A TWO-LEVEL ENRICHED FINITE ELEMENT METHOD FOR A MIXED PROBLEM

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Abstract. The simplest pair of spaces $\mathbb{P}_1/\mathbb{P}_0$ is made inf-sup stable for the mixed form of the Darcy equation. The key ingredient is enhance the finite element spaces inside a Petrov-Galerkin framework with functions satisfying element-wise local Darcy problems with right hand sides depending on the residuals over elements and edges. The enriched method is symmetric, locally mass conservative and keeps the degrees of freedom of the original interpolation spaces. First, we assume local enrichments exactly computed and we prove uniqueness and optimal error estimates in natural norms. Then, a low cost two-level finite element method is proposed to effectively obtain enhancing basis functions. The approach lays on a two-scales numerical analysis and shows that well-posedness and optimality is kept despite of the second level numerical approximation. Several numerical experiments validate the theoretical results and compares (favourably in some cases) our results with the classical Raviart-Thomas element.

1. Introduction

The selection of finite dimensional spaces for the Galerkin method demands careful attention when it comes to solve the weak form of mixed boundary value problems. In fact, mixed problems can be handled by polynomial interpolations if the pair of spaces fulfill the well-know inf-sup condition [11]. This leads some very popular choices such as equal order interpolations spaces and the simplest element $\mathbb{P}_1/\mathbb{P}_0$ out of reach, or it prevents nodal values to be chosen as degrees of freedom if some physical properties (such as local conservation of mass) are to be satisfied by the numerical method (cf. [9]).

On the quest to systematically build stable and accurate finite element methods, numerical solutions have been formally decomposed into a solved and an unsolved part with respect to a fixed mesh. Roughly, the resolved part solves the original weak form on a given finite

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element space regardless any numerical drawback, while the unsolved part comes into play to make the complete solution to be free of numerical troubles and recover missed physical properties as well. The final methods are said to incorporate "missing scales" not captured by the mesh, and so, named multi-scale methods. Different approaches to model the unsolved scales have been proposed over the past years, among which we might mention the Variational Multi-Scale method (VMS) [24] and variations of it [3, 16], the Residual-Free-Bubble approach (RFB) [14, 27, 13] and its practical implementation using a two-level method [22], the Heterogeneous Multi-Scale Methods (HMM) [18], and recently, Petrov-Galerkin Enriched Methods (PGEM) (see [21, 20, 5], and [1] for a survey). Overall, algorithms are closely related by making the unsolved contribution dependent on geometrical aspects of the mesh, the interpolation space choice and the boundary value problem itself. In parallel, RFB and PGEM have been systematically related to stabilized methods and stabilization parameters been obtained with respect the mean value of enriching basis functions [4, 12, 7, 8, 2].

This work addresses the subject for the Darcy equation, a model that appears in porous media, in the Petrov-Galerkin enriched framework. The PGEM has been introduced as a way to incorporate edge residual contribution into the unsolved scale modeling, an aspect neglected by the RFB and responsible for the non-physical oscillations in the numerical solutions. In [5] the standard trial and test finite element spaces $\mathbb{P}_1/\mathbb{P}_0$ are differently enriched. The latter remains been enhanced with bubble functions as in the RFB method, but now the trial space incorporates functions driven by the Darcy operator depending on the residual of the equation in each element and on each edge of the triangulation. The boundary conditions for the local problems are set in such a way they ensure the continuity of both the normal component of the enriched velocity (strongly) and pressure (weakly). Thereby, the desired features of the $\mathbb{P}_1/\mathbb{P}_0$ element are preserved, namely its simplicity along with its nodal-valued degrees of freedom for the velocity and the locally mass conservative velocity field.

Two non symmetric enriched methods were proposed in [5] and extensive numerical validations have been carried out for one of the methods in which the solution to the local problems is known analytically. However, neither numerical analysis nor numerical validations have been proposed for the original (two level) method from [5]. This work aims to overcome these shortcomings and introduce, moreover, a symmetric method. Keeping the simplest element $\mathbb{P}_1/\mathbb{P}_0$ as the target spaces, and assuming that enriching local problems are exactly solved, we prove existence and uniqueness for both the original non symmetric and the symmetric methods, as well as optimal error estimates in natural norms. In fact, we show the leading error between both methods tends faster to zero than the error itself when the characteristic length of the mesh goes to zero.

Next, the semi discrete method analyzed before is completely discretized using a two-level approach incorporating numerically computed unsolved scales into the enriched method. The order of convergence is not affected by this approximation under mild conditions on the fine scale discretization. It is worth mentioning that, up to our knowledge, few works incorporate fine scales approximation in the numerical analysis. Moreover, a low-cost procedure has been proposed to effectively incorporate the subscales. As a matter of fact, as fine scale mesh, one single $\mathbb{P}_1$ element is used throughout all the numerical experiments, keeping optimal convergence. Although not theoretically proved, numerical results highlight quadratic convergence for the velocity in the L^2 norm, a feature that is clearly not expected for the Raviart-Thomas element [26, 11]. The comparison with the standard Raviart-Thomas method is pushed further and allow us to outline the main features of the two-level enriched method, namely:

- has lower number of degrees of freedom for a fixed mesh;
- induces a symmetric linear system (if we were disposed to relax the symmetry requirement, then an equivalent non symmetric but positive definite system may be proposed);
- keeps nodal-valued degrees of freedom for the velocity;
- is locally mass conservative.

The remainder of the paper is as follows: the current section ends with notations and preliminary results; for the sake of completeness the derivation of the PGEM in [5] is revisited in Section 2, and then, the resulting methods are mathematically analyzed in Section 3. Section 4 is devoted to the two-level enriched method which is numerically validated in Section 5. Conclusions are drawn in Section 6.

1.1. **Notations.** This section introduces definitions and notations used throughout. In what follows, Ω denotes an open bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$, and $\mathbf{x} = (x_1, x_2)$ is a typical point in Ω. As usual, $L^2(\Omega)$ is the space of square integrable functions over Ω, $L^2_0(\Omega)$ represents functions belonging to $L^2(\Omega)$ with zero average in Ω, and $H(div,\Omega)$ ($H(curl,\Omega)$) is composed by functions that belong to $L^2(\Omega)^2$ with divergence (curl) in $L^2(\Omega)$. The space $H_0(div,\Omega)$ stands for the space of functions belonging to $H(div,\Omega)$ which have normal component vanishing on $\partial\Omega$. Finally, $(\cdot\,,\cdot)_D$ stands for the inner product in $L^2(D)$ (or in $L^2(D)^2$, when necessary), and $\|\cdot\|_{s,D}$ ($|\cdot|_{s,D}$) the norm (seminorm) in $H^s(D)$ (or $H^s(D)^2$), $|\cdot|_{0,D} = \|\cdot\|_{0,D}$, and $\|\cdot\|_{div,D}$ the norm in $H(div,D)$.

From now on we denote by $\{\mathcal{T}_H\}$ a family of regular triangulations of $\overline{\Omega}$ built up using triangles K with boundary ∂K composed by edges F. The set of internal edges of the triangulation $\mathcal{T}_H$ is denoted by $\mathcal{E}_H$. The characteristic length of K and F are denoted by H_K and H_F, respectively, and $H := \max\{H_K : K \in \mathcal{T}_H\}$, and due to the mesh regularity there exists a positive constant C such that $H_F \leq H_K \leq C\, H_F$, for all $F \subseteq \partial K$. Also, for each $F = K \cap K' \in \mathcal{E}_H$ we choose, once and for all, an unit normal vector $\boldsymbol{n}$ which coincides with the unit outward normal vector when $F \subseteq \partial\Omega$. The standard outward normal vector at the edge F with respect to the element K is denoted by $\boldsymbol{n}_F^K$. Moreover, for a function q, $[\![q]\!]$ denotes its jump, defined by (see Figure 1):

$$[\![q]\!](\mathbf{x}) := \lim_{\delta\to 0^+} q(\mathbf{x} + \delta\boldsymbol{n}) - \lim_{\delta\to 0^-} q(\mathbf{x} + \delta\boldsymbol{n})\,, \tag{1}$$

and $[\![q]\!] = 0$ if $F \subseteq \partial\Omega$. We finally introduce the following broken spaces:

$$H_0(div, \mathcal{T}_H) := \{\boldsymbol{v} \in L^2(\Omega)^2 \,:\, \boldsymbol{v}\big|_K \in H_0(div, K)\,\forall K \in \mathcal{T}_H\}\,, \tag{2}$$

$$L_0^2(\mathcal{T}_H) := \{q \in L^2(\Omega) \,:\, q\big|_K \in L_0^2(K)\,\forall K \in \mathcal{T}_H\}\,. \tag{3}$$

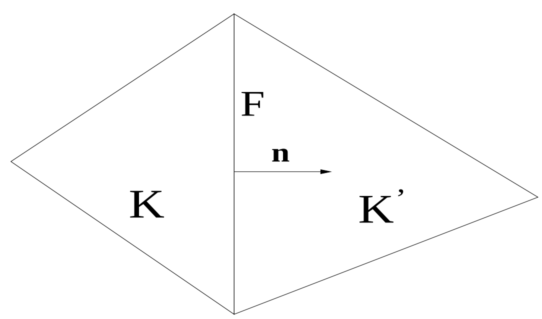

FIGURE 1. The normal vector.

1.2. **Preliminaries.** In this work we consider the following Darcy problem: Find $(\boldsymbol{u}, p)$ such that

$$\begin{gathered} \sigma\,\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u} = g \quad \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4}$$

where $\sigma = \frac{\mu}{\kappa} \in \mathbb{R}^+$ is assumed constant in Ω, with μ and κ denoting the viscosity and permeability, respectively. Here, $\boldsymbol{u}$ is the so-called Darcy velocity, p is the pressure, $\boldsymbol{f}$ and g are given source terms. We suppose $\boldsymbol{f}$ piecewise constant since it is usually related to the gravity force (see the numerical experiments for an extension to the general case). Moreover, we assume that the usual compatibility condition

$$\int_\Omega g = 0,$$

holds.

The standard mixed variational formulation associated with (4) reads: *Find* $(\boldsymbol{u}, p) \in H_0(div, \Omega) \times L^2_0(\Omega)$ *such that*

$$\mathbf{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{L}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in H_0(div, \Omega) \times L^2_0(\Omega), \tag{5}$$

where

$$\mathbf{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) := (\sigma\, \boldsymbol{u}, \boldsymbol{v})_\Omega - (p, \nabla\cdot \boldsymbol{v})_\Omega - (q, \nabla\cdot \boldsymbol{u})_\Omega, \quad \boldsymbol{L}(\boldsymbol{v}, q) := (\boldsymbol{f}, \boldsymbol{v})_\Omega - (g, q)_\Omega.$$

The well-posedness of (5) follows from the classical Babuska-Brezzi theory for variational problems with constraints (see [11] for details).

2. The enriched finite element method

We start generalizing the derivation carried out in [5]. We introduce the standard finite element space $\mathbf{V}_H := [V_H]^2 \cap H_0(div, \Omega)$ for the velocity variable, where

$$V_H := \{v \in C^0\big(\overline{\Omega}\big) : v|_K \in \mathbb{P}_1(K),\, \forall K \in \mathcal{T}_H\}, \tag{6}$$

whereas the pressure is discretized using the space

$$Q_H := \{q_0 \in L^2_0(\Omega) : q_0|_K \in \mathbb{P}_0(K)\; \forall\, K \in \mathcal{T}_H\}\,. \tag{7}$$

Our starting point is the following Petrov-Galerkin scheme: *Find* $\boldsymbol{u}_H = \boldsymbol{u}_1 + \boldsymbol{u}_e \in \mathbf{V}_H + H_0(div, \Omega)$ *and* $p_H = p_0 + p_e \in Q_H \oplus L^2_0(\mathcal{T}_H)$ *such that*

$$\mathbf{A}((\boldsymbol{u}_H, p_H), (\boldsymbol{v}_H, q_H)) = \boldsymbol{L}(\boldsymbol{v}_H, q_H), \tag{8}$$

for all $\boldsymbol{v}_H = \boldsymbol{v}_1 + \boldsymbol{v}_b \in \mathbf{V}_H \oplus H_0(div, \mathcal{T}_H)$ and for all $q_H = q_0 + q_b \in Q_H \oplus L^2_0(\mathcal{T}_H)$.

Thanks to the choice made for the test function enrichment space, we can split (8) as the following system:

$$\mathbf{A}((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_0 + p_e), (\boldsymbol{v}_1, q_0)) = \boldsymbol{L}(\boldsymbol{v}_1, q_0) \qquad \forall\, (\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H, \tag{9}$$

$$\mathbf{A}((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_0 + p_e), (\boldsymbol{v}_b, q_b)) = \boldsymbol{L}(\boldsymbol{v}_b, q_b) \qquad \forall\, (\boldsymbol{v}_b, q_b) \in H_0(div, \mathcal{T}_H) \times L^2_0(\mathcal{T}_H). \tag{10}$$

First, considering test functions $(\boldsymbol{v}_b, q_b)$ supported in a single element, we see that (10) is the weak form of the following strong problem for $(\boldsymbol{u}_e, p_e)$:

$$\sigma\, \boldsymbol{u}_e + \nabla p_e \;=\; \boldsymbol{f} - \sigma \boldsymbol{u}_1, \quad \nabla\cdot \boldsymbol{u}_e = g + C_K \quad \text{in } K, \tag{11}$$

where $C_K \in \mathbb{R}$ is, a priori, free. Now, to close this problem we impose the following boundary condition on $\boldsymbol{u}_e$ (see [5]):

$$\sigma\, \boldsymbol{u}_e \cdot \boldsymbol{n} = \alpha_F \int_F [\![p_0]\!]\,, \tag{12}$$

on each $F \subseteq \partial K \cap \Omega$, and $\boldsymbol{u}_e \cdot \boldsymbol{n} = 0$ on $F \subseteq \partial\Omega$. Here, α_F stands for a constant not depending on H or σ, but that can vary on each $F \in \mathcal{E}_H$. This choice for boundary condition leads us to fix the constant C_K, which is given by

$$C_K = \overline{[\![p_0]\!]} - \Pi_K(g) := \frac{1}{|K|} \sum_{i=1}^{3} \frac{\alpha_{F_i}\, H_{F_i}}{\sigma} \int_{F_i} [\![p_0]\!]\, \boldsymbol{n} \cdot \boldsymbol{n}_{F_i}^K - \frac{\int_K g}{|K|}\,.$$

Remark. An alternative method was also derived in [5] by proposing a different boundary condition (see also [6] for the convergence analysis of a related approach). □

Now, we can split $(\boldsymbol{u}_e, p_e) = (\boldsymbol{u}_e^M, p_e^M) + (\boldsymbol{u}_e^D, p_e^D) + (\boldsymbol{u}_e^g, p_e^g)$, solutions of

$$\begin{aligned} \sigma \boldsymbol{u}_e^M + \nabla p_e^M &= \boldsymbol{f} - \sigma \boldsymbol{u}_1\,, \quad \nabla \cdot \boldsymbol{u}_e^M = 0 \quad \text{in } K, \\ \sigma\, \boldsymbol{u}_e^M \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \text{each } F \subseteq \partial K, \end{aligned} \tag{13}$$

$$\begin{aligned} \sigma\, \boldsymbol{u}_e^D + \nabla p_e^D &= \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{u}_e^D = \overline{[\![p_0]\!]} \quad \text{in } K, \\ \sigma\, \boldsymbol{u}_e^D \cdot \boldsymbol{n} &= \alpha_F \int_F [\![p_0]\!] \quad \text{on} \quad \text{each } F \subseteq \partial K\,, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \sigma\, \boldsymbol{u}_e^g + \nabla p_e^g &= \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{u}_e^g = g - \Pi_K(g) \quad \text{in } K, \\ \sigma\, \boldsymbol{u}_e^g \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \text{each } F \subseteq \partial K\,, \end{aligned} \tag{15}$$

respectively. Also, we define the solution operators associated to the above problems such that we can write

$$(\boldsymbol{u}_e^M, p_e^M) = (\mathcal{M}_K^u(\boldsymbol{f} - \sigma \boldsymbol{u}_1), \mathcal{M}_K^p(\boldsymbol{f} - \sigma \boldsymbol{u}_1))\,, \tag{16}$$

$$(\boldsymbol{u}_e^D, p_e^D) = (\mathcal{D}_K^u([\![p_0]\!]), \mathcal{D}_K^p([\![p_0]\!]))\,, \tag{17}$$

$$(\boldsymbol{u}_e^g, p_e^g) = (\mathcal{G}_K^u(g - \Pi_K(g)), \mathcal{G}_K^p(g - \Pi_K(g)))\,, \tag{18}$$

and we remark that the local problem (14) may be solved analytically (cf. [5]), which leads to the following explicit expression for $\boldsymbol{u}_e^D$:

$$\boldsymbol{u}_e^D = \mathcal{D}_K^u([\![p_0]\!]) = \sum_{F \subseteq \partial K} \frac{\alpha_F H_F}{\sigma} [\![p_0]\!]_F\, \boldsymbol{\varphi}_F\,, \tag{19}$$

where $\boldsymbol{\varphi}_F$ is the Raviart-Thomas' basis function defined by

$$\boldsymbol{\varphi}_F(\mathbf{x}) = \pm\frac{H_F}{2|K|}(\mathbf{x}-\mathbf{x}_F), \tag{20}$$

$\mathbf{x}_F$ denotes the node opposite to the edge F and in which the sign on the Raviart-Thomas basis function $\boldsymbol{\varphi}_F$ depends on whether the normal vector $\boldsymbol{n}$ on $F \subseteq \partial K$ points inwards or outwards K.

