

Thinking About Maths

**A Review of Issues in
Teaching Number from 5 to 14 years**

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This publication has been written to stimulate and inform debate on an important educational issue. The views expressed are those of the authors and are not necessarily the views of Learning and Teaching Scotland.

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Preface

We have written this document as a contribution to the National Debate on Mathematics Education in Scotland. We have written it with the age range P1 to S2 in mind, and our emphasis has been on the acquisition of basic numeracy. Scotland has made some outstanding contributions to mathematics in the past. We have no doubt that Scottish mathematicians will make many great contributions in the future. However, our concern in writing this document has been to encourage universal numeracy, rather than to help produce a large pool of mathematical talent. For this reason, we focus on number rather than on the broad mathematics curriculum. We have taken as our starting point the issues raised by various documents written by the Scottish Executive Education Department (SEED) and Her Majesty's Inspectorate of Education (HMIE) over the past decade. We have summarised the history to the debate in Scotland in Section 2. Sections 3–7 each deal with a substantive issue in the teaching of maths. Sections 8 and 9 summarise our discussions. Each section has been written as an independent unit, so it is not necessary to read them in linear order. You can, if you wish, dip in and read each section separately in any order you choose.

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1. Introduction

Over the past decade, there has been an increased interest in the teaching and learning of number. Programmes of instruction in mathematics have been formulated and reformulated according to shifting banks of evidence as to what 'works' best in both a national and an individual sense. Many changes have been proposed – in teaching methods, in curriculum content, in the meaning of teaching and learning mathematics, and in responsibility for policy and instructional decisions. Suggestions for change are often based on complex research evidence.

However, these changes are unlikely to improve our teaching if:

- we cannot understand what the changes actually mean in terms of our practice in the classroom
- the changes do not agree with our beliefs about what mathematics is and how best to teach it.

Real change in our practice can only take place if the suggested changes feel educationally appropriate and defensible. In considering this issue and the implications for our own practice as teachers, a first step is to reflect on those issues (described in section 2) highlighted by recent SEED and HMIE documents as being significant in effective mathematical learning. This text is intended to support such professional reflection by:

- providing a broad background to the current changes
- discussing issues that have been raised in recent documents on mathematics for the years 5–14
- contributing in a positive yet challenging way to discussion of the issues.

This document is a focused discussion that draws on high-quality research evidence that is conceptually relevant in the Scottish context. Our aim is to be interesting, relevant, and thought provoking.

2. Background

2.1 The international context

Wholesale changes in mathematics teaching are often initiated by international comparisons that show a nation's children to be achieving less than children in comparable countries. A series of international comparisons of children's abilities in maths and science has set the scene for intensive scrutiny of the maths curriculum in the English-speaking world.¹ In the 1980s, American educationalists found that Chinese, Taiwanese and Japanese children outperformed American children on a wide range of mathematical skills. Subsequent research identified the organisation of maths teaching in these countries as the source of this difference.²

2.2 The national context

In a parallel line of discovery, British educationalists found in the 1990s that children in some European countries were very much better at maths than were British children. Two research groups focused on bringing maths teaching practices to Britain from these countries in the 1990s. One was in Exeter, one in London. Both began their programmes in 1995. Charitable foundations and industry provided initial funding for both these programmes.

2.2.1 Mathematics Enhancement Programme (University of Exeter, 1995–present)³

This programme was derived from the best practice observed in Hungary. Professor David Burghes, an educationalist at the Institute, designed it. The programme was initially designed for secondary maths teaching, and then followed by a primary extension. The programme uses the Central European model of early education. That is, there is an emphasis on the education of the senses and the attentional system, and there is no attempt to teach written text or

¹ Medrich, Elliot, and Griffith, Jeanne, *International Mathematics and Science Assessments: What have we learned?*, Washington, DC: National Center for Education Statistics, 1992

² See, for example, Stigler, James W., and Perry, Michelle, 'Mathematics Learning in Japanese, Chinese, and American Classrooms', in G. B. Saxe and M. Gearhart (eds), *Children's Mathematics*, New Directions in Child Development, No. 41, San Francisco: Jossey-Bass, 1988, pp. 27–54.

³ Burghes, D., *Mathematics Enhancement Programme*, 1998. Available at: <http://www.ex.ac.uk/cimt/mep/meppinfo.pdf>

number skills until the age of 6 or 7. The approach was proved effective through trials conducted with 70 primary schools in England, although its introduction was not without problems.⁴

2.2.2 Improving Primary Mathematics (National Institute of Economic and Social Research, 1995–present)

This programme was derived from the best practice observed in Zurich. Professor Sig Prais, an economist at the Institute, designed it. The approach emphasises oral rather than written mathematics. The teaching pattern is a ‘warm-up’ mental maths session, then demonstrations, questioning and discussion, around a single topic, followed by maths games and workbooks. The programme is based on a discursive principle: the maths lesson is dominated by talk and demonstration rather than by work on paper. There are two discursive rationales for taking the entire class through the same experience (rather than grouping the children). One is that the children must have something in common to discuss. The other is that if the class breaks up into groups doing different tasks, then the teacher cannot teach entirely through an oral approach. The programme was proved effective through trials in six schools in Barking and Dagenham Education Authority and a further 50 schools across four more boroughs.⁵

2.2.3 National Numeracy Project (DfEE, 1996–1998)

The National Numeracy Project produced the ‘Framework for Teaching Mathematics’ that was trialled in a number of primary schools.⁶ Summer schools on numeracy for 11-year-olds became an essential part of the strategy. These were designed to update the maths abilities of children entering secondary school without the experience of a maths curriculum based on an oral and mental approach.

2.2.4 National Numeracy Strategy (DfES, 1998–present)⁷

The National Numeracy Strategy is a coordinated government-led development for England and Wales, based on the successes of previous numeracy schemes. It has a prescribed content that is linked

⁴ Burghes, D., *Mathematics Enhancement Programme (MEP): The first three years*, 2000. Available at: <http://www.ex.ac.uk/cimt/mep/intrep00.pdf>

⁵ Whitburn, Julia, ‘Mathematical Attainments in Primary Schooling: Raising standards and reducing diversity’, *National Institute Economic Review*, No. 179, London: National Institute of Economic and Social Research, 2002

⁶ Department for Education and Employment, *National Numeracy Project: Progress report 1996–1998*, London: DfEE, 1999

⁷ Department for Education and Skills, *Implementing the National Numeracy Strategy: The final report of the Numeracy Task Force*, London: DfES, 1998

to a national coordinating network and training scheme. It is optional, and not legally part of the National Curriculum for England and Wales.⁸ However, schools opting not to join the strategy must show in their SATS results and OFSTED reports that they have implemented a viable alternative. Results of the National Numeracy Strategy are monitored through the national testing programme. Over the last few years, these programmes have achieved a radical change in mathematics teaching in England and Wales. They have shifted the emphasis onto organised, whole class teaching in which talk and mental activity is prioritised over the written worksheet.

2.3 The Scottish context

The Third International Mathematics and Science Study (1996, 1997)⁹ is an international comparison of over 40 countries that showed that average maths scores of Scottish 9- and 13-year-olds were well below those of pupils in higher achieving countries. This evidence, and the example of the English National Numeracy Strategy, has produced pressure in Scotland to update approaches to number teaching. However, the resources, training and teacher support associated with the English National Numeracy Strategy have not been available in Scotland. The National Numeracy Project and Strategy were coherent programmes of training and resources that set out to equip schools to teach maths in a different way. In Scotland, some documents have been produced, but schools and education authorities have been left to decide what changes to implement, and to find the resources for these changes. Recommendations in key documents have echoed the changes happening south of the border. The eight key documents are as follows.

2.3.1 Improving Mathematics Education 5–14 (1997)

This document was written in response to concerns about the international standing of Scottish maths education. It was based on a study of schools in Scotland, Singapore, Korea, Japan, Switzerland, and Hungary. It suggested that:

- increased attention be paid to mental calculation
- greater use should be made of whole-class teaching
- teacher expectations of pupils should be raised

⁸ ATL: Available at: http://www.askatl.org.uk/issues/Key_Stage_3/KS3_latest_news/latest_news.htm

⁹ National Academy of Sciences, *The Third International Mathematics and Science Study*, National Research Council, Washington, DC: Mathematical Sciences Education Board, 1996

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- common textbooks should be adopted and variations between schools in the pace of work lessened
 - schools should move to setting in S1.

