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# TRADE IN BILATERAL OLIGOPOLY WITH ENDOGENOUS MARKET FORMATION

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# Trade in bilateral oligopoly with endogenous market formation

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**Abstract**

We study a strategic market game in which traders are endowed with both a good and money and can choose whether to buy or sell the good. We derive conditions under which a non-autarkic equilibrium exists and when the only equilibrium is autarky. Autarky is 'nice' (robust to small perturbations in the game) when it is the only equilibrium, and 'very nice' (robust to large perturbations) when no gains from trade exist. We characterize economies where autarky is nice but not very nice; that is, when gains from trade exist and yet no trade takes place.

*Key words:* Bilateral oligopoly, strategic market game, trade.
*JEL codes:* C72, D43, D51, L13.

## 1 Introduction

There is a small but important literature concerned with the observation that, in a market where all agents are allowed to behave strategically and trade is organized via a strategic market game of the type originally introduced by Shapley and Shubik [12], there are economies in which gains from trade exist and yet there is no equilibrium in which trade takes place. Autarky is always trivially a Nash equilibrium of the game, so interest is focussed on when it is a 'legitimate' equilibrium. Cordella and Gabszewicz [3] coin autarky 'nice' if it is robust to a small external bid and offer being placed on the market (i.e. it is an 'equilibrium point' of the game) and provide an example of an economy in which autarky exhibits this characteristic. Busetto and Codognato [2] call autarky 'very nice' if it is robust to large perturbations in the game (i.e. it can be supported by a 'virtual price'), and provide an example of an economy in which autarky is nice but not very

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nice. We contribute to this literature by demonstrating that autarky is nice iff it is the only equilibrium in the game, and very nice iff no gains from trade exist. In a general model of bilateral oligopoly we derive conditions on the primitives of the economy under which autarky has these characteristics, which allows us to deduce when autarky is nice but not very nice; that is, when gains from trade exist and yet no trade takes place.

Interest in strategic market games stems both from the desire to provide a strategic foundation for the price-taking assumptions in the Walrasian model of exchange, and to provide a prediction of outcomes in markets in which all traders have market power. Restricting attention to the market for a single commodity in which there is also a commodity money and signals are quantity-based, there is a trading post to which agents, being endowed with both the good and money, can offer some of their endowment of the good to be exchanged for money, and may bid some of their money to be exchanged for the good. The rate of exchange of the good for money is determined by the ratio of aggregate bid to aggregate offer. One modelling assumption permits agents to both buy *and* sell the good simultaneously, allowing agents to make wash trades. In this paper, we focus on the case where agents are restricted to act on only a single side of the market in a buy *or* sell game. Existence results for the buy and sell game are well known (see, for instance, [5] and [9]), but there is an inherent multiplicity of equilibria due to the fact that agents can make wash trades. Ray [10] has noted that for a given economy equilibrium outcomes will be different in the two games. The buy or sell game corresponds to Shapley's 'variant I', and in an early paper he commented: "our goal is a theorem that asserts [for the game without wash trades] that a NE exists in which each commodity has an actual price ... if actively traded, or a virtual price [so the market is legitimately inactive] ... if not." [11, pp 171] Excepting the literature cited, little attention has been devoted to the study of Shapley's 'variant I', and to the best of our knowledge no there is no theorem that elucidates the conditions under which a Shapley equilibrium exists.

We have previously studied the existence of non-autarkic Nash equilibria in a model of bilateral oligopoly in which agents have corner endowments, in which case whether an agent is a buyer or seller is exogenously specified [4]. This analysis culminates in strategic versions of supply and demand, intersections of which correspond to Nash equilibria in the game. We begin in this paper by extending this analysis to the case of interior endowments where agents can choose on which side of the market they want to act, and this choice depends on the actions of others. We show that a similar analysis utilizing strategic supply and demand functions can be used, and that strategic supply is defined only for prices above some cutoff price $P^{\mathrm{S}}$, and strategic demand only below a cutoff price $P^{\mathrm{B}}$, so there is a non-autarkic Nash equilibrium if and only if $P^{\mathrm{S}} < P^{\mathrm{B}}$. $P^{\mathrm{S}}$ and $P^{\mathrm{B}}$ depend only on traders' preferences and their endowments.

If $P^{\mathrm{S}} \geq P^{\mathrm{B}}$ then the only equilibrium is autarky. Since no trader in the game can unilaterally open the market, autarky is always an equilibrium even if a non-autarkic Nash equilibrium exists, and has therefore been considered a trivial consequence of the trading process. Noting in an example that autarky may be the only equilibrium in the game, Cordella and Gabszewicz [3] coined autarky 'nice' if it is an equilibrium point in the game, that is if, when a small external bid and offer are made to the market, the equilibrium remains close to autarky. Put another way, autarky is nice if it is a legitimate equilibrium in the context of the game. We show that autarky is nice if and only if it is the only equilibrium in the game. Busetto and Codognato [2], in light of Shapley's previously cited challenge, coined autarky 'very nice' if there exists a virtual price for the equilibrium, that is, if autarky remains an equilibrium when any external bid and offer are made to the market (so long as they are in the proportion of the virtual price). We show that autarky is very nice if and only if there are no gains from trade in the economy, so it is legitimate in the context of the economy.

We also address the question of whether, in the presence of gains from trade, trade will always take place in equilibrium. When wash trades are allowed the answer to this question is affirmative [9]. However, we show that when wash trades are not allowed there is a non-trivial set of economic environments in which autarky is nice but not very nice, so despite the fact that gains from trade exist trade will not take place. Rather, for a *non*-autarkic equilibrium to exist there must be 'sufficient' gains from trade. This generalizes the example of Busetto and Codognato [2] that illustrates the phenomenon. A direct corollary gives conditions on the primitives of the economy under which a 'Shapley equilibrium' exists in our model, providing a theorem that achieves Shapley's goal for bilateral oligopoly.

The rest of the paper is structured as follows. The next section outlines the economic model and the strategic market game. Our analysis of non-autarkic equilibria proceeds in Section 3, following which we characterize when autarky is nice and very nice, and the relationship between these concepts, which culminates in the conclusion that there are economic environments in which gains from trade exist and yet the only equilibrium is autarky. Section 5 contains our concluding remarks. All proofs are collected in an appendix.

