Some Mental Notes on Mental Addition

Abstract
Popular descriptions of how children secure the answers to addition operations may be contributing to confused thinking. Distinctions appear to be made between adding 'on paper', 'with a calculator' and 'with concrete materials'. Such distinctions are real only in the sense that they distinguish between the trappings of addition. This may well be deflecting us from the more essential issue, which is the nature of the mental strategies children use when adding.

Introduction
There are at least two characteristics of arithmetic which can contribute to difficulties in learning about number.
The first of these is that arithmetic is an abstract system of ideas and symbols, which has been developed by successive generations of learned persons. What this means, as Skemp (1971) points out, is that arithmetic cannot be learned directly from the everyday environment but has to be learned indirectly, through the mediation of teachers. The second, and related, characteristic is that within the system, many of the ideas are hierarchically tied to each other. Thus, learning to add two-digit numbers depends on mastery of some number bonds which, in turn, depends on a knowledge of counting. When basic ideas are poorly understood, the more sophisticated ideas which build on the basic ideas will have little meaning for learners.

Because of these particular characteristics of arithmetic there is, of necessity, a need for the teacher to sequence and pace the arithmetical ideas such that learners can make meaningful and confident progress. While it is, of course, desirable for all learners to make good progress, it is particularly important that learners who have been identified as having specific difficulties do not experience poor teaching that can add to their difficulties.
There is nothing particularly radical in what has been claimed so far: those of us with interests in special educational needs have always been concerned to structure the learning tasks, materials and environment to obviate learning difficulties where possible and to ameliorate those which have become entrenched.

Paradoxically, however, in our enthusiasm to portray the power and utility of arithmetic as a means of understanding and participating in the world, we are on the verge of believing in, and becoming confused by, the rhetoric which has emerged to describe learners' achievements in arithmetic. One fairly potent example of this confusion is associated with the concept of addition. It is ubiquitously expressed as desirable - during informal conversation, in curriculum policy statements and in school textbooks - that learners should be able to 'add mentally'. The phrase 'add mentally' is probably so entrenched in every teacher's thinking that few may think to question it. But what does it mean to 'add mentally'? This is actually quite a difficult question to grapple with, though popular attempts to answer it contrast adding which is done 'in the head' with adding which is done 'on paper', 'with a calculator' or 'using concrete materials'. Let us consider each of these in turn.

Adding 'in the head' against adding 'on paper'
Such a simplistic distinction fails to recognise that the point of adding 'on paper' is to record the steps which are being carried out 'in the head' and so help to keep control over what might otherwise be an unmanageable task. Recording the computation on paper does not make the task non-mental or indeed less mental.

The traditional emphasis put on recording (in the context of school mathematics) raises the interesting question, 'For whose purpose is the recording being done?' If the recording is being done by the individual 'computer' because he/she finds such recording useful, it is likely that such activity is relieving working memory's load and monitoring capacity (Baddeley, 1976) and so is an integral and meaningful part of the being done because the teacher requires of the learner some evidence of the product (and possibly the processes) of the assigned computation, the value of such activity may be suspect, at least to the learner. It is common in the rhetoric of teachers to hear them complain that learners do not, but should, 'show their working'. It is almost as common, and indeed entertaining, to hear tales (for example, Skemp, 1971; Ginsburg, 1977; Resnick, 1982**)** of learners who, not realising the import of visible working, effect the computation by one means and then record the working according to the teacher's requirements whilst somehow excusing the teacher's idiosyncratic predilections!

Adding 'in the head' against adding 'with a calculator'
This is a variant of the confusion of adding 'on paper'. The point of using a calculator is to relieve working memory of load and of the tedium which can attend repeated calculations. Again, using a calculator does not make the task non-mental or less mental. Indeed, fairly sophisticated mental powers of estimation are required in order to judge whether the sum of the addends is a reasonable one. Without this interfacing mental activity, the calculator's contribution to learning is about as valuable as leafing through a book in which the text is inaccessible: there may be some learning but its nature and incidence are unclear.