2.3.2 Achievement for All (1997)

This document addresses the issue of setting children by attainment. It recommends this form of organisation (particularly for mathematics) in S1/S2 and in P6/P7. The rationale for this is to reduce the time teachers spend managing resources and explaining tasks. This gives them more time for direct teaching, which the authors define as questioning, giving explanations and providing feedback, checking understanding, sharing objectives, monitoring progress and identifying next steps in learning. The document makes it clear that HMI sees classroom organisation as a key factor in raising attainment.

2.3.3 Standards and Quality in Scottish Schools 1995 to 1998 (1999)

This document is part of a regular series of reports, and covers the whole of the curriculum in primary and secondary sectors. Its recommendations for maths in the primary sector included the advice to raise attainment in mental calculations, fractions and decimals, and to raise attainment in problem solving and enquiry in 65 per cent of schools. It was left to teachers to exercise their professional judgement on how they should raise attainment in these areas.

2.3.4 Standards and Quality in Secondary Schools 1995–1999: Mathematics (1999)

This document was based on 92 schools, and found that many S1/S2 maths courses were not well planned. These had weaknesses that included paying too little attention to mental maths and problem solving, relying too much on commercial schemes, and lack of coordination. Pupils coped with the work, but much of it was familiar and undemanding. Improved performance is required in mental calculation, fractions, decimals and percentages, algebra and solving non-routine problems. Problems in S1/S2 came from an emphasis on individualised learning, which often gave slow progress. Pupils need more opportunities to discuss problems and develop independent thinking. There were particular problems in assessment as part of teaching in S1/S2, with assessment of mental calculation being a particular difficulty. Development planning in secondary maths departments was a notable weakness, with 70 per cent of departments having important weaknesses.

2.3.5 Assessment of Achievement Programme Sixth Survey of Mathematics (2000)¹⁰

This survey established that pupil performance was satisfactory in all categories in P4 and P7. In S2, pupil performance was unsatisfactory in fractions, percentages and ratio. Comparisons over time showed that in P4, P7 and S2, performance had improved in comparison to the 1997 survey. Performance in problem solving was lower than in all other categories, and continued to need attention.

2.3.6 Standards and Quality in Primary Schools: Mathematics 1998–2001 (2002)

This document identifies the main areas for improvement as problem solving, the pace of progress from P4 to P7, challenges for high-attaining pupils, opportunities to use ICT in mathematics and the use of assessment information in planning. In contrast to the 1995–98 study (2.3.3), it states that 90 per cent of primary schools inspected had satisfactory programmes in number, money, and measure. It states that only 50 per cent of primary schools inspected had satisfactory programmes in problem solving. In schools with poor programmes, there was over-reliance on the textbook, and specific strategies were taught, leaving the children no understanding of how to apply these strategies in their maths work. Children often needed support when problem solving in an unfamiliar context, and there were particular weaknesses in recording and reporting problem-solving activities. The characteristics of the least effective programmes were that:

- mental maths sessions missed opportunities for shared thinking
- homework focused on number work and didn't include problem solving
- teachers relied too heavily on maths scheme materials
- pupils worked on repetitive exercises with little discussion
- teachers did not make systematic use of assessment evidence in their planning.

There were few problems in attainment at the early stages. Problems tended to emerge in the upper primary between P4 and P7. Routine calculations did not pose many problems. Difficulties more often arose in solving word problems and dealing with fractions, decimals, and percentages.

¹⁰ Scottish Executive Education Department, *Sixth Survey of Mathematics, 2000, Findings and Issues*, Edinburgh: SEED, 2001

2.3.7 Early Intervention: 1998–2000 (2002)

This document reports on early intervention in mathematics in 24 primary schools from 17 of the 32 authorities. Most of these schools had increased their time allocation for maths in P1 and P2 to provide mental maths lessons. In some of the schools, the increased emphasis on number was accompanied by a decreased emphasis on problem solving and enquiry. Although assessment was generally good, teachers tended not to be confident about identifying next steps for learning for children that are more able. Despite some shortcomings in provision, overall attainment was good or very good in all the schools.

2.3.8 The National Strategy for Numeracy (2002)

This document has been produced to help educators at two levels. It will help education authorities with the work of publishing annual statements of improvement objectives and reports of success. It will help schools prepare their development plans and their annual report on progress. The document is focused on improving standards of attainment in numeracy in schools. It summarises a number of reports and documents, and outlines the framework in Scotland for raising attainment.

2.4 Overview

Recent changes in emphasis in Scottish primary maths have the same sources as the National Numeracy Strategy in England and Wales. The implementation of change has been very different in Scotland, and there is no guarantee that changing Scottish teachers' style will replicate the effects of the National Numeracy Strategy. It may be that additional resources and training are essential for raised attainment. The pressure for change follows an assumption that the correct conclusions about teaching have been drawn from international comparisons of attainment. These conclusions are problematic because comparative data are very hard to interpret. There are, for instance, some notable exceptions to the general rule that systematic whole-class teaching is related to high attainment at ages 9 and 13. There is also the possibility that marked differences in early education can interact with later provision. Education systems are usually rooted in their context of culture and language, and do not readily transplant. The National Numeracy Strategy is reported to have raised attainment in computational skills. This may have been at the expense of problem-solving skills.¹¹ There may be echoes of this trend in Scotland. While

¹¹ Brown, Margaret, *What are the Key Aspects of the National Numeracy Strategy?*, Plenary Address to British Congress of Mathematics Education, 1999. Available at: <http://www.edweb.co.uk/bcme/proceedings/plenary/brown.htm>

the 2002 SEED document on quality reports a large improvement in teaching of number, it reports no real improvements in problem solving. This is perhaps unsurprising, given that problem solving is presented in the *Mathematics: 5–14 National Guidelines* as a procedure to be taught rather than as a context for learning. Difficulties in maths seem to emerge at the higher levels of primary and at early secondary, not at the initial primary school stages. This may well be because conceptual accuracy is not an issue in the assessment of early maths, but cannot be escaped at later stages. Comments throughout the most recent SEED documents characterise poor quality maths teaching by slavish adherence to schemes of work, repetitive exercises, lack of shared discussion and a lack of focused assessment information.

2.4.1 The major issues

Although mathematics from 5 to 14 has been through a decade of major change, there are still unresolved issues about the teaching and learning of maths in Scotland. Basing our conclusions on the documents we have reviewed above, we think that the major issues relate to five interconnected themes. These themes are: problem solving, conceptual (as opposed to procedural) learning, mental maths, commercial schemes, and assessment. In the rest of the document, we have devoted a section to each of these themes. We believe that:

- the way problem solving is framed in the *Mathematics: 5–14 National Guidelines* poses major difficulties. We discuss this more fully in section 3
- an imbalance between early procedural and conceptual teaching can lead to later failure. Children can often do their ‘sums’ without understanding number. It is easy to teach basic calculations as a set of procedures, but this does not provide an adequate conceptual framework for later learning – especially for word problems, fractions, decimals, and percentages. We discuss this more fully in section 4
- a wholehearted change to teaching mental rather than written strategies is quite difficult to achieve. We discuss this more fully in section 5
- maths schemes and associated homework frameworks can be problematic. We discuss this more fully in section 6
- there are major contradictions in the assessments of pupils that teachers are currently required to carry out. We discuss this more fully in section 7.

In each of the sections 3–7, we look at these five issues in relation to the conflicting constraints and demands on teachers. We then try to suggest ways to gain a better understanding of the major issues. In each section, the ‘Points to consider’ and ‘Things to try’ are designed for whole-school policy review.

3. Problem solving

Many children who are able to ‘do their sums’ with reasonable accuracy and proficiency and could be said to have computational skills struggle when faced with a problem. This is a worry because problem solving is regarded as a major way in which children learn. Many teachers ask ‘why is it that children *learn* best in problem-solving situations yet *function* best when presented with straightforward calculations?’¹² The obvious answer is that children are not learning in a way that maximises their problem-solving behaviour. The main reason for this is that ‘problem solving’ as outlined in *Mathematics: 5–14 National Guidelines* is seen as a procedure rather than as a context for learning.

