Comment on "Application of $(\frac{G'}{G})$-expansion method to travelling wave solutions of three nonlinear evolution equation" [Computers & Fluids 2010;39:1957–1963]

E.J. Parkes [a]

[a] *Department of Mathematics & Statistics, University of Strathclyde, Livingstone Tower, Richmond Street,Glasgow G1 1XH, UK*

Abstract

In a recent paper [Abazari R. Application of $(\frac{G'}{G})$-expansion method to travelling wave solutions of three nonlinear evolution equation. Computers & Fluids 2010;39:1957–1963], the (G'/G)-expansion method was used to find travelling-wave solutions to three nonlinear evolution equations that arise in the mathematical modelling of fluids. The author claimed that the method delivers more general forms of solution than other methods. In this note we point out that not only is this claim false but that the delivered solutions are cumbersome and misleading. The extended tanh-function expansion method, for example, is not only entirely equivalent to the (G'/G)-expansion method but is more efficient and user-friendly, and delivers solutions in a compact and elegant form.

Key words: Nonlinear evolution equations; Travelling-wave solutions; $(\frac{G'}{G})$-expansion method; Tanh-function expansion method.
PACS: 02.30.Jr.

Over the past two decades or so several methods for finding travelling-wave solutions to nonlinear evolution equations have been proposed, developed and extended. The solutions to dozens of equations have been found by one or other of these methods. References [1–12] and some of the references therein mention some of this activity.

One recent method that has proved to be popular is the (G'/G)-expansion method originally proposed by Wang et al. [1]. The solutions delivered by this method look rather cumbersome; furthermore, they appear to have more free parameters than solutions delivered by other methods. This has led to two unfortunate phenomena:

Email address: `e.j.parkes@strath.ac.uk` (E.J. Parkes).

(i) some authors claim that the (G'/G)-expansion method delivers solutions that are 'more general' than those delivered by other methods;
(ii) some authors go on to claim that they have found 'new' solutions when often the truth is that the so-called 'new' solutions are merely disguised versions of previously known solutions that may be found by more efficient methods.

The claim in phenomenon (i) was shown to be false independently in three recent papers [2–4]; in each of these papers it is shown that the (G'/G)-expansion method is entirely equivalent to the extended tanh-function expansion method originally proposed by Fan [5]. Furthermore, in [4] we showed that the latter method is more efficient and more user-friendly than the former method. This was illustrated with reference to the KdV equation.

To illustrate both phenomena we focus on a paper that appeared recently in this journal [6]. In [6], the (G'/G)-expansion method was applied to three nonlinear evolution equations that arise in the mathematical modelling of fluids. For illustrative purposes we consider the solutions only for the first of these equations, namely the transformed reduced Ostrovsky equation (dubbed the Vakhnenko–Parkes equation in [6])

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0; \tag{1}$$

similar observations may be made about the solutions to the other two equations, namely the regularized long wave (RLW) equation and the symmetric RLW equation.

In [7] we summarized the derivation of nine solutions to Eq. (1) that are delivered efficiently by the extended tanh-function expansion method, namely

$$u_{11} = 6K^2 \operatorname{sech}^2(K\eta), \tag{2}$$
$$u_{12} = -6K^2 \operatorname{cosech}^2(K\eta), \tag{3}$$
$$u_{13} = -6K^2 \sec^2(K\eta), \tag{4}$$
$$u_{14} = -6K^2 \operatorname{cosec}^2(K\eta), \tag{5}$$
$$u_{21} = -4K^2 + 6K^2 \operatorname{sech}^2(k\eta), \tag{6}$$
$$u_{22} = -4K^2 - 6K^2 \operatorname{cosech}^2(k\eta), \tag{7}$$
$$u_{23} = 4K^2 - 6K^2 \sec^2(K\eta), \tag{8}$$
$$u_{24} = 4K^2 - 6K^2 \operatorname{cosec}^2(K\eta), \tag{9}$$
$$u_3 = -6/\eta^2, \tag{10}$$

where $\eta := x - ct - x_0$, and K, c and x_0 are arbitrary real constants. (Note that use of a computer takes the drugery out of applying expansion methods by hand. For example, the basic tanh-function method may be applied with minimal effort by use of the automated tanh-function method [8] which uses the Mathematica package `ATFM`.) In [7], we also described where these solutions have appeared before in the literature.

In [6] it is shown that the solution to Eq. (1) is given by

$$u(x,t) = -6k^2 \left[\left(\frac{G'}{G} \right)^2 + \lambda \left(\frac{G'}{G} \right) + \mu \right], \tag{11}$$

where k, λ and μ are arbitrary constants, and (G'/G) has three different forms depending on whether $\lambda^2 - 4\mu$ is positive, negative or zero. For example, when $\lambda^2 - 4\mu > 0$,

$$\frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{C_1 \sinh\theta + C_2 \cosh\theta}{C_1 \cosh\theta + C_2 \sinh\theta} \right) - \frac{\lambda}{2}, \tag{12}$$

where $\theta = \sqrt{\lambda^2 - 4\mu}\,\xi/2$, $\xi = kx + \omega t$, and ω, C_1 and C_2 are arbitrary constants. In this case (11) becomes (18a) in [6], namely

$$u(x,t) = \frac{3k^2(\lambda^2 - 4\mu)(C_1^2 - C_2^2)}{2(C_2 \sinh\theta + C_1 \cosh\theta)^2}. \tag{13}$$

If $C_1^2 > C_2^2$, then (13) may be rewritten as

$$u(x,t) = \frac{3}{2} k^2(\lambda^2 - 4\mu) \operatorname{sech}^2(\theta + \theta_0), \tag{14}$$

where $\tanh\theta_0 := C_2/C_1$; if $C_1^2 < C_2^2$, then (13) may be rewritten as

$$u(x,t) = -\frac{3}{2} k^2(\lambda^2 - 4\mu) \operatorname{cosech}^2(\theta + \theta_0), \tag{15}$$

where $\coth\theta_0 := C_2/C_1$.