## 2 The model

We wish to model trade in the market for a single commodity in which all agents are allowed to behave strategically, actions are quantity based and whether an agent is a buyer or seller in equilibrium is not pre-specified but determined endogenously with market outcomes. Formally, the economy

$\mathcal{E} = \left\{ \left( e_i, u_i, \mathbb{R}^2_+ \right) : i \in I \right\}$ consists of two commodities; a consumption good ($x_1$) and a commodity money ($x_2$). The set of agents is $I = \{1, \ldots, i, \ldots, n\}$ and each agent $i \in I$ has an endowment given by $(e_i, m_i) \geq \mathbf{0}$. Agents may initially possess *both* the good and money (in which case they have an 'interior' endowment), or they may be endowed with either the good or money (a 'corner' endowment). The preferences of agent $i \in I$ over final consumption bundles of the two commodities are assumed to be representable by a utility function $u_i : \mathbb{R}^2_+ \to \mathbb{R}$, which we assume to be monotonic and twice differentiable, and we write $\partial_i (x_1, x_2)$ for the marginal rate of substitution with $x_1$ units of the good and $x_2$ units of money. We assume preferences are binormal which, *inter alia*, implies (competitive) income expansion paths are non-decreasing.

**Assumption 1** *For each agent $i \in I$, $x_1 \leq x'_1, x_2 \geq x'_2 \Rightarrow \partial_i (x) \geq \partial_i (x')$ where the final inequality is strict if $x_1 < x'_1$ and $x_2 > 0$; that is, marginal rates of substitution increase under moves to the north-west.*

**Remark 1 (On gains from trade)** *If $e_i, m_i > 0$ for every $i \in I$ then there exist gains from trade whenever there are two agents $i \neq j \in I$ for whom $\partial_i (e_i, m_i) \neq \partial_j (e_j, m_j)$, i.e. whenever there is heterogeneity amongst agents. When corner endowments are also permitted there may be agents i and j for whom $\partial_i (e_i, m_i) < \partial_j (e_j, m_j)$ but no profitable exchanges between these agents can be made because agent i has no endowment of the good or j no endowment of money. The condition for gains from trade to exist is thus:*

$$\min_{\{j \in I : e_j \neq 0\}} \{\partial_j (e_j, m_j)\} < \max_{\{j \in I : m_j \neq 0\}} \{\partial_j (e_j, m_j)\}.$$

Exchange takes place according to the rules of a strategic market game initially introduced in the research program of Shapley and Shubik (see [11] and [12], among others). There is a trading post at which agents can *offer* some of their endowment of the good to be exchanged for money and place a *bid* of money to be exchanged for the good. Denoting bids and offers by $b_i$ and $q_i$ respectively the set of strategies available to each $i \in I$ is

$$S_i = \{(b, q) : 0 \leq b \leq m_i, 0 \leq q \leq e_i\}.$$

Given a profile of bids and offers $(\mathbf{b}, \mathbf{q}) \in S = \bigcap_{j \in I} S_j$ the trading post aggregates bids to $B = \sum_{j \in I} b_j$ and offers to $Q = \sum_{j \in I} q_j$, and the (market clearing) price is determined as

$$p = \frac{B}{Q}.$$

If either $B = 0$ or $Q = 0$ or both the market is deemed closed, no trade takes place and any bids or offers made to the market are confiscated. Otherwise, the market is open and, having chosen a strategy $(b, q) \in S_i$, the final allocation awarded to agent $i$ is

$$(x_1, x_2) = (e_i - q + b/p, m_i - b + qp) . \tag{1}$$

The specified trading rules constitute a well-defined game and the equilibrium concept we use is Nash equilibrium (in pure strategies), in which each trader may be seen as solving the problem

$$\max_{(b,q)\in S_i} u_i \left(e_i - q + b/p, m_i - b + qp\right) , \tag{2}$$

taking the actions of other traders as given.

Following Shapley's 'variant I' [11, pp 167] we impose a further restriction that each agent can enter the market as either a seller or a buyer or neither, but not both. Thus, the strategy set of trader $i \in I$ is restricted to

$$\tilde{S}_i = \{(b, q) \in S_i : b \cdot q = 0\} . \tag{3}$$

This variant is known as the buy *or* sell strategic market game.

Whilst existence results for the buy *and* sell game are well-known (see [5] and [9], for example) much less attention has been devoted to the existence of equilibrium when the trading rules prevent wash trading, as in the buy or sell game. Shapley laid down the challenge in his original contribution: "our goal is a theorem that asserts [for the buy or sell game] that a NE exists in which each commodity has an actual price ... if actively traded, or a virtual price [so the market is legitimately inactive] ... if not." [11, pp 171] Busetto and Codognato [2] have made some progress in addressing this question by providing an example of an economy in which no such 'Shapley equilibrium' exists. Our aim is to determine, in the general model of bilateral oligopoly with heterogeneous traders outlined in this section, the conditions on the primitives of the model under which a non-autarkic equilibrium exists, which will indirectly reveal when the only equilibrium is autarky, after which we determine when it is legitimate.

## 3 Non-autarkic equilibria

Inspection of the rules of the game reveal that there is always an autarkic equilibrium in which all traders use their null strategies of bidding and offering zero to the market: if any agent believes all others will be inactive then either a bid or offer will be confiscated, so $(0, 0)$ is a best response confirming $(\mathbf{0}, \mathbf{0})$ is an equilibrium profile. In this section we consider when, in addition to the autarkic equilibrium, a *non*-autarkic equilibrium also exists.

Agents choose between their available strategies with the purpose of maximizing their payoff in the game given the actions of other traders. Each trader can be seen as solving (2) but where $(b,q) \in \tilde{S}_i$, so a Nash equilibrium will be characterized by

$$\partial_i\left(e_i - q + \frac{b}{B}Q, m_i - b + \frac{q}{Q}B\right) = \frac{1-\frac{q}{Q}}{1-\frac{b}{B}}\frac{B}{Q} \tag{4}$$

for each $i \in I$, with the additional restriction that $b \cdot q = 0$. To take into account solutions at the boundaries of the strategy set, the equality is replaced by $\leq$ (resp. $\geq$) if $b = 0$ (resp. $q = 0$), but these inequalities must hold with equality if $q > 0$ (resp. $b > 0$).

In a previous paper [4] we studied the existence of non-autarkic equilibrium in bilateral oligopoly where agents are endowed with either the good *or* money (so that the side of the market on which a player acts is specified exogenously) by deriving strategic versions of supply and demand functions, intersections of which are in one-to-one correspondence with non-autarkic Nash equilibria in the game. Here, whether an agent becomes a buyer or seller is endogenous to market outcomes, and in this section we extend our previous work to allow for the fact that agents can choose on which side of the market to act.