Now, after the full characterization of $(\boldsymbol{u}_e, p_e)$ given by (16)-(18), we come back to (9). First, since $p_e \in L^2_0(K)$ and $\nabla\cdot \boldsymbol{v}_1\,|_K \in \mathbb{R}$ we obtain

$$(p_e, \nabla\cdot \boldsymbol{v}_1)_K = 0 \quad \text{for all } K \in \mathcal{T}_H. \tag{21}$$

Therefore, the problem (9) becomes: *Find* $(\boldsymbol{u}_1, p_0) \in \mathbf{V}_H \times Q_H$ *such that*

$$\mathbf{A}((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_0), (\boldsymbol{v}_1, q_0)) = \boldsymbol{L}(\boldsymbol{v}_1, q_0) \quad \forall (\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H, \tag{22}$$

or, equivalently using that $\nabla\cdot \boldsymbol{u}_e^M = 0$ and that $\nabla\cdot \boldsymbol{u}_e^g = g - \Pi_K(g)$ is orthogonal to any constant in K,

(23)
$$\sigma\,(\boldsymbol{u}_1 + \boldsymbol{u}_e^M + \boldsymbol{u}_e^D + \boldsymbol{u}_e^g, \boldsymbol{v}_1)_\Omega - (p_0, \nabla\cdot \boldsymbol{v}_1)_\Omega - (q_0, \nabla\cdot \boldsymbol{u}_1)_\Omega - (q_0, \nabla\cdot \boldsymbol{u}_e^D)_\Omega = \boldsymbol{L}(\boldsymbol{v}_1, q_0)\,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$. Next, integrating by parts in each $K \in \mathcal{T}_H$ and using the boundary condition (12) we obtain

$$\sum_{K\in\mathcal{T}_H} (q_0, \nabla\cdot \boldsymbol{u}_e^D)_K = \sum_{F\in\mathcal{E}_H} \tau_F\,([\![p_0]\!], [\![q_0]\!])_F\,, \tag{24}$$

where

$$\tau_F := \frac{\alpha_F H_F}{\sigma}\,. \tag{25}$$

Remark. The term related to $\boldsymbol{f}$ in (16) vanishes. Indeed, since $\boldsymbol{f}$ is constant in K then it is easy to realize that $\mathcal{M}_K^u \boldsymbol{f} = 0$, which leads to $\boldsymbol{u}_e^M = -\sigma\mathcal{M}_K^u(\boldsymbol{u}_1)$. $\square$

Finally, based on the previous remark, and replacing (24) and (16)-(18) in (22), we arrive at the following form of our enriched method: *Find* $(\boldsymbol{u}_1, p_0) \in \mathbf{V}_H \times Q_H$ *such that*

$$\mathbf{A}((\boldsymbol{u}_1 - \sigma\mathcal{M}_K^u(\boldsymbol{u}_1), p_0), (\boldsymbol{v}_1, q_0)) + \sum_{K\in\mathcal{T}_H} (\mathcal{D}_K^u([\![p_0]\!]), \sigma\,\boldsymbol{v}_1)_K$$
$$- \sum_{F\in\mathcal{E}_H} \tau_F\,([\![p_0]\!], [\![q_0]\!])_F = \boldsymbol{L}(\boldsymbol{v}_1, q_0) - \sum_{K\in\mathcal{T}_H} (\mathcal{G}_K^u(g - \Pi_K(g)), \sigma\boldsymbol{v}_1)_K,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$. Since our aim is to derive a symmetric method, in the following Lemma we further explore the properties of the operator $\mathcal{M}_K^u$.

Lemma 1. The linear operator $\sigma\mathcal{M}_K^u$ is an orthogonal projection with respect to the $L^2(K)^2$ inner product. More precisely, for all $\boldsymbol{v} \in L^2(K)^2$

$$(\boldsymbol{v} - \sigma\mathcal{M}_K^u(\boldsymbol{v}), \boldsymbol{w})_K = 0\,, \tag{26}$$

for all $\boldsymbol{w} \in H_0(div, K)$ such that $\nabla \cdot \boldsymbol{w} = 0$ in K. Moreover,

$$(\mathcal{M}_K^u(\boldsymbol{v}), \nabla\psi)_K = 0 \qquad \forall\, \psi \in H^1(K)\,. \tag{27}$$

Proof. To prove (26) we multiply (13) by a function $\boldsymbol{w} \in H_0(div, K)$ such that $\nabla \cdot \boldsymbol{w} = 0$ in K and integrate by parts to obtain

$$(\sigma\mathcal{M}_K^u(\boldsymbol{v}), \boldsymbol{w})_K - (p_e^M, \nabla \cdot \boldsymbol{w})_K + (p_e^M, \boldsymbol{w} \cdot \boldsymbol{n})_{\partial K} \;=\; (\boldsymbol{v}, \boldsymbol{w})_K\,,$$

and the result follows applying the properties of $\boldsymbol{w}$. Finally, integrating by parts and using that $\mathcal{M}_K^u(\boldsymbol{v})$ has a vanishing divergence and normal component (27) follows. $\square$

Remark. Since $p_e^M \in L_0^2(K)$, then (26) is also valid if $\nabla \cdot \boldsymbol{w} \in \mathbb{R}$ in K. $\square$

Using the previous lemma, and the fact that all Raviart-Thomas' functions are gradients, we can give the following presentation for our method: *Find* $(\boldsymbol{u}_1, p_0) \in \mathbf{V}_H \times Q_H$ *such that*

$$\boldsymbol{B}_{ns}((\boldsymbol{u}_1, p_0), (\boldsymbol{v}_1, q_0)) = \boldsymbol{L}(\boldsymbol{v}_1, q_0) - \sum_{K\in\mathcal{T}_H} (\mathcal{G}_K^u(g - \Pi_K(g)), \sigma\boldsymbol{v}_1)_K, \tag{28}$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$, where

$$\begin{aligned}\boldsymbol{B}_{ns}((\boldsymbol{u}_1, p_0), (\boldsymbol{v}_1, q_0)) :=\; &\mathbf{A}((\rho(\boldsymbol{u}_1), p_0), (\rho(\boldsymbol{v}_1), q_0)) \\ &+ \sum_{K\in\mathcal{T}_H} (\mathcal{D}_K^u([\![p_0]\!]), \sigma\, \rho(\boldsymbol{v}_1))_K - \sum_{F\in\mathcal{E}_H} \tau_F\, ([\![p_0]\!], [\![q_0]\!])_F\,.\end{aligned}$$

To simplify the notation we have noted

$$\rho(\boldsymbol{v})|_K := (\boldsymbol{I} - \sigma\mathcal{M}_K^u)(\boldsymbol{v})\,, \tag{29}$$

where $\boldsymbol{I}$ stands for the identity operator.

Remark. Let $(\boldsymbol{u}_H, p_H) := (\boldsymbol{u}_1 + \boldsymbol{u}_e, p_0 + p_e)$. Then, (28) implies that

$$(\boldsymbol{u}_H, \boldsymbol{v}_1)_\Omega - (p_H, \nabla \cdot \boldsymbol{v}_1)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega \qquad \forall\, \boldsymbol{v}_1 \in \mathbf{V}_H\,,$$

and hence, integrating by parts and using that $\boldsymbol{u}_H + \nabla p_H = \boldsymbol{f}$ in each K we see that p_H satisfies

$$\sum_{F\in\mathcal{E}_H} ([\![p_H]\!], \boldsymbol{v}_1 \cdot \boldsymbol{n})_F = 0 \qquad \forall \boldsymbol{v}_1 \in \mathbf{V}_H\,. \tag{30}$$

Then, when enhanced with p_e, the discrete pressure is weakly continuous. On the other hand, since we have also $[\![\boldsymbol{u}_H \cdot \boldsymbol{n}]\!] = \mathbf{0}$ we see that $[\![\nabla p_H \cdot \boldsymbol{n}]\!] = 0$ on the internal edges. Hence, our method satisfies naturally the conditions requested in [10] in the discontinuous Galerkin framework. $\square$

To complete the derivation we only have to notice that we can neglect the nonsymmetric term. This is ensured by the next lemma. As a matter of fact, the following result ensures us that the nonsymmetric term is of a smaller size than the rest of the terms, and then in Lemma 11 below we will show that the solution of (28) and the solution of the symmetric method (32) below are superclose in the natural norms.

Lemma 2. There exists a positive constant C such that

$$\sum_{K\in\mathcal{T}_H} (\mathcal{D}_K^u([\![q_0]\!]), \sigma\, \rho(\boldsymbol{v}_1))_K \le C \left\{ \sum_{F\in\mathcal{E}_H} \tau_F\, \|[\![q_0]\!]\|_{0,F}^2 \right\}^{\frac{1}{2}} \sqrt{\sigma\alpha}\, H\, \|\rho(\boldsymbol{v}_1)\|_{0,\Omega}\,, \tag{31}$$

for all $\boldsymbol{v}_1 \in \mathbf{V}_H$ and $q_0 \in Q_H$,where $\alpha := \max\{\alpha_F : F \in \mathcal{E}_H\}$.

Proof. Using successively the Cauchy-Schwarz inequality, (19), $\|\boldsymbol{\varphi}_F\|_{0,K} \le C\, H_F$, the definition of τ_F (cf. (25)) and the mesh regularity we get

$$\begin{aligned}
\sum_{K\in\mathcal{T}_H} (\mathcal{D}_K^u([\![q_0]\!]), \sigma\, \rho(\boldsymbol{v}_1))_\Omega &\le \sum_{K\in\mathcal{T}_H} \|\mathcal{D}_K^u([\![q_0]\!])\|_{0,K}\, \sigma\, \|\rho(\boldsymbol{v}_1)\|_{0,K} \\
&\le \sum_{K\in\mathcal{T}_H} \sum_{F\subseteq\partial K} \frac{\alpha_F}{\sigma} \int_F |[\![q_0]\!]|\, \|\boldsymbol{\varphi}_F\|_{0,K}\, \sigma\, \|\rho(\boldsymbol{v}_1)\|_{0,K} \\
&\le C \sum_{K\in\mathcal{T}_H} \sum_{F\subseteq\partial K} \tau_F\, H_F^{\frac{1}{2}}\, \|[\![q_0]\!]\|_{0,F}\, \sigma\, \|\rho(\boldsymbol{v}_1)\|_{0,K} \\
&\le C \left\{ \sum_{F\in\mathcal{E}_H} \tau_F\, \|[\![q_0]\!]\|_{0,F}^2 \right\}^{\frac{1}{2}} H\, \sqrt{\sigma\,\alpha}\, \|\rho(\boldsymbol{v}_1)\|_{0,\Omega}\,,
\end{aligned}$$

which ends the proof. $\square$

Hence, using this result we arrive at the following symmetric (and final) form of our Petrov-Galerkin Enriched Method: *Find $(\boldsymbol{u}_1, p_0) \in \mathbf{V}_H \times Q_H$ such that*

$$\boldsymbol{B}((\boldsymbol{u}_1, p_0), (\boldsymbol{v}_1, q_0)) = \mathbf{F}(\boldsymbol{v}_1, q_0)\,, \tag{32}$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$, where

$$\boldsymbol{B}((\boldsymbol{u}_1, p_0), (\boldsymbol{v}_1, q_0)) := \mathbf{A}((\rho(\boldsymbol{u}_1), p_0), (\rho(\boldsymbol{v}_1), q_0)) - \sum_{F\in\mathcal{E}_H} \tau_F\, ([\![p_0]\!], [\![q_0]\!])_F\,, \tag{33}$$

$$\mathbf{F}(\boldsymbol{v}_1, q_0) := \boldsymbol{L}(\boldsymbol{v}_1, q_0) - \sum_{K\in\mathcal{T}_H} (\mathcal{G}_K^u(g - \Pi_K(g)), \sigma\boldsymbol{v}_1)_K\,, \tag{34}$$

ρ is the operator defined in (29), and the coefficient τ_F is defined in (25).

We end this section by presenting the local mass conservation result. The proof of this Lemma is a direct application of the results from [6], §2.3, and hence we skip the details. We only stress here the fact that the techniques developed here may be applied to any jump-based stabilized finite element method for the Darcy equation, and then, every low order method (e.g. the one from [15]) may be easily post-processed in order to get a locally conservative velocity field.

Lemma 3. Let $\boldsymbol{u}_1$ be the solution of (32) and $\boldsymbol{u}_e^D$ given by (19). Then

$$\int_K \nabla\cdot(\boldsymbol{u}_1 + \boldsymbol{u}_e^D) - g \;=\; 0 \qquad \forall\, K \in \mathcal{T}_H\,.$$

3. Error analysis of the semidiscrete problem

In the sequel C denotes a generic positive constant, independent of H or σ, with values that may vary in each occurrence. From now on, and just for simplicity of the presentation, we will assume that $\alpha_F = \alpha$ for all $F \in \mathcal{E}_H$.

3.1. **Preliminairies.** We start by presenting the Clément interpolation operator (cf. [17, 23, 19]) $\mathcal{C}_H : H^1(\Omega) \to V_H$ (with the obvious extension to vector-valued functions), satisfying, for all $K \in \mathcal{T}_H$ and all $F \in \mathcal{E}_H$,

$$\|\mathcal{C}_H(v)\|_{1,\Omega} \;\le\; C_{cle}\, \|v\|_{1,\Omega} \qquad \forall v \in H^1(\Omega)\,, \tag{35}$$

$$\|v - \mathcal{C}_H(v)\|_{m,K} \;\le\; C_{cle}\, H_K^{t-m}\, |v|_{t,\omega_K} \qquad \forall v \in H^t(\omega_K)\,, \tag{36}$$

$$\|v - \mathcal{C}_H(v)\|_{0,F} \;\le\; C_{cle}\, H_F^{t-\frac{1}{2}}\, |v|_{t,\omega_F} \qquad \forall v \in H^t(\omega_F)\,, \tag{37}$$

for $t = 1, 2$, $m = 0,\, 1$, where $\omega_K = \{K' \in \mathcal{T}_H : K \cap K' \neq \emptyset\}$ and $\omega_F = \{K \in \mathcal{T}_H : K \cap F \neq \emptyset\}$.