The solutions given by (14) and (15) are equivalent to (18b) and (18c) in [6]. Note that, apparently, these two solutions have six free parameters, namely k, ω, λ, μ, C_1 and C_2. However, if we introduce the quantities c, K and x_0 defined by

$$c = -\frac{\omega}{k}, \quad K = \frac{k\sqrt{\lambda^2 - 4\mu}}{2}, \quad Kx_0 = -\theta_0, \tag{16}$$

then (14) and (15) reduce to u_{11} and u_{12}, respectively. In this form the solutions have only three free parameters, namely c, K and x_0. Similarly, with

$$c = -\frac{\omega}{k}, \quad K = \frac{k\sqrt{4\mu - \lambda^2}}{2}, \quad Kx_0 = -\theta_0, \tag{17}$$

the solutions given by (19b) and (19c) in [6] can be rewritten as u_{13} and u_{14}, respectively. A fifth solution is given by (20) in [6]; apparently it has four free parameters. However, with $c = -\omega/k$ and $x_0 = -C_1/(kC_2)$, the solution reduces to u_3 which has only two free parameters, namely c and x_0. Solutions corresponding to u_{21}, u_{22}, u_{23} and u_{24} were not presented in [6].

The foregoing illustrative discussion exemplifies two deficiencies of the (G'/G)-expansion method: firstly, that the method delivers solutions in a cumbersome form (see (11) with (12), or (13), for example) and secondly, that the solutions appear to contain more free parameters than is actually the case. Typically, each solution can be manipulated into a neater form which displays the correct number of free parameters. These neat forms are the ones that are delivered directly and more efficiently by the extended tanh-function method.

It is of interest to note that the introduction of the parameter k into the (G'/G)-expansion method as presented in [6] is superfluous. This embellishment is not present in the descriptions of the method in [1–4]. Furthermore, as has been pointed out in [9], the parameter λ is superfluous at least as far as the basic method is concerned. These observations are exemplified by setting $k = 1$ and $\lambda = 0$ in the above argument: the generality of the solutions is unaffected.

Finally, we make some additional pertinent comments regarding Eq. (1). In [10], the Exp-function method was used to derive two solutions to Eq. (1). These solutions are rather cumbersome but, as shown in [11], they may be reduced to u_{11} and u_{21} respectively. Reference to [10] is also made in [6]; in the latter it is shown that, by use of particular values of some of the parameters, (14) reduces to a special case of u_{11} which is also one of the special cases mentioned in [10]. In [12], an auxiliary equation method was used to solve Eq. (1). Twenty eight solutions were derived including 'many new' solutions. However, in [11] we showed that all twenty eight solutions may be reduced to u_{11} or u_{12}. In [13], the solutions u_{11}, u_{21} and u_3 were derived via a method involving Laurent series.

In [13], periodic-wave solutions to Eq. (1) in terms of the Jacobi elliptic cn function were derived by direct integration. These solutions may also be derived easily by the Jacobi elliptic-function expansion method as outlined in [14], for example. We have used the semi-automated procedure that we described in [14] to obtain the aforementioned solutions in a more user-friendly form, namely

$$u(x,t) = 6mK^2\,\mathrm{cn}^2(K\eta|m) + 2\left(1 - 2m \pm \sqrt{1 - m + m^2}\right)K^2, \tag{18}$$

where K, c and x_0 are arbitrary, and m is a parameter such that $0 < m \leq 1$. In the limit $m \to 1$, the two solutions given by (18) reduce straightforwardly to u_{11} and u_{21}, respectively.

Acknowledgement

I am grateful to a referee for drawing my attention to Ref. [13].

References

[1] Wang M, Li X, Zhang J. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys Lett A 2008;372:417–423.

[2] Liu CP. (G'/G)-expansion method equivalent to extended tanh function method. Commun Theor Phys 2009;51:985–988.

[3] Kudryashov NA. A note on the G'/G-expansion method. Appl Math Comput 2010;217:1755-1758.

[4] Parkes EJ. Observations on the basic (G'/G)-expansion method for finding solutions to nonlinear evolution equations. Appl Math Comput 2010;217:1759–1763.

[5] Fan E. Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 2000;277:212–218.

[6] Abazari R. Application of $(\frac{G'}{G})$-expansion method to travelling wave solutions of three nonlinear evolution equation. Comput Fluids 2010;39:1957–1963.

[7] Parkes EJ. A note on "New travelling wave solutions to the Ostrovsky equation". Appl Math Comput 2010;217:3575–3577.

[8] Parkes EJ, Duffy BR. An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations. Comput Phys Commun 1996;98:288–300.

[9] Aslan I. A note on the (G'/G)-expansion method again. Appl Math Comput 2010;217:937–938.

[10] Köroğlu C, Öziş T. A novel traveling wave solution for Ostrovsky equation using Exp-function method. Comput Math Appl 2009;58:2142–2146.

[11] Parkes EJ. A note on solitary travelling-wave solutions to the transformed reduced Ostrovsky equation. Commun Nonlinear Sci Numer Simulat 2010;15:2769–2771.

[12] Kangalgil F, Ayaz F. New exact travelling wave solutions for the Ostrovsky equation. Phys Lett A 2008;372:1831–1835.

[13] Kudryashov NA. Meromorphic solutions of nonlinear ordinary differential equations. Commun Nonlinear Sci Numer Simulat 2010;15:2778–2790.

[14] Parkes EJ, Duffy BR, Abbott PC. The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations. Phys Lett A 2002;295:280–286.