Rather than studying best responses, we seek behavior consistent with a Nash equilibrium in which the *aggregate* offer and bid, including those of the player in question, take particular values. For any $B, Q > 0$ define

$$R_i\left(B,Q\right) = (b,q) \in \left\{ \underset{(q',b')\in\tilde{S}_i}{\arg\max}\,\partial_i\left(e_i - q' + \frac{b'}{B-b+b'}\left(Q-q+q'\right),\right.\right.$$
$$\left.\left. m_i - b' + \frac{q'}{Q-q+q'}\left(B-b+b'\right)\right) = \frac{1-\frac{q'}{Q-q+q'}}{1-\frac{b'}{B-b+b'}}\frac{B-b+b'}{Q-q+q'}\right\}.$$

The vector $R_i\left(B,Q\right)$ gives the bid and offer combination of trader $i \in I$ (one of which must be zero) consistent with a non-autarkic Nash equilibrium in which the aggregate bid is $B > 0$ and the aggregate offer is $Q > 0$. Denoting by $r_i^b$ the first component and by $r_i^q$ the second, a Nash equilibrium in which the aggregate bid is $\hat{B} > 0$ and the aggregate offer is $\hat{Q} > 0$ is fixed by the requirement that individual bids and offers sum to their respective aggregates:

$$\sum_{i\in I} r_i^b\left(\hat{B},\hat{Q}\right) = \hat{B} \qquad \& \qquad \sum_{i\in I} r_i^q\left(\hat{B},\hat{Q}\right) = \hat{Q}.$$

To deduce whether there are a pair of aggregates that satisfy the above equations simultaneously, we must identify which agents will act on which side of the market for given values of the aggregates. We proceed by fixing the price $p$ at a non-autarkic Nash equilibrium, for as the next lemma shows, there is then a natural separation of agents into those that are buyers and those that are sellers in a Nash equilibrium with this price, should one exist. This fact will assist in identifying non-autarkic Nash equilibria, since having determined the 'market structure' we can then determine levels of the aggregate bid and offer consistent with an equilibrium with this price. Then, if there is a price whereby the supply equals the implied demand (equal to the ratio of aggregate bid to price) there will be a non-autarkic Nash equilibrium with this price.

**Lemma 1** *Suppose the preferences of all agents are binormal and there is a non-autarkic Nash equilibrium in which the price is* $p$*. Then if* $p \geq \partial_i (e_i, m_i)$ *agent* $i$ *will not be a buyer in the equilibrium, and if* $p \leq \partial_i (e_i, m_i)$ *agent* $i$ *will not be a seller.*

At the price $p$ we call those agents for whom $\partial_i (e_i, m_i) < p$ *potential sellers* and those for whom $\partial_i (e_i, m_i) > p$ *potential buyers*, and we define

$$p_i^* = \partial_i (e_i, m_i)$$

as the critical price at which agent $i$ switches between being a potential seller and a potential buyer in a Nash equilibrium with this price. If $p = p_i^*$ trader $i$ will be inactive in a Nash equilibrium with price $p$.

We now seek to determine individual behavior consistent with equilibrium, following which we deduce consistent aggregate behavior. Consider a typical agent $i \in I$. If there is a non-autarkic Nash equilibrium with price $p > p_i^*$ this trader is a potential seller in equilibrium. If the aggregate offer of all potential sellers is $Q > 0$ then it can be deduced from (4) that trader $i$'s offer consistent with this equilibrium is given by $q = Qs_i^{\mathrm{S}} (p; Q)$, where $s_i^{\mathrm{S}} (p; Q) = \min \{\sigma, e_i/Q\}$ and $\sigma$ is such that

$$\partial_i (e_i - \sigma Q, m_i + \sigma Q p) \geq (1 - \sigma) p,$$

with equality if $\sigma > 0$.

If there is an equilibrium with price $p < p_i^*$ then agent $i$ is a potential buyer at such an equilibrium and (4) can again be used to deduce that if the bid of all potential buyers is $B > 0$ then the bid of trader $i$ consistent with the Nash equilibrium is given by $b = Bs_i^{\mathrm{B}} (p; B)$ where $s_i^{\mathrm{B}} (p; B) = \min \{\sigma, m_i/B\}$ and $\sigma$ is such that

$$\partial_i (e_i + \sigma B/p, m_i - \sigma B) \leq (1 - \sigma)^{-1} p,$$

with equality if $\sigma > 0$.

The functions $s_i^{\mathrm{S}} (p; Q)$ (resp. $s_i^{\mathrm{B}} (p; B)$) represent agent $i$'s share of the aggregate offer (resp. bid) consistent with a non-autarkic Nash equilibrium in which the aggregate offer (resp. bid) forthcoming from all potential sellers (resp. buyers) is $Q$ (resp. $B$) and the price is $p > p_i^*$ (resp. $p < p_i^*$). It follows from our assumptions on preferences that these characterizations, that we call *share functions*, have several *desiderata*.

**Lemma 2** *Suppose the preferences of all agents are binormal. Then for each agent $i \in I$ there exist two functions $s_i^{\mathrm{S}}(p;Q)$ and $s_i^{\mathrm{B}}(p;B)$ as previously defined that represent the behavior of that agent consistent with an equilibrium in which the price is $p > (<)\, p_i^*$ and the aggregate offer (bid) of all agents $j \in I$ for whom $p_j^* < (>)\, p$ is $Q$ ($B$). Moreover,*

**a)** $s_i^{\mathrm{S}}(p;Q)$ is defined for all $p > p_i^*$ and $Q > 0$ where it is positive, continuous and strictly decreasing in $Q$ with the properties $\lim_{Q\to 0} s_i^{\mathrm{S}}(p;Q) = 1 - \frac{p_i^*}{p}$ and $\lim_{Q\to\infty} s_i^{\mathrm{S}}(p;Q) = 0$.

**b)** $s_i^{\mathrm{B}}(p;B)$ is defined for all $p < p_i^*$ and $B > 0$ where it is positive, continuous and strictly decreasing in $B > 0$ with $\lim_{B\to 0} s_i^{\mathrm{B}}(p;B) = 1 - \frac{p}{p_i^*}$ and $\lim_{B\to\infty} s_i^{\mathrm{B}}(p;B) = 0$. In addition, with a fixed ratio $B/p$, the function $s_i^{B}(p; p[B/p])$ is strictly decreasing in $p$.

If there is a non-autarkic Nash equilibrium with a price $p$ such that $p_i^* < p$ for agent $i$ and the aggregate offer of all agents $j \in I$ for whom $p_j^* < p$ is $Q > 0$ then the offer of trader $i$ consistent with this equilibrium is $Qs_i^{\mathrm{S}}(p;Q)$. Likewise, if there is a non-autarkic Nash equilibrium with price $p$ such that $p_i^* > p$ and the aggregate bid of all agents $j \in I$ for whom $p_j^* > p$ is $B > 0$ then the bid of trader $i$ consistent with equilibrium is $Bs_i^{\mathrm{B}}(p;B)$. Having established behavior consistent with equilibrium at the individual level we now proceed to determine, for each price, consistent behavior at the aggregate level.