3.1 What we know

The way in which problem solving is taught in schools could be described as a ‘strategy-driven’ approach. *Mathematics: 5–14 National Guidelines* presents schools with a list of nine suggested strategies, which have become the focus of most problem-solving programmes. Consequently, the product rather than the process can become the focus and teachers may find themselves setting problems, explaining the method of solution and then telling the children whether they got the problems right.

What teachers say

‘I feel I’m ticking boxes to say the children can use a particular strategy. Sometimes their own ways of solving the problem are better but I don’t have time to explore these.’

Primary 5 teacher

This approach has also led to a tendency within some mathematics programmes to treat problem solving as a separate element of work. When this happens, skills and strategies can be developed in isolation from the rest of the mathematics curriculum. For example, it is quite common for schools to have ‘problem-solving days’. Here the mathematics time for that day is devoted to solving problems chosen for their relation to the next problem-solving strategy to be developed. However, research suggests that problem solving is not teachable and our attempts to make it so may be misguided.

¹² Atkinson, S. (ed.), *Mathematics with Reason*, London: Hodder and Stoughton, 1992

What teachers say

‘My school focuses on developing strategies. I could spend the whole week teaching measure and then on a Friday, which is problem solving day, have to give the children a shape problem.’

Primary 6 teacher

By presenting children with rules and strategies, and asking them to apply these to a range of problems, we are demanding that children engage in quite a sophisticated form of reasoning. It is known as *deductive* reasoning and does not develop until quite late in childhood. Even adults find this form of reasoning difficult. One example of deductive reasoning, familiar to many, is being taught a set of rules for the grammar of a second language and then being expected to apply them.

Not all problem solving needs to be deductive. An alternative approach is one in which teachers present children with a range of particular problems and support them as they work out for themselves the strategies and rules for solving them. This approach gives children opportunities to engage in *inductive* learning, and it is more developmentally appropriate for young children. An example of *inductive* learning is learning the grammar of our first language. When we first learned to speak, we were not explicitly taught rules or strategies. We worked out the rules for ourselves through exposure to language and the modelling and support of adults.

‘As human beings we appear to be very able to engage in the process of *induction* (inferring general rules or patterns from a range of particular cases), but relatively less well-equipped for *deductive* reasoning (the opposite process of inferring particular cases from a general rule).’¹³

The most natural way for children and adults to learn is to search for patterns and regularities within the variety of their experience in order to make sense of new experiences.¹⁴ We all expect to find connections between new experiences and what we know already. Consider, for example, how we might tackle setting up a newly purchased home

¹³ Whitebread, D., ‘Emergent Mathematics or How to Help Young Children Become Confident Mathematicians’ in Anghileri, J. (ed.), *Children’s Mathematical Thinking in the Primary Years*, London: Cassell, 1995

¹⁴ Whitebread, D., ‘Emergent Mathematics or How to Help Young Children Become Confident Mathematicians’ in Anghileri, J. (ed.), *Children’s Mathematical Thinking in the Primary Years*, London: Cassell, 1995

computer. It is likely that most of us would launch in, and try to set it up using the experience we have of computers at work or at home (familiarity may even have played an essential part in our selection – we usually stick with what we know). A more laborious and cognitively demanding approach is to read the instructions and try to use and apply the suggested strategies and procedures. Most people do not do this. In the same way, children do not often apply general rules to a particular problem. This scientific way of thinking does not come naturally; it has to be specifically taught. Children need to be presented with specific problems designed to encourage the development of *deductive* reasoning. The way in which children make sense of new experiences has major implications for the types of problems that we routinely use and the kinds of reasoning they provoke.

As teachers, we should be aware that deductive reasoning is important and children should be provided with opportunities to engage in this type of reasoning. However, we must guard against pushing children prematurely towards deductive reasoning. The exclusive emphasis in the early and middle stages of primary school should be on inductive learning processes, which are developmentally more appropriate at these stages. The shift to deductive reasoning will be made for most children in the upper stages of primary school and after the transition into secondary school. Our assessment of children's learning will help us determine whether children are ready to make this shift.

The teacher's role

The teacher's role in developing children's logical abilities may shift from that of teacher to facilitator and supporter. He or she can facilitate children's learning by providing opportunities for children to explore, creating situations to enable discussion and supporting children as they participate in these. It is important to create situations that will enable discussion, since the research is clear that children learn most effectively when they are talking to adults.¹⁵ Discussions with adults help children to make sense of the mathematics they are doing and increase their confidence in their own mathematical ability. Discussions are also of benefit to teachers. They allow them to assess the extent of children's understanding and to use this information to plan and teach subsequent sessions.¹⁶

¹⁵ Atkinson, S. (ed.), *Mathematics with Reason*, London: Hodder and Stoughton, 1992

¹⁶ Whitebread, D., 'Emergent Mathematics or How to Help Young Children Become Confident Mathematicians' in Anghileri, J. (ed.), *Children's Mathematical Thinking in the Primary Years*. London: Cassell, 1995

3.1.1 Types of problems

There is a prevalent view that teachers should use 'real-life' problems. This view is perpetuated by a recent HMI report,¹⁷ which states: 'effective problem-solving programmes include a wide range of types of problem in varied real-life contexts.' Some researchers argue that many of the problems we use are in fact not 'real' to the children. For example, 'Tom has £2 to spend. How many 35p bars of chocolate can he buy?' may reflect children's lives, but children will not necessarily see it as a problem they want to solve.¹⁸

Other 'real-world' problems are so contrived that they become unrealistic. For example, Noelting's orange juice problems:¹⁹

Problem 1: If we mix the following amounts of orange concentrate and water, will the two mixtures taste the same or will one taste stronger (more concentrated)?

4 cups of orange concentrate mixed with 2 cups of water

2 cups of concentrate mixed with 1 cup of water

Problem 2: If we mix the following amounts of orange concentrate and water, will the two mixtures taste the same or will one taste stronger (more concentrated)?

3 cups of orange concentrate mixed with 2 cups of water

2 cups of concentrate mixed with 3 cups of water

Problems of this type have been criticised²⁰ because in real life children (and adults) are unlikely to solve the problem using computation or calculation. They are more likely to taste the juice and vary the concentration until it tastes good!²¹

If we are to provide 'real-life' problems, we should perhaps reflect on the extent to which they are real to the children. For example, projects

¹⁷ HMI, *Standards and Quality in Primary Schools: Mathematics 1998–2001*, Edinburgh: HMI, 2001

¹⁸ Tregaskis, O., 'Real "Real Problem-Solving"' in Atkinson, S. (ed.), *Mathematics with Reason*, London: Hodder and Stoughton, 1992

¹⁹ Wood, D., *How Children Think and Learn*, Oxford: Blackwell Publishers, 2001

²⁰ Nunes, T., and Bryant, P., *Children Doing Mathematics*, Oxford: Blackwell Publishers, 1996

²¹ Wood, D., *How Children Think and Learn*, Oxford: Blackwell Publishers, 2001

that involve children in organising their own school sports day or class party are 'real' in that they are relevant to the children and provide problems that they own.²² When we design and select problems we should remember that what is a problem for one child may not necessarily be problematic for another. Some researchers argue that there should be a balance of 'real-life' and abstract numerical problems. In the first type, 'the outcome is of practical use or relevance' and in the second, 'the outcome is of little apparent practical significance'.²³ The following is an example of an abstract numerical problem:

Put numbers into the matrix so that all rows and columns add up to 5.

(Adapted from Haylock and Cockburn, 1997)

There are two main arguments for providing children with both types of problem.

- The two types represent different ends of the mathematical spectrum. 'Real-life' problems could be considered to represent 'applied' mathematics, whereas the more abstract numerical problems represent 'pure' mathematics. In providing a balance between both types we are promoting mathematical thinking at different levels.

