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THERMAL PROPERTIES OF HEAT EXCHANGER FOULING

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ABSTRACT

The heat transfer capabilities of industrial heat exchangers are reduced by the build-up of insulating deposits (fouling) on their surfaces. This has an adverse environmental impact due to the necessary increase in energy consumption and the subsequent depletion of nonrenewable fuel sources. Studies of the microstructure of material obtained from heat exchanger surfaces in pulverised coal combustion plants, highlight their geometrical self-similarity over a range of length scales. We will discuss a methodology for estimating the thermal properties of these materials which utilises these self-similarity properties using fractal analysis and renormalisation. The self-similar microstructure of the fouling material is captured by a family of random fractals called shuffled Sierpinski carpets (SSC). The thermal conductivity of the SSC can then be predicted both from its box counting fractal dimension and via a generalised real space renormalisation method. This latter approach also affords an analysis of the percolation threshold of two phase fractal media.

1 INTRODUCTION

The build-up of insulating deposits on the surfaces of industrial heat exchangers seriously inhibits their heat transfer capabilities. This has an adverse environmental impact due to the necessary increase in energy consumption and the subsequent depletion of nonrenewable fuel sources [1, 2]. The work reported here is part of a recent collaboration funded within the framework of the JOULE programme aimed at improving our understanding of the formation mechanisms of fouling and its physical properties [3]. The fouling deposits arise from the interaction of various transport and chemical processes. In this paper we will focus on porous fouling material composed of small particles which have been transported to the surface by a gas stream. Studies of the microstructure of material obtained from heat exchanger surfaces in pulverised coal combustion plants, highlight their geometrical self-similarity over a range of length scales [4]. In section 2 we will discuss a methodology for estimating the thermal properties of these materials which utilises these self-similarity properties using fractal analysis and renormalisation. The *in situ* determination of the physical properties of these materials is intrinsically difficult and so, to test our hypotheses, we have simulated the fouling material using a computerised Monte Carlo type method. Their thermal properties are then estimated by numerically solving the steady-state heat equation using finite differences. This approach is computationally very intensive however. This is due both to the large matrices involved and the high number of iterations required to obtain reasonable accuracy. Therefore, this approach is limited to only a few orders of magnitude size range in the particles. We provide a brief report on our recent findings in section 3.

2 REAL SPACE RENORMALISATION GROUP THEORY

In this section we describe a class of random fractals known as shuffled Sierpinski carpets (SSC) which can adequately capture the self-similar microstructure of the fouling material. The thermal conductivity of these fractals can be determined both by a real space renormalisation approach and from its box counting fractal dimension. The former approach also affords an analysis of the percolation threshold of two phase fractal media.

We start by defining the deterministic Sierpinski Carpet (SC) [5] set. The initial configuration or pre-fractal is the unit square which we denote by $E_0 \in I\!R^2$. We divide E_0 into nine squares of side length one third and remove the middle square. The union of the eight remaining squares we denote by E_1. Each element of E_1 is treated in a similar fashion, whereby a central square of side length one ninth is removed from each of them. Repeating this process we get a decreasing sequence of nonempty compact sets $E_{n+1} \subset E_n$. The Sierpinski Carpet set F is then given by, $F = \bigcap_{n=0}^{\infty} E_n$. We denote the length scale of the remaining squares at generation level n by $\delta = (1/3)^n$ and the number of such squares as

$N(\delta) = 8^n$. The box counting dimension can be shown to equal $dim_B F = \lim_{\delta \to 0} log(N(\delta))/log(1/\delta)$.

