Fractional Calculus of Periodic Distributions

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Abstract

Two approaches for defining fractional derivatives of periodic distributions are presented. The first is a distributional version of the Weyl fractional derivative in which a derivative of arbitrary order of a periodic distribution is defined via Fourier series. The second is based on the Grünwald-Letnikov formula for defining a fractional derivative as a limit of a fractional difference quotient. The equivalence of the two approaches is established and an application to a fractional diffusion equation posed in a space of periodic distributions is also discussed.

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1 Introduction

It is generally regarded that the modern theory of fractional calculus began with the work of Riemann and Liouville. Motivated by the fact that $D^n(e^{ax}) = a^n e^{ax}$, $n = 0, 1, 2, \dots$, Liouville, in 1832, defined the fractional derivative $D^\alpha \phi$ of the function

$$\phi(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$$

by

$$(D^\alpha \phi)(x) = \sum_{k=0}^{\infty} c_k a_k^\alpha e^{a_k x}, \quad (1.1)$$

provided the series converges. Other pioneering ideas due to Liouville which influenced later work on fractional calculus include the fractional integration formula

$$(D^{-\alpha}\phi)(x) = \frac{1}{(-1)^\alpha \Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \phi(t) dt, \quad x \in \mathbb{R}, \alpha > 0, \quad (1.2)$$

and also the notion of a fractional derivative as a limit of a fractional difference quotient. In 1847 Riemann produced an alternative formula to (1.2) for a fractional integral of order α , namely

$$(L^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi(t) dt, \quad x > 0, \alpha > 0. \quad (1.3)$$

In modern terminology, (1.3) is referred to as the Riemann-Liouville fractional integral of order α , while (1.2) (with the factor $(-1)^\alpha$ omitted) is the Weyl fractional integral of order α .

Later, Grünwald (1867) and Letnikov (1868) pursued Liouville's earlier idea of obtaining a theory of fractional differentiation via the limit of some fractional difference quotient. Both used the formula

$$(D^\alpha\phi)(x) = \lim_{t \rightarrow 0} \frac{(\Delta_t^\alpha\phi)(x)}{t^\alpha}, \quad (1.4)$$

where Δ_t^α is a fractional difference operator defined by

$$(\Delta_t^\alpha\phi)(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \phi(x - tj), \quad \alpha > 0. \quad (1.5)$$

Liouville's first formula, (1.1), can also be regarded as the forerunner of a theory of fractional calculus introduced by Weyl in 1917. In this case, the α^{th} fractional integral and α^{th} fractional derivative of a 2π -periodic function ϕ are defined, respectively, by

$$(D^{-\alpha}\phi)(x) = \sum_{l=-\infty}^{\infty} (il)^{-\alpha} \hat{\phi}(l) e^{ilx}, \quad (D^\alpha\phi)(x) = \sum_{l=-\infty}^{\infty} (il)^\alpha \hat{\phi}(l) e^{ilx}, \quad (1.6)$$

where $\alpha > 0$ and

$$\hat{\phi}(l) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ilt} \phi(t) dt, \quad l = \pm 1, \pm 2, \dots, \quad \hat{\phi}(0) = 0.$$

This use of Fourier series, or, equivalently, the finite (or discrete) Fourier transform can be regarded as the most natural way of defining fractional integrals and derivatives of a periodic function ϕ . Moreover, if we consider formula (1.3) for the case when ϕ is a 2π -periodic function, and proceed formally, then we obtain

$$L^\alpha\phi \sim \sum_{l=-\infty}^{\infty} \hat{\phi}(l) L^\alpha\psi_l, \quad (1.7)$$

where

$$\psi_l(x) = e^{ilx}. \quad (1.8)$$

From results in [2, pp.420-427], when $0 < \alpha < 1$ and the lower limit of integration in the definition of $L^\alpha\phi$ is replaced by $-\infty$, it can be shown that

$$L^\alpha\psi_l = (il)^{-\alpha}\psi_l, \quad \forall l \in \mathbb{Z} : l \neq 0. \quad (1.9)$$

Hence, if we define

$$\widehat{L^\alpha\phi}(0) = 0,$$

then it follows that

$$L^\alpha \phi \sim \sum_{l \neq 0} (il)^{-\alpha} \hat{\phi}(l) \psi_l, \quad (1.10)$$

showing that the definition of $D^{-\alpha} \phi$ given in (1.6) coincides with $L^\alpha \phi$ for $0 < \alpha < 1$. Thus the formulae in (1.6) emerge as the natural candidates for defining fractional derivatives and integrals of periodic functions for any $\alpha > 0$.

A rigorous treatment of the Weyl approach to fractional calculus, based on (1.6), can be carried out in the Banach spaces

$$L_{2\pi}^p := \{ \phi \text{ is } 2\pi\text{-periodic on } \mathbb{R} \text{ and } \|\phi\|_{L_{2\pi}^p} < \infty \} \quad (1.11)$$

where

$$\|\phi\|_{L_{2\pi}^p} := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\phi(x)|^p dx \right\}^{1/p}, \quad 1 \leq p < \infty. \quad (1.12)$$

The factor $1/2\pi$ is convenient for Fourier analysis. As described in [3], formula (1.10) can be used to motivate an alternative definition of a fractional integral, $I^\alpha \phi$, expressed as a convolution integral, that will make sense for any $\alpha > 0$ and $\phi \in L_{2\pi}^p$. The key to obtaining a formula for I^α is to rewrite the right-hand side of (1.10) as

$$\sum_{l=-\infty}^{\infty} \widehat{\eta}_\alpha(l) \hat{\phi}(l) \psi_l \quad (1.13)$$

where

$$\widehat{\eta}_\alpha(l) = \begin{cases} (il)^{-\alpha} & \text{when } l \neq 0, \\ 0 & \text{when } l = 0. \end{cases}$$

The corresponding function η_α which has the above sequence $\{\widehat{\eta}_\alpha\}$ as its Fourier coefficients is given by

$$\eta_\alpha(x) := \sum_{l \neq 0} (il)^{-\alpha} e^{ilx}. \quad (1.14)$$

Since $\eta_\alpha \in L_{2\pi}^1$ for all $\alpha > 0$, it can be used as the kernel for an associated convolution integral operator, I^α , defined by

$$(I^\alpha \phi)(x) = (\eta_\alpha * \phi)(x) = \frac{1}{2\pi} \int_0^{2\pi} \eta_\alpha(x-u) \phi(u) du. \quad (1.15)$$

It follows from [2, pp.8-10] that I^α is a bounded linear operator on $L_{2\pi}^p$ for all $p \in [1, \infty)$ with

$$\|I^\alpha \phi\|_{L_{2\pi}^p} \leq \|\eta_\alpha\|_{L_{2\pi}^1} \|\phi\|_{L_{2\pi}^p} \quad \forall \phi \in L_{2\pi}^p.$$

Moreover, from the convolution theorem for periodic functions [2, Theorem 4.1.3], we obtain

$$\begin{aligned} \widehat{(\eta_\alpha * \phi)}(l) &= \widehat{\eta}_\alpha(l) \hat{\phi}(l) \quad \forall l \in \mathbb{Z} \\ &= \begin{cases} (il)^{-\alpha} \hat{\phi}(l) & \text{when } l \neq 0, \\ 0 & \text{when } l = 0. \end{cases} \end{aligned} \quad (1.16)$$

On comparing this with (1.6) and (1.10), it follows that I^α can justifiably be regarded as a fractional integral operator on periodic functions and can be defined by

$$(I^\alpha \phi)(x) = \sum_{l \neq 0} (il)^{-\alpha} \hat{\phi}(l) \psi_l. \quad (1.17)$$

Although $I^\alpha \phi$ is well defined as a function in $L_{2\pi}^p$ for all $\phi \in L_{2\pi}^p$, the same cannot be said for the Weyl fractional derivative of order α , given by

$$D^\alpha \phi \sim \sum_{l=-\infty}^{\infty} (il)^\alpha \hat{\phi}(l) \psi_l, \quad \alpha > 0, \quad (1.18)$$

where $(il)^\alpha$ is taken to be 0 when $l = 0$. For example, in the particular case $p = 2$, when we have the Hilbert space $L_{2\pi}^2$, Parseval's equation can be used to show that the maximal domain for the fractional differential operator in (1.18) is

$$\text{Dom}(D^\alpha) := \{\phi \in L_{2\pi}^2 : D^\alpha \phi \in L_{2\pi}^2\} = \left\{ \phi \in L_{2\pi}^2 : \sum_{l=-\infty}^{\infty} l^{2\alpha} |\hat{\phi}(l)|^2 < \infty \right\}. \quad (1.19)$$

As we shall establish later, one way of extending the class of functions on which D^α is defined is to work in a distributional setting.