Consistency of aggregate supply at price $p$ requires the sum of the individual offers of potential sellers to be equal to the aggregate offer, or for the sum of share functions of potential sellers to be equal to unity. Denote by $\mathcal{X}_1^{\mathrm{S}}(p)$ the consistent aggregate supply of potential sellers at price $p$, which we call *strategic supply.* Strategic supply is fixed by the requirement that

$$\sum_{\left\{j\in I: p_j^* < p\right\}} s_j^{\mathrm{S}}\left(p; \mathcal{X}_1^{\mathrm{S}}(p)\right) = 1.$$

Strategic supply will not be defined for all prices. In particular, if $p < \min_{j\in I}\left\{p_j^*\right\}$ then there are no potential sellers and so strategic supply will be undefined. Interestingly, however, strategic supply does not necessarily become positive immediately as $p$ exceeds $\min_{j\in I}\left\{p_j^*\right\}$. Recall that at price $p$ we seek the level of $Q$ where $\sum_{\left\{j\in I: p_j^* < p\right\}} s_j^{\mathrm{S}}(p;Q) = 1$. From Lemma 2 we know that individual share functions are strictly decreasing in $Q > 0$, and any sum of share functions inherits this property. As such, a necessary (and, since share functions are continuous, sufficient) condition to find a consistent level of $Q > 0$ requires $p$ to be such that $\lim_{Q\to 0}\sum_{\left\{j\in I: p_j^* < p\right\}} s_j^{\mathrm{S}}(p;Q) > 1$. Since $\lim_{Q\to 0} s_i^{\mathrm{S}}(p;Q) = 1 - \frac{p_i^*}{p}$, strategic supply is only defined for prices such that $\sum_{\left\{j\in I:p_j^*<p\right\}} 1-\frac{p_j^*}{p}>1$. Since this sum is increasing in $p$ ($1-\frac{p_i^*}{p}$ increases in $p$ for each seller, and more agents are included in the sum as $p$ increases), there exists a unique value of $P^{\mathrm{S}}$ such that

$$\sum_{\left\{j\in I:p_j^*<P^{\mathrm{S}}\right\}} 1-\frac{p_j^*}{P^{\mathrm{S}}}=1, \tag{5}$$

and strategic supply is defined and positive for all prices exceeding $P^{\mathrm{S}}$.

Similar considerations apply for potential buyers. Consistency of the aggregate bid at price $p$ requires the sum of the individual bids of potential buyers to be equal to the aggregate bid, or for the sum of share functions of potential buyers to be equal to one. Rather than express consistent aggregate behavior in terms of the aggregate bid it is more convenient to consider the level of aggregate demand resulting from such a bid, given by the ratio of the bid to the price. We call this *strategic demand* which at price $p$ is given by $\mathcal{X}_1^{\mathrm{B}}(p)$, where

$$\sum_{\left\{j\in I:p_j^*>p\right\}} s_j^{\mathrm{B}}\left(p;p\mathcal{X}_1^{\mathrm{B}}(p)\right)=1.$$

Strategic demand is undefined for prices exceeding $\max_{j\in I}\left\{p_j^*\right\}$, but arguments that parallel those presented above for potential sellers reveals $p<\max_{j\in I}\left\{p_j^*\right\}$ is not sufficient to guarantee positive strategic demand. Rather, if we (uniquely) define $P^{\mathrm{B}}$ by

$$\sum_{\left\{j\in I:p_j^*>P^{\mathrm{B}}\right\}} 1-\frac{P^{\mathrm{B}}}{p_j^*}=1, \tag{6}$$

which is the price above which the sum of share functions of potential buyers is less than one for all $B>0$, then strategic demand is defined and positive only for $0<p<P^{\mathrm{B}}$, where it is also continuous in $p$.

We now have representations of aggregate behavior consistent with a non-autarkic Nash equilibrium at a given price, taking into account the fact that a) there are strategic effects permeating the economy and b) the agents that act on each side of the market in a Nash equilibrium are determined endogenously. It is readily checked by definition-chasing that there is a non-autarkic Nash equilibrium in the game at price $p$ if and only if strategic supply equals strategic demand at this price. If $\mathcal{X}_1^{\mathrm{S}}(\hat{p})=\mathcal{X}_1^{\mathrm{B}}(\hat{p})$ then the equilibrium strategy of trader $i\in I$ is $\hat{b}_i=0$, $\hat{q}_i=\mathcal{X}_1^{\mathrm{S}}(\hat{p})\,s_i^{\mathrm{S}}\left(\hat{p};X_1^{\mathrm{S}}(\hat{p})\right)$ if $p_i^*<\hat{p}$ and $\hat{b}_i=\hat{p}\mathcal{X}_1^{\mathrm{B}}(\hat{p})\,s_i^{\mathrm{B}}\left(\hat{p};\hat{p}\mathcal{X}_1^{\mathrm{B}}(\hat{p})\right)$, $\hat{q}_i=0$ if $p_i^*>\hat{p}$. If $p_i^*$ exactly equals $\hat{p}$ trader $i$ will be inactive in equilibrium.

As our arguments in this section made clear, $P^{\mathrm{S}} < P^{\mathrm{B}}$ is a necessary condition for strategic supply and demand to intersect, and therefore for a non-autarkic Nash equilibrium to exist. When all traders' preferences are binormal continuity of strategic supply and demand functions can be deduced by slight modifications of the arguments in [4, Lemmas 3.2 and 3.4]. As such, this condition is also sufficient for the existence of an intersection of strategic supply and demand.

**Proposition 3** *Suppose the preferences of all agents are binormal. Then there is a non-autarkic Nash equilibrium in the buy or sell strategic market game if and only if* $P^{\mathrm{S}} < P^{\mathrm{B}}$.

## 4 Autarky: nice or very nice?

The previous section analyzed the existence of a non-autarkic equilibrium but, as noted, there is also always an autarkic equilibrium in the market game. When it exists alongside another non-autarkic equilibrium (i.e. when $P^{\mathrm{S}} < P^{\mathrm{B}}$) it may be thought of as a trivial consequence of the trading rules. This could be formally justified by appealing to the coalition-proof refinement [1] of the set of equilibria that would exclude autarky if a non-autarkic equilibrium exists. Additional support comes from a recent experiment involving a strategic market game [6] in which autarky was not observed as an outcome of trade in an economy in which there is a non-autarkic equilibrium. However, Theorem 3 implies that when $P^{\mathrm{S}} \geq P^{\mathrm{B}}$ autarky is the *only* equilibrium in the game and so it cannot be classed as trivial. In this section we investigate the relationship between the conditions under which autarky is the only equilibrium and the conditions required for gains from trade to exist.