We now stipulate that the removed squares (or *tremas*) are conducting particles and that the set F forms an insulating pore network. This basic model can then be generalised by allowing the tremas to occupy (non-overlapping) random locations, that is we *shuffle* or mix the conducting particles in the insulating matrix F. In this way the tremas can come into contact with each other and hence a conducting (or percolation) path through this two-phase medium can be established. By varying δ and n a range of particle size distributions can be easily prescribed. The area of F is given by the sum of an infinite series as $1 - \delta^2/(1 - N\delta^2)$ for $N\delta^2 < 1$, and so by requiring this to equal zero, as in the deterministic SC, we obtain

$$N(\delta) = \frac{1}{\delta^2} - 1. \tag{1}$$

Since the corresponding box-counting dimension is given by $dim_B F = Log(\frac{1}{\delta^2} - 1)/Log(1/\delta)$ we get the following relationship between $dim_B F$ and δ,

$$\delta^2 + \delta^{2 - dim_B F} - 1 = 0. \tag{2}$$

It can be seen therefore that we get the full range of dimensions $dim_B F \in [1, 2]$ for $\delta \in [0, (-1 + \sqrt{5})/2]$. We can use this framework to simulate a family of SSC's , governed by Eq. (1), on computer (see Figure 1).

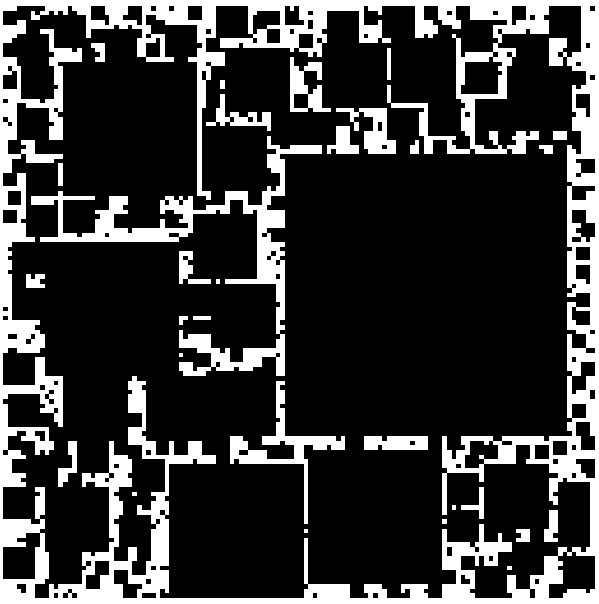

Figure 1: A Shuffled Sierpinski Carpet (SSC) with phase A (white) conductivity $\sigma_A = 0$ and phase B (black) conductivity $\sigma_B = 1$ and $\delta = 0.475$, $n = 6$, $W = 2^\gamma$, $\gamma = 7$, $p_B^{SSC} = 0.784371$, $W\delta^i = \{61, 29, 14, 7, 3, 1\}$, $N^i = \{1, 4, 12, 41, 139, 477\}$ and $dim_B F = 1.66$.

The SSC can of course yield to a standard real space renormalisation group (RSRG) approach [6] for the calculation of its thermal conductivity. However, we will show that the estimates obtained can be improved upon by revising the probability structure. Denote the set F by phase A and the tremas by phase B, which have volume fraction p_B^{SSC}, say. Also denote the intrinsic conductances of the pure substances A and B as σ_A and σ_B respectively. A square grid is placed over the structure where the size of the grid elements corresponds to the smallest monophase particle. The grid is then divided up into groups of 2×2 cells or renormalisation clusters to form the next tessellation β_1. In general β_w is renormalised to β_{w+1} by replacing every renormalisation cluster in β_w by a cell whose effective conductivity is equivalent to the effective conductivity of that renormalisation cluster. The process continues until the mean cluster size tends to zero and the effective conductivity is obtained from the single cell which remains. The effective conductivity of a cluster is estimated by Kirchoff's laws via, $\sigma^e = \sigma_1\sigma_2/(\sigma_1 + \sigma_2) + \sigma_3\sigma_4/(\sigma_3 + \sigma_4)$. Initially there are 16 possible cluster configurations and so the number of phases will grow exponentially as the renormalisation transformation is repeatedly applied, each with effective conductances lying between σ_A and σ_B. To enable some analytic headway the phases are averaged into two phases at each renormalisation iteration [6]. We have compared the effective conductivity calculated in this way and that obtained numerically by allowing the multi-phase distribution to persist [4] and the results show that the RSRG approach deviates quite substantially from the non-averaged value (see Figure 2). This is because the RSRG approach is appropriate for materials where each of the 16 cluster configurations occurs with equal probability. However the size distribution of the tremas in the SSC greatly affects these probabilities. This leads to a revised probability structure which is detailed in Table 1.