The Banach spaces $L_{2\pi}^p$ were also used by Butzer and Westphal [3] in their investigations into the difference quotient approach to fractional derivatives, where a (strong) Liouville-Grünwald fractional derivative $D^{<\alpha>}$ of order α is defined by

$$D^{<\alpha>} \phi := s - \lim_{t \rightarrow 0^+} \frac{\Delta_t^\alpha \phi}{t^\alpha}, \quad (1.20)$$

provided that the limit exists in $L_{2\pi}^p$. One of the main results obtained in [3] is the following theorem.

Theorem 1.1 *The following statements are equivalent for $\phi \in L_{2\pi}^p, 1 \leq p < \infty$, and $\alpha > 0$.*

- (i) $D^{<\alpha>} \phi \in L_{2\pi}^p$.
- (ii) There exists $\psi \in L_{2\pi}^p$ such that $\hat{\psi}(l) = (il)^\alpha \hat{\phi}(l)$, $l \in \mathbb{Z}$.
- (iii) There exists $\psi \in L_{2\pi}^p$ such that $I^\alpha \psi = \phi - \hat{\phi}(0)$ almost everywhere, where I^α is defined by (1.15).

If (i) is satisfied, then $D^{<\alpha>} \phi = \psi$, where $\psi \in L_{2\pi}^p$ is the function defined in (ii). Similarly if (ii) is satisfied then $D^{<\alpha>} \phi = \psi$.

Proof: See [3, Theorem 4.1]. □

Our main aim in this paper is to extend the work of Butzer and Westphal in [3] to the case of periodic distributions. Distributional versions of the Riemann-Liouville and Weyl fractional integrals (and the related Erdélyi-Kober operators) have been produced by Erdélyi [4] and McBride [7]. However, it appears that no attempt has been made to define fractional derivatives of distributions via fractional difference quotients. Here we adopt a strategy that we applied successfully in [6] to produce a distributional theory of the fractional Fourier transform on the space of tempered distributions. Hence we begin with an appropriate symmetric differential operator T with a complete orthonormal system of smooth eigenfunctions $\{\psi_j\}_{j \in J}$ in a Hilbert space H . We then follow the theory developed by Zemanian in [11, Chapter 9] to construct a space $\mathcal{A}'_{2\pi}$ of generalised functions in which all elements can be represented by Fourier expansions in terms of $\{\psi_j\}_{j \in \mathbb{Z}}$, with the series converging in the weak* topology.

In Section 2, we introduce the spaces of test-functions $\mathcal{A}_{2\pi}$ and distributions $\mathcal{A}'_{2\pi}$ and demonstrate that $\mathcal{A}'_{2\pi}$ can be identified with the space $P'_{2\pi}$ of π -periodic distributions defined in [10, Chapter 11].

Sections 3 and 4 are concerned with the the Fourier series-based approach and difference quotient approach to fractional calculus on $\mathcal{A}_{2\pi}$ and $\mathcal{A}'_{2\pi}$. By modifying the arguments presented in [3], we establish that both approaches lead to equivalent definitions of the α^{th} fractional derivative of a 2π -periodic distribution f . Finally, in Section 5, we consider fractional diffusion equations involving distributional initial conditions.

2 Spaces of Periodic Test-Functions and Distributions

In [11, Chapter 9], Zemanian discusses the convergence of orthonormal series of eigenfunctions in a complete multinormed space $\mathcal{A}$, constructed around a symmetric differential operator T in the Hilbert space $L^2(I)$, where $I \subseteq \mathbb{R}$ is an interval. We consider the case when

$$I = (0, 2\pi), T = -iD, \psi_l(x) = e^{ilx}, l \in \mathbb{Z}, \text{ and } \lambda_l = l, \quad (2.1)$$

given in [11, Example 9.2-1]. Note that in [11, Example 9.2-1], a normalising factor of $1/\sqrt{2\pi}$ appears in the definition of ψ_l . This is unnecessary here as we choose to work with a weighted version, $L^2_{2\pi}(I)$, of the usual $L^2(I)$ space in which the inner product and norm are given by

$$(\phi, \psi)_{L^2_{2\pi}(I)} := \frac{1}{2\pi} \int_0^{2\pi} \phi(x) \overline{\psi(x)} dx, \quad \|\phi\|_{L^2_{2\pi}(I)} := \left[\frac{1}{2\pi} \int_0^{2\pi} |\phi(x)|^2 dx \right]^{1/2}. \quad (2.2)$$

Note that

$$(\phi, \psi)_{L^2_{2\pi}(I)} = (\phi, \psi)_{L^2_{2\pi}} \text{ for all periodic functions } \phi, \psi \in L^2_{2\pi}.$$

The domain, $D(T)$, of the operator T is given by

$$D(T) := \left\{ \phi \in C^\infty(I) : T^k \phi \in L^2_{2\pi}(I), (T^k \phi, \psi_l)_{L^2_{2\pi}(I)} = (\phi, T^k \psi_l)_{L^2_{2\pi}(I)}, \forall k, l = 0, 1, \dots \right\}. \quad (2.3)$$

A Fréchet space, $\mathcal{A}$, is obtained from $D(T)$ by imposing on $D(T)$ the topology generated by the countable multinorm $\{\beta_k\}_{k=0}^\infty$, where

$$\beta_k(\phi) := \left(\frac{1}{2\pi} \int_0^{2\pi} |(-i)^k D^k \phi(x)|^2 dx \right)^{1/2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\phi^{(k)}(x)|^2 dx \right)^{1/2}.$$

We now define a space of periodic test-functions $\mathcal{A}_{2\pi}$ and corresponding space of periodic distributions $\mathcal{A}'_{2\pi}$ that are closely related to the spaces $P_{2\pi}$ and $P'_{2\pi}$ defined in [10, Chapter 11]. The space $P_{2\pi}$ consists of 2π -periodic test-functions, and is equipped with the topology generated by multinorms $\{\eta_k\}_{k=0}^\infty$, where

$$\eta_k(\phi) := \sup_{x \in \mathbb{R}} |(D^k \phi)(x)| = \|D^k \phi\|_\infty, \quad k \in \mathbb{N}_0, \phi \in C_{2\pi}^\infty,$$

and $C_{2\pi}^\infty$ is the space of infinitely differentiable, 2π -periodic functions. As stated in [11, pp.256 - 257], each $\phi \in \mathcal{A}$ can be extended to a smooth periodic function $\phi^{per} \in P_{2\pi}$, via

$$\phi^{per}(x) := \phi(x), \quad 0 < x < 2\pi, \quad (2.4)$$

$$\phi^{per}(0) := \lim_{x \rightarrow 0^+} \phi(x), \quad \phi^{per}(2\pi) := \lim_{x \rightarrow 2\pi^-} \phi(x), \quad (2.5)$$

$$\phi^{per}(x + 2\pi) := \phi^{per}(x), \quad -\infty < x < \infty. \quad (2.6)$$

Also $\{\phi_n\}$ converges in $\mathcal{A}$ if and only if $\{\phi_n^{per}\}$ converges in $P_{2\pi}$. Consequently, the test-function space $\mathcal{A}$ can be identified through this extension with $P_{2\pi}$.

Motivated by the fact that each $\phi \in \mathcal{A}$ gives rise to an infinitely differentiable, periodic function on $\mathbb{R}$, we introduce the following “periodic version” of $\mathcal{A}$, denoted by $\mathcal{A}_{2\pi}$.

Definition 2.1 We define $\mathcal{A}_{2\pi}$ to be the vector space $C_{2\pi}^\infty$ equipped with the topology generated by the family of seminorms $\{\rho_k\}_{k=0}^\infty$, where

$$\rho_k(\varphi) := \|T^k \varphi\|_{L_{2\pi}^2} = \|D^k \varphi\|_{L_{2\pi}^2}, \quad \phi \in \mathcal{A}_{2\pi}, \quad T = -iD. \quad (2.7)$$

Clearly, if $\varphi \in \mathcal{A}_{2\pi}$ then the function φ restricted to the interval $[0, 2\pi]$, denoted by $\varphi|_{[0, 2\pi]}$ or φ^{res} , lies in $\mathcal{A}$. It is easy to show that the mapping $\mathcal{R} : \mathcal{A}_{2\pi} \rightarrow \mathcal{A}$, defined by $\mathcal{R}\varphi := \varphi^{res}$, is a homeomorphism from $\mathcal{A}_{2\pi}$ onto $\mathcal{A}$ with inverse $\mathcal{R}^{-1} : \mathcal{A} \rightarrow \mathcal{A}_{2\pi}$ defined by $\mathcal{R}^{-1}\psi = \psi^{per}$. Consequently, the spaces $\mathcal{A}$ and $\mathcal{A}_{2\pi}$ are homeomorphic and this means that all the properties we require of $\mathcal{A}_{2\pi}$ are inherited from $\mathcal{A}$. Therefore, from the results presented in [11, pp.253-254], $\mathcal{A}_{2\pi}$ is not only a Fréchet space but also a test-function space.