²² Tregaskis, O., 'Real "Real Problem-Solving"' in Atkinson, S. (ed.), *Mathematics with Reason*, London: Hodder and Stoughton, 1992

²³ Haylock, D., and Cockburn, A., *Understanding Mathematics in the Lower Primary Years*, London: Paul Chapman Publishing, 1997

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- As teachers we know how crucial it is to cater for the different needs and interests of children. Whilst some children will be motivated by 'real-life' problems, others will prefer activities that are more abstract. It is therefore unfair to focus on one problem type to the exclusion of the other.

What the researchers say



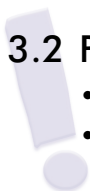
'I do not object to topics involving the application of mathematics to real-life situations being included in the mathematics curricula. The view I wish to challenge is that such applications are essential for understanding and that mathematics curricula should consist predominantly of such applications.'²⁴

Perhaps we should consider shifting the emphasis from supposedly 'real-life' problems. We could work to achieve a balance between these and other, more abstract, numerical problems. This may make it easier for teachers to source or design problems that are more closely matched to the children's ongoing class work.

3.1.1.1 Summary

HMI's recent conclusion²⁵ that problem solving is an area of weakness in 50 per cent of recently inspected mathematics programmes suggests there is a need for change. Effecting such a change demands a shift from the current approach of problem solving as strategy driven, to one of problem solving as a context for learning. For such a change to happen, we need to attend to what we know about how children learn. We would also need to reflect on the types of problem we routinely provide, the kinds of reasoning they provoke, and the extent to which they motivate children to think.

3.2 Points to consider



- To what extent do we use problem solving to develop target concepts?
- To what extent do we provide problem-solving tasks that are related to the ongoing mathematics programme?
- Reflect on an aspect of mathematics you have taught this session. Do an audit of the problem-solving tasks you used. Your audit may include the following headings:

²⁴ Wright, R., 'The Role of Counting in the Numerical Development of Young Children', *The Australian Journal of Early Childhood*, 1994, 16 (2), pp. 43–48

²⁵ HMI, *Standards and Quality in Primary Schools: Mathematics 1998–2001*, Edinburgh: HMI, 2001

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- Name/Description of Problem
 - Source
 - Type of Problem – Abstract (A)/Real-Life (R)
 - Type of Reasoning – Inductive (I)/Deductive (D)

Possible questions:

- Is there a balance between scheme-related problems and problems from other sources?
- Is there a balance between abstract and real-life problems?
- Is there a relationship between types of problem and the sources they come from?

Discuss the source column with your colleagues. Make a list of all sources noted and highlight any which you have not yet used.

What type of reasoning do the problems provided provoke?

- To what extent are the problems you provide based on the nine strategies provided in the *Mathematics: 5–14 National Guidelines*?

3.3 To try:²⁶

- Present problems with a balance of real-life and abstract numerical problems that are meaningful to the class and related to their ongoing work.
- Provide opportunities for children to engage in inductive reasoning; present them with a problem without suggesting possible strategies. Encourage children to develop their own strategies for solving given problems. Encourage children to explain their own ideas.
- Encourage children to represent their thinking in a variety of ways, e.g. verbally and graphically.
- Involve children in a variety of kinds of dialogue that encourage reflection on mathematical processes.

Select some of these points to try as a whole-school exercise.

²⁶ Adapted from Whitebread, D., 'Emergent Mathematics or How to Help Young Children Become Confident Mathematicians' in Anghileri, J. (ed.), *Children's Mathematical Thinking in the Primary Years*, London: Cassell, 1995

4. Conceptual understanding

Conceptual understanding means creating relationships between two or more pieces of existing numerical knowledge or between existing knowledge and new knowledge. Relationships can be between small pieces of information. For instance, you can connect the need to line up decimal points when adding and subtracting to the value of the digits on the right of the point. Alternatively, relationships can be between larger pieces of information which are already networked. For example, you might connect your understanding of how to add decimal fractions to your understanding of how to add common fractions.

However, many pupils have poorly developed conceptual understanding – so much so that they have difficulty in determining which operations to use when faced with problems such as the following.

John is saving up for a new bicycle that costs £120. He has already saved £72 and has a part-time job from which he can save £7 each week. How many complete weeks will it be before he can buy the bike?

Difficulties with reading are often reported as an explanation for poor performance in problems like these. Low-level reading skill and lack of familiarity with the specifically mathematical meaning of some of our language may be part of the explanation. However, significant numbers of pupils cannot make sense of situations that contextualise computations, because they:

- have difficulty with decomposing/recomposing in multi-digit operations
- have a poor grasp of place value when working with decimal fractions
- view common fractions as pairs of whole numbers.^{27, 28}

Pupils' use of very primitive strategies is a further barrier to conceptual understanding. They often fail to choose numbers and operations according to the meaning in a problem. One primitive strategy is to try all the basic operations on the numbers and then choose the answer that seems most reasonable. Another strategy, equally primitive, is to focus on key words such as 'more' or 'difference' to decide whether the operation required is addition or subtraction.²⁹

²⁷ Muth, K., 'Solving Arithmetic Word Problems: Role of reading and computational skills', *Journal of Educational Psychology*, 1984, 76(2), pp. 205–10

²⁸ Siegler, R., *Children's Thinking*, New Jersey: Prentice Hall, 1998

²⁹ Sowder, L., 'Children's Solutions to Story Problems', *Journal of Mathematical Behaviour*, 1988, 7(3), pp. 227–238

These strategies mean that pupils may be 'solving problems' without very much understanding. They can often find success by using procedures that bypass understanding. If pupils are genuinely to engage in problem solving (as the *Mathematics: 5–14 National Guidelines* imply), we must recognise the variations in pupil experiences. Any given mathematical situation may be impossible for one pupil, a routine exercise for a second and a problem for a third. For example, consider the following:

The local photography club had its annual fete last week. Admission to the fete was £3 for adults and £2 for persons under 12. Total attendance was 100 persons and £232 was collected. How many adults and how many under-12s attended the fete?

In terms of computational expertise, this task needs no more than the operations of addition and subtraction. For the pupil who has knowledge of algebra and can solve algebraic equations, the task would be routine. Without algebra, however, pupils have to be more innovative in using strategies such as 'guess and check' or 'make a list'.

4.1 What we know

Conceptual understanding is often contrasted with procedural (or algorithmic) knowledge, and in maths both are equally important. We need to remember procedures, but procedural knowledge is quite limited unless it is connected to a conceptual knowledge base. Pupils need to develop the way their number knowledge is organised. This conceptual organisation lets pupils use their procedural knowledge flexibly.³⁰ We know that conceptual knowledge is an essential aid to procedural knowledge, but we do not know which of these develops first. It is not clear whether procedural knowledge develops before or after conceptual understanding, or whether both develop concurrently.³¹ It is quite possible that there is an interaction between the two. That is, increases in one lead to increases in the other and then further increases in the first.

³⁰ Hiebert, J., *Conceptual and Procedural Knowledge*, New Jersey: Lawrence Erlbaum Associates, 1986

³¹ Rittle-Johnson, B., and Siegler, R. S., 'The Relationship between Conceptual and Procedural Knowledge in Learning Mathematics: A review' in C. Donlan (ed.), *The Development of Mathematical Skills*, Hove, East Sussex: Psychology Press, 1998, ch. 4, pp.75–110

A focus on word problems helps to develop connections between procedural and conceptual knowledge. It is important to help pupils move on from primitive strategies such as trying all the operations (adding, subtracting, multiplying or dividing) and choosing the 'reasonable' answer, or focusing on key words to decide on the operation to use.

4.1.1.1 Summary

If pupils are to develop conceptual understanding, they must link different pieces or networks of mathematical knowledge. They need to engage in learning activities that encourage them to make such linkages. Discussion is an important mechanism through which pupils can make links between different pieces of knowledge.

4.2 Points to consider

- How important is it to you that pupils have conceptual understanding of (as distinct from procedural proficiency in) the mathematics they are expected to learn?
- How well do your beliefs about what is important match with your practices in the classroom?
- In your experience of teaching maths, how do you see the development of the conceptual understanding relationship between procedural and conceptual knowledge of number?
- What is your view on why pupils have difficulties in solving problems and what ways would you suggest of responding to such difficulties?