We will also use the $L^2(\Omega)$ projection onto Q_H which is denoted by $\Pi_H : L^2(\Omega) \to Q_H$. This projection satisfies (cf. [19])

$$\left\{ \sum_{K\in\mathcal{T}_H} \|q - \Pi_H(q)\|_{m,K}^2 \right\}^{\frac{1}{2}} \;\le\; C\, H^{t-m}\, |q|_{t,\Omega} \qquad \forall\, q \in H^t(\Omega)\,, \tag{38}$$

for $0 \leq m \leq t \leq 1$. Moreover, using (38) and the following local trace inequality: there exists C_t such that for all $K \in \mathcal{T}_H$ and all $v \in H^1(K)$

$$\|v\|_{0,\partial K}^2 \leq C_t \left(H_K^{-1} \|v\|_{0,K}^2 + H_K |v|_{1,K}^2 \right), \tag{39}$$

we obtain

$$\Big[\sum_{F \in \mathcal{E}_H} H_F \, \|[\![q - \Pi_H(q)]\!]\|_{0,F}^2 \Big]^{1/2} \leq C\, H\, |q|_{1,\Omega}. \tag{40}$$

Before heading to stability, two auxiliary results are stated next.

Lemma 4. There exist two constants $C, C' > 0$ such that, for all $K \in \mathcal{T}_H$ and all $\boldsymbol{v}_1 \in \mathbb{P}_1(K)^2$

$$C H_K^{\frac{1}{2}} \|\boldsymbol{v}_1 \cdot \boldsymbol{n}\|_{0,\partial K} \leq \|\boldsymbol{v}_1\|_{0,K} \leq C' H_K^{\frac{1}{2}} \|\boldsymbol{v}_1 \cdot \boldsymbol{n}\|_{0,\partial K}\,.$$

Proof. First, using (39) and an inverse inequality it follows that

$$\|\boldsymbol{v}_1 \cdot \boldsymbol{n}\|_{0,\partial K}^2 \leq C_t \,(H_K^{-1} \|\boldsymbol{v}_1\|_{0,K}^2 + H_K |\boldsymbol{v}_1|_{1,K}^2) \leq C H_K^{-1} \|\boldsymbol{v}_1\|_{0,K}^2\,, \tag{41}$$

and the first inequality follows. Let now $\hat{K}$ be the standard reference element of vertices $(0,0), (1,0)$ and $(0,1)$. Since in $\mathbb{P}_1(\hat{K})^2$ both quantities $\|\hat{\boldsymbol{w}}_1\|_{0,\hat{K}}$ and $\|\hat{\boldsymbol{w}}_1 \cdot \boldsymbol{n}\|_{0,\partial\hat{K}}$ define norms, there exists $C > 0$ such that

$$\|\hat{\boldsymbol{w}}_1\|_{0,\hat{K}} \leq C \|\hat{\boldsymbol{w}}_1 \cdot \hat{\boldsymbol{n}}\|_{0,\partial\hat{K}}\,, \tag{42}$$

for all $\hat{\boldsymbol{w}}_1 \in \mathbb{P}_1(\hat{K})^2$. Let now $\hat{\boldsymbol{v}}_1$ be the Piola transform of $\boldsymbol{v}_1$ (cf. [11]). Using the definition of the Piola transform, (42) and the fact that $\boldsymbol{v}_1 \cdot \boldsymbol{n} = H_F^{-1} \hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{n}}$, we get

$$\begin{aligned}
\|\boldsymbol{v}_1\|_{0,K}^2 &\leq C\, \|\hat{\boldsymbol{v}}_1\|_{0,\hat{K}}^2 \\
&\leq C\, \|\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{n}}\|_{0,\partial\hat{K}}^2 \\
&= C \sum_{\hat{F} \subseteq \partial\hat{K}} \int_{\hat{F}} (\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{n}})^2 \\
&= C \sum_{F \subseteq \partial K} \int_F H_F^{-1} H_F^2 (\boldsymbol{v}_1 \cdot \boldsymbol{n})^2 \\
&\leq C H_K \, \|\boldsymbol{v}_1 \cdot \boldsymbol{n}\|_{0,\partial K}^2\,,
\end{aligned}$$

and the result follows. □

We now define the following mesh-dependent norm

$$\|(\boldsymbol{v}, q)\|_H^2 := \sigma\, \|\boldsymbol{v}\|_{div,\Omega}^2 + \frac{\alpha}{\sigma}\, \|q\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \tau_F\, \|[\![q]\!]\|_{0,F}^2\,, \tag{43}$$

and present the following result which will be fundamental in the proof of the inf-sup condition below.

Lemma 5. There exists $C > 0$ such that, for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$,

$$\|(\boldsymbol{v}_1, q_0)\|_H \;\leq\; C\,\|(\rho(\boldsymbol{v}_1), q_0)\|_H\,.$$

Proof. First, using an inverse inequality, the mesh regularity, $\rho(\boldsymbol{v}_1)\cdot\boldsymbol{n} = \boldsymbol{v}_1\cdot\boldsymbol{n}$, and the fact that $\|\boldsymbol{v}_1\cdot\boldsymbol{n}\|_{-1/2,\partial K} \leq \|\boldsymbol{v}_1\|_{div,K}$ we obtain

$$\begin{aligned}
\|\boldsymbol{v}_1\cdot\boldsymbol{n}\|^2_{0,\partial K} &= \sum_{F\subseteq\partial K} \|\boldsymbol{v}_1\cdot\boldsymbol{n}\|^2_{0,F}\\
&\leq\; C \sum_{F\subseteq\partial K} H_F^{-1}\,\|\boldsymbol{v}_1\cdot\boldsymbol{n}\|^2_{-1/2,F}\\
&\leq\; 3C\,H_K^{-1}\,\|\boldsymbol{v}_1\cdot\boldsymbol{n}\|^2_{-1/2,\partial K}\\
&=\; C\,H_K^{-1}\,\|\rho(\boldsymbol{v}_1)\cdot\boldsymbol{n}\|^2_{-1/2,\partial K}\\
&\leq\; C\,H_K^{-1}\,(\,\|\rho(\boldsymbol{v}_1)\|^2_{0,K} + \|\nabla\cdot\boldsymbol{v}_1\|^2_{0,K})\,,
\end{aligned}$$

and then

$$H_K\,\|\boldsymbol{v}_1\cdot\boldsymbol{n}\|^2_{0,\partial K} \;\leq\; C\,(\,\|\rho(\boldsymbol{v}_1)\|^2_{0,K} + \|\nabla\cdot\boldsymbol{v}_1\|^2_{0,K})\,. \tag{44}$$

The result follows applying Lemma 4, the definition of the norm $\|\cdot\|_H$ and the fact that $\mathcal{M}^u_K(\boldsymbol{v}_1)$ is a solenoidal function. □

We end this section by proving some technical results involving the operators $\sigma\mathcal{M}^u_K$ and $\rho|_K = \boldsymbol{I} - \sigma\mathcal{M}^u_K$.

Lemma 6. Let $K \in \mathcal{T}_H$. Then, for all $\boldsymbol{v} \in H^1(K)^2$ there holds

i) $(\rho(\boldsymbol{v}), \boldsymbol{v})_K = \|\rho(\boldsymbol{v})\|^2_{0,K}$;
ii) $\|\rho(\boldsymbol{v})\|^2_{0,K} = \|\boldsymbol{v}\|^2_{0,K} - \|\boldsymbol{v} - \rho(\boldsymbol{v})\|^2_{0,K}$;
iii) $\|\sigma\mathcal{M}^u_K(\boldsymbol{v})\|_{0,K} \leq \|\boldsymbol{v}\|_{0,K}$; $\|\rho(\boldsymbol{v})\|_{0,K} \;\leq\; \|\boldsymbol{v}\|_{0,K}$;
iv) $\|\boldsymbol{v} - \rho(\boldsymbol{v})\|_{0,K} \leq \frac{H_K}{\pi}\,|\boldsymbol{v}|_{1,K}$.

Proof. The first three items follow directly from the fact that $\sigma\mathcal{M}^u_K$ (and hence, ρ) is an orthogonal projection with respect to the $L^2(K)^2$ inner product. To prove iv), let $\boldsymbol{v} \in H^1(K)^2$ and let us denote $\boldsymbol{v}_0 = \Pi_K(\boldsymbol{v})$. Since $\boldsymbol{v}_0$ is a constant in each element, there holds that $\mathcal{M}^u_K(\boldsymbol{v}_0) = \boldsymbol{0}$ and then using iii) we arrive at

$$\|\boldsymbol{v} - \rho(\boldsymbol{v})\|_{0,K} \;=\; \|\sigma\,\mathcal{M}^u_K(\boldsymbol{v})\|_{0,K} = \|\sigma\,\mathcal{M}^u_K(\boldsymbol{v} - \boldsymbol{v}_0)\|_{0,K} \leq \|\boldsymbol{v} - \boldsymbol{v}_0\|_{0,K}\,.$$

Finally, in [25] the following optimal Poincaré inequality is proved

$$\inf_{c\in\mathbb{R}} \|v-c\|_{0,K} \le \frac{H_K}{\pi}\,|v|_{1,K} \qquad \forall\, v\in H^1(K)\,, \tag{45}$$

and iv) follows. □

3.2. **Stability and convergence.** Before proving the stability we recall that, for all $q_0 \in Q_H$ there exists (cf. [23]) $\boldsymbol{w}\in H_0^1(\Omega)^2$ such that

$$\nabla\cdot\boldsymbol{w} = -\frac{1}{\sigma}\,q_0 \quad \text{in } \Omega \quad \text{ and } \quad \|\boldsymbol{w}\|_{1,\Omega} \le \frac{C_1}{\sigma}\|q_0\|_{0,\Omega}\,, \tag{46}$$

where C_1 depends only on Ω.

Theorem 7. *Let us suppose that $\alpha \le \min\{\frac{1}{2C_t},\frac{3}{8}\}$. Then, there exists $\beta>0$, independent of H, σ and α, such that*

$$\sup_{(\boldsymbol{w}_1,t_0)\in\mathbf{V}_H\times Q_H-\{\mathbf{0}\}} \frac{\boldsymbol{B}((\boldsymbol{v}_1,q_0),(\boldsymbol{w}_1,t_0))}{\|(\boldsymbol{w}_1,t_0)\|_H} \ge \beta\,\|(\boldsymbol{v}_1,q_0)\|_H\,,$$

for all $(\boldsymbol{v}_1,q_0)\in\mathbf{V}_H\times Q_H$, and the problem (32) is well posed.

Proof: Let $(\boldsymbol{v}_1,q_0)\in\mathbf{V}_H\times Q_H$. Then, from the definition of $\boldsymbol{B}$ it follows that

$$\boldsymbol{B}((\boldsymbol{v}_1,q_0),(\boldsymbol{v}_1,-q_0)) = \sigma\,\|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 + \sum_{F\in\mathcal{E}_H}\tau_F\,\|[\![q_0]\!]\|_{0,F}^2\,. \tag{47}$$

Next, we see that since $\nabla\cdot\boldsymbol{v}_1\in Q_H$ for all $\boldsymbol{v}_1\in\mathbf{V}_H$, then we can take $-\sigma\,\nabla\cdot\boldsymbol{v}_1$ as test function leading to

$$\begin{aligned}\boldsymbol{B}((\boldsymbol{v}_1,q_0),(\mathbf{0},-\sigma\nabla\cdot\boldsymbol{v}_1)) &= \sigma\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\Omega}^2 + \sigma\sum_{F\in\mathcal{E}_H}\tau_F([\![q_0]\!],[\![\nabla\cdot\boldsymbol{v}_1]\!])_F\\ &\ge \sigma\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\Omega}^2 - \frac{\sigma^2}{2}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![\nabla\cdot\boldsymbol{v}_1]\!]\|_{0,F}^2 - \frac{1}{2}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![q_0]\!]\|_{0,F}^2\,. \end{aligned}\tag{48}$$

Using (39) we obtain

$$\begin{aligned}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![\nabla\cdot\boldsymbol{v}_1]\!]\|_{0,F}^2 &\le \frac{2\alpha}{\sigma}\sum_{K\in\mathcal{T}_H}H_K\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\partial K}^2\\ &\le \frac{2C_t\alpha}{\sigma}\sum_{K\in\mathcal{T}_H}\|\nabla\cdot\boldsymbol{v}_1\|_{0,K}^2 = \frac{2C_t\alpha}{\sigma}\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\Omega}^2\,,\end{aligned}$$

and then (48) becomes

$$\boldsymbol{B}((\boldsymbol{v}_1,q_0),(\mathbf{0},-\sigma\nabla\cdot\boldsymbol{v}_1)) \ge \sigma\,(1-C_t\alpha)\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\Omega}^2 - \frac{1}{2}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![q_0]\!]\|_{0,F}^2\,. \tag{49}$$

Let now $\boldsymbol{w} \in H_0^1(\Omega)^2$ be given by (46) and let $\boldsymbol{w}_1 := \mathcal{C}_H(\boldsymbol{w})$. Integrating by parts, using (37), the mesh regularity and (46) we arrive at

$$\begin{aligned}
\frac{1}{\sigma}\,\|q_0\|_{0,\Omega}^2 &= -(q_0, \nabla\cdot\boldsymbol{w})_\Omega \\
&= -(q_0, \nabla\cdot(\boldsymbol{w}-\boldsymbol{w}_1))_\Omega - (q_0, \nabla\cdot\boldsymbol{w}_1)_\Omega \\
&= -\sum_{F\in\mathcal{E}_H} ([\![q_0]\!], (\boldsymbol{w}-\boldsymbol{w}_1)\cdot\boldsymbol{n})_F - (q_0, \nabla\cdot\boldsymbol{w}_1)_\Omega \\
&\le C_{cle} \sum_{F\in\mathcal{E}_H} H_F^{\frac{1}{2}}\,\|[\![q_0]\!]\|_{0,F}\,|\boldsymbol{w}|_{1,\omega_F} - (q_0, \nabla\cdot\boldsymbol{w}_1)_\Omega \\
&\le \sqrt{3}C_{cle}C_1 \left\{\sum_{F\in\mathcal{E}_H} \frac{H_F}{\sigma}\,\|[\![q_0]\!]\|_{0,F}^2\right\}^{\frac{1}{2}} \frac{\|q_0\|_{0,\Omega}}{\sqrt{\sigma}} - (q_0, \nabla\cdot\boldsymbol{w}_1)_\Omega\,,
\end{aligned}$$

and then

$$-(q_0, \nabla\cdot\boldsymbol{w}_1)_\Omega \;\ge\; \frac{1}{2\sigma}\,\|q_0\|_{0,\Omega}^2 - \frac{C_2}{\alpha}\sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![q_0]\!]\|_{0,F}^2\,, \tag{50}$$

where $C_2 = \frac{3C_{cle}^2 C_1^2}{2}$. From Lemma 6-iii), (35) and (46), we then obtain

$$\begin{aligned}
\boldsymbol{B}((\boldsymbol{v}_1, q_0), (\boldsymbol{w}_1, 0)) &= \sigma\,(\rho(\boldsymbol{v}_1), \rho(\boldsymbol{w}_1))_\Omega - (q_0, \nabla\cdot\boldsymbol{w}_1)_\Omega \\
&\ge -\gamma_1^{-1}\sigma\,\|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 - \sigma\,\gamma_1\,\|\rho(\boldsymbol{w}_1)\|_{0,\Omega}^2 \\
&\quad + \frac{1}{2\sigma}\|q_0\|_{0,\Omega}^2 - \frac{C_2}{\alpha}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![q_0]\!]\|_{0,F}^2 \\
&\ge \frac{1}{\sigma}\left(\frac{1}{2} - C_1^2 C_{cle}^2\,\gamma_1\right)\|q_0\|_{0,\Omega}^2 - \gamma_1^{-1}\sigma\,\|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 - \frac{C_2}{\alpha}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![q_0]\!]\|_{0,F}^2 \\
&= \frac{1}{4\sigma}\,\|q_0\|_{0,\Omega}^2 - 4C_{cle}^2C_1^2\sigma\,\|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 - \frac{C_2}{\alpha}\sum_{F\in\mathcal{E}_H}\tau_F\|[\![q_0]\!]\|_{0,F}^2\,, \qquad (51)
\end{aligned}$$

choosing $\gamma_1 = \frac{1}{4C_{cle}^2C_1^2}$.