Lemma 2.2 The operators T^r and D^r are continuous linear operators from $\mathcal{A}_{2\pi}$ into $\mathcal{A}_{2\pi}$ for each $r \in \mathbb{N}$; i.e $T^r \in L(\mathcal{A}_{2\pi})$ and $D^r \in L(\mathcal{A}_{2\pi})$, where $L(\mathcal{A}_{2\pi})$ denotes the vector space of continuous linear operators from $\mathcal{A}_{2\pi}$ into $\mathcal{A}_{2\pi}$.

Proof: The proof follows from the fact that the restricted operators T^r and D^r are continuous on $\mathcal{A}$, as the topologies defined on $\mathcal{A}$ and $\mathcal{A}_{2\pi}$ are defined via the same expressions. $\square$

Lemma 2.3 Let K be the convolution integral operator defined on $\mathcal{A}_{2\pi}$ by

$$(K\phi)(x) = \frac{1}{2\pi} \int_0^{2\pi} k(x-u)\phi(u) du = \frac{1}{2\pi} \int_0^{2\pi} k(u)\phi(x-u) du. \quad (2.8)$$

If $k \in L_{2\pi}^1$, then $K \in L(\mathcal{A}_{2\pi})$.

Proof: It can be shown that

$$D^r K\phi = K D^r \phi, \forall \phi \in \mathcal{A}_{2\pi} \text{ and } r = 0, 1, 2, \dots$$

Hence, by [2, pp.8-10],

$$\rho_r(K\phi) = \|K D^r \phi\|_{L_{2\pi}^2} \leq \|k\|_{L_{2\pi}^1} \|D^r \phi\|_{L_{2\pi}^2} = \|k\|_{L_{2\pi}^1} \rho_r(\phi).$$

$\square$

The following results on the summability of functions in $\mathcal{A}_{2\pi}$ can be established by using the same arguments as for the theory presented in [11, pp.254-255].

Lemma 2.4

(i) The functions $\{(\varphi, \psi_l)_{L_{2\pi}^2} \psi_l\}_{l \in \mathbb{Z}}$ are summable to φ in $\mathcal{A}_{2\pi}$ for each $\varphi \in \mathcal{A}_{2\pi}$.

(ii) Let $\{a_l\}_{l \in \mathbb{Z}}$ be a family of scalars. Then the functions $\{a_l \psi_l\}_{l \in \mathbb{Z}}$ are summable to some function $\phi \in \mathcal{A}_{2\pi}$ if and only if $\sum_{l \in \mathbb{Z}} l^{2k} |a_l|^2 < \infty$ for each $k \in \mathbb{N}_0$.

$\square$

We now turn to the space $\mathcal{A}'_{2\pi}$, the dual of $\mathcal{A}_{2\pi}$, and equip $\mathcal{A}'_{2\pi}$ with the weak* topology. Each $f \in \mathcal{A}'_{2\pi}$ assigns a unique complex number $\langle f, \phi \rangle$ to each $\phi \in \mathcal{A}_{2\pi}$. In the following, it is convenient to use the notation

$$2\pi(f, \phi) := \langle f, \bar{\phi} \rangle, \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}.$$

Note that each $\varphi \in L_{2\pi}^2$ generates a regular generalised function $\tilde{\varphi} \in \mathcal{A}'_{2\pi}$ defined by

$$\langle \tilde{\varphi}, \bar{\phi} \rangle = 2\pi(\tilde{\varphi}, \phi) := 2\pi(\varphi, \phi)_{L_{2\pi}^2} = \int_0^{2\pi} \varphi(x) \overline{\phi(x)} dx. \quad (2.9)$$

Also $L_{2\pi}^2$ is continuously imbedded in $\mathcal{A}'_{2\pi}$. Since $\psi_l \in \mathcal{A}_{2\pi}$, $\{\tilde{\psi}_l\}_{l \in \mathbb{Z}}$ is a sequence of regular generalised functions in $\mathcal{A}'_{2\pi}$. Consequently we can discuss the convergence in $\mathcal{A}'_{2\pi}$ of infinite series of the form $\sum_{l=-\infty}^{\infty} b_l \tilde{\psi}_l$.

Theorem 2.5

- (i) Each $f \in \mathcal{A}'_{2\pi}$ can be expressed as $f = \sum_{l \in \mathbb{Z}} (f, \psi_l) \tilde{\psi}_l = \lim_{N \rightarrow \infty} \sum_{l=-N}^N (f, \psi_l) \tilde{\psi}_l$ where the series converges in $\mathcal{A}'_{2\pi}$.
- (ii) Let $\{b_l\}_{l \in \mathbb{Z}} \subset \mathbb{C}$. Then $\sum_{n \in \mathbb{Z}} b_l \tilde{\psi}_l = f$ in $\mathcal{A}'_{2\pi}$ if there exists $q \in \mathbb{N}_0$ such that $\sum_{l \neq 0} l^{-2q} |b_l|^2$ is convergent in $\mathbb{R}$.

Proof: The proof is analogous to that of [11, Theorem 9.6-1]. □

For $\tilde{T}$ to be an extension of T defined on $\mathcal{A}'_{2\pi}$, we require $\tilde{T}\tilde{\varphi} = \widetilde{T\varphi}$, for all $\varphi \in D(T)$. Then

$$(\tilde{T}\tilde{\varphi}, \phi) = (\widetilde{T\varphi}, \phi) = (T\varphi, \phi)_{L_{2\pi}^2} = (\varphi, T\phi)_{L_{2\pi}^2} = (\tilde{\varphi}, T\phi).$$

Therefore the operator $\tilde{T}$ on $\mathcal{A}'_{2\pi}$ is defined by

$$(\tilde{T}f, \phi) := (f, T\phi) \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}. \quad (2.10)$$

Similarly, the generalised differential operator $\tilde{D}$ on $\mathcal{A}'_{2\pi}$ is defined via

$$(\tilde{D}f, \phi) := (f, -D\phi) \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}. \quad (2.11)$$

Theorem 2.6 For each $r \in \mathbb{N}$, $\tilde{T}^r, \tilde{D}^r \in L(\mathcal{A}'_{2\pi})$.

Proof: For each $r \in \mathbb{N}$, $\widetilde{T^r} = \tilde{T}^r$ and $\widetilde{(-D)^r} = \tilde{D}^r$. It follows that $\tilde{T}^r$ and $\tilde{D}^r$ are the adjoints of T^r and $(-D)^r$ respectively. As T^r and $(-D)^r$ are in $L(\mathcal{A}_{2\pi})$, the result follows from [11, Theorem 1.10-1]. □

The above discussion enables us to develop a theory of fractional calculus on the spaces of test-functions $\mathcal{A}_{2\pi}$ and of periodic distributions $\mathcal{A}'_{2\pi}$.

3 Fractional Calculus of Periodic Test Functions

In this section, we produce a general theory of fractional calculus operators defined on the space of test-functions $\mathcal{A}_{2\pi}$. By continuity of D^r on $\mathcal{A}_{2\pi}$ for each $r \in \mathbb{N}$, we have

$$D^r \phi = \sum_{l \in \mathbb{Z}} (\phi, \psi_l)_{L_{2\pi}^2} D^r \psi_l = \sum_{l \in \mathbb{Z}} (\phi, \psi_l)_{L_{2\pi}^2} (il)^r \psi_l, \quad \forall \phi \in \mathcal{A}_{2\pi}. \quad (3.1)$$

Motivated by (3.1), we define the fractional derivative D^α of order $\alpha > 0$ on $\mathcal{A}_{2\pi}$ by

$$D^\alpha \phi = \sum_{l \in \mathbb{Z}} (\phi, \psi_l)_{L_{2\pi}^2} (il)^\alpha \psi_l. \quad (3.2)$$

This definition is identical to the Weyl formula (1.18) for D^α on $L_{2\pi}^2$. In the following lemmas, we discuss the properties of D^α on the space $\mathcal{A}_{2\pi}$.

Theorem 3.1 For each $\alpha > 0$, $D^\alpha \in L(\mathcal{A}_{2\pi})$.