4.3 To try

Outlined below are some strategies to support discussion.

Teaching Action	Teaching Point
Before pupils begin working on problems, the teacher can:	
<ul style="list-style-type: none">• read the problem to the class or have a pupil read the problem• discuss words or phrases that pupils may not understand• use a whole-class discussion to enable pupils to understand what the problem is• use a whole-class discussion to consider possible solution strategies	<ul style="list-style-type: none">• illustrate the importance of reading problems carefully.• focus on words that have particular mathematical meaning• focus on important data in the problem.• clarify what is unclear• elicit ideas for possible ways to solve the problem
While pupils are working on the problem the teacher can circulate to:	
<ul style="list-style-type: none">• observe and question pupils to determine where they are in the problem-solving process• provide hints as needed• Provide problem extensions	<ul style="list-style-type: none">• diagnose pupils' strengths and weaknesses in problem solving• help pupils to overcome blockages in solving a problem• challenge early finishers to generalise to a similar problem
Finally , the teacher can require/encourage pupils to:	
<ul style="list-style-type: none">• check that a solution 'answers the question'• show, share and discuss solutions• relate the problem to previously solved problems.• have pupils solve extensions of the problem• discuss special features of the problem, such as a picture	<ul style="list-style-type: none">• require pupils to check that their answer makes sense• name and illustrate the different successful strategies• demonstrate that strategies are not problem-specific• draw attention to the influences of missing, extraneous or contradictory information on our thinking about a problem

Select some of these points to try as a whole-school exercise.

5. Mental maths

‘I’ve been teaching for a long time. Mental maths isn’t new; it’s what we used to do years ago before 5–14 came along.’

Primary 3 teacher

The term mental maths is currently in vogue, but can you point to any maths that is not ‘mental’ in its entirety? The whole point of mathematics is that it is done entirely ‘in the mind’, which is what ‘mental’ means. Where else would one do maths? The meaning comes from an implicit contrast. Actually, there are two possible meanings because there are two implicit contrasts. The first is the contrast between written maths and maths done in the head. The second is the contrast between passive mental recall (remembering, recognising) and active mental construction (calculating, puzzling, modelling, visualising). The two meanings are often conflated: we assume that written maths is the product of passive mental recall, and that maths in the head is the product of active generation. Sometimes, however, mental maths can be passive – relying on rote learned facts that have not been well understood. Reliable number constructions tend to be generated rather than passively viewed in the mind. How are teachers supposed to tell the difference in how children have produced the maths? And what are they supposed to do about these differences? Some initial work on the topic has been done in *Thinking Numbers*,³² a document in the same series as this one.

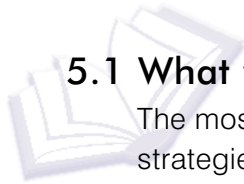
‘Teaching mental maths to the class is good for most of the children but the less able kids can’t keep up. They pretend to be thinking about the answer or try to look at their neighbour’s number fan. I feel I’m letting them down.’

Primary 5 teacher

³² Wilson, G., *Thinking Numbers: A discussion on mental mathematics 5-14*, Dundee: Scottish CCC, 1999



5.1 What we know



The most important aspect of mental maths is the (often hidden) strategies that pupils use for arriving at answers to calculations.

5.1.1 The nature of children's arithmetical strategies

Currently we know quite a lot about the strategies that children develop for mental calculation.³³ In the early stages of learning number, children progress from 'count all' to 'count on' strategies. Children's arithmetical strategies are firmly related to their understanding of number, and cannot be taught as procedures. When children start to progress to a further understanding of number, they develop yet more sophisticated strategies. Ian Thompson³⁴ has described the levels of Year 4/Year 5 children's strategies for two-digit addition and subtraction as follows:

1. counting (counting strategies with no use of number facts)
2. manipulating digits (ignoring quantities represented by digits)
3. partitioning (splitting both numbers and operating separately on multiples of tens and ones)
4. sequencing and compensating (retaining the first number as a whole and adding or subtracting chunks of the second).

Strategies 2 and 3 are directly related to the written algorithms we teach young children. Strategies 1 and 4 form a progression that would extend from P1 to S2. We need to ensure that children are comfortable within one stage before we try to move them to the next. If we accelerate children too quickly through the stages then their understanding will be deficient and the quality of their later learning will be compromised. Thompson's list is a simplification of a complex and sometimes rather untidy development. For instance, there are often bridging stages between these major stages, and children do not always use the most sophisticated strategy available to them. In addition, when asked to describe how they calculated an answer, children do not always report the strategy that they actually used – they sometimes report a strategy that is less sophisticated than the one they actually use. The most sophisticated strategy (4) involves two steps rather than three, and is very efficient. However, children need to

³³ Wright, R., Martland, J., and Stafford, A., *Early Numeracy: Assessment for teaching and intervention*, London: Paul Chapman Publishing, 2000

³⁴ Thompson, I., and Smith, F., *An Investigation of the Mental Calculation Strategies Used by Children in Year 4 and Year 5 for the Addition and Subtraction of Two-digit Numbers: Report to the Nuffield Foundation*, 1999

be taught this strategy explicitly, or they tend not to develop it. In order to teach it effectively, we need to determine whether a child has begun to understand the decade structure of numbers. It is clear from the research that strategies are an integral part of a child's number understanding and that they cannot be taught as procedures in isolation from this understanding. We cannot assume that an S1 child will necessarily have a more sophisticated strategy for double-digit subtraction, say, than a P3 child may have.

5.1.2 The relation between mental strategies and written algorithms

Children's strategies for mental calculation are not always related to the kinds of strategies used for written number work. Written number strategies, or conventional algorithms, are routinely taught as 'the way to do sums'. However, they can conflict with the sort of mental strategies that children may develop to work out sums in their head.³⁵ In particular, the more advanced strategy of 'sequencing' is quite different from the written algorithm. The 'partitioning' strategy fits well with the written algorithm. If children are not to manipulate digits independently of meaning, they need to understand place value before they use this strategy.

The 'empty number line' was devised as an alternative to Dienes blocks or Unifix cubes, which were thought to encourage dependence on concrete materials and passivity in young children's calculating strategies. It is used as a support to develop skill at the sort of strategy that allows children to move from partitioning to sequencing, the most efficient strategy. Written work on the empty number line has the function of recording the strategies that are chosen.³⁶ For instance:



This is a structured empty number line that can be used to teach children how to 'cross ten' without partitioning.



This is a completely empty number line that older children use to record sequencing strategies.

³⁵ See, for example, Thompson I., 'Mental and Written Algorithms: Can the gap be bridged?' in I. Thompson (ed.), *Teaching and Learning Early Number*, Buckingham: Open University Press, 1997.

³⁶ Beishuizen, M., 'The Empty Number Line as a New Model' in I. Thompson (ed.), *Issues in Teaching Numeracy in Primary Schools*, Buckingham: Open University Press, 1999



What the researchers say

Julia Anghileri³⁷ has compared the English and Dutch approaches to mental maths. The comparison is an interesting one because the Dutch are about ten years ahead of the English on this particular road, and have researched the effects of their emphasis on mental maths. They discovered, predictably, that mental recall and strategy use are not isolated from each other; remembering number facts allows children to progress to strategies that are more efficient. They found that merely focusing on a variety of strategies did not improve attainment; they had to make some pedagogical decisions. The hardest decision was to delay the teaching of formal arithmetical algorithms until children had developed the strategies to deal with two-digit addition and subtraction mentally.

The Dutch are ahead of us in the way they have organised the curriculum so that children develop a real understanding of place value. In contrast, British and US curricula have been structured by the notion of written place value as an essential mathematical principle that must be introduced very early. The Dutch have postponed the teaching of written algorithms and adopted the empty number line model up to Year 4. This approach has improved their children's level of attainment. In the initial years of primary school, they put the emphasis exclusively on mental maths. This allows children to develop number concepts and number logic independently of any written algorithms. When children are eventually taught formal written algorithms, they add these to a meaningful framework of number knowledge. Consequently, they tend not to manipulate symbols independently of meaning – a primitive strategy that has been observed many times in the context of British and American curricula.³⁸

³⁷ Anghileri, Julia, and Beishuizen, M., 'Which Mental Strategies in the Early Number Curriculum?: A comparison of British Ideas and Dutch views', *British Educational Research Journal*, 1998

³⁸ See, for example, Hughes, M., *Children and Number*, Oxford: Blackwell, 1986.