Finally, let $(\mathbf{z}_1, t_0) := (\boldsymbol{v}_1, -q_0) + \delta_1(\mathbf{0}, -\sigma\nabla \cdot \boldsymbol{v}_1) + \delta_2(\boldsymbol{w}_1, 0)$ with $\delta_1, \delta_2 > 0$. Then, collecting (47), (49) and (51) we obtain

$$
\begin{aligned}
\boldsymbol{B}((\boldsymbol{v}_1, q_0), (\mathbf{z}_1, t_0)) \geq\ & \sigma\, \|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 + \sum_{F\in\mathcal{E}_H} \tau_F \|[\![q_0]\!]\|_{0,F}^2 \\
& + \delta_1\sigma\ (1 - C_t\alpha)\ \|\nabla \cdot \boldsymbol{v}_1\|_{0,\Omega}^2 - \frac{\delta_1}{2} \sum_{F\in\mathcal{E}_H} \tau_F \|[\![q_0]\!]\|_{0,F}^2 \\
& + \frac{\delta_2}{4\sigma}\, \|q_0\|_{0,\Omega}^2 - 4\delta_2 C_{cle}^2 C_1^2 \sigma\, \|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 - \frac{\delta_2 C_2}{\alpha} \sum_{F\in\mathcal{E}_H} \tau_F \|[\![q_0]\!]\|_{0,F}^2 \\
=\ & \sigma \left(1 - 4\delta_2 C_{cle}^2 C_1^2\right)\ \|\rho(\boldsymbol{v}_1)\|_{0,\Omega}^2 + \delta_1\sigma\,(1 - C_t\alpha)\ \|\nabla \cdot \boldsymbol{v}_1\|_{0,\Omega}^2 + \frac{\delta_2}{4\sigma}\|q_0\|_{0,\Omega}^2 \\
& + \left(1 - \frac{\delta_1}{2} - \frac{\delta_2 C_2}{\alpha}\right) \sum_{F\in\mathcal{E}_H} \tau_F \|[\![q_0]\!]\|_{0,F}^2 \\
\geq\ & C\, \|(\rho(\boldsymbol{v}_1), q_0)\|_H^2\,, \qquad (52)
\end{aligned}
$$

if $\alpha \leq \frac{1}{2C_t}$, $\delta_1 \leq \frac{1}{2}$ and $\delta_2 = \frac{\alpha}{2C_2}$, thus guarantying that $C > 0$ is independent of α. The result follows then using Lemma 5 and the fact that, thanks to the choice of δ_2, $\|(\mathbf{z}_1, t_0)\|_H \leq C\, \|(\boldsymbol{v}_1, q_0)\|_H$, where C does not depend on α. $\square$

Remark. If we look carefully at the proof of the last result, we may see that, since the quantity $\|\rho(\boldsymbol{v}_1)\|_{0,\Omega}$ defines a norm in $\mathbf{V}_H$, the well-posedeness of (32) follows directly from (47), independently of the value of α. The reason to prove an inf-sup condition is the control in the norm of the divergence, which, thanks to Lemma 5 allow us to prove an error estimate for $\boldsymbol{u} - \boldsymbol{u}_1$, instead of $\boldsymbol{u} - \rho(\boldsymbol{u}_1)$, that would arise naturally from (47). $\square$.

Next, we present the following consistency result.

Lemma 8. Let $(\boldsymbol{u}, p) \in H_0(div, \Omega) \times [H^1(\Omega) \cap L_0^2(\Omega)]$ be the weak solution of (5) and $(\boldsymbol{u}_1, p_0)$ the solution of (32), respectively. Then,

$$
\boldsymbol{B}\big((\boldsymbol{u} - \boldsymbol{u}_1, p - p_0), (\boldsymbol{v}_1, q_0)\big)\ \leq\ C\,\sigma\, H\, |\boldsymbol{u}|_{1,\Omega} \|\boldsymbol{v}_1\|_{0,\Omega}\,,
$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$.

Proof. Noting that $[\![p]\!] = 0$ a.e. across all the internal edges, and from the definition of $\boldsymbol{B}$ we easily see that

$$\begin{aligned}
\boldsymbol{B}\big((\boldsymbol{u},p),(\boldsymbol{v}_1,q_0)\big) &= \mathbf{A}((\boldsymbol{u},p),(\boldsymbol{v}_1,q_0)) - \sum_{K\in\mathcal{T}_H} \sigma\,(\sigma\mathcal{M}_K^u(\boldsymbol{u}),\boldsymbol{v}_1)_K \\
&\quad + \sum_{K\in\mathcal{T}_H} (\mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}) - g + \Pi_K(g)),\sigma\boldsymbol{v}_1)_K \\
&= \mathbf{F}(\boldsymbol{v}_1,q_0) - \sum_{K\in\mathcal{T}_H} \sigma\,(\sigma\mathcal{M}_K^u(\boldsymbol{u}),\boldsymbol{v}_1)_K + \sum_{K\in\mathcal{T}_H} (\mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u})),\sigma\boldsymbol{v}_1)_K \\
&= \boldsymbol{B}((\boldsymbol{u}_1,p_0),(\boldsymbol{v}_1,q_0)) - \sum_{K\in\mathcal{T}_H} \sigma\,(\sigma\mathcal{M}_K^u(\boldsymbol{u}),\boldsymbol{v}_1)_K + \sum_{K\in\mathcal{T}_H} (\mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u})),\sigma\boldsymbol{v}_1)_K\,,
\end{aligned}$$

and then

$$\boldsymbol{B}\big((\boldsymbol{u}-\boldsymbol{u}_1,p-p_0),(\boldsymbol{v}_1,q_0)\big) \le \sum_{K\in\mathcal{T}_H} \left(\|\sigma\mathcal{M}_K^u(\boldsymbol{u})\|_{0,K} + \|\mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}))\|_{0,K}\right)\sigma\|\boldsymbol{v}_1\|_{0,K}\,. \tag{53}$$

Next, to bound the term $\|\mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}))\|_{0,K}$ we follow very closely the results from [6], Appendix A. First, we recall that, from (18), the problem satisfied by $\boldsymbol{w} := \mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}))$ is given by

$$\begin{gathered}
\boldsymbol{w} + \nabla\eta = \boldsymbol{0} \quad , \quad \nabla\cdot\boldsymbol{w} = \nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}) \quad \text{in } K\,, \\
\boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial K\,,
\end{gathered} \tag{54}$$

where $\eta \in L_0^2(K)$. Now, multiplying the first equation in (54) by $\boldsymbol{w}$, the second by η, adding both and integrating by parts we arrive at

$$\sigma\,\|\boldsymbol{w}\|_{0,K}^2 = (\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}),\eta)_K \le \|\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u})\|_{0,K}\|\eta\|_{0,K}\,,$$

and since $\eta \in L_0^2(K)$ then $\|\eta\|_{0,K} \le \frac{H_K}{\pi}|\eta|_{1,K} = \frac{H_K}{\pi}\|\boldsymbol{w}\|_{0,K}$ and we get

$$\sigma\,\|\boldsymbol{w}\|_{0,K}^2 \le \frac{H_K}{\pi}\,\|\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u})\|_{0,K}|\eta|_{1,K} \le \frac{H_K}{\pi}\,\|\nabla\cdot\boldsymbol{u}\|_{0,K}\|\boldsymbol{w}\|_{0,K}\,,$$

which leads to

$$\|\mathcal{G}_K^u(\nabla\cdot\boldsymbol{u} - \Pi_K(\nabla\cdot\boldsymbol{u}))\|_{0,K} \le \frac{H_K}{\sigma\pi}\,\|\nabla\cdot\boldsymbol{u}\|_{0,K}\,.$$

The result follows from (53) and Lemma 6-iv). $\square$

With this result in mind we can present the following convergence result.

Theorem 9. *Let us suppose that* $(\boldsymbol{u}, p)$*, solution of* (5) *belongs to* $H^2(\Omega)^2 \times H^1(\Omega)$*, and let* $(\boldsymbol{u}_1, p_0)$ *be the solution of* (32)*. Then, there exists* $C > 0$*, independent of* H, σ *and* α*, such that*

$$\|(\boldsymbol{u} - \boldsymbol{u}_1, p - p_0)\|_H \le CH \left(\sqrt{\frac{\sigma}{\alpha}}\, \|\boldsymbol{u}\|_{2,\Omega} + \frac{1}{\sqrt{\sigma}}\, |p|_{1,\Omega} \right).$$

Proof: First, let $(\boldsymbol{v}_1, q_0) := (\mathcal{C}_H(\boldsymbol{u}), \Pi_H(p))$. Then, from Theorem 7 there exists $(\boldsymbol{w}_1, t_0) \in \mathbf{V}_H \times Q_H$ such that $\|(\boldsymbol{w}_1, t_0)\|_H = 1$ and

$$\beta\, \|(\boldsymbol{u}_1 - \boldsymbol{v}_1, p_0 - q_0)\|_H \le \boldsymbol{B}((\boldsymbol{u}_1 - \boldsymbol{v}_1, p_0 - q_0), (\boldsymbol{w}_1, t_0))\,.$$

Now, using Lemmas 5 and 8, the Cauchy-Schwarz's inequality, the fact that $(p - q_0, \nabla\cdot \boldsymbol{w}_1)_\Omega = 0$ and (36)-(38), we obtain

$$\begin{aligned}
&\beta\, \|(\boldsymbol{u}_1 - \boldsymbol{v}_1, p_0 - q_0)\|_H \le \boldsymbol{B}((\boldsymbol{u} - \boldsymbol{v}_1, p - q_0), (\boldsymbol{w}_1, t_0)) + \boldsymbol{B}((\boldsymbol{u}_1 - \boldsymbol{u}, p_0 - p), (\boldsymbol{w}_1, t_0)) \\
&= \sigma\, (\rho(\boldsymbol{u} - \boldsymbol{v}_1), \rho(\boldsymbol{w}_1))_\Omega - (p - q_0, \nabla\cdot \boldsymbol{w}_1)_\Omega - (t_0, \nabla\cdot(\boldsymbol{u} - \boldsymbol{v}_1))_\Omega \\
&\quad - \sum_{F \in \mathcal{E}_H} \tau_F([\![p - q_0]\!], [\![t_0]\!])_F + \boldsymbol{B}((\boldsymbol{u}_1 - \boldsymbol{u}, p_0 - p), (\boldsymbol{w}_1, t_0)) \\
&\le C \left\{ \sigma\, \|\rho(\boldsymbol{u} - \boldsymbol{v}_1)\|_{0,\Omega}^2 + \frac{\sigma}{\alpha}\, \|\nabla\cdot(\boldsymbol{u} - \boldsymbol{v}_1)\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \tau_F \|[\![p - q_0]\!]\|_{0,F}^2 + \sigma H^2\, |\boldsymbol{u}|_{1,\Omega}^2 \right\}^{\frac{1}{2}} \\
&\quad \left\{ \sigma\, \|\rho(\boldsymbol{w}_1)\|_{0,\Omega}^2 + \frac{\alpha}{\sigma} \|t_0\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \tau_F \|[\![t_0]\!]\|_{0,F}^2 + \sigma\, \|\boldsymbol{w}_1\|_{0,\Omega}^2 \right\}^{\frac{1}{2}} \\
&\le C \left\{ \frac{\sigma}{\alpha}\, \|\boldsymbol{u} - \boldsymbol{v}_1\|_{div,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \tau_F \|[\![p - q_0]\!]\|_{0,F}^2 + \sigma H^2\, |\boldsymbol{u}|_{1,\Omega}^2 \right\}^{\frac{1}{2}} \\
&\le CH \left(\sqrt{\frac{\sigma}{\alpha}}\, \|\boldsymbol{u}\|_{2,\Omega} + \frac{\alpha}{\sqrt{\sigma}}\, |p|_{1,\Omega} \right),
\end{aligned}$$

and the result follows using the triangle inequality. □

Next, as it was mentioned in the previous section, in order to provide a mass conservative velocity field we must enhance $\boldsymbol{u}_1$ with the Raviart-Thomas' field $\boldsymbol{u}_e^D$. The next result shows that this fact does not undermine the convergence of the method.

Corollary 10. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_1, p_0)$ be the solutions of (5) and (32), respectively. Then, under the hypothesis of the previous Theorem there exists C such that

$$\|\boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_e^D\|_{div,\Omega} \leq C\, H \left(\frac{1}{\sqrt{\alpha}} \|\boldsymbol{u}\|_{2,\Omega} + \frac{1}{\sigma} |p|_{1,\Omega} \right) , \tag{55}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_e^M - \boldsymbol{u}_e^D\|_{div,\Omega} \leq C\, H \left(\frac{1}{\sqrt{\alpha}} \|\boldsymbol{u}\|_{2,\Omega} + \frac{1}{\sigma} |p|_{1,\Omega} \right) , \tag{56}$$

where $\boldsymbol{u}_e^D$ is given by (19) and $\boldsymbol{u}_e^M|_K = -\sigma \mathcal{M}_K^u(\boldsymbol{u}_1)$.

Proof. We use the local mass conservation feature to prove (55). In fact, from Lemma 3 we obtain that

$$\int_K \nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e^D) = \int_K g \qquad \forall\, K \in \mathcal{T}_H\,, \tag{57}$$

and then $\nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e^D)|_K = \Pi_K(g)$ in each K, which leads to

$$\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_e^D)\|_{0,K} \;=\; \|g - \Pi_K(g)\|_{0,K} \;\leq\; \frac{H_K}{\pi} \,|\nabla \cdot \boldsymbol{u}|_{1,K}\,. \tag{58}$$

Following the same arguments from the proof of Lemma 2 we can prove that

$$\|\boldsymbol{u}_e^D\|_{0,\Omega} \leq C\, \sigma^{-\frac{1}{2}} H \left\{ \sum_{F \in \mathcal{E}_H} \tau_F \, \|[\![p - p_0]\!]\|_{0,F}^2 \right\}^{\frac{1}{2}} , \tag{59}$$

and then using the previous theorem we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_e^D\|_{0,\Omega} \leq C\, H \left(\frac{1}{\sqrt{\alpha}} \|\boldsymbol{u}\|_{2,\Omega} + \frac{1}{\sigma} |p|_{1,\Omega} \right) , \tag{60}$$

and (55) follows from (58), (60) and Theorem 9.

Next, to prove (56) we recall that $\boldsymbol{u}_e^M = -\sigma \mathcal{M}_K^u(\boldsymbol{u}_1) = -\boldsymbol{u}_1 + \rho(\boldsymbol{u}_1)$, and then using (59) and Lemma 6

$$\begin{aligned} \|\boldsymbol{u} - \rho(\boldsymbol{u}_1) - \boldsymbol{u}_e^D\|_{0,\Omega} &\leq \|\boldsymbol{u} - \rho(\boldsymbol{u})\|_{0,\Omega} + \|\rho(\boldsymbol{u}) - \rho(\boldsymbol{u}_1)\|_{0,\Omega} + \|\boldsymbol{u}_e^D\|_{0,\Omega} \\ &\leq C\, H\, |\boldsymbol{u}|_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_1\|_{0,\Omega} + C\, \frac{H}{\sqrt{\sigma}} \|(\boldsymbol{u} - \boldsymbol{u}_1, p - p_0)\|_H\,, \end{aligned}$$

and the result follows using the previous theorem and the fact that $\boldsymbol{u}_e^M$ is solenoidal. $\square$

We end this section by explaining more in depth why we can actually neglect the non-symmetric term.