Proof: We can write $D^\alpha \phi = \sum_{l \in \mathbb{Z}} a_l \psi_l$, where $a_l = (\phi, \psi_l)_{L_{2\pi}^2} (il)^\alpha$. From Theorem 2.4, $\sum_{l \in \mathbb{Z}} a_l \psi_l$ converges in $\mathcal{A}_{2\pi}$ if and only if $\sum_{l \in \mathbb{Z}} |a_l|^2 |l|^{2k}$ converges for each $k \in \mathbb{N}_0$. Now, for each $\phi \in \mathcal{A}_{2\pi}$, $\sum_{l \in \mathbb{Z}} |(\phi, \psi_l)|^2 l^{2N}$ converges for any $N \in \mathbb{N}_0$. Suppose for any given $\alpha > 0$ and $k \in \mathbb{N}_0$, we choose N such that $(\alpha + k) < N$. Then

$$\sum_{l \in \mathbb{Z}} |(\phi, \psi_l)_{L_{2\pi}^2}|^2 l^{2(\alpha+k)} \leq \sum_{l \in \mathbb{Z}} |(\phi, \psi_l)_{L_{2\pi}^2}|^2 l^{2N} < \infty.$$

Linearity of D^α on $\mathcal{A}_{2\pi}$ is clear from the definition. To prove continuity of D^α , we proceed as follows. Given $\alpha > 0$, there exists $N \in \mathbb{N}_0$ such that $\alpha \in (N, N+1]$, and so

$$[\rho_k(D^\alpha \phi)]^2 = \left\| \sum_{l \in \mathbb{Z}} (\phi, \psi_l)_{L_{2\pi}^2} (il)^{\alpha+k} \psi_l \right\|_{L_{2\pi}^2}^2 \leq \sum_{l \in \mathbb{Z}} |(\phi, \psi_l)_{L_{2\pi}^2}|^2 |l|^{2(N+1)+2k} = \|D^{k+N+1} \phi\|_{L_{2\pi}^2}^2.$$

Therefore

$$\rho_k(D^\alpha \phi) \leq \rho_{k+N+1}(\phi),$$

and the result follows. $\square$

We now express I^α in the form of a Fourier series as

$$I^\alpha \phi := \sum_{l \neq 0} (il)^{-\alpha} (\phi, \psi_l)_{L_{2\pi}^2} \psi_l. \quad (3.3)$$

It is clear from Theorem 2.4 that for each $\phi \in \mathcal{A}_{2\pi}$, the series in (3.3) converges in $\mathcal{A}_{2\pi}$. The operator I^α can also be expressed as the convolution integral operator (1.15). From Lemma 2.3, we deduce that $I^\alpha \in L(\mathcal{A}_{2\pi})$. The following theorem shows that I^α and D^α satisfy the expected properties of fractional integral and differential operators on the space of test-functions $\mathcal{A}_{2\pi}$.

Theorem 3.2 For all $\alpha, \beta > 0$ and $\phi \in \mathcal{A}_{2\pi}$

$$(i) \quad I^\alpha I^\beta \phi = I^{\alpha+\beta} \phi, \quad D^\alpha D^\beta \phi = D^{\alpha+\beta} \phi,$$

$$(ii) \quad D^\alpha I^\alpha \phi = I^\alpha D^\alpha \phi = \phi - \hat{\phi}(0).$$

Proof: The proof is similar to that of [3, Proposition 4.1]. $\square$

The fact that D^α is defined as a continuous linear operator on the entire space $\mathcal{A}_{2\pi}$ is one advantage of working within this Fréchet space framework. Recall that in the $L_{2\pi}^2$ setting, D^α is not continuous and is not defined on all of $L_{2\pi}^2$. Unfortunately, to achieve these preferred properties of D^α , we have severely restricted the class of functions on which D^α acts. However, this drawback can be overcome by extending the theory to the generalised functions in $\mathcal{A}'_{2\pi}$. As we shall again use the adjoint method to define the distributional versions of the operators, we must first introduce the fractional differential operator $(-D)^\alpha$, given by

$$(-D)^\alpha \phi = \sum_{l \in \mathbb{Z}} (-il)^\alpha (\phi, \psi_l)_{L_{2\pi}^2} \psi_l, \quad \phi \in \mathcal{A}_{2\pi}. \quad (3.4)$$

An argument similar to that used in the proof of Theorem 3.1 shows that $(-D)^\alpha \in L(\mathcal{A}_{2\pi})$ for each $\alpha > 0$.

Similarly, we can define a fractional integral operator $(-I)^\alpha$ by

$$(-I)^\alpha \phi = \sum_{l \neq 0} (-il)^{-\alpha} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l, \quad \phi \in \mathcal{A}_{2\pi}. \quad (3.5)$$

It is straightforward to show that $(-I)^\alpha$ and $(-D)^\alpha$ satisfy the usual properties (analogous to those listed in Theorem 3.2) of fractional integral and differential operators on the space of test-functions $\mathcal{A}_{2\pi}$.

One of the key results obtained by Butzer and Westphal in [3] is a connection between (1.18) and (1.20). The first step towards this is an explicit formula for the each Fourier coefficient of the fractional difference $\Delta_t^\alpha \phi$, namely

$$(\Delta_t^\alpha \phi)^\wedge(l) := \frac{1}{2\pi} \int_0^{2\pi} \phi(x) e^{-ilx} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} e^{-iltj} dx = \left(\frac{1 - e^{-ilt}}{t} \right)^\alpha \hat{\phi}(l), \quad \phi \in L^p_{2\pi}, l \in \mathbb{Z}. \quad (3.6)$$

Consequently,

$$\lim_{t \rightarrow 0^+} \left(\frac{\Delta_t^\alpha \phi}{t^\alpha} \right)^\wedge(l) = (il)^\alpha \hat{\phi}(l). \quad (3.7)$$

To establish the results stated in Theorem 1.1 for the strong Liouville-Grünwald fractional derivative defined on $L^p_{2\pi}$ ($1 \leq p < \infty$) by (1.20), Butzer and Westphal made use of the function

$$p_\alpha(x) := \begin{cases} \frac{1}{\Gamma(\alpha)} \sum_{0 \leq j < x} (-1)^j \binom{\alpha}{j} (x-j)^{\alpha-1}, & (0 < x < \infty) \\ 0 & (-\infty < x < 0) \end{cases} \quad (3.8)$$

where $\alpha > 0$. This function has the following properties for each $\alpha > 0$ (see [3, Proposition 3.1]):

$$p_\alpha \in L^1(-\infty, \infty), \quad \int_{-\infty}^{\infty} p_\alpha(u) du = 1, \quad (3.9)$$

and

$$(\mathcal{F}p_\alpha)(v) = \begin{cases} (iv)^{-\alpha} (1 - e^{-iv})^\alpha, & v \neq 0 \\ 1 & v = 0, \end{cases} \quad (3.10)$$

where $\mathcal{F}p_\alpha$ denotes the Fourier transform of p_α . In addition, p_α can be used to construct a related function χ_α defined by

$$\chi_\alpha(x; t) = 2\pi \sum_{\frac{-x}{2\pi} < j < \infty} \frac{p_\alpha\left(\frac{x+2\pi j}{t}\right)}{t} \quad (t > 0).$$

Then, from [2] and [3, Lemma 3.1], for each $\alpha > 0$

$$\chi_\alpha(\cdot; t) \in L^1_{2\pi}, \quad \int_0^{2\pi} \chi_\alpha(u, t) du = 2\pi, \quad (3.11)$$

$$\widehat{\chi_\alpha(\cdot; t)}(k) = \begin{cases} (ikt)^{-\alpha} (1 - e^{-ikt})^\alpha, & k \neq 0 \\ 1 & k = 0, \end{cases}$$

and

$$\chi_\alpha(x; t) = t^{-\alpha} (\Delta_t^\alpha \eta_\alpha)(x) + 1 = t^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \eta_\alpha(x - tj) + 1, \quad (3.12)$$

where η_α is defined by (1.14). Moreover, on setting

$$(J_{t,\alpha}\phi)(x) := \frac{1}{2\pi} \int_0^{2\pi} \chi_\alpha(x-u;t)\phi(u) du,$$

we have

$$\lim_{t \rightarrow 0^+} \|J_{t,\alpha}\phi - \phi\|_{L_{2\pi}^p} = 0, \quad \forall \phi \in L_{2\pi}^p. \quad (3.13)$$

We now take a modified version of the Liouville-Grünwald fractional derivative $D^{<\alpha>}$ in $\mathcal{A}_{2\pi}$, where the limit in (1.20) is with respect to the topology on $\mathcal{A}_{2\pi}$. Note that if $\Delta_t^\alpha \phi / t^\alpha \rightarrow \phi$ in $\mathcal{A}_{2\pi}$ as $t \rightarrow 0^+$ then $\Delta_t^\alpha \phi / t^\alpha \rightarrow \phi$ in $L_{2\pi}^2$ as $t \rightarrow 0^+$. Therefore, we can write

$$\lim_{t \rightarrow 0^+} \frac{\Delta_t^\alpha \phi}{t^\alpha} = D^{<\alpha>} \phi, \quad \alpha > 0, \phi \in \mathcal{A}_{2\pi}$$

whenever the limit on the left-hand side exists in $\mathcal{A}_{2\pi}$.

In the subsequent discussion, we shall require the following extension of equation (3.13).

Lemma 3.3 *If $\phi \in \mathcal{A}_{2\pi}$, then $\lim_{t \rightarrow 0^+} J_{t,\alpha}\phi = \phi$ in $\mathcal{A}_{2\pi}$.*

Proof: Let $\phi \in \mathcal{A}_{2\pi}$. Then, from the definition of ρ_k , we obtain

$$\rho_k(J_{t,\alpha}\phi - \phi) = \|D^k(J_{t,\alpha}\phi) - D^k\phi\|_{L_{2\pi}^2} = \|J_{t,\alpha}(D^k\phi) - D^k\phi\|_{L_{2\pi}^2}.$$

Since $D^k\phi \in L_{2\pi}^2$ for all $k \in \mathbb{N}_0$, it follows from (3.13) that

$$\rho_k(J_{t,\alpha}\phi - \phi) \rightarrow 0 \quad \text{as } t \rightarrow 0^+.$$

□

The following theorem is analogous to Theorem 1.1.