5.2 Points to consider

‘My class enjoy mental maths but sometimes I feel I’m working with a ‘one size fits all’ approach. Some children aren’t ready for the strategies I introduce and others have been using them for ages.’

Primary 6 teacher

- Can children be taught to use strategies before they are conceptually ready for them?

‘I don’t know how strict to be with the children when it comes to mental maths. Should I let them jot down their workings, or is ‘real’ mental doing everything in your head?’

Primary 6 teacher

- How can mental maths be integrated with the teaching of written mathematics?

‘I feel I’m ticking boxes to say the children can use a particular strategy. Sometimes their own ways of solving the problem are better, but I don’t have time to explore these.’

Primary 5 teacher

- How can teachers begin to explore the way that children think?

An awareness of these strategies and the understandings and skills that underpin them, combined with the ability to recognise each of them from a child’s explanation, are important prerequisites for any teacher endeavouring to understand an individual’s mathematical thinking. This is an important first step that needs to be backed up by training in observation techniques and ongoing support in this type of formative and diagnostic assessment.

Ian Thompson, 2001

- How can schools support maths teachers in developing appropriate diagnostic skills?

5.3 To try

1. Take one child and discuss with them a calculation appropriate to their level.
 - Can you elicit the kind of explanations of strategies that Thompson describes?
 - Does this child produce a range of strategies?
 - What is the most sophisticated strategy that this child uses?
2. Invite a group of children to explain their addition and subtraction (four-, three-, two- or one-digit depending on their age).
 - Can you identify any strategies from the framework outlined in 5.1.1?
 - What range of strategies do you have in your classroom?
3. Invite discussion during classroom mental maths sessions.
 - Do the children with less advanced strategies benefit from listening to this discussion?
 - Do the children with more advanced strategies benefit from explaining themselves?
4. Invite explicit discussion of strategies (rather than answers).
 - Does the focus on 'how' rather than on the 'right answer' reduce tension?
 - Do the children find it easy to describe and discuss their strategies?

Select some of these points to try as a whole-school exercise.

6. Commercial schemes

Part of the allure of commercial schemes may be their broad coverage. They make planning easier because they provide a framework of development. They also provide a wide range of materials and activities from booklets. These can be used for both class work and homework. Commercial schemes should ideally be offering teachers a bank of activities and ideas, from which they can select the most appropriate for their children and their situation. However, it seems that these schemes can put teachers under pressure to complete every section.

‘I feel under a lot of pressure to get through the pages of the workbooks before the children move class in August.’

Primary 1 teacher

‘There’s just too much in the scheme. I’m finding it a real struggle to push my class through.’

Primary 1 teacher

The pressure to complete extends to other scheme-related tasks such as homework. Teachers feel under pressure to use homework exercises linked to the commercial scheme, as these are often part of the school’s maths programme. Deviating from the scheme-related exercises and accessing a greater variety of tasks would place demands on their time.

Teachers are also constrained by limitations on resources and on their time.

‘We stick to the homework suggested in our forward plans. It’s just another book to get through.’

Primary 4 teacher

‘When I think of homework, I think about how long it takes to mark and give feedback to each child.’

Primary 6 teacher

‘I would love to make up games to give as homework. I just don’t have time.’

Primary 4 teacher

The reliance on mathematical schemes can perhaps be attributed to the constraints and pressures under which teachers work. Teachers are under pressure from within the school and from the local authority to meet set targets. They are also under pressure from parents, senior management, and Inspectors to provide evidence of work completed. Teachers in the twenty-first century face the challenge of moving from a mathematics curriculum that is 'scheme-dominated' to one that might be described as 'scheme as resource'.³⁹ For many, this will require a major shift if the teacher, not the published scheme, is to be in control.

'Parents expect every page to be done when they come up at parents' night or when books are sent home.'

Primary 1 teacher

'If the children do the workbook, it's there as evidence for anyone who wants to see it.'

Primary 2 teacher

6.1 What we know

'There is evidence that many primary teachers still experience feelings of panic and anxiety when faced with unfamiliar mathematical tasks,⁴⁰ that they are muddled in their thinking about many of the basic mathematical concepts which underpin the material they teach to children, and that they are all too aware of their personal inadequacies in mathematics.'⁴¹

Ninety per cent of Scottish primary schools base their mathematics curriculum on a leading mathematics scheme. Furthermore, a large proportion of secondary schools continue to use this same scheme in S1 and S2. The widespread use of the scheme could be considered beneficial for a number of reasons. It provides compatibility with other schools. So, if a child moves from one school to another, and they both use the same scheme, there should be little difficulty in progression. Within schools, the use of the scheme should promote continuity through the stages. The scheme should make the transition from one stage to the next easier for children and should make communication

³⁹ Hopkins, C., Gifford, S. and Pepperell, S., *Mathematics in the Primary School: A sense of progression*, London: David Fulton Publishers, 1998

⁴⁰ Briggs, M., 'Bags and Baggage Revisited', *Mathematics Education Review*, 1993, vol. 2, pp. 16–20

⁴¹ Haylock, D., and Cockburn, A., *Understanding Early Years Mathematics*, London: Paul Chapman Publishing, 1989

among teachers more straightforward. This also applies to the move from primary to secondary, where use of the same scheme will ease the transition. There may be another reason why commercial schemes are so widely used. This is simply that many teachers are deferring to the scheme authors, who are perceived as experts. Many primary teachers feel anxious about mathematics and are not confident in their subject knowledge. For those teachers especially, the commercial scheme provides reassurance that the mathematics they teach is at least age and stage appropriate.

'It is not uncommon for mathematics to be treated very differently from other aspects of the curriculum. The learning experience of children can, in some cases, consist almost exclusively of working through a series of textbooks. This approach would horrify if applied across the curriculum...yet it is considered normal for mathematics.'⁴²

In one study,⁴³ 80 per cent of the children's time was spent either working on paper and pencil exercises or being carefully prepared to work on such exercises.

If teachers are to take control, they have to be equipped with a sound knowledge of the mathematics curriculum. They also need a good understanding of the important targets and activities within that curriculum. Armed with this knowledge, the teachers can make informed decisions about the types of tasks to be provided and the range of teaching approaches to be employed. A recent HMI report notes that the quality of learning and teaching was very good where a balance of practical, oral and written tasks was provided. Quality was also very good where teachers used an 'appropriate blend of whole class, group teaching and individual work.'⁴⁴

The place of the published scheme is most important. It is crucial that it is 'seen as just one part of the children's experience rather than most of it'.⁴⁵ Taking control of the mathematics curriculum means controlling our use of the scheme. This means selectively drawing on the detailed

⁴² Hopkins, C., Gifford, S. and Pepperell, S., *Mathematics in the Primary School: A sense of progression*, London: David Fulton Publishers, 1998

⁴³ Desforges, C., and Cockburn, A., *Understanding the Mathematics Teacher*, Lewes: The Falmer Press, 1987

⁴⁴ HMI, *Standards and Quality in Primary Schools: Mathematics 1998–2001*, Edinburgh: HMI, 2001

⁴⁵ Hopkins, C., Gifford, S. and Pepperell, S., *Mathematics in the Primary School: A sense of progression*, London: David Fulton Publishers, 1998

activities provided by the scheme authors and using these alongside a variety of teacher-prepared resources. It also means using ideas from other sources to create stimulating activities that will support children's learning and sustain their interest. More discriminatory use of the scheme will mean children are not slavishly completing endless workbook pages and '... practising skills they have already mastered fully'.⁴⁶ Rather, children will experience a broader and more balanced mathematics programme.

In order to bring about change teachers will need:

- a broad overview of the mathematics curriculum and existing programmes within their school
- partnership with colleagues in order to discuss the range of approaches to teaching and learning mathematics and to share resources
- support from headteachers to inform parents about the 'scheme-as-resource' curriculum
- support from headteachers and local authorities to shift the emphasis of forward-planning frameworks from 'scheme-dominated' to 'scheme as resource'.