Lemma 11. Let us suppose that α is small enough. Then, there exists $\beta_1 > 0$, independent of H and α, such that

$$\beta_1 \,\|(\boldsymbol{v}_1, q_0)\|_H \;\le \sup_{(\boldsymbol{w}_1,t_0)\in \mathbf{V}_H\times Q_H-\{\mathbf{0}\}} \frac{\boldsymbol{B}_{ns}((\boldsymbol{v}_1, q_0),(\boldsymbol{w}_1, t_0))}{\|(\boldsymbol{w}_1, t_0)\|_H}\,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$. Furthermore, there exists $C > 0$ independent of H, σ and α such that

$$\|(\boldsymbol{u}_1 - \hat{\boldsymbol{u}}_1, p_0 - \hat{p}_0)\|_H \;\le\; C\, H^2 \left(\sqrt{\sigma}\,|\boldsymbol{u}|_{2,\Omega} + \frac{1}{\sqrt{\sigma}}\,|p|_{1,\Omega}\right)\,,$$

where $(\boldsymbol{u}_1, p_0)$ and $(\hat{\boldsymbol{u}}_1, \hat{p}_0)$ are the solutions of (32) and (28), respectively.

Proof. For the inf-sup condition we start noting that, from the definition of $\boldsymbol{B}_{ns}$ and Lemma 2 there follows that

$$\begin{aligned}
\boldsymbol{B}_{ns}((\boldsymbol{v}_1, q_0),(\boldsymbol{v}_1, -q_0)) &= \sigma\,\|\rho(\boldsymbol{v}_1)\|^2_{0,K} + \sum_{K\in\mathcal{T}_H} (\mathcal{D}^u_K([\![q_0]\!]), \rho(\boldsymbol{v}_1))_K + \sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![q_0]\!]\|^2_{0,F} \\
&\ge \frac{\sigma}{2}\left(1 - \alpha\, C_*^2 H^2\right)\,\|\rho(\boldsymbol{v}_1)\|^2_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![q_0]\!]\|^2_{0,F} \\
&\ge \frac{3\sigma}{8}\,\|\rho(\boldsymbol{v}_1)\|^2_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![q_0]\!]\|^2_{0,F}\,. \qquad (61)
\end{aligned}$$

if we suppose that α satisfies $\alpha \le \frac{1}{4C_*^2H^2}$, where C_* is the constant from Lemma 2. The remaining part of the proof of the inf-sup condition is completely analogous to the proof of Theorem 7, and then we skip the details. To prove the error estimate, we see that, from the inf-sup condition and using the definition of $\boldsymbol{B}$ and $\boldsymbol{B}_{ns}$ and Lemma 2 we have

$$\begin{aligned}
\beta_1\,\|(\boldsymbol{u}_1 - \hat{\boldsymbol{u}}_1, p_0 - \hat{p}_0)\|_H &\le \sup_{(\boldsymbol{w}_1,t_0)\in \mathbf{V}_H\times Q_H-\{\mathbf{0}\}} \frac{\boldsymbol{B}_{ns}((\boldsymbol{u}_1 - \hat{\boldsymbol{u}}_1, p_0 - \hat{p}_0),(\boldsymbol{w}_1, t_0))}{\|(\boldsymbol{w}_1, t_0)\|_H} \\
&= \sup_{(\boldsymbol{w}_1,t_0)\in \mathbf{V}_H\times Q_H-\{\mathbf{0}\}} \frac{\boldsymbol{B}_{ns}((\boldsymbol{u}_1, p_0),(\boldsymbol{w}_1, t_0)) - \mathbf{F}(\boldsymbol{w}_1, t_0)}{\|(\boldsymbol{w}_1, t_0)\|_H} \\
&= \sup_{(\boldsymbol{w}_1,t_0)\in \mathbf{V}_H\times Q_H-\{\mathbf{0}\}} \frac{\sum_{K\in\mathcal{T}_H} (\mathcal{D}^u_K([\![p_0]\!]), \sigma\boldsymbol{w}_1)_K}{\|(\boldsymbol{w}_1, t_0)\|_H} \\
&\le \sqrt{\alpha}\, CH \left\{\sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![p - p_0]\!]\|^2_{0,F}\right\}^{\frac{1}{2}} \\
&\le \sqrt{\alpha}\, CH\,\|(\boldsymbol{u} - \boldsymbol{u}_1, p - p_0)\|_H\,,
\end{aligned}$$

and the result follows using Theorem 9. $\square$

4. A two-level finite element method

First we start remarking that, from the definition of $\mathcal{M}_K$ and $\mathcal{G}_K$ (cf. (16)-(18)) it follows that

$$\rho(\boldsymbol{v}_1) = \boldsymbol{v}_1 - \sigma\,\mathcal{M}_K^u(\boldsymbol{v}_1) = \nabla\mathcal{M}_K^p(\boldsymbol{v}_1) \quad \text{and} \quad \sigma\,\mathcal{G}_K^u(g-\Pi_K(g)) = -\nabla\mathcal{G}_K^p(g-\Pi_K(g))\,, \tag{62}$$

and then the method (32) may be rewritten in the following equivalent way

$$\begin{aligned}\sum_{K\in\mathcal{T}_H}\sigma\,(\nabla\mathcal{M}_K^p(\boldsymbol{u}_1),\nabla\mathcal{M}_K^p(\boldsymbol{v}_1))_K-(p_0,\nabla\cdot\boldsymbol{v}_1)_\Omega-(q_0,\nabla\cdot\boldsymbol{u}_1)_\Omega-\sum_{F\in\mathcal{E}_H}\tau_F\,([\![p_0]\!],[\![q_0]\!])_F\\ =(\boldsymbol{f},\boldsymbol{v}_1)_\Omega-(g,q_0)_\Omega+\sum_{K\in\mathcal{T}_H}(\nabla\mathcal{G}_K^p(g-\Pi_K(g)),\boldsymbol{v}_1)_K\,,\end{aligned} \tag{63}$$

for all $(\boldsymbol{v}_1,q_0)\in\mathbf{V}_H\times Q_H$. Let us further remark that, from (13) and (15) the functions $p_e^M(\boldsymbol{v}_1)=\mathcal{M}_K^p(\boldsymbol{v}_1)$ and $p_e^g=\mathcal{G}_K^p(g-\Pi_K(g))$ may be computed by solving the following Neumann problems in each $K\in\mathcal{T}_H$

$$\begin{aligned}-\Delta p_e^M(\boldsymbol{v}_1) &= -\nabla\cdot\boldsymbol{v}_1 \qquad \text{in } K\,,\\ \partial_{\boldsymbol{n}}p_e^M(\boldsymbol{v}_1) &= \boldsymbol{v}_1\cdot\boldsymbol{n} \qquad \text{on } \partial K\,,\end{aligned} \tag{64}$$

and

$$\begin{aligned}-\Delta p_e^g &= \sigma\,(g-\Pi_K(g)) \qquad \text{in } K\,,\\ \partial_{\boldsymbol{n}}p_e^g &= 0 \qquad \text{on } \partial K\,,\end{aligned} \tag{65}$$

respectively. Using these writings for the local problems a two-level finite element method arises by replacing p_e^M and p_e^g in (63) by suitable finite element approximations. To do this, let, for each $K\in\mathcal{T}_H$, $\{\mathcal{T}_h^K\}_{h>0}$ be a regular family of triangulations of K built using triangles $\bar{K}\subseteq K$ with diameter less or equal than h (the value for h may vary from one element to another, but for simplicity of the presentation it will always be denoted by h), and let

$$\mathcal{R}_h^K := \{\xi_h\in\mathcal{C}^0(K)\,:\,\xi_h|_{\bar K}\in\mathbb{P}_l(\bar K)\,,\,\forall\bar K\in\mathcal{T}_h^K\}\,, \tag{66}$$

where $l\ge 1$. Hence, we propose the following discretizations for (64) and (65): *Find* $p_h(\boldsymbol{v}_1)\in\mathcal{R}_h^K$ *such that*

$$\int_K\nabla p_h(\boldsymbol{v}_1)\cdot\nabla\xi_h=\int_K\boldsymbol{v}_1\cdot\nabla\xi_h \qquad \forall\xi_h\in\mathcal{R}_h^K\,, \tag{67}$$

and: *Find* $p_h^g\in\mathcal{R}_h^K$ *such that*

$$\int_K\nabla p_h^g\cdot\nabla\xi_h=\sigma\int_K(g-\Pi_K(g))\xi_h \qquad \forall\xi_h\in\mathcal{R}_h^K\,, \tag{68}$$

respectively. With these approximations we introduce then our two-level finite element method: *Find* $(\boldsymbol{u}_{1,h}, p_{0,h}) \in \mathbf{V}_H \times Q_H$ *such that:*

$$\boldsymbol{B}_h((\boldsymbol{u}_{1,h}, p_{0,h}), (\boldsymbol{v}_1, q_0)) = \mathbf{F}_h(\boldsymbol{v}_1, q_0) \qquad \forall\, (\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H \,, \tag{69}$$

where

$$\begin{aligned} \boldsymbol{B}_h((\boldsymbol{v}_1, q_0), (\boldsymbol{w}_1, t_0)) := \sum_{K\in\mathcal{T}_H} \sigma\,(\nabla p_h(\boldsymbol{v}_1), \nabla p_h(\boldsymbol{w}_1))_K - (q_0, \nabla\cdot \boldsymbol{w}_1)_\Omega - (t_0, \nabla\cdot \boldsymbol{v}_1)_\Omega \\ - \sum_{F\in\mathcal{E}_H} \tau_F\,([\![q_0]\!], [\![t_0]\!])_F \,, \end{aligned} \tag{70}$$

and

$$\mathbf{F}_h(\boldsymbol{v}_1, q_0) := (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega - (g, q_0)_\Omega + \sum_{K\in\mathcal{T}_H} (\nabla p_h^g, \boldsymbol{v}_1)_K \,, \tag{71}$$

respectively.

4.1. **Numerical analysis of the fully discrete method.** To prove stability we start proving the following lemma.

Lemma 12. There exists $C > 0$, independent of H, h, σ and K, such that

$$|p_e^M(\boldsymbol{v})|_{1,K} \leq \|\boldsymbol{v}\|_{0,K} \,, \tag{72}$$

$$|p_e^M(\boldsymbol{v}) - p_h(\boldsymbol{v})|_{1,K} \leq Ch\,|\boldsymbol{v}|_{1,K} \,, \tag{73}$$

$$|p_e^g - p_h^g|_{1,K} \leq \frac{Ch\,H_K^t}{\sigma}\,|g|_{t,K} \,, \tag{74}$$

for all $\boldsymbol{v} \in H^1(K)^2$, and for $t = 0, 1$.

Proof. For the first estimate we consider $\xi_h = p_h(\boldsymbol{v})$ in (67) and apply the Cauchy-Schwarz's inequality to prove that $|p_h(\boldsymbol{v})|_{1,K}^2 \leq \|\boldsymbol{v}\|_{0,K} |p_h(\boldsymbol{v})|_{1,K}$.

For the remaining parts, we start by stating the following result from [23], Theorem 3.9, p. 55 (which is also valid in two space dimensions): If $\boldsymbol{w} \in H_0(div, K) \cap H(curl, K)$, then $\boldsymbol{w} \in H^1(K)^2$ and satisfies

$$|\boldsymbol{w}|_{1,K} \leq \|\nabla\cdot \boldsymbol{w}\|_{0,K} + \|curl(\boldsymbol{w})\|_{0,K} \,. \tag{75}$$

Then, recalling the mixed form of the problem satisfied by $p_e^M(\boldsymbol{v})$, i.e.,:

$$\begin{aligned} \boldsymbol{u}_e^M(\boldsymbol{v}) + \nabla p_e^M(\boldsymbol{v}) = \boldsymbol{v} \quad , \quad \nabla\cdot \boldsymbol{u}_e^M(\boldsymbol{v}) = 0 \quad \text{in } K \,, \\ \boldsymbol{u}_e^M(\boldsymbol{v})\cdot \boldsymbol{n} = 0 \quad \text{on } \partial K \,, \end{aligned} \tag{76}$$

and taking the *curl* from the first equation in (76) we see that $\boldsymbol{u}_e^M(\boldsymbol{v}) \in H_0(div, K) \cap H(curl, K)$ which implies $\boldsymbol{u}_e^M(\boldsymbol{v}) \in H^1(K)^2$, and satisfies (75). From (76) we see then that $p_e^M(\boldsymbol{v}) \in H^2(K)$ and satisfies

$$|p_e^M(\boldsymbol{v})|_{2,K} \leq |\boldsymbol{u}_e^M(\boldsymbol{v})|_{1,K} + |\boldsymbol{v}|_{1,K} \leq \|curl(\boldsymbol{v})\|_{0,K} + |\boldsymbol{v}|_{1,K}\,, \tag{77}$$

which leads to

$$|p_e^M(\boldsymbol{v}) - p_h(\boldsymbol{v})|_{1,K} \leq Ch\,|p_e^M(\boldsymbol{v})|_{2,K} \leq Ch\,|\boldsymbol{v}|_{1,K}\,,$$

where we have used the fact that $l \geq 1$ and standard finite element estimates (see, e.g., [19]). Following analogous steps we can prove (74). □

The next result may be seen as a fully discrete version of Lemma 5.

Lemma 13. Let $\|\cdot\|_h$ be the mesh-dependent norm given by

$$\|(\boldsymbol{v}_1, q_0)\|_h^2 := \sum_{K\in\mathcal{T}_H} \sigma\,\|\nabla p_h(\boldsymbol{v}_1)\|_{0,K}^2 + \sigma\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\Omega}^2 + \frac{\alpha}{\sigma}\,\|q_0\|_{0,\Omega}^2 + \sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![q_0]\!]\|_{0,F}^2\,,$$

and let us suppose that there exists $C_0 > 0$ such that $h \leq C_0 H_K$, for all $K \in \mathcal{T}_H$. Then, there exists $C > 0$ independent of H, h, σ or α such that

$$\|(\boldsymbol{v}_1, q_0)\|_H \leq C\,\|(\boldsymbol{v}_1, q_0)\|_h\,, \tag{78}$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$.

Proof. From Lemma 5 we know that there exists $C > 0$ such that

$$\begin{aligned}
\|(\boldsymbol{v}_1, q_0)\|_H^2 &\leq C\,\|(\rho(\boldsymbol{v}_1), q_0)\|_H^2 \\
&= C\left\{\sum_{K\in\mathcal{T}_H} \sigma\,\|\nabla p_e^M(\boldsymbol{v}_1)\|_{0,K}^2 + \sigma\,\|\nabla\cdot\boldsymbol{v}_1\|_{0,\Omega}^2 + \frac{\alpha}{\sigma}\,\|q_0\|_{0,\Omega}^2 + \sum_{F\in\mathcal{E}_H} \tau_F\,\|[\![q_0]\!]\|_{0,F}^2\right\} \\
&\leq C\left\{\sum_{K\in\mathcal{T}_H} \sigma\,|p_e^M(\boldsymbol{v}_1) - p_h(\boldsymbol{v}_1)|_{1,K}^2 + \|(\boldsymbol{v}_1, q_0)\|_h^2\right\}. \qquad (79)
\end{aligned}$$

Now, using Lemma 12 and an inverse inequality we obtain

$$|p_e^M(\boldsymbol{v}_1) - p_h(\boldsymbol{v}_1)|_{1,K}^2 \leq Ch^2\,|p_e^M(\boldsymbol{v}_1)|_{2,K}^2 \leq Ch^2\,|\boldsymbol{v}_1|_{1,K}^2 \;\leq\; Ch^2 H_K^{-2}\,\|\boldsymbol{v}_1\|_{0,K}^2\,,$$

and then (79) becomes

$$\|(\boldsymbol{v}_1, q_0)\|_H^2 \leq C\,\sigma \sum_{K\in\mathcal{T}_H} h^2 H_K^{-2}\,\|\boldsymbol{v}_1\|_{0,K}^2 + C\,\|(\boldsymbol{v}_1, q_0)\|_h^2\,,$$

and the result follows supposing that $h \leq \frac{H_K}{\sqrt{2C}}$. □

Remark. As will be clear after the following lemma, the last result shows us, in particular, that it is enough to choose in advance one type of mesh to solve the local problems in each element, without the need to refine the subgrid mesh if the coarse mesh is refined, and independently of having a coarse mesh with very different sizes. Hence, the computation of $p_h(\boldsymbol{v}_1)$ has the same cost over all the elements and it can be indeed inexpensive. □

Lemma 14. Under the hypothesis of the previous lemma, there exists $\beta_2 > 0$ independent of H, h and α such that

$$\sup_{(\boldsymbol{w}_1,t_0)\in\mathbf{V}_H\times Q_H} \frac{\boldsymbol{B}_h((\boldsymbol{v}_1,q_0),(\boldsymbol{w}_1,t_0))}{\|(\boldsymbol{w}_1,t_0)\|_H} \geq \beta_2 \|(\boldsymbol{v}_1,q_0)\|_H , \tag{80}$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$.