Table 1. Probability distribution of cluster classes in the Shuffled Sierpinski Carpet.

Class		Probability $I\!P_i^{SSC}$
I	A A / A A	$(1-\theta)^4 p_4$
II	A A / A B	$4\theta(1-\theta)^3 p_4$
III	B A / B A	$4\theta^2(1-\theta)^2 p_4 + 1/2(1-\theta)^2 p_2$
IV	A A / B B	$2\theta^2(1-\theta)^2 p_4 + 1/2(1-\theta)^2 p_2$
V	B A / B B	$4\theta^3(1-\theta)p_4 + 2\theta(1-\theta)p_2$
VI	B B / B B	$\theta^4 p_4 + \theta^2 p_2 + p_1$

We generate the SSC on a square grid of dimensions $W \times W$, where $W = 2^\gamma$, as this allows easy identification of the renormalisation clusters at each renormalisation level γ. The smallest trema in the SSC dictates the initial grid size and so $\gamma = \lceil -nlog\delta/log2 \rceil$ (where $\lceil x \rceil$ denotes the least integer greater than or equal to x). The overall side length of the SSC is $[W\delta^n]$ grid cells (where $[x]$ denotes the nearest integer to x). Let p_1 denote the probability that a given cell is in phase B and is part of a trema of cell dimension greater than or equal to 2. Let p_2 denote the probability of being in a cell bordering one of the tremas associated with p_1. Finally, let p_4 denote the probability of a cell being in the remaining area, that is $p_4 = 1 - p_1 - p_2$. The density of the unitary (1×1) phase B cells in phase A is denoted by θ. The derivation of the probability structure in Table 1 is described elsewhere [4] but numerical investigations show that it provides a better model of the SSC than the standard structure [6]. We can utilise this probability structure to examine the dependence of the percolation threshold [7] of the SSC on δ and n.

Theorem. *The number of generation levels n^* required to achieve percolation threshold in a SSC, with length generator δ is*

$$n^* = 1 - \frac{log(2 - \delta^2 - 2\delta^4)}{log(1 - \delta^2)} \tag{3}$$

where the volume fraction of phase B is,

$$p_B^{*SSC} = (\frac{5}{2}\delta^4 - \delta^8 - \delta^2 - \frac{1}{2})(1 - \delta^2)^{2n-2} - (\frac{1}{2} - \delta^2 + \frac{1}{2}\delta^4)(1 - \delta^2)^{n-1} + 1. \tag{4}$$

Proof. The percolation threshold is achieved at the (unstable) fixed point of the renormalisation group transformation given by $p_B^{\gamma+1} = p_B^\gamma$. That is,

$$\begin{aligned} p_B^{\gamma+1} &= \sum_{i=IV}^{VI} IP_i^\gamma \\ &= 1/4IP_{II}^\gamma + 1/2IP_{III}^\gamma + 1/2IP_{IV}^\gamma + 3/4IP_V^\gamma + IP_{VI}^\gamma \\ &= p_B^\gamma \end{aligned}$$

That is,

$$1/4IP_{II}^\gamma + 1/2IP_{III}^\gamma - 1/2IP_{IV}^\gamma - 1/2IP_V^\gamma = 0$$

i.e.

$$((1-\theta)^2 + \theta(1-\theta) - 2\theta^2)p_4 - p_2 = 0. \quad (*)$$

Now for large W we have $W = \delta^{-n}$, $N(\delta) = 1/\delta^2 - 1$ and so

$$p_1 = \sum_{i=1}^{n-1} \frac{[W\delta^i]^2 \lceil N(\delta)^{i-1} \rceil}{W^2} = 1 - (1 - \delta^2)^{n-1}.$$