6.2 Points to consider

To what extent could your mathematics programme be described as 'scheme-dominated'?

Focus on an aspect of mathematics that you have taught this session.

- How many of the children's tasks were scheme-based?
- How does this compare to tasks which involved teacher-prepared resources and/or ideas from other sources?
- Were there any parts of the scheme that you felt under pressure to complete solely as a source of evidence?

Focus on an aspect of mathematics you will teach next session.

- Consider how you might teach this aspect of mathematics using only the teacher's guide from the commercial scheme as a support.

How might you inform parents about, and persuade them of, the scheme-as-resource curriculum?

⁴⁶ HMI, *Standards and Quality in Primary Schools: Mathematics 1998–2001*, Edinburgh: HMI, 2001

6.3 To try

Consider planning as a whole school to have a day that is 'scheme-free'. Discuss the implications of this with your colleagues. What preparation will be necessary in terms of resource provision, etc? Could these days be extended if they proved a success?

7. Assessment

There is no standard usage of the term 'assessment'. It is used in different ways, in different contexts and for different purposes. It can mean almost anything. In this document, we use the term to refer to the full range of information gathered in classrooms to help understand pupils and monitor teaching. This meaning of assessment is the same as the term as used in the Assessment is for Learning programme.⁴⁷ This programme aims to provide a streamlined and coherent system of assessment. It also aims to ensure that parents, teachers and other professionals have the feedback they need on pupils' learning and development needs.

Assessment is a powerful tool to promote effective learning. However, it has to be used in particular ways if it is to promote learning. For example, there is no evidence that increasing the amount of assessment will enhance learning – although it might satisfy demands for accountability. Rather, we need to use assessment as part of teaching and learning to enable pupils to learn and to achieve. Our concern is therefore with formative rather than summative assessment.

Essentially, formative assessment is interactive since it underscores the processes of thinking and learning. It is an active procedure. Both pupil and teacher engage in dialogue to explore the extent to which the pupil's performance can be enhanced by effort or intervention. Formative assessment is embedded in instructional episodes. Pupils learn from exposure to assessment. There are also situations in which assessment constitutes the entirety of the instructional episode. This is quite different from summative assessment. Summative assessment is grounded in pupils' instructional experiences but it provides no instructional value.

It is easy to become confused about what might constitute appropriate assessment practice in the mathematics curriculum. This is partly because the term 'formative assessment' is open to a variety of interpretations. It is also partly because of the historical dominance of summative assessment,

⁴⁷ Information is available on the Assessment is for Learning website at www.LTScotland.com/assess



7.1 What we know

Formative assessment includes:

- the provision of effective feedback
- the active involvement of pupils in their own learning
- adjusting teaching to take account of the results of assessment
- a recognition of the profound influence assessment has on the motivation and self-esteem of pupils, both of which are crucial influences on learning
- the need for pupils to be able to assess themselves and understand how to improve.

Recent research has identified a number of assessment practices that inhibit the promotion of effective learning. There is a tendency to:

- assess the quantity and presentation of work rather than the quality of learning
- give greater attention to marking and grading (which can lower self-esteem) rather than to providing advice for improvement
- compare pupils with each other (which can demoralise the less successful learners)
- use feedback for the management of social behaviour rather than the management of learning
- be insufficiently aware of pupils' learning needs.⁴⁸

Engaging in practices such as those listed above may contribute to the summative assessment of a pupil. They are not the focus of formative assessment.

You may find that the following model helps to clarify the different stages of the assessment process. Formative assessment is not mechanistic. There are three phases of teacher activity:

1. Posing specific assessment questions

- What is the specific purpose of the assessment (for example, is it to gather information about particular achievements or to diagnose strengths and weaknesses in some particular aspect of mathematics)?
- Will the assessment task(s) yield the necessary information clearly and unambiguously?
- Is the desired information already being gathered in some other way?

⁴⁸ Black, P., and Wiliam, D., 'Assessment and Classroom Learning', *Assessment in Education*, 1998, 5(1), pp. 7–74



2. Designing ways of answering the questions

- What tasks/methods can be used to obtain the required information?
- Is there a need for alternative or supplementary tasks?
- How likely is it that these tasks/methods provide adequate and accurate information?
- How good is the quality of information gathered? Is this information sufficient to determine next steps in mathematical learning?
- Is there additional information that might enhance the assessment?

3. Speculating on what the findings might mean

- How should this assessment information be interpreted?
- What are some alternative interpretations of the information? How plausible are these interpretations?
- What other questions are there about the achievements of particular pupils?

By working systematically through the three phases and reflecting on our answers to the questions in each phase, we can identify what needs to be done. For example, if the desired information is already available, the need to continue with a particular assessment might be redundant. The second of these three phases – the design phase – may seem to be the most important because of its immediate and practical nature. However, it is no more important than reflecting on the reason for the assessment, the validity of the methods and the meaning of the results.

7.2 Points to consider

- Given the importance currently placed on teacher accountability, how constrained do you feel by the requirements of summative assessment? What does this mean for your practice in teaching maths? What does this mean for your practice in assessing maths?
- Think particularly about one pupil's mathematical performance. In giving feedback, what might you say to focus on the particular qualities of the pupil's work and to advise on how to improve?
- What pupil self-assessment do you currently use in the mathematics curriculum? How might you extend this to enable pupils to appreciate what they need to do to improve?
- What opportunities do you design into your teaching so that pupils can express their understanding of the mathematical ideas? How do you respond to the pupils during these opportunities?

7.3 To try

There are several well-known methods for gathering assessment information.

- Make observations. If these are done systematically, they can allow us to make inferences about the status of mathematical learning and achievement.
- Use open-ended questions. These allow pupils to describe their reasoning and their use of strategies.
- Set tasks that require the use of certain skills or the application of particular ideas.
- Ask pupils to represent their thinking through drawings, concept maps and artefacts, as well as with mathematical symbols. This shows the extent of their understanding as distinct from whether they get the right answer.
- Discuss words and their specialised mathematical meanings. This teases out conceptions and misconceptions.

It is important to use the information to improve learning. Any of the above methods could yield information on which to build success and correct weakness. The procedure below can help to diagnose what a pupil is doing when tackling word problems.

Procedure	Purpose
1. Read the question to me. If you do not know a word, leave it out.	To identify reading errors
2. Tell me what the question is asking you to do.	To identify errors in comprehension
3. Tell me how you are going to find the answer.	To identify transformation errors
4. Show me what to do to get the answer. Tell me what you are going to do as you work.	To identify errors in the use of procedures
5. Write down the answer.	To identify encoding errors

Select some of these points to try as a whole-school exercise.

8. Reflective review

8.1 Summary

We have reviewed problem solving, conceptual learning, mental maths, commercial schemes and assessment. In each of these sections, we have derived a major conclusion from the research – both the international academic research and the local research that assesses the progress of Scottish children’s learning. These conclusions are as follows.

1. Due to the inductive nature of children’s learning, problem solving is a context for learning rather than a procedure to be learnt. In most schools, it is framed as a set of procedures to be learnt, and this is not helpful for teachers. It is particularly unhelpful at the lower end of the primary school.
2. Procedural knowledge is quite limited unless it is connected to a conceptual knowledge base. Pupils often use very primitive strategies when they tackle contextualised computations. They often fail to choose numbers and operations according to the meaning in a problem. Using word problems as a context for learning can help to develop connections between procedural and conceptual knowledge. It can also help pupils move on from primitive strategies.
3. Children’s understanding of number develops from a perception of single units to mentally segmenting by tens, then hundreds, then thousands. The strategies they use in mental maths are an integral part of their number understanding and cannot be taught as procedures in isolation from this understanding. Written number strategies, or conventional algorithms, can conflict with the sort of mental strategies that children may develop to work out sums in their head. Much progress has been made in European countries by introducing written algorithms in the fourth year instead of in the second year.
4. Commercial schemes can be like corsets. When they are first used, they support and strengthen the back. However, over-use leads to atrophy of the muscles due to lack of natural exercise. In time, the corset can come to restrict breathing and digestion. Commercial schemes can offer external markers of ‘progress’, and the authority of the authors is reassuring for anxious teachers. We need to exercise professional judgement. Our own pupils’ understanding

should be the measure of progress, not the performance of the 'average pupil' at whom the scheme will be aimed.