Adding 'in the head' against adding 'using concrete materials'
The emphasis on concretisation in primary school arithmetic is largely derived from orthodox and neo-Piagetian theory. Such theory builds on the constructivist assumption that arithmetical knowledge, like all knowledge, is not directly absorbed by the learner from the teacher but is actively constructed by each individual learner. The realisation that for learning to be effective, it had to be active, manifested itself in a veritable explosion of 'activity methods' and 'learning by doing' in the (mistaken) belief that unless children were physically busy they could not be learning. However, as Schwebel & Raph (1973) and Finn (1992) point out, in Piagetian theory the nature of the activity is critical. For the activity to facilitate learning, it must cause *intellectual change* in the learner. According to this criterion, the activity could be one of reflection, advanced abstraction or verbal manipulation (Finn, 1992). The use of concrete materials can only contribute to learning then, if, in addition, there is some attempt *to* strip the 'noise' from the activity and extract the underlying mental meaning.

It is not being argued that the use of pencil and paper, calculators or concrete materials is wrong or undesirable. What is being argued is that these adjunct aids must not take on a life of their own. This is very easy to say but probably much more difficult to honour. It is easy to understand how the hard-pressed teacher can be deflected by the plethora of superficially attractive tasks and materials which for any one particular learner may turn out to be meaningless, disembedded or purposeless. With the best of intentions the teacher will bring such tasks and materials into service in order to offer the experiences from which the learners will abstract the concept(s), to enable learners to develop useful algorithms, and to assist each learner to co-ordinate algorithmic skill with conceptual understanding. However, that algorithmic skill and conceptual understanding are not the same thing (Bell, Costello & Kuchemann, 1983) is a significant point which we may easily overlook in the daily frisson of classroom life. Furthermore, trying to institute both pieces of learning simultaneously may be difficult, if not impossible (Smith, 1987).

It is precisely this lack of co-ordination between algorithmic skill and conceptual understanding which is reflected in the perceived tension between adding 'in the head' and adding 'on paper'. **As** Plunkett **(1979)** points out, written algorithms are of a permanent and standardised form (which renders them 'correctable'), are efficient and automatic (which renders them amenable to use even if they are not understood), and are generalisable (which renders them capable of application to any domain of number but need not have any articulation with the way(s) in which people think about number). Conceptual understanding, on the other hand, may be fleeting and flexible and has to be achieved by each individual learner (Cockcroft, 1982). This in turn means that at any one time, understanding, far from being an all-or-nothing affair, may be partial, incomplete and incorrect. Thus the desire or intention to achieve a match between conceptual understanding and algorithmic performance may not find a correspondence: if the written algorithms have themselves not been learned they are not available for the individual to use; conversely, if the algorithms have themselves been learned but are not tied to any conceptual structures, their potential for use will not be recognised by the individual. If it is not helpful to consider addition in dichotomous terms, as something which happens with or without particular adjunct aids, how should we be viewing it? The real issue is *the sense that children make of addition tasks.* Above all else, it is with this that teachers should be concerned if they are really working from the beliefs and orientations of constructivism.

While the end product of learning to add is being able to retrieve 'addition facts' or 'number bonds' from memory, this is very much an adult strategy to which normally developing children move only gradually (Resnick, 1989**).** Until children can make full use of the retrieval strategy (and for some, perfectly normal, children this may not be until they are 11 or 12 years of age), they deploy the only other strategy which will enable them to find the sum of addends: that of counting (Groen & Parkman, 1972, Fuson, 1982; Secada, Fuson & Hall, 1983). Counting is not to be thought of as a mechanistic rehearsal of number names but an altogether more sophisticated activity of being able to match each one of a collection of entities with a stable series of number words and knowing that the last number name represents the total numerosity of the collection (Schaeffer, Eggleston & Scott, 1974; Gelman & Gallistel, 1978). Because counting is itself composed of a number of component skills, the activity of counting is susceptible to error and children with learning difficulties seem prone to making counting errors (Baroody, 1986; McEvoy & McConkey, 1990).

An intermediate position between making exclusive use of counting and being able to retrieve addition facts from memory is to derive new facts from one's existing repertoire (Fuson, 1982; Fuson & Fuson, 1992). Thus, for example, the child might calculate that the sum of 4 and 5 is 9 because 5 is one more than 4 and the fact that 4 and 4 are 8 is already in memory. Progress towards the mature strategy of retrieval, then, is a process of the gradual abandonment of counting with a complementary gradual institution of the number facts. While many children at the end of primary education will use the retrieval strategy predominantly when they are performing addition operations, it is, nevertheless, perfectly possible for others to spend the entire period of primary education using a counting strategy to obtain solutions to addition operations. It follows that children who have difficulty with number operations may well rely on a counting strategy for an even longer period of time.