Hence, $p_2 = p_1(1 - p_1) = (1 - (1 - \delta^2)^{n-1})(1 - \delta^2)^{n-1}$, $p_4 = (1 - \delta^2)^{2n-2}$ and

$$\theta = \frac{N(\delta)^{n-1}/W^2}{(1 - p_1)} = \frac{(1/\delta^2 - 1)^{n-1}/\delta^{-2n}}{(1 - \delta^2)^{n-1}} = \delta^2.$$

Therefore $(*)$ can now be written,

$$(1 - \delta^2)^{n-1}(2 - \delta^2 - 2\delta^4) = 1$$

and hence

$$n^* = 1 - \frac{log(2 - \delta^2 - 2\delta^4)}{log(1 - \delta^2)}.$$

Now $p_B^{*SSC} = \sum_{i=IV}^{VI} IP_i^\gamma = (2\theta^2 - \theta^4)p_4 + (1 - 1/2(1 - \theta)^2)p_2 + p_1$ and substituting using the above we obtain the result.

So for example if $\delta = 0.2$, then $n^* = 18$. Note that to test this hypothesis would be difficult using a numerical method such as the Hoshen-Kopelman algorithm [8] since the corresponding matrix containing the original tessellation would have some 10^{25} elements ! Here $p_B^{*SSC} = 0.636$ which is significantly higher than that for the non-fractal case ($p_B^{*Random} = 0.618 = (-1 + \sqrt{5})/2$). We can view the SSC as a poorly mixed or lumpy counterpart of the non-fractal case and hence requires a far larger phase B volume fraction for a percolation path to be achieved.

We can take advantage of this by utilising the simpler RSRG structure of the non-fractal case to calculate the effective thermal conductivity of the SSC by using an equivalent volume fraction $\bar{p}_B^{SSC}$. Thus we reduce the volume fraction of the SSC by a factor determined by the difference between p_B^{*SSC} and $p_B^{*Random}$. That is, $\bar{p}_B^{SSC} = (1 - (p_B^{*SSC} - p_B^{*Random}))p_B^{SSC}$. The revised model gives a better estimate of the thermal conductivity for the SSC (see Figure 2) particularly as we near percolation threshold. Note that as $\delta \to 0^+$, $dim_B F \to 2$ and $n^* \to \infty$. So we see that as F fills two dimensional space more effectively we require exponentially more generation levels of conducting tremas to form a spanning cluster.

3 THERMAL CONDUCTIVITY OF SHUFFLED SIERPINSKI CARPETS

In this section we use the SSC geometry as a model for the fouling geometry discussed in section 1. As such the tremas now act as the insulator whilst the set F acts as the conducting medium. Of course it is still possible to use a RSRG approach to estimate the thermal conductivity but here we adopt a second approach which provides an estimate for the thermal conductivity as a function of $dim_B F$.

By utilising bounds on the thermal conductivity of random media in the limit of low porosity Thovert et al [9] were able to derive an expression for the thermal conductivity σ of regular fractals, By stipulating that the constituent particles of F have unit conductivity we get the following estimate for the thermal conductivity of a SSC,

$$\sigma = (1 - \frac{1 - \delta^{(2-dim_B F)}}{1 - 2g})^n, \qquad (5)$$

where n is the number of generation levels in the construction of the fractal and g is a morphological parameter characterising the pore shape (as the structures are defined on a square lattice we take $g = 0.27$ [9]).