5. Assessment is a powerful tool for promoting effective learning when it is an integral part of teaching. This is the underpinning rationale of the Assessment is for Learning programme. Care must be taken that increases in required assessments do not have a negative effect on learning and teaching. A number of assessment practices can actually inhibit effective learning. Teachers need support in using assessment as a tool to promote understanding.

Two important and inter-related ideas have recurred in our discussion of these five issues. The first idea is that basic mathematical concepts are learned intuitively. The second is that school maths should be learned with understanding.

8.2 Intuitive learning

Basic numeracy – the ability to count, add, subtract and solve practical problems with numbers – develops intuitively (that is, it is learned inductively). Number systems are part of the wider culture that children begin to absorb before they enter school. If children come to school with varying degrees of informal number knowledge, then formal instruction should allow them to build on this. For example, if counting is the only calculation strategy that a child understands, then more sophisticated strategies will not replace it when the teacher demonstrates them. Of course, this does not mean that pupils should not be challenged to make progress. Teachers have the job of knowing just how to challenge their class so that real progress is made. We should not reinforce intuitive thinking to the extent that children do not progress. We are in the business of promoting sophisticated mathematical learning. Without schooling, children will not achieve scientific and mathematical literacy. However, we must recognise the power of each developmental stage. Children learn spontaneously through induction – they experience something and then generalise from this experience to the world at large. Deduction – learning a general rule and then applying the rule to particular cases – is a more powerful way of thinking. It is fundamental to science and mathematics, but it is a way of thinking that has to be specifically taught. It does not come naturally. Logic (like reading the instructions) is a last resort, not an instinctively applied tool. However, you cannot teach deductive thinking in a didactic manner, as you would teach a topic. The practised teacher will encourage the development of

deductive thinking by presenting pupils with suitable problems that *cannot* be solved in any other way.

8.3 Learning with understanding

We understand something if we see how it is related or connected to other things that we know. Understanding is the other side of the coin of the inductive learning we outlined above. We understand things that we learn inductively. Deduction is useful for prediction, certainty, precision and reliability. However, knowledge gained deductively is often counter-intuitive; it doesn't 'feel' right, even though we know it is. Deductive thinking also develops quite late. Because of this schism between two types of knowing, it is best if children leave primary school able to think deductively but confidently able to learn inductively. They will understand content that they have learnt inductively and will feel satisfaction in, and ownership of, such knowledge.

8.4 Understanding through reflecting and communicating

Reflection is what happens when we consciously think about our experiences. It means turning ideas over in our mind, thinking about things from different points of view, stepping back to look at things again, and consciously thinking about what we are doing. Communicating can involve talking, listening, writing, demonstrating, or watching. It means interacting with and sharing ideas with others. It allows us to challenge each other's ideas and to ask for clarification and explanation. It also allows us to think more deeply about our own ideas. Communication works together with reflection to produce new connections between different bits of knowledge. Pupils who reflect on what they do and communicate with others about it are in the best position to understand the maths they are learning. Learning computational skills and developing conceptual understanding are often seen as competing aims. We believe that this is wrong. It is not necessary to sacrifice skills for understanding, nor understanding for skills. They should develop together. For this to happen, procedural skills must be learned with understanding. Pupils who are asked to work out their own ways of calculating answers to arithmetic problems and to share these with others will be reflecting and communicating. The process of developing problem-solving procedures, rather than copying the textbook procedure, will help children to focus on mathematical meaning rather than on the solution.

9. Conclusion

Recent changes in Scottish maths teaching may appear to follow international trends, but they have not been accompanied at a national level, with resources for schools and teachers that would help to ensure a genuine, that long-lasting shift in emphasis in the teaching of maths. It may be that raised attainment in maths in the early primary school is being bought at the expense of deeper mathematical understanding in the later years of primary and early secondary school. The nature and amount of summative assessments required of some schools and teachers can impact adversely on the quality of teaching and learning. Ongoing dependence on maths schemes tend to supplant teachers' understanding of maths development. The recent emphasis on mental maths is unlikely to produce increased understanding unless the conflict with written algorithms in the curriculum is solved. The low levels of attainment in problem solving are unlikely to increase until there is a shift in perceptions. We need recognition that problem solving is a means to learning and not a procedure to be taught.

There have been considerable changes in the teaching of maths, and the content of such changes has often produced conflict for teachers. Demands for more assessments can produce an 'industrial' approach to curriculum design and delivery. The requirement for pupils to demonstrate achievement encourages schools to use pre-designed commercial materials that can de-skill teachers. When teachers no longer plan and control their work, important skills can be lost. These skills include determining relevant goals, establishing content, designing lessons, determining teaching strategies, and customising the curriculum for specific educational needs. Teachers' professional judgement is rooted in their knowledge of children's mental development and in their knowledge of what it means to learn mathematics. Under circumstances of change, it is very important that teachers have autonomy and professional power. If they do then they can use their knowledge to determine the most appropriate changes for maths teaching in their schools.

10. Annotated Bibliography

10.1.1 Background

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This document gives a clear description of the National Numeracy Strategy in England and Wales.

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This paper gives a critical and informed perspective on the National Numeracy Strategy.

MacNab, Donald, 'Raising Standards in Mathematics Education', *Educational Studies in Mathematics*, 2000, 42, pp. 61–80.

This paper discusses international changes in mathematics education from a Scottish perspective.

National Academy of Sciences, *The Third International Mathematics and Science Study*, National Research Council, Washington, DC: Mathematical Sciences Education Board, 1996.

TIMMS is a landmark study of maths and science education in more than 40 countries. This paper outlines what international studies such as TIMMS can tell us, how the study was conducted, and what information we might be able to glean from it. The authors discuss the notion of intended and implemented curricula, and list a number of related questions that TIMMS may answer.

10.1.2 Problem solving

Whitebread, D., 'Emergent Mathematics or How to Help Young Children Become Confident Mathematicians' in Anghileri, J. (ed.), *Children's Mathematical Thinking in the Primary Years*, London: Cassell, 1995.

This chapter considers the way children learn and the implications of this for early mathematics teaching.

10.1.3 Conceptual understanding

Siegler, Robert, 'The Development of Academic Skills' in *Children's Thinking*, New Jersey: Prentice Hall, 1998, ch. 9, pp. 285–297.

This chapter gives a succinct account of the development of children's learning in arithmetic skill, complex arithmetic algebra and computer programming. Siegler also recommends further reading.

10.1.4 Mental maths

1. Thompson, I., and Smith, F., *An Investigation of the Mental Calculation Strategies Used by Children in Year 4 and Year 5 for the Addition and Subtraction of Two-digit Numbers: Report to Nuffield Foundation*, 1999.

This paper describes the type of strategies that 144 children from Year 4 and Year 5 used for mental calculation.

2. Anghileri, Julia, and Beishuizen, M., 'Which Mental Strategies in the Early Number Curriculum? A comparison of British ideas and Dutch views', *British Educational Research Journal*, 1998.

This paper describes the Dutch 'empty number line' in the context of the development of informal mental strategies and the debate over teaching place value.

10.1.5 Commercial schemes

Hopkins, C., Gifford, S., and Pepperell, S., *Mathematics in the Primary School: A sense of progression*, London: David Fulton, 1998.

This book challenges the view of a mathematics curriculum that is 'scheme-dominated' and proposes a move to a curriculum that might be described as 'scheme as resource'.

10.1.6 Assessment

1. Black, P., and Wiliam, D., 'Assessment and Classroom Learning', *Assessment in Education*, 1998, 5(1), pp. 7–74.

This paper lays the foundations for many of the ideas that are promoted in the Assessment is for Learning programme.

2. Black, P., and Wiliam, D., *Inside the Black Box*, London: King's College London, 1998.

A summarised version of the ideas in the previous reference are to be found here.

3. Assessment Reform Group, *Assessment for Learning*, Cambridge: University of Cambridge School of Education, 1999.

This pamphlet is a follow-up to reference 2 above.

