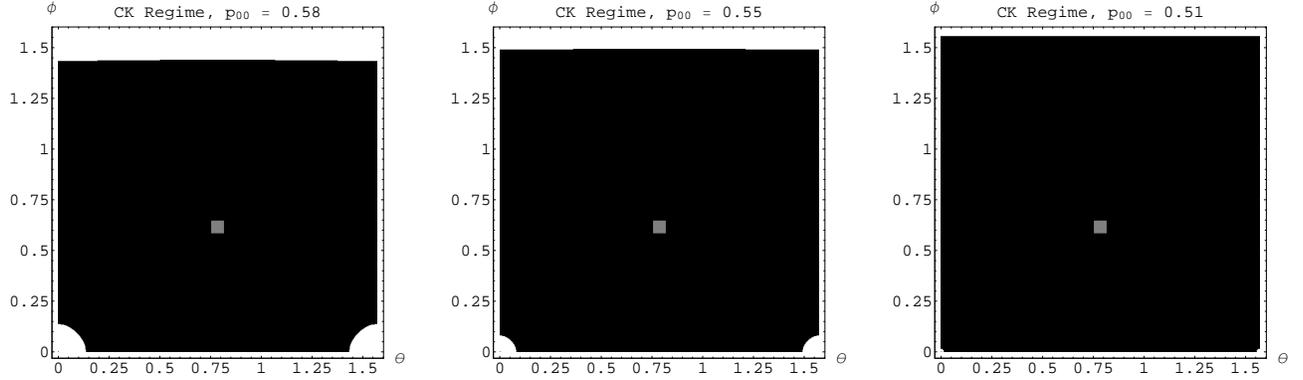[17] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

FIG. 1: Comparison of secure regions for the protocol for different values of p_{00} under an incoherent attack. White regions in the plot represent states that are secure against incoherent attacks by Eve in the scenario when Alice and Bob do not attempt AD or QED (CK regime). When $p_{00} \to \frac{1}{2}$ the white areas disappear with the exception of the certain points that never become black. These points correspond to the states $p_{00}|z_{00}\rangle\langle z_{00}| + p_{01}|z_{01}\rangle\langle z_{01}|$ ($\theta = 0$, $\phi = 0$), $p_{00}|z_{00}\rangle\langle z_{00}| + p_{10}|z_{10}\rangle\langle z_{10}|$ ($\theta = \frac{\pi}{2}$, $\phi = 0$) and $p_{00}|z_{00}\rangle\langle z_{00}| + p_{11}|z_{11}\rangle\langle z_{11}|$ ($\phi = \frac{\pi}{2}$). These states are resistant to any incoherent attack. As reference, the grey areas (exaggerated in the figure) indicate Werner states.
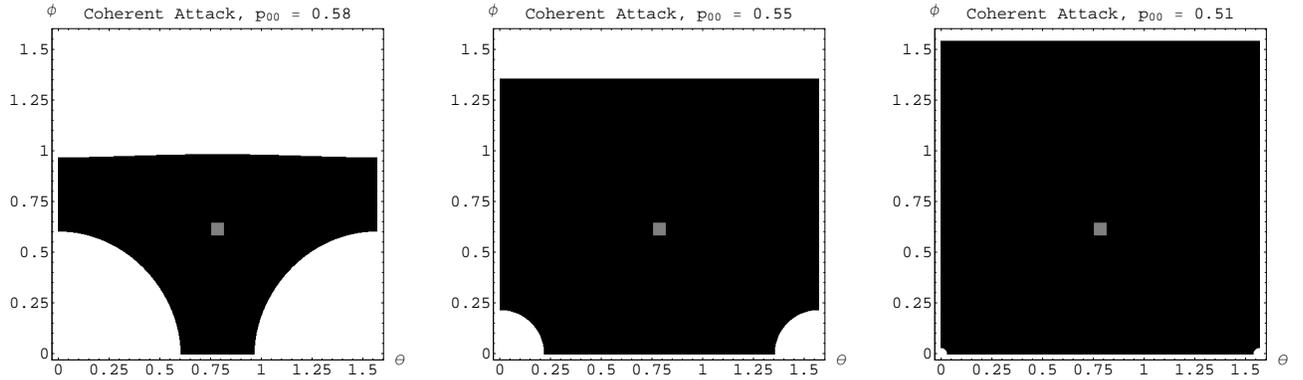

FIG. 2: Comparison of secure regions in advantage distillation for different values of p_{00} under a coherent attack. White regions in the plot represent states that are secure against coherent attacks by Eve in the scenario when Alice and Bob perform AD. Black regions correspond to states for which AD fails under coherent attack. As the state becomes more mixed ($p_{00} \to \frac{1}{2}$), the white areas disappear with the exception of the certain points that never become black. As with the CK regime for $p_{00} \to \frac{1}{2}$, the surviving states are $p_{00}|z_{00}\rangle\langle z_{00}| + p_{01}|z_{01}\rangle\langle z_{01}|$, $p_{00}|z_{00}\rangle\langle z_{00}| + p_{10}|z_{10}\rangle\langle z_{10}|$ and $p_{00}|z_{00}\rangle\langle z_{00}| + p_{11}|z_{11}\rangle\langle z_{11}|$. In comparison, with only incoherent attacks all states with $p_{00} > \frac{1}{2}$ are secure. As reference, the grey areas (exaggerated in the figure) refer to Werner states.