Theorem 3.4 *Let $\phi \in \mathcal{A}_{2\pi}$, $\alpha > 0$. Then the following statements are equivalent.*

- (i) $D^{<\alpha>}\phi \in \mathcal{A}_{2\pi}$.
- (ii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $\hat{\psi}(l) = (il)^\alpha \hat{\phi}(l)$, $l \in \mathbb{Z}$.
- (iii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $(I^\alpha \psi)(x) = \phi(x) - \hat{\phi}(0)$.

If (i) is satisfied, then $D^{<\alpha>}\phi = \psi$, where $\psi \in \mathcal{A}_{2\pi}$ is the function defined in (ii). Similarly if (ii) is satisfied then $D^{<\alpha>}\phi = \psi$.

Proof: Let $\phi \in \mathcal{A}_{2\pi}$ be such that $D^{<\alpha>}\phi = \psi$ in $\mathcal{A}_{2\pi}$. Then $D^{<\alpha>}\phi = \psi$ in $L_{2\pi}^2$ and so, as in Theorem 1.1,

$$\hat{\psi}(l) = (D^{<\alpha>}\phi)(l) = (il)^\alpha \hat{\phi}(l), \quad l \in \mathbb{Z}.$$

Now we suppose that (ii) holds. Then $\psi \in \mathcal{A}_{2\pi}$ is such that

$$\hat{\psi}(l) = (il)^\alpha \hat{\phi}(l), \quad l \in \mathbb{Z}.$$

Since $\psi, \phi \in L_{2\pi}^2$, it follows from Theorem 1.1 that $(I^\alpha \psi)(x) = \phi(x) - \hat{\phi}(0)$ almost everywhere.

Let (iii) be true so that

$$(I^\alpha \psi)(x) = \phi(x) - \hat{\phi}(0) \quad \text{a.e.}$$

Since $(I^\alpha \psi)^\wedge(0) = 0$, we can assume that $\hat{\psi}(0) = 0$. Then, as in the proof of Theorem 1.1, given in [3, Theorem 4.1],

$$\frac{\Delta_t^\alpha \phi}{t^\alpha} = J_{t,\alpha} \psi.$$

Hence from Lemma 3.3, it follows that

$$\lim_{t \rightarrow 0^+} \frac{\Delta_t^\alpha \phi}{t^\alpha} = \lim_{t \rightarrow 0^+} J_{t,\alpha} \psi = \psi$$

in $\mathcal{A}_{2\pi}$. □

We now obtain the following result.

Theorem 3.5 *Let $\phi \in \mathcal{A}_{2\pi}$. Then $D^{<\alpha>} \phi \in \mathcal{A}_{2\pi}$ for all $\alpha > 0$. Also $D^\alpha \phi = D^{<\alpha>} \phi$ and so $D^{<\alpha>} \in L(\mathcal{A}_{2\pi})$.*

Proof: Let $\phi \in \mathcal{A}_{2\pi}$. Then

$$\sum_{l \in \mathbb{Z}} |\hat{\phi}(l)|^2 l^{2k} < \infty, \quad k \in \mathbb{N}_0.$$

Therefore, for each $\alpha > 0$ and $k \in \mathbb{N}_0$,

$$\sum_{l \in \mathbb{Z}} |\hat{\phi}(l)|^2 l^{2\alpha+2k} < \infty.$$

Hence, by Theorem 2.4, $\sum_{l \in \mathbb{Z}} (il)^\alpha \hat{\phi}(l) \psi_l$ exists in $\mathcal{A}_{2\pi}$. Thus, it follows from Theorem 3.4 that $D^{<\alpha>} \phi$ exists in $\mathcal{A}_{2\pi}$ and

$$D^{<\alpha>} \phi = \sum_{l \in \mathbb{Z}} (il)^\alpha \hat{\phi}(l) \psi_l = D^\alpha \phi \quad (\text{from (3.2)}).$$

□

For the distributional treatment of the Liouville-Grünwald fractional derivative, we also require properties of an operator that is closely related to $D^{<\alpha>}$, namely $(-D)^{<\alpha>}$. The latter is defined by

$$(-D)^{<\alpha>} \phi := \lim_{t \rightarrow 0^+} \frac{\Delta_{-t}^\alpha \phi}{t^\alpha}, \quad \phi \in \mathcal{A}_{2\pi}, \quad (3.14)$$

where the limit is with respect to the topology on $\mathcal{A}_{2\pi}$ and

$$(\Delta_{-t}^\alpha \phi)(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \phi(x + tj). \quad (3.15)$$

Then, following the analysis of $D^{<\alpha>}$ given in [3], we obtain

$$(\Delta_{-t}^\alpha \phi)^\wedge(l) = (1 - e^{ilt})^\alpha \hat{\phi}(l), \quad \phi \in \mathcal{A}_{2\pi}, l \in \mathbb{Z},$$

which gives

$$\lim_{t \rightarrow 0^+} \left(\frac{\Delta_{-t}^\alpha \phi}{t^\alpha} \right)^\wedge(l) = \lim_{t \rightarrow 0^+} \left(\frac{1 - e^{ilt}}{t} \right)^\alpha \hat{\phi}(l) = (-il)^\alpha \hat{\phi}(l). \quad (3.16)$$

Let p_α^* be the function defined by

$$p_\alpha^*(u) := p_\alpha(-u),$$

where p_α is given by (3.8), and define Λ_α by

$$\Lambda_\alpha(x; t) := 2\pi \sum_{j=-\infty}^{\infty} \frac{p_\alpha^*\left(\frac{x+2\pi j}{t}\right)}{t}, \quad t > 0.$$

Then

$$\widehat{\Lambda_\alpha(\cdot; t)}(l) = (\mathcal{F}p_\alpha^*)(lt) = \begin{cases} (-ilt)^{-\alpha}(1 - e^{ilt})^\alpha, & l \neq 0 \\ 1 & l = 0, \end{cases}$$

and therefore

$$\Lambda_\alpha(x; t) = t^{-\alpha}(\Delta_{-t}^\alpha \eta_\alpha^*)(x) + 1, \quad (3.17)$$

where

$$\eta_\alpha^*(x) = \overline{\eta_\alpha(-x)} = \sum_{l \neq 0} (-il)^{-\alpha} e^{ilx}.$$

We can now present the following result.

Theorem 3.6 *The following statements are equivalent for $\phi \in \mathcal{A}_{2\pi}$, $\alpha > 0$.*

- (i) $(-D)^{<\alpha>} \phi \in \mathcal{A}_{2\pi}$.
- (ii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $\hat{\psi}(l) = (-il)^\alpha \hat{\phi}(l)$, $\forall l \in \mathbb{Z}$.
- (iii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $((-I)^\alpha \psi)(x) = \phi(x) - \hat{\phi}(0)$ almost everywhere, where $(-I)^\alpha$ is defined by (3.5).

If (i) is satisfied, then $(-D)^{<\alpha>} \phi = \psi$, where $\psi \in \mathcal{A}_{2\pi}$ is the function defined in (ii). Similarly if (ii) is satisfied then $(-D)^{<\alpha>} \phi = \psi$.

Proof: The proof is similar to that of Theorem 3.4. □

Lemma 3.7 *Let $\phi \in \mathcal{A}_{2\pi}$. Then, $(-D)^{<\alpha>} \phi \in \mathcal{A}_{2\pi}$ for all $\alpha > 0$. Also $(-D)^\alpha \phi = (-D)^{<\alpha>} \phi$ and so $(-D)^{<\alpha>} \in L(\mathcal{A}_{2\pi})$.*

Proof: The proof is a consequence of Theorem 3.6 and similar to that of Theorem 3.5 and hence is omitted. □

4 Fractional Calculus of Periodic Distributions

We now define an operator $\widetilde{D}^\alpha$ on $\mathcal{A}'_{2\pi}$ which can be interpreted as an extension of $D^\alpha \in L(\mathcal{A}_{2\pi})$ to the space of periodic distributions $\mathcal{A}'_{2\pi}$. For this purpose, we apply (2.11) to obtain

$$(\widetilde{D}^r f, \phi) = (f, (-D)^r \phi) \quad \forall r \in \mathbb{N}, f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}.$$

This motivates us to define $\widetilde{D}^\alpha$, for $\alpha > 0$, as

$$(\widetilde{D}^\alpha f, \phi) := (f, (-D)^\alpha \phi), \quad (4.1)$$

where $(-D)^\alpha$ is defined by (3.4). Therefore, for $\phi \in \mathcal{A}_{2\pi}$, we have

$$\begin{aligned} (\widetilde{D}^\alpha f, \phi) &:= (f, (-D)^\alpha \phi) = (f, \sum_{l \in \mathbb{Z}} (-il)^\alpha (\phi, \psi_l)_{L^2_{2\pi}} \psi_l) \\ &= \sum_{l \in \mathbb{Z}} (il)^\alpha (\widetilde{\psi}_l, \phi) (f, \psi_l) = \left(\sum_{l \in \mathbb{Z}} (il)^\alpha (f, \psi_l) \widetilde{\psi}_l, \phi \right). \end{aligned}$$

Hence the generalised differential operator $\widetilde{D}^\alpha$ may also be represented by

$$\widetilde{D}^\alpha f := \sum_{l \in \mathbb{Z}} (il)^\alpha (f, \psi_l) \widetilde{\psi}_l, \quad \text{for all } f \in \mathcal{A}'_{2\pi}, \quad (4.2)$$

where the infinite series converges in $\mathcal{A}'_{2\pi}$.