In applying this formula to naturally occurring deposits, or indeed Monte Carlo simulations, we must determine the parameters δ, $dim_B F$ and n. The $W \times W$ binary matrix, which describes the SSC geometry, can be recoded using the Hoshen-Kopelman algorithm [8] to assign a distinct integer to each of the d individual tremas. The size distribution of these pores $\psi = \{(\delta_i, N_i) : i = 1, ..., d\}$ can then be readily determined, where the pore size δ is the square root of the number of cells in the particular pore. The remaining surface area ϕ_i at length scale δ_i follows the scaling law $\phi_i \propto \delta_i^{2-dim_B F}$, where $\phi_i = 1 - \sum_{j=1}^{i} \delta_j^2 N_j$. A least squares estimate of the gradient of a log-log plot of ϕ_i versus δ_i, allied with Eq.(2) provides an estimate for the length scale generator $\delta^\star$. Hence, an estimate for the number of generation levels is given by $n^\star = log(\delta^\star \mathrm{Min}\{\delta_i\}/\mathrm{Max}\{\delta_i\})/log\delta^\star$. This technique has been successfully tested on objects of known fractal parameters and good least squares fits have been obtained with random deposit structures [4].

Due to the paucity of experimental cross-sectional images we have employed Monte Carlo simulations to provide more realistic geometries of fouling deposits. Here we have the added benefit that we have knowledge of and control over the various particle transport mechanisms and deposit formation mechanisms. Such simulations can incorporate a range of transport mechanisms, particle sticking probabilities and particle size distributions, and also interface easily with bulk flow calculations at the boundary layer interface [4]. A typical simulated deposit is shown is Figure 3.

We have tested Eq.(5) by first solving the Laplace equation in F and then, from Fourier's law, calculating the thermal conductivity [10]. This can obviously only be conducted for low generation levels of the SSC or Monte Carlo simulations consisting of only a few tens of thousands of particles. A full description of the methodology and results is presented in [4]. In summary, the results are very encouraging and this improves as the generation level increases. Unfortunately the computer memory requirements preclude high generation level calculations. This approach seems to provide good estimates for the thermal conductivity of the SSC and the algorithm used to recover the underlying length scale generator and fractal dimension is robust.

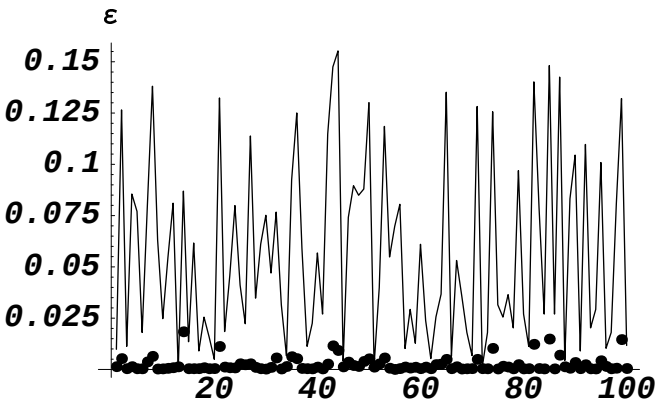

Figure 2: Relative errors ϵ in estimates for the thermal conductivity σ of the Shuffled Sierpinski Carpet using renormalisation with the standard probability distribution ($-$) and that using renormalisation with the revised probability structure ($\bullet$). The calculations are shown for 100 different SSC which satisfy $\delta \in [0.3, 0.4]$, $n \in \{1, ..., 5\}$, $\sigma_A = 0.05$ and $\sigma_B = 0.8$ (the estimate for each particular SSC is averaged over 100 realisations).