We approach this by determining the conditions under which autarky is 'nice' and 'very nice', and we recall these definitions next.

**Definition 4 (Dubey and Shubik [5])** *Consider a modified game labelled* $\Gamma^{\varepsilon}$ *in which there is an external bid and offer of* $\varepsilon$ *placed in the market, and consider a sequence* $\{\varepsilon^n\}$ *with* $\varepsilon^n \to 0$ *as* $n \to \infty$. *Then a Nash equilibrium of the original game* $\left(\hat{\mathbf{b}}, \hat{\mathbf{q}}\right)$ *is* 'nice' *if and only if there is a sequence of equilibria in the games* $\Gamma^{\varepsilon^n}$ *denoted* $\left(\hat{\mathbf{b}}^n, \hat{\mathbf{q}}^n\right)$ *such that as* $n \to \infty$ $\left(\hat{\mathbf{b}}^n, \hat{\mathbf{q}}^n\right) \to \left(\hat{\mathbf{b}}, \hat{\mathbf{q}}\right)$.

Thus, a Nash equilibrium is nice if it is an 'equilibrium point' of the game, in the sense that if a small external bid and offer are made to the market there is not a discontinuity in traders' behavior compared to when these are absent.

**Definition 5 (Busetto and Codognato [2])** *A Nash equilibrium* $\left(\hat{\mathbf{b}},\hat{\mathbf{q}}\right)$ *is '*very nice*' if either a)* $\left(\hat{\mathbf{b}},\hat{\mathbf{q}}\right) \gg (\mathbf{0},\mathbf{0})$ *or b)* $\left(\hat{\mathbf{b}},\hat{\mathbf{q}}\right) = (\mathbf{0},\mathbf{0})$ *and there exists a virtual price for the equilibrium, in the sense that there exists a* $v > 0$ *such that for all* $\delta > 0$*,* $(\mathbf{0},\mathbf{0})$ *remains a Nash equilibrium when an external bid of* $\delta v$ *and offer of* $\delta$ *are made to the market.*

Thus, the autarkic equilibrium is nice if it is an equilibrium point in the game, and it is very nice if there exists a virtual price associated with the equilibrium. Both concepts require an outside bid and offer to be placed on the market; their difference lies in the fact that autarky is nice if, when a small bid and offer are made to the market the equilibrium remains close to autarky, whereas for autarky to be very nice it must remain an equilibrium regardless of how large the bid and offer are (so long as they are in the same proportion, $v$).

## 4.1 Nash equilibria with external bids and offers

The study of equilibrium points and very nice equilibria require analysis of modified games in which an external bid and offer are made, and we now provide an analysis of these games utilizing strategic supply and demand functions. Define $\Gamma(\beta,\gamma)$ as the game in which an external bid of $\beta > 0$ and an external offer of $\gamma > 0$ are made to the market. Unlike in the unmodified game (that we denote $\Gamma$) it is not the case that autarky is always an equilibrium in $\Gamma(\beta,\gamma)$. Any trader can now unilaterally enter the market to marginally alter her allocation at a rate of exchange of $\beta/\gamma$ by trading with the external agency without requiring opposing preferences from another agent. As such, autarky will only be an equilibrium in the modified game if no trader wants to unilaterally make a positive bid or offer.

With a slight abuse of notation, write $\mathcal{X}_1^{\mathrm{S}}(p;\gamma)$ for the strategic supply in the game $\Gamma(\beta,\gamma)$. Strategic supply is fixed by the requirement that the sum of individual offers at price $p$ with aggregate supply $\mathcal{X}_1^{\mathrm{S}}(p;\gamma)+\gamma$ plus the external offer exactly equals the aggregate supply, or in share function terms that

$$\frac{\gamma}{\mathcal{X}_1^{\mathrm{S}}(p;\gamma)+\gamma} + \sum_{\left\{j\in I:p_j^*<p\right\}} s_j^{\mathrm{S}}\left(p;\mathcal{X}_1^{\mathrm{S}}(p;\gamma)+\gamma\right) = 1. \tag{7}$$

Lemma 2 implies that $\frac{\gamma}{Q+\gamma} + \sum_{\left\{j\in I:p_j^*<p\right\}} s_j^{\mathrm{S}}(p;Q+\gamma)$ is strictly decreasing in $Q$, so a necessary and sufficient condition for $\mathcal{X}_1^{\mathrm{S}}(p;\gamma)$ to be defined at price $p$ is for the limit as $Q\to 0$ of this expression to exceed one. But since $\lim_{Q\to 0}\frac{\gamma}{Q+\gamma}=1$, this holds whenever there is a trader with a positive value of $s_i^{\mathrm{S}}$ in this same limit, and Lemma 2 can again be used to deduce that this is the case when there is an $i\in I$ for whom $e_i>0$ and $p_i^*<p$, i.e.

when $p > \min_{\{j \in I: e_j \neq 0\}} \left\{ p_j^* \right\}$. Thus, in $\Gamma(\beta, \gamma)$ strategic supply $\mathcal{X}_1^{\mathrm{S}}(p; \gamma)$ is defined and positive for all prices exceeding $\min_{\{j \in I: e_j \neq 0\}} \left\{ p_j^* \right\}$ (where it will also be continuous).

Likewise, strategic demand in $\Gamma(\beta, \gamma)$, that we write $\mathcal{X}_1^{\mathrm{B}}(p; \beta)$, is fixed by the requirement that

$$\frac{\beta}{p \mathcal{X}_1^{\mathrm{B}}(p; \beta) + \beta} + \sum_{\{j \in I: p_j^* > p\}} s_j^{\mathrm{B}} \left( p; p \mathcal{X}_1^{\mathrm{B}}(p; \beta) + \beta \right) = 1. \tag{8}$$

Arguments that parallel those presented above for strategic supply can be used to deduce that strategic demand is defined, positive and continuous for all prices less than $\max_{\{j \in I: m_j \neq 0\}} \left\{ p_j^* \right\}$.

Definition-chasing reveals that there is a Nash equilibrium in the game $\Gamma(\beta, \gamma)$ with price $\hat{p}$ if and only if

$$\mathcal{X}_1^{\mathrm{S}}(\hat{p}; \gamma) + \gamma = \mathcal{X}_1^{\mathrm{B}}(\hat{p}; \beta) + \beta / \hat{p}. \tag{9}$$

If either $\mathcal{X}_1^{\mathrm{S}}(\hat{p}; \gamma) > 0$ or $\mathcal{X}_1^{\mathrm{B}}(\hat{p}; \beta) > 0$ then the identified Nash equilibrium is non-autarkic: at least some traders make exchanges away from their endowments. If $\mathcal{X}_1^{\mathrm{S}}(\hat{p}; \gamma) = \mathcal{X}_1^{\mathrm{B}}(\hat{p}; \beta) = 0$ then the Nash equilibrium is autarky since the final allocation of every trader is their initial endowment.