Proof. Let $(\boldsymbol{v}_1, q_0) \in \mathbf{V}_H \times Q_H$. Following exactly the same arguments from Theorem 7 (using this time (72) in (51)), we can build $(\mathbf{z}_1, t_0) \in \mathbf{V}_H \times Q_H$ such that $\|(\mathbf{z}_1, t_0)\|_h \leq C \|(\boldsymbol{v}_1, q_0)\|_h$ with C independent of H, h, σ and α, and such that

$$\boldsymbol{B}_h((\boldsymbol{v}_1,q_0),(\mathbf{z}_1,t_0)) \geq C \|(\boldsymbol{v}_1,q_0)\|_h^2 , \tag{81}$$

and the result follows from Lemma 13. □

We end this section by proving the main error estimate for the method (69).

Theorem 15. *Under all the previous hypothesis, there exists $C > 0$ independent of H, h, σ and α such that*

$$\|(\boldsymbol{u}-\boldsymbol{u}_{1,h}, p-p_{0,h})\|_H \leq C \left(\frac{hH^t}{\sqrt{\sigma}} |g|_{t,\Omega} + \sqrt{\frac{\sigma}{\alpha}}(H+h) \|\boldsymbol{u}\|_{2,\Omega} + \frac{H}{\sqrt{\sigma}} |p|_{1,\Omega} \right) , \tag{82}$$

for $t = 0, 1$.

Proof. Let $(\boldsymbol{v}_1, q_0) = (\mathcal{C}_H(\boldsymbol{u}), \Pi_H(p))$, then

$$\|(\boldsymbol{u}-\boldsymbol{u}_{1,h}, p-p_{0,h})\|_H \leq \|(\boldsymbol{u}-\boldsymbol{v}_1, p-q_0)\|_H + \|(\boldsymbol{u}_{1,h}-\boldsymbol{v}_1, p_{0,h}-q_0)\|_H .$$

The first term is easily estimated using (36), (38) and (40). For the second one, from Lemma 14 there exists $(\boldsymbol{w}_1, t_0) \in \mathbf{V}_H \times Q_H$ satisfying $\|(\boldsymbol{w}_1, t_0)\|_H = 1$ and

$$\begin{aligned}\beta_2 \|(\boldsymbol{u}_{1,h}-\boldsymbol{v}_1, p_{0,h}-q_0)\|_H &\leq \boldsymbol{B}_h((\boldsymbol{u}_{1,h}-\boldsymbol{v}_1, p_{0,h}-q_0),(\boldsymbol{w}_1,t_0)) \\ &= \boldsymbol{B}_h((\boldsymbol{u}_{1,h}-\boldsymbol{u}, p_{0,h}-p),(\boldsymbol{w}_1,t_0)) + \boldsymbol{B}_h((\boldsymbol{u}-\boldsymbol{v}_1, p-q_0),(\boldsymbol{w}_1,t_0)) .\end{aligned}$$

Now, since $(p-q_0,\nabla\cdot\boldsymbol{w}_1)_\Omega=0$, it follows that

$$
\begin{aligned}
&\boldsymbol{B}_h((\boldsymbol{u}-\boldsymbol{v}_1,p-q_0),(\boldsymbol{w}_1,t_0)) = \\
&= \sum_{K\in\mathcal{T}_H}\sigma\,(\nabla p_h(\boldsymbol{u}-\boldsymbol{v}_1),\nabla p_h(\boldsymbol{w}_1))_K-(t_0,\nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_1))_\Omega-\sum_{K\in\mathcal{T}_H}\tau_F\,([\![p-q_0]\!],[\![t_0]\!])_F \\
&\le \sum_{K\in\mathcal{T}_H}\sigma\,|p_h(\boldsymbol{u}-\boldsymbol{v}_1)|_{1,K}|p_h(\boldsymbol{w}_1)|_{1,K}\;+\;\|t_0\|_{0,\Omega}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_1)\|_{0,\Omega} \\
&\quad+\sum_{F\in\mathcal{E}_H}\tau_F\,\|[\![p-q_0]\!]\|_{0,F}\|[\![t_0]\!]\|_{0,F}\,.
\end{aligned}
$$

Hence, using (72), (36), (38),(40) and $\|(\boldsymbol{w}_1,t_0)\|_H=1$ we arrive at

$$
\begin{aligned}
\boldsymbol{B}_h((\boldsymbol{u}-\boldsymbol{v}_1,p-q_0),(\boldsymbol{w}_1,t_0)) &\le \sum_{K\in\mathcal{T}_H}\sigma\,\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}\|\boldsymbol{w}_1\|_{0,K}\;+\;\|t_0\|_{0,\Omega}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_1)\|_{0,\Omega} \\
&\quad+\sum_{F\in\mathcal{E}_H}\tau_F\,\|[\![p-q_0]\!]\|_{0,F}\|[\![t_0]\!]\|_{0,F} \\
&\le C\left(\sum_{K\in\mathcal{T}_H}\sigma\,H_K^4|\boldsymbol{u}|^2_{2,\omega_K}+\frac{H^2\,\sigma}{\alpha}\,|\boldsymbol{u}|^2_{2,\Omega}+\frac{H^2}{\sigma}\,|p|^2_{1,\Omega}\right)^{\frac{1}{2}}.
\end{aligned}
$$

To bound the remaining term, we use that $[\![p]\!]=0$ and the definition of $\boldsymbol{B}_h$ and $\boldsymbol{B}$ to obtain

$$
\begin{aligned}
&\boldsymbol{B}_h((\boldsymbol{u}_{1,h}-\boldsymbol{u},p_{0,h}-p),(\boldsymbol{w}_1,t_0))\;=\;\boldsymbol{B}_h((\boldsymbol{u}_{1,h},p_{0,h}),(\boldsymbol{w}_1,t_0))-\boldsymbol{B}_h((\boldsymbol{u},p),(\boldsymbol{w}_1,t_0)) \\
&=\;\mathbf{F}_h(\boldsymbol{w}_1,t_0)-\left\{\sum_{K\in\mathcal{T}_H}\sigma\,(\nabla p_h(\boldsymbol{u}),\nabla p_h(\boldsymbol{w}_1))_K-(p,\nabla\cdot\boldsymbol{w}_1)_\Omega-(t_0,\nabla\cdot\boldsymbol{u})_\Omega\right\} \\
&=\;\mathbf{F}_h(\boldsymbol{w}_1,t_0)-\left\{\sum_{K\in\mathcal{T}_H}\sigma\,(\nabla p_e^M(\boldsymbol{u}),\nabla p_e^M(\boldsymbol{w}_1))_K-(p,\nabla\cdot\boldsymbol{w}_1)_\Omega-(t_0,\nabla\cdot\boldsymbol{u})_\Omega\right\} \\
&\quad+\sum_{K\in\mathcal{T}_H}\sigma\,\left[(\nabla p_e^M(\boldsymbol{u}),\nabla p_e^M(\boldsymbol{w}_1))_K-(\nabla p_h(\boldsymbol{u}),\nabla p_h(\boldsymbol{w}_1))_K\right] \\
&=\;\mathbf{F}_h(\boldsymbol{w}_1,t_0)-\boldsymbol{B}((\boldsymbol{u},p),(\boldsymbol{w}_1,t_0)) \\
&\quad+\sum_{K\in\mathcal{T}_H}\sigma\,\left[(\nabla p_e^M(\boldsymbol{u})-\nabla p_h(\boldsymbol{u}),\nabla p_e^M(\boldsymbol{w}_1))_K+(\nabla p_h(\boldsymbol{u}),\nabla p_e^M(\boldsymbol{w}_1)-\nabla p_h(\boldsymbol{w}_1))_K\right].
\end{aligned}
$$

Next, using the Galerkin orthogonality in each element K we get $(\nabla p_h(\boldsymbol{u}), \nabla p_e^M(\boldsymbol{w}_1) - \nabla p_h(\boldsymbol{w}_1))_K = 0$, and then from Lemmas 8 and 12 we arrive at

$$
\begin{aligned}
&\boldsymbol{B}_h((\boldsymbol{u}_{1,h} - \boldsymbol{u}, p_{0,h} - p), (\boldsymbol{w}_1, t_0)) = \mathbf{F}_h(\boldsymbol{w}_1, t_0) - \mathbf{F}(\boldsymbol{w}_1, t_0) - \boldsymbol{B}((\boldsymbol{u} - \boldsymbol{u}_1, p - p_0), (\boldsymbol{w}_1, t_0)) \\
&\quad + \sum_{K\in\mathcal{T}_H} \sigma\,(\nabla p_e^M(\boldsymbol{u}) - \nabla p_h(\boldsymbol{u}), \nabla p_e^M(\boldsymbol{w}_1))_K \\
&= \sum_{K\in\mathcal{T}_H} \Big[(\nabla p_e^g - \nabla p_h^g, \boldsymbol{w}_1)_K + \sigma\,(\nabla p_e^M(\boldsymbol{u}) - \nabla p_h(\boldsymbol{u}), \nabla p_e^M(\boldsymbol{w}_1))_K\Big] - \boldsymbol{B}((\boldsymbol{u} - \boldsymbol{u}_1, p - p_0), (\boldsymbol{w}_1, t_0)) \\
&\le \sum_{K\in\mathcal{T}_H} \Big[|p_e^g - p_h^g|_{1,K}\|\boldsymbol{w}_1\|_{0,K} + \sigma\,|p_e^M(\boldsymbol{u}) - p_h(\boldsymbol{u})|_{1,K}|p_e^M(\boldsymbol{w}_1)|_{1,K}\Big] + C\,\sigma\,H\,|\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{w}_1\|_{0,\Omega} \\
&\le \sum_{K\in\mathcal{T}_H} C\,\Big[\sigma^{-1}h\,|p_e^g|_{2,K}\|\boldsymbol{w}_1\|_{0,K} + \sigma\,h\,|p_e^M(\boldsymbol{u})|_{2,K}\|\boldsymbol{w}_1\|_{0,K}\Big] + C\,\sigma\,H\,|\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{w}_1\|_{0,\Omega} \\
&\le C\left(\frac{hH^t}{\sqrt{\sigma}}\,|g|_{t,\Omega} + \sqrt{\sigma}\,(H+h)\,|\boldsymbol{u}|_{1,\Omega}\right),
\end{aligned}
$$

and the result follows. □

5. Numerical Experiments

Now, we are interested in the numerical validation of the fully discrete method (69). The validations are performed trough three series of numerical tests. The first two experiments aim to compare the solution provided by (69) with available analytical solutions. For both tests the local mass conservation feature is verified and an analysis of sensitivity with respect to α_F is performed. Finally, the robustness of the method to face out of the scope problems is assessed in the final test by solving the so-called five-spot benchmark.

5.1. Analytical solution: first study.

The domain is $\Omega = (0,1)\times(0,1)$ for the first test and for all remaining tests as well. Moreover, we set $\sigma = 1$ and set the exact pressure equals to $p(x,y) = \cos(2\pi x)\cos(2\pi y)$, $\boldsymbol{u} = -\nabla p$ and thus $b = 0$ and $g = \nabla\cdot\boldsymbol{u} = 8\pi^2\cos(2\pi x)\cos(2\pi y)$. In Figure 2 we report the errors on velocity and pressure in a sequence of structured meshes using $\alpha_F = 0.1$, and observe optimal convergence of all quantities as $H \to 0$ in their respective natural norms, which is in accordance with the theoretical results. For all the examples, we use the notation

$$
\|[\![p_0]\!]\|_J = \left\{\sum_{F\in\mathcal{E}_H} H_F\,\|[\![p_0]\!]\|_{0,F}^2\right\}^{\frac{1}{2}}.
$$

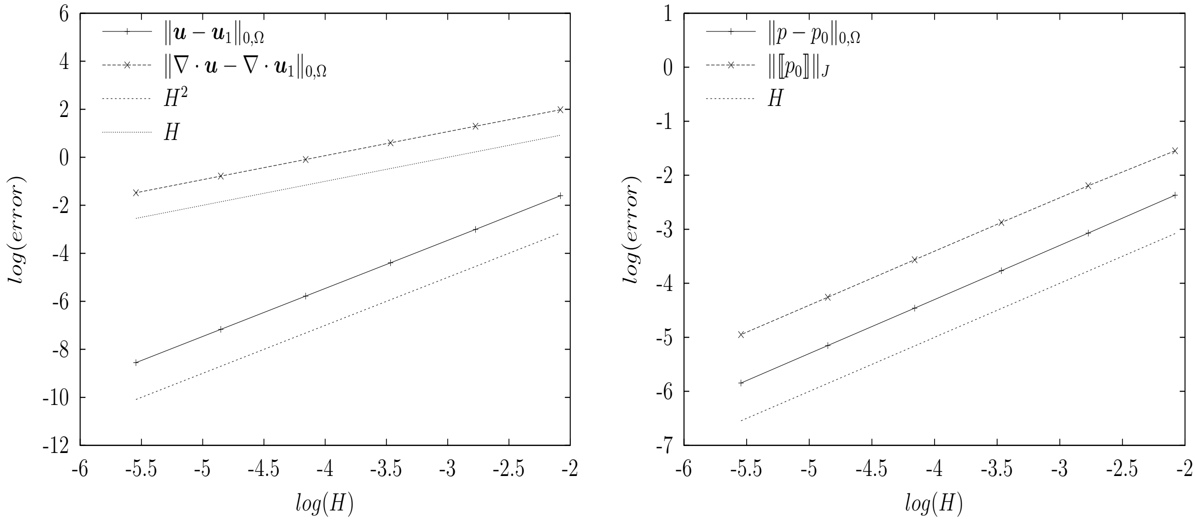

FIGURE 2. Example I: Convergence history for the velocity field and its divergence (left) and the pressure and jump (right).

Furthermore, in Table 1 we study the local mass conservation feature for $\boldsymbol{u}_1+\boldsymbol{u}_e^D$. For that we define the quantity

$$M_e := \max_{K\in\mathcal{T}_H} \frac{|\int_K(\nabla\cdot(\boldsymbol{u}_1+\boldsymbol{u}_e^D)-g)dx|}{|K|}\,, \tag{83}$$

and observe that we recover the local mass conservation property updating the linear velocity field by the multiscale velocity $\boldsymbol{u}_e^D$.

TABLE 1. Example I: Relative local mass conservation error.