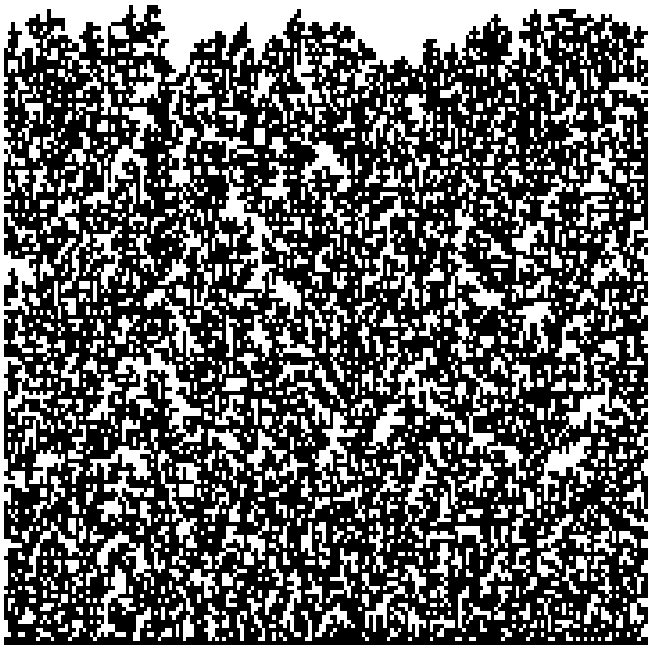

Figure 3: A Monte Carlo simulation of fouling geometry (2.5×10^4 mono-dispersed particles)

4 CONCLUSIONS

Recent experimental evidence suggests that the geom-

etry and thermal conductivity of fouling deposits growing in industrial heat exchanger environments can be described by a fractal model. We have developed theoretical models for the thermal conductivity of two dimensional cross-sections of such materials, based on renormalisation group techniques and fractal methodologies. We restrict our attention to two phase composites whose particle size distribution we model using a Shuffled Sierpinski Carpet (SSC) structure. This model led to a revised probability structure in a Real Space Renormalisation analysis of the structure. In this way we derived estimates of the thermal properties but this framework also affords an analytical hold on the percolation threshold behaviour of finite generation SSC's. The SSC can be viewed as a poorly mixed counterpart of a mono-dispersed, random two-phase media. As such the volume fraction of conducting tremas at percolation threshold is always higher. We were able to derive an equation relating the generation level of the SSC to its length scale generator at percolation threshold. We have shown that the thermal conductivity estimates compare favourably with those obtained using a standard RSRG approach.

In section 3 we presented a second approach based on bounds for the conductivity in the limit of high volume fraction and high contrast in the intrinsic conductivity of both constituents. We used this to develop a recursion relationship for the thermal conductivity which has an explicit dependency on the box counting dimension of the structure. Here the supporting medium F is a conductor and the tremas are perfect insulators. Of course to use this method with naturally occurring deposits we need to model the geometry as an SSC. We have therefore given details of how one can extract the length scale generator δ and generation level n from a given geometry. Importantly, this approach is valid for objects which are fractal over a finite range of length scales, which is a prerequisite for the correct modelling of any natural fractal. As there is a paucity of experimental information on explicit pore structure in relation to thermal conductivity we have relied on Monte Carlo simulation of realistic fouling geometries to test this approach. For low generation levels we have solved the relevant field equation using finite differences and the results show good agreement with the model predictions. In general the RSRG methods underestimate the thermal conductivity, although this approach has the advantage that there is no restriction on the relative intrinsic conductivities of the two phases. The recursive approach does perform better but at present this is restricted to composites with a high ratio between the intrinsic conductivities of the constituents. The relaxation of this latter point is the subject of our current investigations.

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NOMENCLATURE

All units have been non-dimensionalised.

Latin

$dim_b F$	box counting dimension of the set F.
d	number of tremas in a random deposit structure.
E	Sierpinski Carpet pre-fractal.
F	(Shuffled) Sierpinski Carpet set.
g	statistical morphological parameter.
$N(\delta)$	number of squares of size δ.
n	fractal generation level.
p	volume fraction or probability.
IP_i	renormalisation cluster configuration probability.
W	SSC grid dimension.

Greek

β	renormalisation tessellation.
δ	fractal length scale generator.
ϵ	relative errors in thermal conductivity estimates.
ϕ	remaining surface area of random deposit.
γ	renormalisation level.
ψ	pore size distribution in random deposit.
σ	thermal conductivity.
θ	density of unitary phase B cells in phase A.

Subscripts

A	phase A.
B	phase B.
e	effective conductivity.
$I, ..., VI$	renormalisation cluster type.

Superscripts

$Random$	mono-dispersed random two-phase composite.
SSC	shuffled Sierpinski carpet.
$*$	at percolation threshold.
$\bar{}$	equivalent volume fraction.
$\star$	estimated fractal generation parameters in a random deposit structure.

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