## 4.2 Nice or very nice?

Using this characterization of equilibrium in modified games in which there are external bids and offers we now turn to derive the conditions on the primitives of the economy under which autarky is nice and very nice.

**Proposition 6** *Autarky is very nice if and only if* $\min_{\{j \in I: e_j \neq 0\}} \left\{ p_j^* \right\} \geq \max_{\{j \in I: m_j \neq 0\}} \left\{ p_j^* \right\}$, *i.e. no gains from trade exist.*

Autarky is therefore very nice when it is a legitimate equilibrium in the context of the economy. As the next proposition shows, autarky is nice when it is a legitimate equilibrium in the context of the game (i.e. small perturbations to the game do not dramatically alter the equilibrium).

**Proposition 7** *Autarky is nice if and only if* $P^{\mathrm{S}} \geq P^{\mathrm{B}}$, *i.e. when autarky is the only Nash equilibrium in the game.*

If there are no gains from trade, we do not expect trade to take place and therefore for a non-autarkic equilibrium to exist. However, recent contributions to the literature that study examples of bilateral oligopoly environments have noted that there exist economies in which the only equilibrium in the market game is autarky even though trade would take place if the rules of a competitive market mechanism were imposed, i.e. gains from trade exist (see, for example, [2], [3] and [7]). Our analysis has allowed us to find conditions on the primitives in a general model of bilateral oligopoly under which the only equilibrium is autarky, thus generalizing these examples. It is well-known that allowing oligopolistic tendencies to permeate an economy via price-mediated trade means that outcomes will generally be Pareto inefficient. Our question is this: far from being just inefficient, is there a non-trivial set of economic environments in which gains from trade exist and yet no trade takes place at all. This question of whether autarky is the only equilibrium in environments in which gains from trade exist can, in the light of Propositions 6 and 7 be re-cast as: are there economic environments in which autarky is nice but not very nice. In an economy populated by a finite number of agents (5) and (6) can be used to deduce that $P^{\mathrm{S}} > \min_{\{j \in I : e_j \neq 0\}} \left\{ p_j^* \right\}$ and $P^{\mathrm{B}} < \max_{\{j \in I : m_j \neq 0\}} \left\{ p_j^* \right\}$, which allows us to draw the conclusion in the following proposition.

**Proposition 8** *If autarky is very nice it is also nice, but there are economic environments in which autarky is nice but not very nice, in particular when* $\min_{\{j \in I : e_j \neq 0\}} \left\{ p_j^* \right\} < \max_{\{j \in I : m_j \neq 0\}} \left\{ p_j^* \right\}$ *but* $P^{\mathrm{S}} \geq P^{\mathrm{B}}$.

This implies that there are economic environments in which gains from trade exist and yet no trade takes place in the buy or sell game due to the strategic behavior of agents. Indeed, the implication is that in order for trade to take place, not only must gains from trade exist, but there need to be *sufficient* gains from trade. We illustrate with an example.

**Example 1** *Suppose there are three agents of type A with* $p_A^* = \alpha\partial$, $\alpha < 1$, *and three agents of type B with* $p_B^* = \partial$. *All agents have interior endowments. A competitive market involves an equilibrium with trade since* $p_A^* \neq p_B^*$ *under our assumption that* $\alpha < 1$. *A measure of the gains from trade is given by* $|p_A^* - p_B^*| = 1 - \alpha$: *the smaller is* $\alpha$ *the higher the potential gains from trade. It is easily verified from (5) and (6) that* $P^{\mathrm{S}} = \frac{3}{2}\alpha\partial$ *whilst* $P^{\mathrm{B}} = \frac{2}{3}\partial$. *Therefore, there is a non-autarkic equilibrium if and only if* $\frac{3}{2}\alpha\partial < \frac{2}{3}\partial \Leftrightarrow \alpha < \frac{4}{9}$, *so 'sufficient' gains from trade need to exist before trade takes place.*

As Busetto and Codognato [2] note, a 'Shapley equilibrium' [11] exists if either an equilibrium with trade exists or if the autarkic equilibrium can be supported by a virtual price (i.e. is very nice). When there is no non-autarkic equilibrium and autarky is merely nice but not very nice a Shapley equilibrium fails to exist. Whilst Busetto and Codognato provided an example, our analysis gives precise conditions on the primitives under which this is the case.

**Corollary 9** *A Shapley equilibrium exists if and only if a)* $P^{\mathrm{S}} < P^{\mathrm{B}}$ *or b)* $\min_{\{j \in I: e_j \neq 0\}} \left\{ p_j^* \right\} \geq \max_{\{j \in I: m_j \neq 0\}} \left\{ p_j^* \right\}$.

The observation that for the buy or sell game there may be economies in which gains from trade exist and yet no trade takes place in equilibrium stands in contrast to the existence results for the buy and sell game. Under certain weak conditions on traders' preferences and endowments an interior Nash equilibrium always exists [5], but there is an indeterminacy due to the inherent multiplicity of equilibria resulting from the ability of agent to make wash trades. However, if traders' endowments are not Pareto efficient, the equilibrium always involves trade taking place. [9] The reason an equilibrium with trade always exists is that a trader can always 'open the market' by simultaneously acting on both sides which ensures that whenever there are gains from trade, no matter how small, trade will take place. Comparing this to our results in the buy or sell game reveals that the ability of agents to make wash trades is crucial to guarantee the existence of a non-autarkic equilibrium in strategic environments: there are economies in which there is no non-autarkic equilibrium in the buy or sell game and yet such an equilibrium exists in the buy and sell game. This generalizes the observations of [10].

# 5 Concluding remarks

Focussing on the buy or sell strategic market game in a simple model of bilateral oligopoly, we have been able to identify non-autarkic Nash equilibria in the game by intersections of strategic versions of supply and demand functions, despite the fact that the side of the market on which an agent wishes to act is endogenous to market outcomes. This allowed us to deduce the conditions on the primitives of the economy under which a non-autarkic equilibrium exists, and when autarky is the only equilibrium so no trade takes place. It was shown that there is a non-trivial set of economic environments in which autarky is nice but not very nice; that is, legitimate in the context of the game but not in the context of the economy. Put differently, there are economic environments in which gains from trade exist and yet no trade takes place via the mechanism of the buy or sell strategic market game. A direct corollary gives the conditions under which a Shapley equilibrium exists. In order to guarantee that trade takes place there must exist 'sufficient' gains from trade in the economy.