Proponents of the 'calculator lobby' (such as Plunkett, 1979; Graham, 1985) might argue that this is just the evidence that they need: if children cannot retrieve the addition facts from memory, they would be better employed in using a calculator to provide the answers. The reasoning underlying such a position is that calculators obviate the need for the individual to perform computational procedures which, if not linked to the mathematical concept, are of questionable educational value.

Now, while electronic calculators are viewed positively in that they can remove the difficulties associated with applying algorithms and, further, can lead to improvements in attitudes towards, and understanding of, mathematics (Cockcroft, 1982; Szetela, 1982), the psychological reality of the learner is a little more complex. According to Siegler & Shrager (1984), children *prefer* to retrieve addition facts from memory, if they possibly can. Retrieving addition facts is not a cognitively difficult thing to do and it can yield a speedy response. However, it seems that children, in trying to retrieve a response from memory, simultaneously prime themselves that a counting strategy may need to be invoked (Greeno, Riley & Gelman, 1984). If the child can quickly retrieve a response, the counting strategy will be aborted but if the child cannot retrieve a response with speed and certainty, the back-up strategy of counting will be used to obtain the answer. If the child is psychologically predisposed to obtain a response either through retrieval or counting (in other words, through some mental mediation), use of the calculator should not, then, be viewed as some sort of alternative strategy but rather as a check on retrieval or counting. To accord a more enhanced status to the calculator (particularly in the context of teaching learners who have difficulties) is to risk impeding the child's attempts to make sense of addition.

The work of Siegler & Shrager might suggest that if children prefer to use the strategy of retrieval, they could be best helped in this achievement by more time being spent on drilling the number facts. Care is needed here, however, to determine *when* drilling may be appropriate. Historically, the mathematics literature has tended to suggest (Bell, Costello & Kuchemann, 1983) that drilling as a teaching strategy is an efficient and effective one; that the issue turns not on whether drilling should take place but on how the drilling should be conducted. However, what such literature may have neglected to make clear is that drilling either before or without some understanding of the meaning of the operations which are being drilled is ineffective because the drilled material is neither retained for any length of time nor available for transfer to new tasks (Ausubel &Youssef, 1965; Ausubel, 1968). Drilling that takes place *after* some meaning has been constructed, however, may very well be appropriate. Both Williams (1971) and Case (1982) argue that when the understanding has been established, the facts contained therein need to be available for immediate recall. If the facts cannot be immediately retrieved when needed, then the individual has to give attention to working them out. In turn, the processing capacity being thus used is unavailable for the mastery of more advanced conceptual material or more sophisticated algorithms.

Drilling then seems to have a value. Perhaps, however, the real value of drilling lies in the opportunity it gives to practise the operation in question. In the case of addition, counting provides children with a reliable method of generating solutions for themselves: a method which will be reinforcing to the children for the success it brings them. Indeed, it may be on the basis of repeated and successful counts that the repertoire of number facts becomes stable and available for more or less automatic access. In other words, counting leads to its own extinction through memory retrieval becoming the dominant strategy.

Summary

In summary, adding as an arithmetical concept must *always* be 'in the head'. Unlike many everyday acts of addition (such as adding an egg to all of the other baking ingredients, which is a physical act of combining), the arithmetical act of addition is concerned with number. But number is a strictly abstract entity. It is the property of a *set* or *collection* of items rather than the property of the individual items within the set or collection. So when we speak of a set of blue plates, the adjective 'blue' describes the plates but when we speak of a set of five plates, the adjective 'five' describes the set, not the plates. The arithmetical act of addition requires us to reason about entities which themselves exist only as mental abstractions! Whether the physical embodiments be 'on paper', 'with a calculator' or 'using concrete materials', the concept which these embodiments illuminate is criterially mental.

What the research now seems to be showing is that in the learning of addition children are using two fairly robust strategies: counting and the retrieval of addition facts from memory. With the passage of time and the experiences of learning, counting gives way to retrieval. It is probably through the experiences of counting that retrieval becomes well established and developed. In any event, there will be a period of years during which counting and retrieval co-exist. For teachers there seems, therefore, to be a need to focus on the strategies that their learners are using. When learners are heavily reliant on counting, such counting needs to be error free. When learners are making use of retrieval, the outcome of such mental activity may need corroboration. It is in the service and promotion of counting and retrieval that adjunct aids should be deployed. By giving primacy to the strategies and recognising the adjunct aids for what they are, it may be possible to unpack some of the confusion and difficulty that hamper learning.

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