Lemma 4.1 *The operator $\widetilde{D}^\alpha$ is a continuous linear mapping from $\mathcal{A}'_{2\pi}$ into $\mathcal{A}'_{2\pi}$.*

Proof: It follows from (4.1) that $\widetilde{D}^\alpha$ is the adjoint operator of $(-D)^\alpha$. Since $(-D)^\alpha$ is a continuous linear operator from $\mathcal{A}_{2\pi}$ into $\mathcal{A}_{2\pi}$, $\widetilde{D}^\alpha$ is a continuous linear mapping from $\mathcal{A}'_{2\pi}$ to $\mathcal{A}'_{2\pi}$. $\square$

A generalised fractional integral operator $\widetilde{I}^\alpha$, an extension of I^α , can also be defined via

$$(\widetilde{I}^\alpha f, \phi) := (f, (I^\alpha)^* \phi) \quad (4.3)$$

where $(I^\alpha)^*$ is given by

$$((I^\alpha)^* \phi)(x) = (\eta_\alpha^* * \phi)(x) = \frac{1}{2\pi} \int_0^{2\pi} \eta_\alpha^*(x - u) \phi(u) du, \quad (4.4)$$

with

$$\eta_\alpha^*(x) := \sum_{l \neq 0} (-il)^{-\alpha} e^{ilx}.$$

Hence $(I^\alpha)^* = (-I)^\alpha$. Therefore, we obtain

$$(\widetilde{I}^\alpha f, \phi) := (f, (-I)^\alpha \phi) = (f, \sum_{l \neq 0} (-il)^{-\alpha} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l) = \left(\sum_{l \neq 0} (il)^{-\alpha} (f, \psi_l) \widetilde{\psi}_l, \phi \right).$$

It follows that $\widetilde{I}^\alpha$ may also be defined by

$$\widetilde{I}^\alpha f := \sum_{l \neq 0} (il)^{-\alpha} (f, \psi_l) \widetilde{\psi}_l \quad \text{for all } f \in \mathcal{A}'_{2\pi}. \quad (4.5)$$

Lemma 4.2 *The operator $\widetilde{I}^\alpha$ is a continuous linear mapping from $\mathcal{A}'_{2\pi}$ into $\mathcal{A}'_{2\pi}$.*

Proof: The integral operator $\widetilde{I}^\alpha$, being the adjoint of $(-I)^\alpha \in L(\mathcal{A}_{2\pi})$, is a continuous linear operator on $\mathcal{A}'_{2\pi}$. $\square$

Properties of these fractional differential and integral operators on $\mathcal{A}'_{2\pi}$ are summarised in the following theorem.

Theorem 4.3

(i) $\widetilde{D}^\alpha \widetilde{D}^\beta f = \widetilde{D^{\alpha+\beta}} f$, $\widetilde{I}^\alpha \widetilde{I}^\beta f = \widetilde{I^{\alpha+\beta}} f$ for all $\alpha, \beta > 0$ and $f \in \mathcal{A}'_{2\pi}$.

(ii) $\widetilde{D}^\alpha \widetilde{I}^\alpha f = \widetilde{I}^\alpha \widetilde{D}^\alpha f = f - \hat{f}(0)$ for all $f \in \mathcal{A}'_{2\pi}$.

Note that $\hat{f}(l) = (f, \psi_l)$, $l \in \mathbb{Z}$.

Proof: The proof for (i) follows directly from (4.1) and (4.3).

(ii) Let $f \in \mathcal{A}'_{2\pi}$. Then f can be expressed uniquely as $f = \sum_{l \in \mathbb{Z}} (f, \psi_l) \widetilde{\psi}_l$, where the series converges in $\mathcal{A}'_{2\pi}$. Similarly, since for all $\alpha > 0$, $g = \widetilde{D}^\alpha \widetilde{I}^\alpha f$ is also in $\mathcal{A}'_{2\pi}$, we can write

$$g = \sum_{l \in \mathbb{Z}} (\widetilde{D}^\alpha \widetilde{I}^\alpha f, \psi_l) \widetilde{\psi}_l.$$

Now, for each $n \in \mathbb{Z}$,

$$(\widetilde{D}^\alpha \widetilde{I}^\alpha f, \psi_l) = (f, (-I)^\alpha (-D)^\alpha \psi_l) = \begin{cases} (f, \psi_l) & l \neq 0 \\ 0 & l = 0. \end{cases}$$

Hence

$$\widetilde{D}^\alpha \widetilde{I}^\alpha f = \sum_{l \in \mathbb{Z}} (f, \psi_l) \widetilde{\psi}_l - (f, \psi_0) \widetilde{\psi}_0 = f - \hat{f}(0).$$

Similarly, it can be shown that

$$\widetilde{I}^\alpha \widetilde{D}^\alpha f = f - \hat{f}(0).$$

□

Now we extend $D^{<\alpha>}$ to a generalised operator $\widetilde{D^{<\alpha>}}$ on $\mathcal{A}'_{2\pi}$ by defining

$$(\widetilde{D^{<\alpha>}} f, \phi) := (f, (-D)^{<\alpha>} \phi) \quad \forall f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}, \quad (4.6)$$

where the operator $(-D)^{<\alpha>}$ is given by (3.14). Therefore, for $f \in \mathcal{A}'_{2\pi}$ and $\alpha > 0$, we obtain

$$\begin{aligned} (\widetilde{D^{<\alpha>}} f, \phi) &:= (f, (-D)^{<\alpha>} \phi) = (f, \lim_{t \rightarrow 0^+} \frac{\Delta_{-t}^\alpha \phi}{t^\alpha}) \\ &= (f, \sum_{l \in \mathbb{Z}} (-il)^\alpha (\phi, \psi_l)_2 \psi_l) \quad (\text{from (3.16)}) \\ &= \sum_{l \in \mathbb{Z}} (il)^\alpha (\psi_l, \phi)_2 (f, \psi_l) = \sum_{l \in \mathbb{Z}} (il)^\alpha (\widetilde{\psi}_l, \phi) (f, \psi_l) \\ &= (\sum_{l \in \mathbb{Z}} (il)^\alpha (f, \psi_l) \widetilde{\psi}_l, \phi) = (\widetilde{D}^\alpha f, \phi) \quad \forall \phi \in \mathcal{A}_{2\pi}. \end{aligned}$$

Note that an equivalent definition of $\widetilde{D^{<\alpha>}} f$ is given by the weak* limit

$$\widetilde{D^{<\alpha>}} f := \lim_{t \rightarrow 0^+} \frac{\widetilde{\Delta_t^\alpha} f}{t^\alpha}, \quad f \in \mathcal{A}'_{2\pi}, \alpha > 0, \quad (4.7)$$

where

$$\widetilde{\Delta}_t^\alpha f := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj)$$

and $f(x - tj)$ represents a translated version of the distribution f . This is a direct consequence of the fact that

$$(\widetilde{\Delta}_t^\alpha f, \phi) = (f, \Delta_{-t}^\alpha \phi), \quad \forall f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}.$$

Since the generalised operators $\widetilde{D^{<\alpha>}}$ and $\widetilde{D^\alpha}$ coincide on $\mathcal{A}'_{2\pi}$, the following results can be deduced immediately from Theorem 4.3.

Theorem 4.4

- (i) $\widetilde{D^{<\alpha>}} \widetilde{D^{<\beta>}} f = \widetilde{D^{<\alpha+\beta>}} f$, for all $\alpha, \beta > 0$ and $f \in \mathcal{A}'_{2\pi}$.
- (ii) $\widetilde{D^{<\alpha>}} \widetilde{I^{<\alpha>}} f = \widetilde{I^{<\alpha>}} \widetilde{D^{<\alpha>}} f = f - \hat{f}(0)$ for all $f \in \mathcal{A}'_{2\pi}$.