H	6.25×10^{-2}	3.12×10^{-2}	1.56×10^{-2}	7.8×10^{-3}	3.9×10^{-3}
M_e	1.40×10^{-12}	1.29×10^{-12}	3.59×10^{-12}	6.87×10^{-12}	7.02×10^{-11}

Next, a study of sensitivity of the numerical error with respect to α_F is performed in Table 2 for a fixed mesh, where we observe that the errors remain independent of the parameter as long as α_F stays of order 1. That agrees with the assumption that τ_F must be at order H_F, as predicted by the theory. We also perform a convergence study for all the variables using different values for α_F. The results are depicted in Figures 3 and 4 where we can see that the errors are practically unaffected by the value of α_F.

Our next objective is to perform a comparison of the performance of (69) with the lowest order Raviart-Thomas' mixed finite element method RT_0/Q_H (cf. [26, 11]). The comparison is performed in Figures 5-6. We can see from the results that clearly (69) is far more accurate than the Raviart-Thomas method in the $L^2(\Omega)^2$ norm of the velocity field, and, thanks to

TABLE 2. Example I: The sensitivity of the errors with respect to α_F.

α_F	$\\|(\boldsymbol{u}-\boldsymbol{u}_1, p-p_0)\\|_H$	$\\|\boldsymbol{u}-\boldsymbol{u}_1\\|_{0,\Omega}$	$\\|\nabla\cdot\boldsymbol{u}-\nabla\cdot\boldsymbol{u}_1\\|_{0,\Omega}$	$\\|p-p_0\\|_{0,\Omega}$	$\\|[\![p_0]\!]\\|_J$
10^{-6}	1.8380	0.012	1.826	0.023	0.056
10^{-4}	1.8380	0.012	1.826	0.023	0.056
10^{-2}	1.8380	0.012	1.826	0.023	0.056
0.1	1.8459	0.012	1.826	0.023	0.056
1	1.9220	0.013	1.830	0.023	0.056

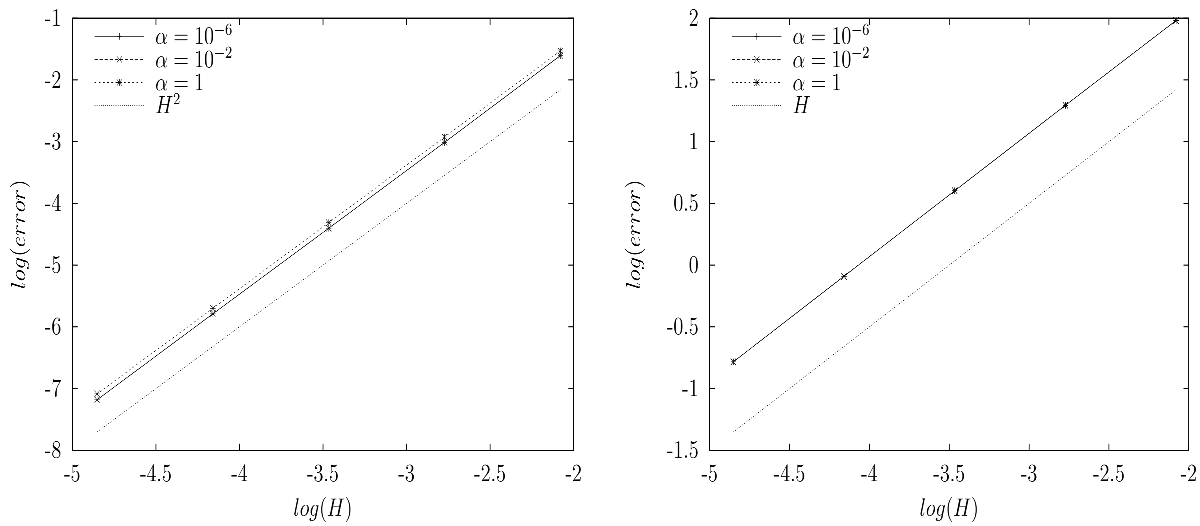

FIGURE 3. Example I: Convergence history for the velocity field (left) and its divergence (right) for different values of α_F.

the mass conservation property for $\boldsymbol{u}_1 + \boldsymbol{u}_e^D$, when the velocity field is updated with $\boldsymbol{u}_e^D$, the errors in the divergence are identical. Both methods seem to perform equally good regarding the errors in the pressure field. We want to stress the fact that the solution of (69) involves less degrees of freedom than RT_0/Q_H when the same mesh is used.

Finally, the unstructured mesh of Figure 7, containing approximately 5000 elements, is adopted. The sensitivity of error in terms of parameter α_F presents a different behavior than before. Nevertheless, since there is a loss of stability when α_F is small, the unexpected robust error behavior when structured meshes are used is no longer preserved. The results are reported in Table 3 where we observe that the individual norms are not independent of α_F, but the whole $\|\cdot\|_H$ norm of the error seems robust with respect to α_F.

5.2. **Second Analytical solution: a divergence-free velocity field.** For the following example we consider a divergence-free velocity field. More precisely, the problem is set up

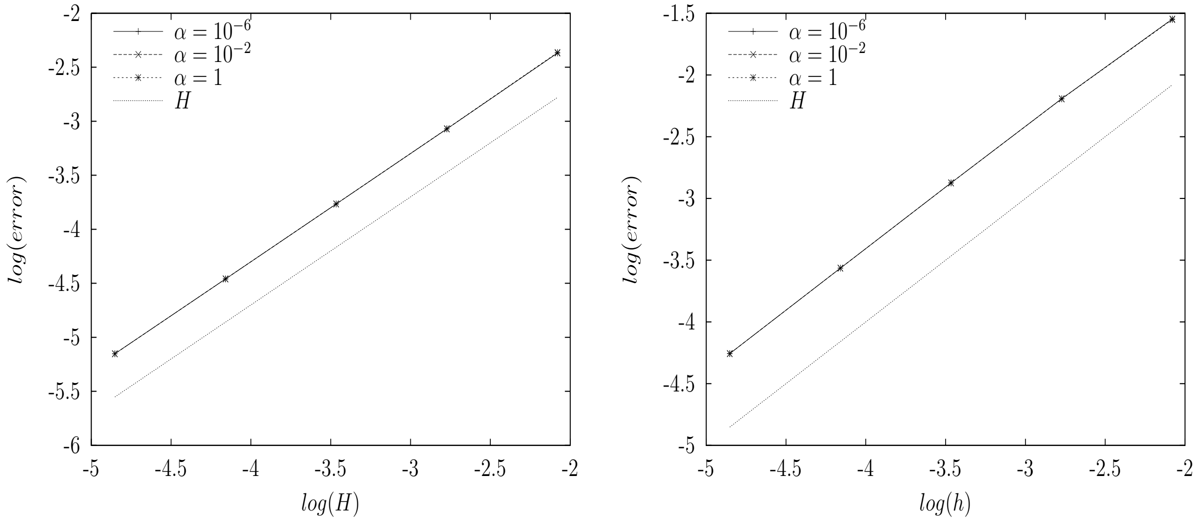

FIGURE 4. Example I: Convergence history for the pressure (left) and jump (right) for different values of α_F.

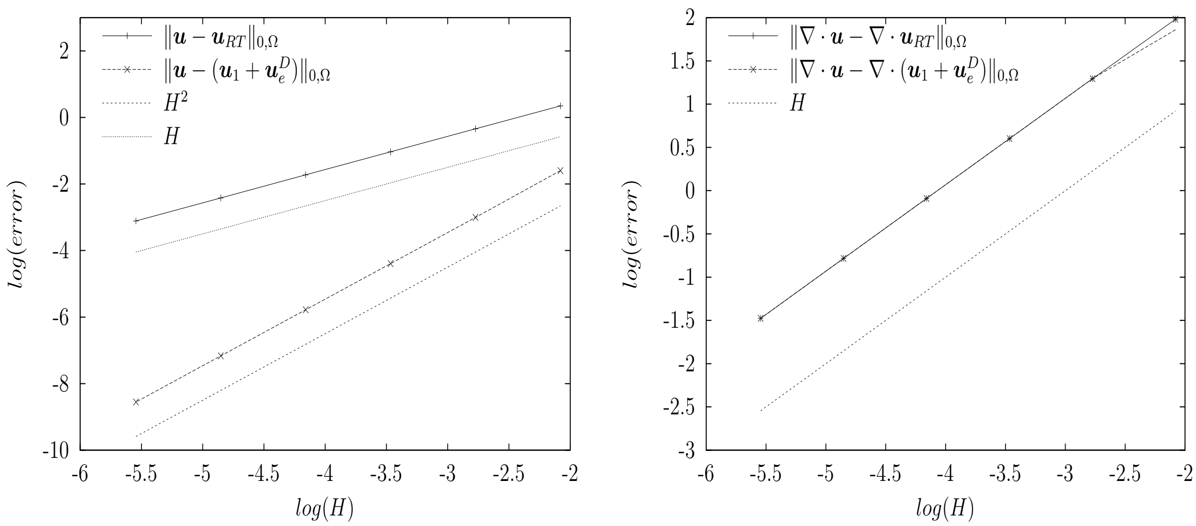

FIGURE 5. Example I: Convergence history for the velocity (left) and its divergence (right) for (69) and RT_0/Q_H.

as in the first test, the exact pressure is now $p(x, y) = x - x^2 - 1/6$ and the velocity field

$$\boldsymbol{u} = (x^2(1-x)^2(2y - 6y^2 + 4y^3), -y^2(1-y)^2(2x - 6x^2 + 4y^3))^t \,.$$

The source term is then

$$\boldsymbol{f} = (x^2(1-x)^2(2y - 6y^2 + 4y^3) + 1 - 2x, -y^2(1-y)^2(2x - 6x^2 + 4y^3))^t \,,$$

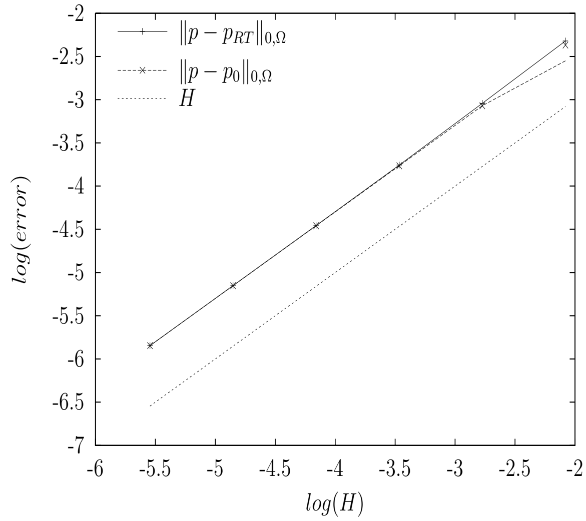

FIGURE 6. Example I: Convergence history for the pressure for (69) and RT_0/Q_H.

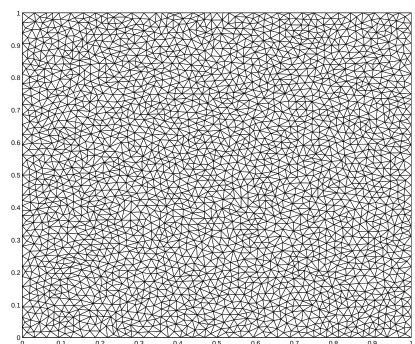

FIGURE 7. The unstructured mesh.

TABLE 3. Example I: The sensitivity of the errors with respect to α_F using an unstructured mesh.

α_F	$\\|\boldsymbol{u} - \boldsymbol{u}_1\\|_{0,\Omega}$	$\\|\nabla \cdot \boldsymbol{u} - \nabla \cdot \boldsymbol{u}_1\\|_{0,\Omega}$	$\\|p - p_0\\|_{0,\Omega}$	$\\|[\![p_0]\!]\\|_J$	$\\|(\boldsymbol{u} - \boldsymbol{u}_1, p - p_0)\\|_H$
10^{-3}	0.6099	1.6741	8.0277	17.6551	1.8843
10^{-2}	0.3017	1.6771	1.8341	2.6921	1.7349
0.1	0.1389	1.6840	0.2585	0.4186	1.6968
1	0.0654	1.7011	0.0373	0.0821	1.7047
10	0.0366	1.8013	0.0228	0.0541	1.8112

and the boundary condition is $b = 0$. Since the source term $\boldsymbol{f}$ is no longer a constant function in K, we must consider the enhancement of $\mathcal{M}_K^u(\boldsymbol{f})$, i.e., we must add the term

$\sum_{K\in\mathcal{T}_H}(\mathcal{M}_K^u(-\boldsymbol{f}),\sigma\boldsymbol{v}_1)_K$ to the right hand side. From the definition of the operator $\mathcal{M}_K^u$ we rewrite the right hand side of the equation as

$$(\boldsymbol{f},\boldsymbol{v}_1)_\Omega+\sum_{K\in\mathcal{T}_H}(\sigma\boldsymbol{u}_e^M(-\boldsymbol{f}),\boldsymbol{v}_1)_K = (\boldsymbol{f},\boldsymbol{v}_1)_\Omega+\sum_{K\in\mathcal{T}_H}(-\boldsymbol{f}-\nabla p_e^M(-\boldsymbol{f}),\boldsymbol{v}_1)_K$$
$$= \sum_{K\in\mathcal{T}_H}(\nabla p_e^M(\boldsymbol{f}),\boldsymbol{v}_1)_K\,,$$

were $p_e^M(\boldsymbol{f})$ is solution of the local problem

$$-\Delta p_e^M=-\nabla\cdot\boldsymbol{f}\quad\text{in}\quad K,\qquad \partial_{\boldsymbol{n}}p_e^M=\boldsymbol{f}\cdot\boldsymbol{n}\quad\text{on}\quad\partial K\,. \tag{84}$$

Now, considering the local problems and the conservation mass property, we have $\nabla\cdot\boldsymbol{u}_1+\nabla\cdot\boldsymbol{u}_e^D=0$ at the element level, leading to

$$\nabla\cdot\boldsymbol{u}_1\Big|_K=-\frac{1}{|K|}\sum_{F\in\partial K}\frac{\alpha_F H_F}{\sigma}\int_F [\![p_0]\!]\boldsymbol{n}\cdot\boldsymbol{n}_F\,. \tag{85}$$

Hence, we do not expect in general that the error for the divergence of the velocity field to have a good behavior with respect to the parameter α_F and we expect a small variation in the norm H since the diverge of the velocity field becomes more important as the parameter α_F is of order one (see Table 5 and Figures 9-10).

The results concerning the errors on velocity and pressure are depicted in Figure 8 using $\alpha_F=0.1$. In there we observe a $H^{3/2}$ convergence for the velocity field in the $H(div,\Omega)$ norm, which is higher than the expected rate of convergence given by the analysis. This is a good thing when we compare to the Raviart-Thomas method, in which the discrete velocity field is exactly divergence-free. Of course, when updated with the enrichment function $\boldsymbol{u}_e^D$, then the velocity field becomes exactly divergence-free (see Table 4 for the mass-conservation results). The sensitivity of the error with respect to α_F is performed in Table 5, and as before, we study the convergence of the method for different choices of α_F and we report the results in Figures 9 and 10 where we observe that the errors in divergence are affected by the value of α_F, while the rest seem fairly independent of α_F.

TABLE 4. Example II: Relative local mass conservation error.

H	6.25×10^{-2}	3.12×10^{-2}	1.56×10^{-2}	7.8×10^{-3}	3.9×10^{-3}
M_e	1.04×10^{-15}	1.03×10^{-15}	9.11×10^{-15}	1.41×10^{-14}	6.30×10^{-15}

Now we perform a comparison of (69) with the lowest order Raviart-Thomas' mixed method RT_0/Q_H where we get better precision for the velocity field, as before, and the errors for the pressure seem very close as well. The results are depicted in Figure 11.