Strategic market games such as that studied here can be used to model, for example, the market for tradable pollution permits when traders are allowed to behave strategically ([8] surveys the current literature). Agents are endowed with permits and money and whether they buy or sell permits depends on the terms of trade. The results in this paper have clear implications for the outcomes in such markets and highlight the importance of the permit allocation mechanism that determines the structure of endowments (and therefore $P^{\mathrm{S}}$ and $P^{\mathrm{B}}$).

# A Proofs

**Proof of Lemma 1.** Suppose to the contrary that $p \geq \partial_i(e_i, m_i)$ and $b > 0$. Then $q = 0$ and $\partial_i(e_i + b/p, m_i - b) = \left(1 - \frac{b}{B}\right)^{-1} p$. But $\left(1 - \frac{b}{B}\right)^{-1} > 1$ so $\partial_i(e_i + b/p, m_i - b) > p$. However, binormality implies $\partial_i(e_i, m_i) > \partial_i(e_i + b/p, m_i - b)$ with the implication that $\partial_i(e_i, m_i) > p$, a contradiction. Thus, $b = 0$ so trader $i$ cannot be a buyer. If $p \leq \partial_i(e_i, m_i)$ and $q > 0$ analogous reasoning reveals a similar contradiction implying that $q = 0$ so trader $i$ will not be a seller in equilibrium. ■

**Proof of Lemma 2.** The properties of the function $s_i^{\mathrm{S}}$ can be deduced from modifications of the arguments in Lemma 3.1 in Dickson and Hartley [4], and those of $s_i^{\mathrm{B}}$ from Lemmas 3.3 and 5.1, where a similar analysis was undertaken with exogenously given sets of buyers and sellers (i.e. where all agents have corner endowments). Since the modifications required are slight, we do not include a formal proof. ■

**Proof of Proposition 6.** First we show that if $\underline{\partial} = \min_{\{j \in I : e_j \neq 0\}} \left\{p_j^*\right\} \geq \max_{\{j \in I : m_j \neq 0\}} \left\{p_j^*\right\} = \bar{\partial}$ then there is a $v$ such that for any $\delta > 0$ autarky is an equilibrium in the game $\Gamma(\delta v, v)$. Inspection of Figure 1 reveals that when $\partial \geq \bar{\partial}$, strategic demand will be zero in equilibrium so long as $\delta v/\bar{\partial} \geq \delta \Leftrightarrow v \geq \bar{\partial}$. Similarly, strategic supply will be zero in equilibrium when $\delta v/\underline{\partial} \leq \delta \Leftrightarrow v \leq \underline{\partial}$. Thus, for any $v \in \left[\bar{\partial}, \underline{\partial}\right]$ autarky is a Nash equilibrium for any $\delta > 0$, and so is very nice. Since $\bar{\partial} \leq \underline{\partial}$ by presumption, such a level of $v$ exists.

Conversely, if $\min_{\{j \in I : e_j \neq 0\}} \left\{p_j^*\right\} < \max_{\{j \in I : m_j \neq 0\}} \left\{p_j^*\right\}$ then for any $\delta, v > 0$ the definition and continuity of $\mathcal{X}_1^{\mathrm{S}}$ and $\mathcal{X}_1^{\mathrm{B}}$ imply that there is a $\tilde{p}$ such that (9) is satisfied with $\beta = \delta v$ and $\gamma = \delta$, and $\tilde{p} \in \left(\min_{\{j \in I : e_j \neq 0\}} \left\{p_j^*\right\}, \max_{\{j \in I : m_j \neq 0\}} \left\{p_j^*\right\}\right)$. Since $\tilde{p} > \min_{\{j \in I : e_j \neq 0\}} \left\{p_j^*\right\}$, $\mathcal{X}_1^{\mathrm{S}}(\tilde{p}; \delta) > 0$ and since $\tilde{p} < \max_{\{j \in I : m_j \neq 0\}} \left\{p_j^*\right\}$ $\mathcal{X}_1^{\mathrm{B}}(\tilde{p}; \delta v) > 0$, confirming that any Nash equilibrium is non-autarkic. Thus, no autarkic equilibrium exists for any $\delta, v > 0$, implying autarky is not very nice. ■

**Proof of Proposition 7.** To prove our claim we must demonstrate that autarky is an equilibrium point when $P^{\mathrm{S}} \geq P^{\mathrm{B}}$, but not when $P^{\mathrm{S}} < P^{\mathrm{B}}$. Consider a sequence of modified games $\{\Gamma(\varepsilon^n, \varepsilon^n)\}$ with $\varepsilon^n \to 0$ monotonically as $n \to \infty$. We first present the following lemma that demonstrates

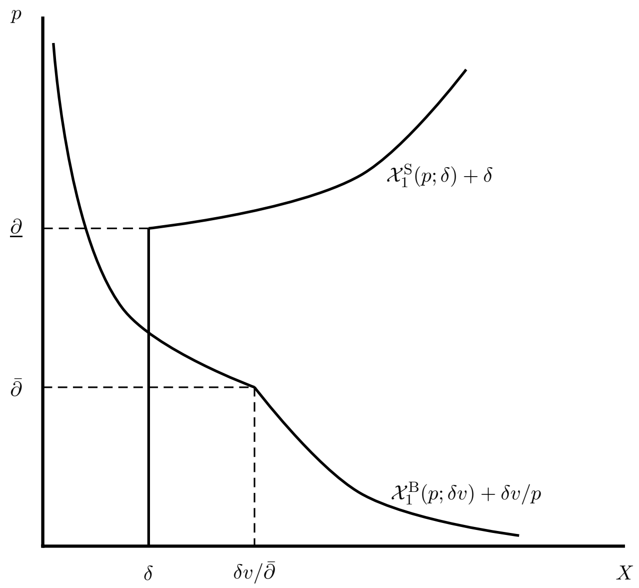


Figure 1: Very nice autarkic equilibrium.

convergence of strategic supply and demand in the modified game to their counterparts in the unmodified game as the external bid and offer reduce to zero.

**Lemma 10** *Consider a sequence of games* $\{\Gamma(\beta^n,\gamma^n)\}$ *with* $\beta^n,\gamma^n \to 0$ *monotonically as* $n \to \infty$. *Then as* $n \to \infty$ $\mathcal{X}_1^{\mathrm{S}}(p;\gamma^n)+\gamma^n \to \mathcal{X}_1^{\mathrm{S}}(p)$ *uniformly in* $p > P^{\mathrm{S}}$ *and* $\mathcal{X}_1^{\mathrm{B}}(p;\beta^n)+\beta^n/p \to \mathcal{X}_1^{\mathrm{B}}(p)$ *uniformly in* $p < P^{\mathrm{B}}$.