□

5 Fractional Diffusion-Type Equations

As an application of the distributional theory of fractional differential operators developed above, we investigate a distributional version of a fractional diffusion-type equation. The classical formulation of this equation is

$$\frac{\partial u(x, t)}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{(\alpha)} u(x, t), \quad 0 \leq x \leq 2\pi, \quad t > 0, \quad 1 < \alpha < 3, \quad (5.1)$$

with u also required to satisfy the periodic boundary conditions

$$\left(\frac{\partial}{\partial x}\right)^{(\alpha-2)} u(0, t) = \left(\frac{\partial}{\partial x}\right)^{(\alpha-2)} u(2\pi, t), \quad \left(\frac{\partial}{\partial x}\right)^{(\alpha-1)} u(0, t) = \left(\frac{\partial}{\partial x}\right)^{(\alpha-1)} u(2\pi, t), \quad t > 0, \quad (5.2)$$

and the initial condition

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 2\pi. \quad (5.3)$$

In [3, Section 9], (5.1) - (5.3) is interpreted as the abstract Cauchy problem (ACP)

$$\frac{du(t)}{dt} = D^\alpha u(t), \quad 1 < \alpha < 3, \quad t > 0; \quad u(0) = u_0, \quad (5.4)$$

posed in the space $L^p_{2\pi}$, where $D^\alpha = D^{<\alpha>}$ is defined via (1.20). The existence and uniqueness of a solution $u(x, t)$ to the ACP (5.4) is proved in [3], with u obtained explicitly in the form

$$u(x, t) = \sum_{l \in \mathbb{Z}} e^{(il)^\alpha t} \widehat{u_0}(l) e^{ilx} = \frac{1}{2\pi} \int_0^{2\pi} q_\alpha(y, t) u_0(x - y) dy, \quad (5.5)$$

where

$$q_\alpha(x, t) := \sum_{l \in \mathbb{Z}} e^{(il)^\alpha t} e^{ilx}.$$

The function u defined by (5.5) is actually a continuous 2π -periodic function of x for each fixed $t > 0$ and any $u_0 \in L^p_{2\pi}$, and is a solution of (5.4), not only for $1 < \alpha < 3$, but also for $4j + 1 < \alpha < 4j + 3$, $j = 0, 1, 2, \dots$

Formula (5.5) can also be used to define a family of operators $\{G_\alpha(t)\}_{t \geq 0}$ on $L_{2\pi}^p$ for each $p \in [1, \infty)$ and $4j + 1 < \alpha < 4j + 3$, $j = 0, 1, 2, \dots$. As before, we shall concentrate on the case when $p = 2$ and, for simplicity, will take $1 < \alpha < 3$. For each such α and $t \geq 0$, we define

$$G_\alpha(t)\phi := \sum_{l \in \mathbb{Z}} e^{(il)^\alpha t} (\phi, \psi_l)_{L_{2\pi}^2} \psi_l, \quad \phi \in L_{2\pi}^2. \quad (5.6)$$

Theorem 5.1 *Let $G_\alpha(t)$ be defined by (5.6). Then, for each $\alpha \in (1, 3)$, $\{G_\alpha(t)\}_{t \geq 0}$ is a strongly continuous semigroup of operators on $L_{2\pi}^2$. Moreover, the infinitesimal generator of $\{G_\alpha(t)\}_{t \geq 0}$ is D^α defined on the domain given by (1.19).*

Proof: For each $\phi \in L_{2\pi}^2$, we have

$$\sum_{l \in \mathbb{Z}} |e^{(il)^\alpha t} (\phi, \psi_l)_{L_{2\pi}^2}|^2 = \sum_{l \in \mathbb{Z}} \left(e^{|l|^\alpha t \cos(\alpha\pi/2)} \right)^2 \left| (\phi, \psi_l)_{L_{2\pi}^2} \right|^2, \quad t \geq 0.$$

Consequently

$$\|G_\alpha(t)\phi\|_{L_{2\pi}^2} \leq \|\phi\|_{L_{2\pi}^2}, \quad 1 < \alpha < 3,$$

showing that $G_\alpha(t)$ is a bounded linear operator on $L_{2\pi}^2$ for each $t \geq 0$.

It follows from (5.6), that the operators $\{G_\alpha(t)\}_{t \geq 0}$ satisfy the algebraic properties of a semigroup, that is $G_\alpha(0) = I$, and $G_\alpha(s)G_\alpha(t) = G_\alpha(s+t) = G_\alpha(t)G_\alpha(s)$, for all $s, t \geq 0$, $1 < \alpha < 3$. Also, for each $\phi \in L_{2\pi}^2$,

$$\|G_\alpha(t)\phi - \phi\|_{L_{2\pi}^2}^2 = \sum_{l \in \mathbb{Z}} |(e^{(il)^\alpha t} - 1)|^2 |(\phi, \psi_l)_{L_{2\pi}^2}|^2.$$

Clearly $|(e^{(il)^\alpha t} - 1)|^2 |(\phi, \psi_l)_{L_{2\pi}^2}|^2 \rightarrow 0$ as $t \rightarrow 0^+$ for all l . Moreover, the restriction that $\alpha \in (1, 3)$ means that

$$\begin{aligned} (Ae^{i|l|^\alpha t \sin(\alpha\pi/2)} - 1)(Ae^{-i|l|^\alpha t \sin(\alpha\pi/2)} - 1) &= A^2 + 1 - 2A \cos(|l|^\alpha t \sin(\alpha\pi/2)) \\ &\leq 4, \end{aligned}$$

where $A = e^{|l|^\alpha t \cos(\alpha\pi/2)}$, and so we have

$$|e^{(il)^\alpha t} - 1|^2 |(\phi, \psi_l)_{L_{2\pi}^2}|^2 \leq 4 |(\phi, \psi_l)_{L_{2\pi}^2}|^2.$$

Therefore it follows from Weierstrass M-test [1, p.438] that

$$\|G_\alpha(t)\phi - \phi\|_{L_{2\pi}^2} \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

Hence $\{G_\alpha(t)\}_{t \geq 0}$ is a strongly continuous semigroup of operators on $L_{2\pi}^2$. Likewise, it can be verified using the same arguments as in [6, Theorem 2.2], that D^α is the infinitesimal generator of $\{G_\alpha(t)\}_{t \geq 0}$ since, for all $\phi \in \text{Dom}(D^\alpha)$, where the latter is given by (1.19), we have

$$\lim_{t \rightarrow 0^+} \left\| \frac{G_\alpha(t)\phi - \phi}{t} - D^\alpha \phi \right\|_{L_{2\pi}^2} = 0.$$

□

It follows that, when $p = 2$, the strong solution of (5.4) is given by $u(t) = G_\alpha(t)u_0$ provided that $u_0 \in \text{Dom}(D^\alpha)$.

To enable us to cater for a larger class of initial data, including cases when u_0 is a non-classical function, we now examine a generalised version of the fractional diffusion ACP (5.4) posed in the

space $\mathcal{A}'_{2\pi}$. Our aim is to establish that a unique solution $u : [0, \infty) \rightarrow \mathcal{A}'_{2\pi}$ can always be found in the form $u(t) = \widetilde{G_\alpha(t)} u_0$ for any $u_0 \in \mathcal{A}'_{2\pi}$, where $\widetilde{G_\alpha(t)}$ is an appropriately defined extension of the operator $G_\alpha(t)$ to $\mathcal{A}'_{2\pi}$. As a first step towards this, we consider the operators $\{G_\alpha(t)\}_{t \geq 0}$ defined via (5.6), but now restricted to the space of test-functions $\mathcal{A}_{2\pi}$.

Theorem 5.2 *The family of operators $\{G_\alpha(t)\}_{t \geq 0}$ is an equicontinuous semigroup on $\mathcal{A}_{2\pi}$ for each fixed $\alpha \in (1, 3)$.*

Proof: We first establish that $G_\alpha(t) \in L(\mathcal{A}_{2\pi})$. For each $\phi \in \mathcal{A}_{2\pi}$,

$$\begin{aligned} \sum_l |l|^{2k} |e^{(il)^\alpha}(\phi, \psi_l)_{L^2_{2\pi}}|^2 &= \sum_l |l|^{2k} \left(e^{|l|^\alpha t \cos(\alpha\pi/2)} \right)^2 \left| (\phi, \psi_l)_{L^2_{2\pi}} \right|^2 \\ &\leq \sum_l |l|^{2k} |(\phi, \psi_l)_{L^2_{2\pi}}|^2 < \infty \quad \forall k = 0, 1, 2, \dots \quad (\text{since } 1 < \alpha < 3), \end{aligned}$$

and therefore, from Lemma 2.4, $G_\alpha(t)\phi \in \mathcal{A}_{2\pi}$. Also, for each $\phi \in \mathcal{A}_{2\pi}$ and $k = 0, 1, \dots$,

$$\begin{aligned} \rho_k(G_\alpha(t)\phi) &= \|D^k G_\alpha(t)\phi\|_{L^2_{2\pi}} = \|G_\alpha(t)D^k\phi\|_{L^2_{2\pi}} \quad (\text{from the linearity and continuity of } D^k \text{ on } \mathcal{A}_{2\pi}) \\ &\leq \|D^k\phi\|_{L^2_{2\pi}} \quad (\text{from Theorem 5.1, since } D^k\phi \in L^2_{2\pi}) \\ &= \rho_k(\phi). \end{aligned}$$