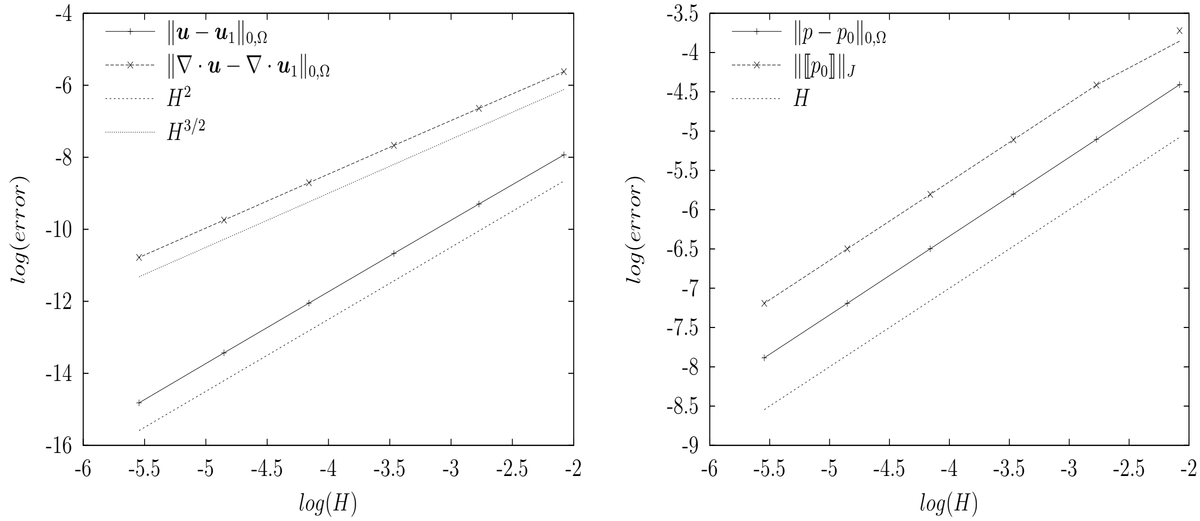

FIGURE 8. Example II: Convergence history for the velocity field and its divergence(left) and the pressure and jump (right).

TABLE 5. Example II: The sensitivity of errors with respect to α_F.

α_F	$\|(\boldsymbol{u}-\boldsymbol{u}_1, p-p_0)\|_H$	$\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,\Omega}$	$\|\nabla\cdot\boldsymbol{u}-\nabla\cdot\boldsymbol{u}_1\|_{0,\Omega}$	$\|p-p_0\|_{0,\Omega}$	$\|[\![p_0]\!]\|_J$
10^{-6}	0.0090	5.7×10^{-5}	4.6×10^{-9}	0.003	0.006
10^{-4}	0.0090	5.7×10^{-5}	4.6×10^{-7}	0.003	0.006
10^{-2}	0.0090	5.7×10^{-5}	4.6×10^{-5}	0.003	0.006
0.1	0.0094	6×10^{-5}	4.6×10^{-4}	0.003	0.006
1	0.0013	1.9×10^{-4}	4.6×10^{-3}	0.003	0.006

5.3. **The five-spot problem.** Due to its practical importance in oil recovery, the quarter five spot problem has served as a paradigm to validate stability and accuracy of numerical methods for the Darcy model. This problem is now addressed considering zero source term $\boldsymbol{f}$ and $\sigma = 1$ in a unit square domain, and instead of modeling injection and production of well by a non-zero source term g, we consider a non-homogeneous boundary condition for the velocity such that its normal component is equal to $\frac{1}{4H_F}$ at points $(0,0)$ and $(1,1)$. This delta of Dirac is linearly approached on the edges sharing such points. The solution obtained is depicted in Figures 12-15 where we observe the total absence of oscillations in the solution. The constant α_F is again fixed equal to 0.1. In Table 6 we study the local mass conservation feature regarding the enhanced method, as soon $\boldsymbol{u}_1$ is updated by $\boldsymbol{u}_1 + \boldsymbol{u}_e$.

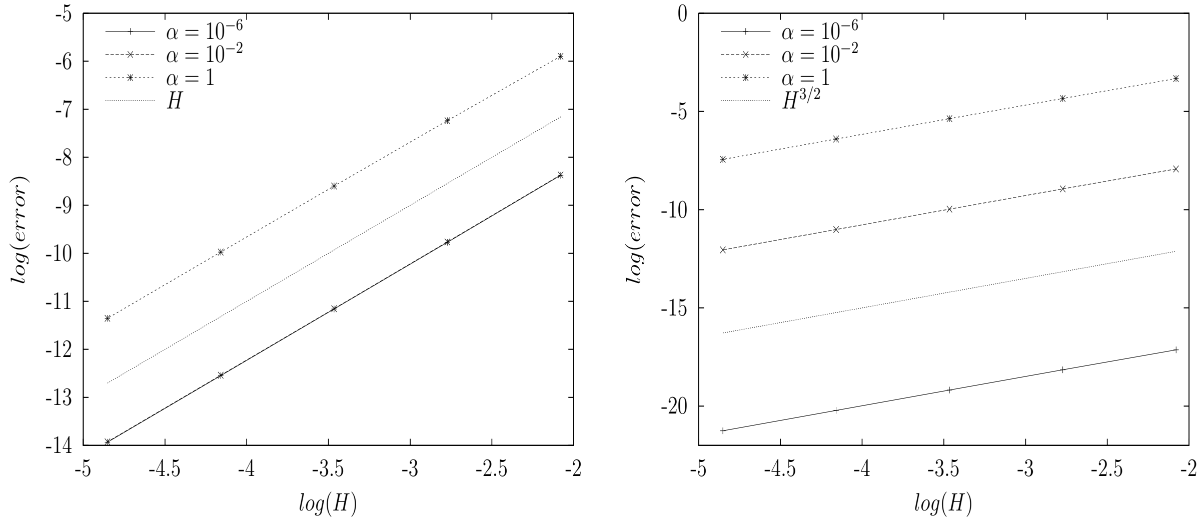

FIGURE 9. Example II: Convergence history for the velocity field (left) and its divergence (right) for different values of α_F.

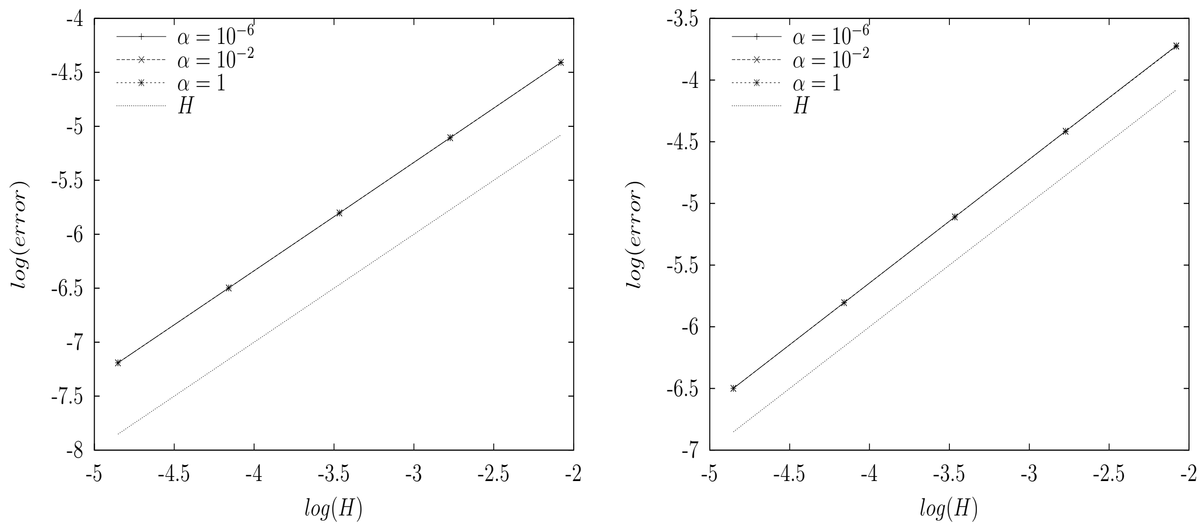

FIGURE 10. Example II: Convergence history for the pressure (left) and jump (right) for different values of α_F.

TABLE 6. Five spot problem: Relative local mass conservation error

h	6.25×10^{-2}	3.12×10^{-2}	1.56×10^{-2}	7.8×10^{-3}	3.9×10^{-3}
M_e	4.11×10^{-12}	2.3×10^{-12}	7.29×10^{-12}	1.34×10^{-11}	1.32×10^{-9}

6. CONCLUSION

To adopt the original operator along with boundary conditions built on imposing continuity for the pressure and the flux appears as the correct form to model unsolved scales in

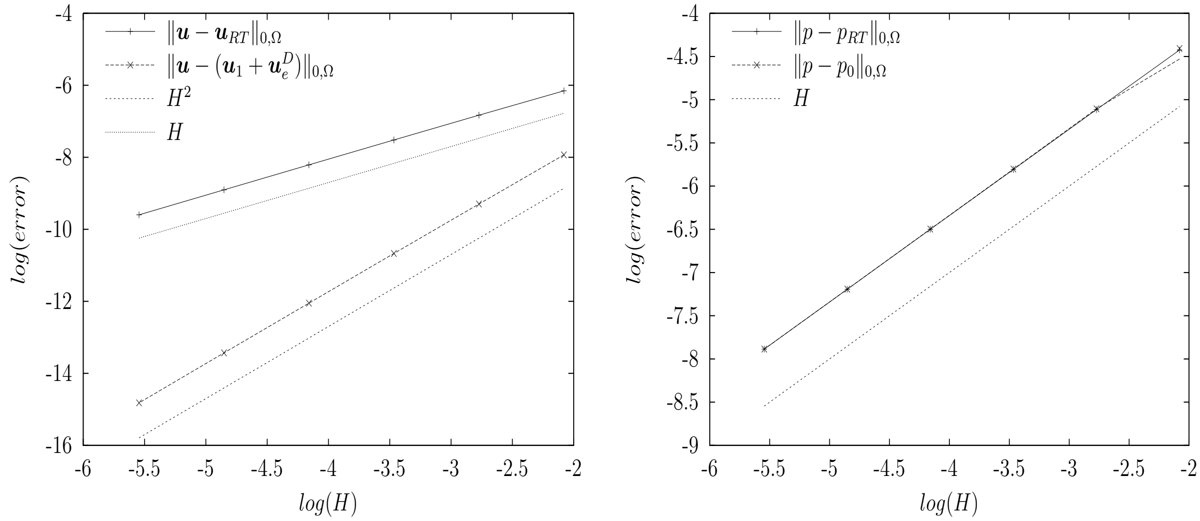

Figure 11. Example II: Convergence history for the velocity (left) and for the pressure (right) for (69) and RT_0/Q_H^0.

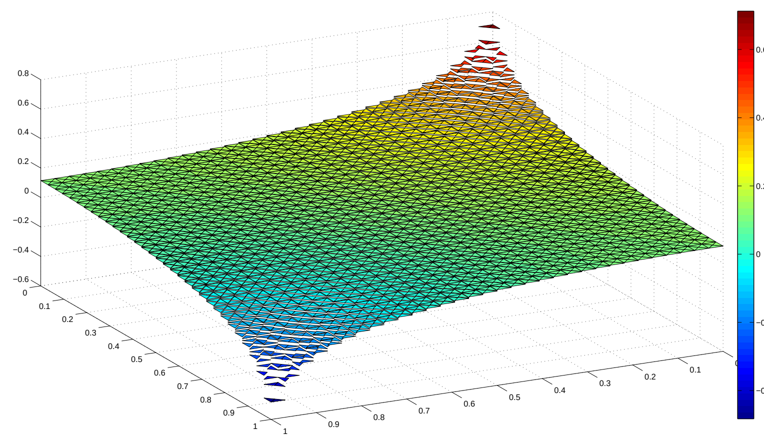

Figure 12. Five spot problem: Profile of pressure.

terms of resolved ones, which means, to recover stability and optimality through the Petrov-Galerkin augmenting space approach. In addition, by incorporating such unsolved scales into the finite element method all the desired features of the original spaces, such as local mass conservation and nodal values for the velocity, are still preserved. When it comes to solve the local problems, the proposed method compromises accuracy with low computational cost. Thus, our method appears as a competitive alternative to tackle more complex flows where analytical solutions are out of reach, such as oscillating coefficients. Higher order pairs of interpolation spaces will also demand a the two-level approach, but in those cases,

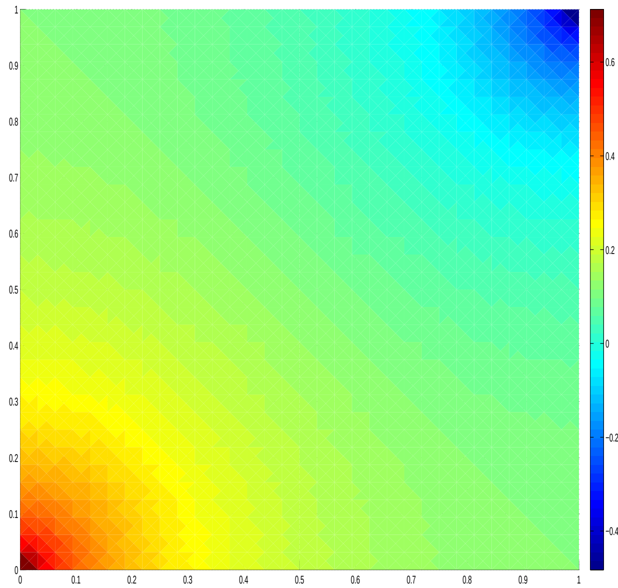

Figure 13. Five spot problem: Isovalues of the pressure.

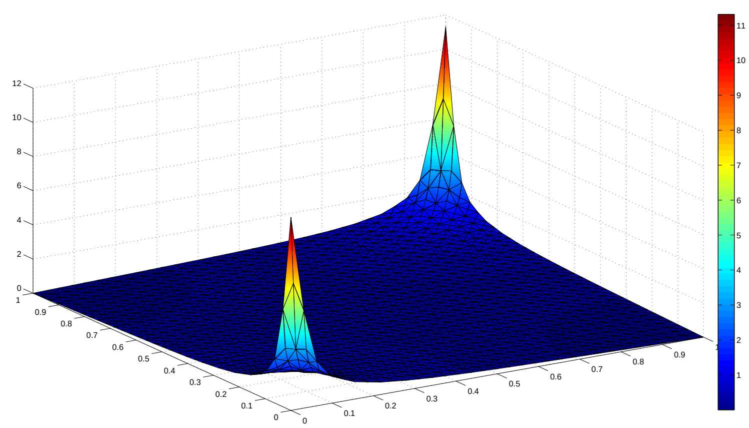

Figure 14. Five spot problem: The 2-norm of the velocity.

the boundary condition for the local problems ought to include further control on the gradient of the pressure. Finally, enriched methods seem to show an intrinsic relationship with some discontinuous finite element methods. This subject should be enrolled in forthcoming works.

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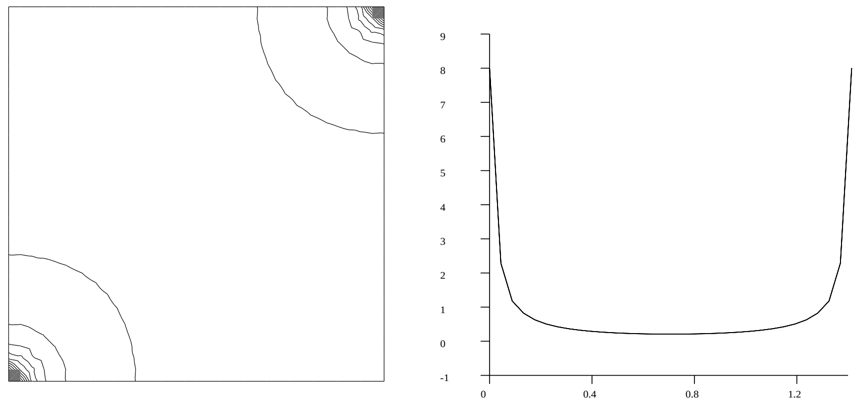

FIGURE 15. Five spot problem: Isovalues for $|\boldsymbol{u}_1|$ and a cross-section along the line $y = x$.

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