**Proof.** We demonstrate our claim for strategic supply only. The proof for strategic demand is similar and omitted. Let $Q^n = \mathcal{X}_1^{\mathrm{S}}(p;\gamma^n)$. Then $Q^n$ must satisfy

$$\frac{\gamma^n}{Q^n+\gamma^n} + \sum_{\{j\in I: p_j^* < p\}} s_j^{\mathrm{S}}(p; Q^n+\gamma^n) = 1. \tag{10}$$

Since $0 \le Q^n+\gamma^n \le \sum_{\{j\in I:p_j^*<p\}} e_j + \bar{\gamma}$ (where $\bar{\gamma} = \sup\{\gamma^n\}$) we conclude that $\{Q^n+\gamma^n\}$ has a limit by taking subsequences if necessary, and we write $X^0 = \lim_{n\to\infty} Q^n+\gamma^n$. For $p > P^{\mathrm{S}}$ we can take limits in (10) (along a subsequence if necessary) to deduce that $\sum_{\{j\in I:p_j^*<p\}} s_j^{\mathrm{S}}(p;X^0) = 1$. Since this is true for all subsequences, it follows that $X^0 = \mathcal{X}_1^{\mathrm{S}}(p)$, and therefore that $\mathcal{X}_1^{\mathrm{S}}(p;\gamma^n)+\gamma^n \to_{n\to\infty} \mathcal{X}_1^{\mathrm{S}}(p)$.

To deduce uniform convergence we must show that the sequence $\{\mathcal{X}_1^{\mathrm{S}}(p;\gamma^n)+\gamma^n\}$ converges monotonically to $\mathcal{X}_1^{\mathrm{S}}(p)$, and we will show that it does so from above. Suppose that $\gamma'' > \gamma'$. Then it follows that

$$\frac{\gamma''}{\mathcal{X}_1^{\mathrm{S}}(p;\gamma') + \gamma'} + \sum_{\{j \in I : p_j^* < p\}} s_j^{\mathrm{S}}\left(p; \mathcal{X}_1^{\mathrm{S}}(p;\gamma') + \gamma'\right) >$$

$$\frac{\gamma'}{\mathcal{X}_1^{\mathrm{S}}(p;\gamma') + \gamma'} + \sum_{\{j \in I : p_j^* < p\}} s_j^{\mathrm{S}}\left(p; \mathcal{X}_1^{\mathrm{S}}(p;\gamma') + \gamma'\right) = 1.$$

But then since $\mathcal{X}_1^{\mathrm{S}}(p;\gamma'') + \gamma''$ is the value of $Q + \gamma$ that makes the left-hand side equal to one, and $\frac{\gamma}{Q+\gamma} + \sum_{\{j \in I : p_j^* < p\}} s_j^{\mathrm{S}}(p; Q + \gamma)$ is decreasing in $Q + \gamma$ (which is easily deduced from Lemma 2), it follows that $\mathcal{X}_1^{\mathrm{S}}(p;\gamma'') + \gamma'' > \mathcal{X}_1^{\mathrm{S}}(p;\gamma') + \gamma'$. This implies convergence is monotonic, and so Dini's theorem can be utilized to conclude that convergence is uniform on compact sets. ■

Since $\varepsilon^n \to 0$ Lemma 10 implies that strategic supply and demand in the modified games $\{\Gamma(\varepsilon^n, \varepsilon^n)\}$ converge to their counterparts in the unmodified game as $n \to \infty$. Our claim can then be substantiated graphically. Refer to Figure 2. When $P^{\mathrm{S}} < P^{\mathrm{B}}$ the conclusion in Lemma 10 implies that there is an $n'$ such that for all $n > n'$ any $p^n$ that satisfies (9) with $\beta = \gamma = \varepsilon^n$ must be such that $p^n \in \left(P^{\mathrm{S}}, P^{\mathrm{B}}\right)$. But then if there is a sequence $p^n$ convergent to $p^0$, say, it follows that $\mathcal{X}_1^{\mathrm{S}}(p^n;\varepsilon^n) \to \mathcal{X}_1^{\mathrm{S}}\left(p^0\right) > 0$ since $p^0 > P^{\mathrm{S}}$, and $\mathcal{X}_1^{\mathrm{B}}(p^n;\varepsilon^n) \to \mathcal{X}_1^{\mathrm{B}}\left(p^0\right) > 0$ since $p^0 < P^{\mathrm{B}}$. As such, any equilibrium point must be non-autarkic.

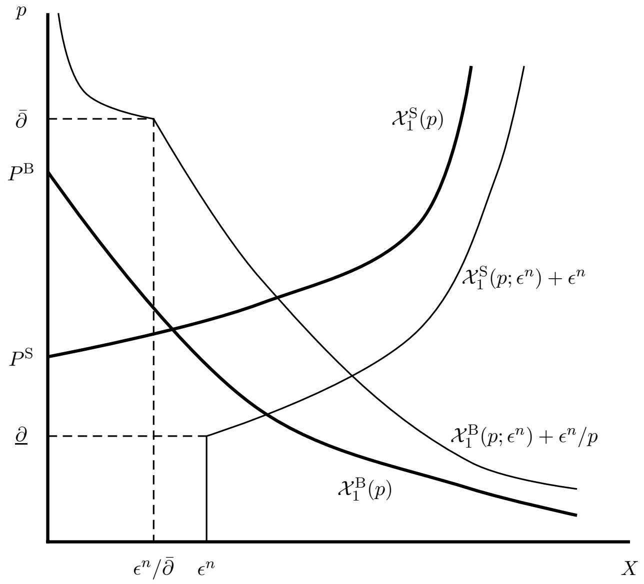


Figure 2: Determining whether autarky is nice.

Conversely, when $P^{\mathrm{S}} \geq P^{\mathrm{B}}$ Lemma 10 implies that there is an $n''$ such that for all $n > n''$ (i.e. for $\varepsilon^n$ small enough) (9) holds with $\beta = \gamma = \varepsilon^n$ only for $p^n \in \left[P^{\mathrm{B}}, P^{\mathrm{S}}\right]$ (refer also to Figure 2, letting $P^{\mathrm{S}} \geq P^{\mathrm{B}}$ in that figure). But then taking limits as $\varepsilon^n \to 0$ in (7) reveals $\mathcal{X}_1^{\mathrm{S}}\left(p^n; \varepsilon^n\right) \to 0$ (since $p \leq P^{\mathrm{S}}$). Moreover, we can conclude that $\mathcal{X}_1^{\mathrm{B}}\left(p^n; \varepsilon^n\right) \to 0$ by taking limits as $\varepsilon^n \to 0$ in (8) (since $p \geq P^{\mathrm{B}}$). Thus, as $n$ increases without bound any convergent sequence of equilibria converges to autarky, so it is an equilibrium point. ■

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