This shows that $G_\alpha(t) \in L(\mathcal{A}_{2\pi})$. Next, it is clear from Theorem 5.1 that $\{G_\alpha(t)\}_{t \geq 0}$ satisfies the algebraic properties of a semigroup because $\mathcal{A}_{2\pi} \subseteq L^2_{2\pi}$. Also, by arguing as above, we obtain

$$\rho_k(G_\alpha(t)\phi - \phi) = \|G_\alpha(t)(D^k\phi) - D^k\phi\|_{L^2_{2\pi}} \rightarrow 0 \quad \text{as } t \rightarrow 0^+ \quad (\text{from Theorem 5.1}).$$

Finally, equicontinuity follows directly from the fact that

$$\rho_k(G_\alpha(t)\phi) \leq \rho_k(\phi), \quad \text{for all } \phi \in \mathcal{A}_{2\pi}, t \geq 0 \text{ and } k = 0, 1, \dots$$

□

The infinitesimal generator of the semigroup $\{G_\alpha(t)\}_{t \geq 0}$ on $\mathcal{A}_{2\pi}$ is the operator $D^\alpha \in L(\mathcal{A}_{2\pi})$ defined by (3.2) (or, equivalently, from Theorem 3.5, $D^{<\alpha>} \in L(\mathcal{A}_{2\pi})$). To establish this, let $\phi \in \mathcal{A}_{2\pi}$ so that $\sum_l |\lambda_l|^{2k} |(\phi, \psi_l)_{L^2_{2\pi}}|^2 < \infty$ for each $k = 0, 1, 2, \dots$. Then

$$\begin{aligned} [\rho_k(\frac{G_\alpha(t)\phi - \phi}{t} - D^\alpha\phi)]^2 &= \left\| D^k \left(\frac{G_\alpha(t)\phi - \phi}{t} \right) - D^k D^\alpha\phi \right\|_{L^2_{2\pi}}^2 \\ &= \left\| \frac{G_\alpha(t)(D^k\phi) - D^k\phi}{t} - D^\alpha(D^k\phi) \right\|_{L^2_{2\pi}}^2 \rightarrow 0 \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

Since this holds for each $k = 0, 1, 2, \dots$, it follows that

$$D^\alpha\phi = \lim_{t \rightarrow 0^+} \frac{G_\alpha(t)\phi - \phi}{t}.$$

Our next task is to extend the operators $\{G_\alpha(t)\}_{t \geq 0}$ to a family of generalised operators $\{\widetilde{G_\alpha(t)}\}_{t \geq 0}$ defined on $\mathcal{A}'_{2\pi}$. This will be achieved through the adjoint-based approach that was used in [6]. Let $\varphi \in L^2_{2\pi}$ and $\phi \in \mathcal{A}_{2\pi}$. Then both φ and $G_\alpha(t)\varphi$ generate regular generalised functions $\widetilde{\varphi}$ and $\widetilde{G_\alpha(t)\varphi}$.

in $\mathcal{A}'_{2\pi}$ via formula (2.9). For $\widetilde{G_\alpha(t)}$ to be an extension of $G_\alpha(t)$, we require $\widetilde{G_\alpha(t)}\widetilde{\varphi} = \widetilde{G_\alpha(t)\varphi}$. This will be true provided that

$$\langle \widetilde{G_\alpha(t)}\widetilde{\varphi}, \widetilde{\phi} \rangle = 2\pi(\widetilde{G_\alpha(t)}\widetilde{\varphi}, \phi) = 2\pi(G_\alpha(t)\varphi, \phi)_{L^2_{2\pi}} = 2\pi(\widetilde{\varphi}, G_\alpha^*(t)\phi) = \langle \widetilde{\varphi}, \widetilde{G_\alpha^*(t)\phi} \rangle,$$

where $G_\alpha^*(t)$, the Hilbert space adjoint of $G_\alpha(t)$, is given by

$$G_\alpha^*(t)\phi = \sum_{l \in \mathbb{Z}} e^{(-il)\alpha t} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l. \quad (5.7)$$

Motivated by this, we define the generalised operator $\widetilde{G_\alpha(t)}$ by

$$(\widetilde{G_\alpha(t)}f, \phi) := (f, G_\alpha^*(t)\phi) \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}. \quad (5.8)$$

Then $\widetilde{G_\alpha(t)} = (G_\alpha^*(t))'$, the $\mathcal{A}_{2\pi} - \mathcal{A}'_{2\pi}$ adjoint of $G_\alpha^*(t)$. Now suppose that $f \in \mathcal{A}'_{2\pi}$. Then

$$(\widetilde{G_\alpha(t)}f, \phi) = (f, G_\alpha^*(t)\phi) = (f, \sum_l e^{(-il)\alpha t} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l) = \sum_l e^{(il)\alpha t} (f, \psi_l) (\widetilde{\psi_l}, \phi).$$

Since the series on the right-hand side converges for each $\phi \in \mathcal{A}_{2\pi}$, we obtain

$$\widetilde{G_\alpha(t)}f = \sum_l e^{(il)\alpha t} (f, \psi_l) \widetilde{\psi_l}, \quad f \in \mathcal{A}'_{2\pi},$$

where the series is weak*-convergent in $\mathcal{A}'_{2\pi}$.

Theorem 5.3 *The family of operators $\{\widetilde{G_\alpha(t)}\}_{t \geq 0}$, defined on $\mathcal{A}'_{2\pi}$ by (5.8), is a weak*-continuous semigroup of linear operators on $\mathcal{A}'_{2\pi}$ for each $\alpha \in (1, 3)$. Moreover the infinitesimal generator of $\{\widetilde{G_\alpha(t)}\}_{t \geq 0}$ is the operator $\widetilde{D}^\alpha \in L(\mathcal{A}'_{2\pi})$ defined by (4.2) (or equivalently, $\widetilde{D^{<\alpha>}} \in L(\mathcal{A}'_{2\pi})$ defined by (4.6)).*

Proof: It follows from (5.7) and Theorem 5.2 that, for each $\alpha \in (1, 3)$, $\{G_\alpha^*(t)\}_{t \geq 0}$ is an equicontinuous semigroup on $\mathcal{A}_{2\pi}$ with infinitesimal generator $(-D)^\alpha$, where $(-D)^\alpha$ is defined by (3.4). From, [5, pp.262-263], it follows that $\{\widetilde{G_\alpha(t)}\}_{t \geq 0} = \{[G_\alpha^*(t)]'\}_{t \geq 0}$ is a weak*-continuous semigroup on $\mathcal{A}'_{2\pi}$ with infinitesimal generator $[(-D)^\alpha]'$. Since

$$(\widetilde{D}^\alpha f, \phi) = \sum_l (il)^\alpha (f, \psi_l) (\widetilde{\psi_l}, \phi) = (f, \sum_l (-il)^\alpha (\phi, \psi_l)_{L^2_{2\pi}} \psi_l) = (f, (-D)^\alpha \phi),$$

for all $f \in \mathcal{A}'_{2\pi}$ and $\phi \in \mathcal{A}_{2\pi}$, we deduce that the generator is $\widetilde{D}^\alpha$. $\square$

An immediate consequence of the above theorem is that the abstract Cauchy problem associated with $\widetilde{D}^\alpha$, namely

$$\frac{du(t)}{dt} = \widetilde{D}^\alpha u(t), \quad 1 < \alpha < 3, t > 0, \quad u(0) = f \in \mathcal{A}'_{2\pi}, \quad (5.9)$$

has a unique weak*-solution given by,

$$u(t) = \widetilde{G_\alpha(t)}f = \sum_{l \in \mathbb{Z}} e^{(il)\alpha t} (f, \psi_l) \widetilde{\psi_l}, \quad t \geq 0, \quad (5.10)$$

for each $\alpha \in (1, 3)$. We conclude this section with the remark that the series (5.10) converges with respect to the weak*-topology in $\mathcal{A}'_{2\pi}$ for any α with $4k + 1 < \alpha < 4k + 3, k = 0, 1, \dots$. Hence the distributional diffusion equation has the solution (5.10), not only for the fractional order $1 < \alpha < 3$, but also for $\alpha \in (4k + 1, 4k + 3)$ for any $k = 0, 1, 2, \dots$.

6 Conclusion

Functionals such as the Dirac delta cannot be treated in a mathematically rigorous manner as classical functions. Consequently, to study problems involving such entities, generalised functions and distributions are required. In this paper we have constructed $\mathcal{A}'_{2\pi}$, the dual of the Fréchet space $\mathcal{A}_{2\pi}$, to study distributional versions of fractional differential and integral operators. This theory allows us to work with a larger class of “functions.” The fractional differential operator is well defined on these distributions and can be considered either as Weyl’s fractional derivative or as Liouville-Grünwald fractional derivative. As an application, a fractional diffusion equation is examined in the space of periodic distributions $\mathcal{A}'_{2\pi}$ where we obtain a weak*-solution of this equation.

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