Design study of a regenerative pump using one-dimensional and three-dimensional numerical techniques

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Abstract

Regenerative pumps are low cost, compact, able to deliver high heads at low flowrates. Furthermore with stable performance characteristics they can operate with very small NPSH. The complexity of the flowfield is a serious challenge for any kind of mathematical modelling. This paper compares an analytical and numerical technique of resolving the performance for a new regenerative pump design. The performance characteristics computed by a CFD approach and a new one-dimensional model are compared and matched to experimental test results. The approaches of both modelling techniques are assessed as potential design tools. The approaches are shown to not only successfully resolve the complex flowfield within the pump; the CFD is also capable of resolving local flow properties to conduct further refinements. The flow field is represented by the CFD as it has never been before. A new design process is suggested. The new regenerative pump design is considered with a comparable duty centrifugal pump, proving that for many high head low flowrate applications the regenerative pump is a better choice.

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Keywords: Regenerative pump, computational fluid dynamics (CFD), helical flow

Nomenclature

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A	Cross sectional area	(m²)	**Superscripts**		
P	Power	(kW)			
Q	Volume flow rate	(m³/s)	$*$	Intermediate variable	
U	Mean fluid velocity	(m/s)	–	Mean value	
ρ	Density	(kg/ m³)			
r	radius	(m)	**Subscripts**		
H	Head	(m)			
p	Pressure	(kN/m²)	o	open channel	
N	Rotational speed	(rev/min)	c	circulatory	
g	Gravitation acceleration	(m/s²)	sp	Suction port	
η	Efficiency		dp	Discharge port	
ω	Angular velocity	(rad/s)	t	tangential	
α	Incidence factor		e	exit	
ζ	Ellipse angle	(rad)	i	inlet	
D	Diameter	(m)	s	Solid body	
θ	Angle measured from inlet		0	Inner channel	
σ	Slip Factor		2	Impeller tip	
Z	Number of impeller blades		3	Outer channel	
ϕ	Flow Coefficient		L	Linear region	
ϑ	Effectiveness of circulatory flow		u	upper	
ψ	Head Coefficient				
K	Loss coefficient				
b	Impeller width	(m)	**Abbreviations**		
c	Impeller tip clearance	(m)			
d	Channel width	(m)	*CFD*	Computational fluid dynamics	
U	Peripheral velocity	(m/s)	*CAD*	Computer Aided Design	
NPSH	Net Positive Suction Head	(m)			

Introduction

Regenerative pumps are rotor-dynamic machines [1], capable of developing high heads at low flowrates with a single impeller [2]. At low specific speeds the pump also shares characteristics with positive displacement pumps and centrifugal pumps. Other benefits with this type of pump include its compact design [3] and the ability to operate at low NPSH. Regenerative pumps, like centrifugal pumps, are kinetic pumps. The head rise is achieved through an exchange in momentum between the impeller and the fluid. There are a limited number of publications for regenerative pumps in comparison to other kinetic devices [4]. In spite of low efficiency, usually less than 50% [3,4], regenerative pumps can offer a more efficient alternative to other (centrifugal) pumps in many applications [5]. Until recently only analytical, one-dimensional models were used to predict the performance characteristics and were limited in representing the complex flow-field within the pump requiring significant experimental correction [6]. CFD analysis and a newly developed one-dimensional model offer a much more precise way of calculating the performance of a regenerative pump. This paper presents these methods which are not dependent on experimental correction, are valid for a range of pump geometries and are considered as design tools to optimise pump performance.
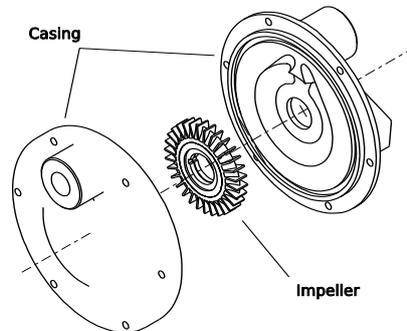
Figure 1: The regenerative pump

A regenerative pump consists of a casing with an annular channel and an impeller, Fig 1. The impeller is a disc with several, usually 20 to 50, radial vanes at the edge of the disc periphery. These vanes rotate in the annular channel. The fluid enters the channel through the suction port and repeatedly circulates through the impeller vanes due to the action of a centrifugal force field. The flow path approximates a toroidal helix, similar to the shape of a corkscrew [3]. This effect of repeated circulation is also described as internal multi-staging [2], for each passage through the vanes may be regarded as a conventional stage [4]. This multi-staging is where the pump derives the name regenerative pump. The fluid exits through the discharge port. The discharge port is located immediately before the stripper, a short section where the annular channel is reduced to a close clearance preventing the fluid from flowing from the discharge to suction port, Fig 2.
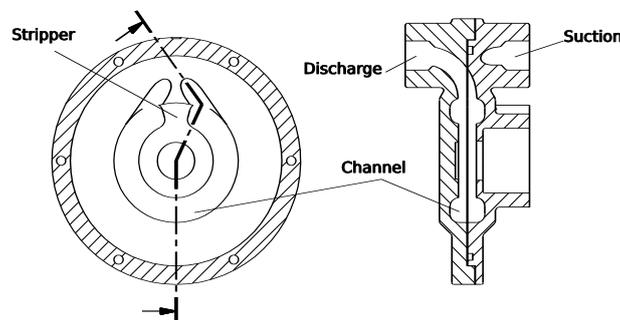
Figure 2: The regenerative pump

The pump detailed in this paper incorporates a new design for the suction port and the discharge port. Previous designs had straight ports leading into the channel in a radial direction. This arrangement has subsequent large losses for the flow into the pump [7]. The new design incorporates the advantages of the conventional geometries (multi-staging) but realigns the fluid flow. The flow enters and exits the flow channel in an axial direction, Fig 2. The ports are aligned in axial direction so that an arrangement with multiple stages is possible by assembling the casings in row using one single shaft. The flow in the ports of this arrangement is lead into a 90° turn and enters the annular channel in a tangential direction centric above the impeller. This permits flow guidance at the suction and discharge ports and minimises axial thrust while the circulatory flow develops[8].

One-dimensional analysis

One-dimensional models, until recently have accounted for the majority of the publications as the preferred method of calculating performance. Engels [9] in succession of Schmiedchen [10] and Ritter [11] suggested that the helical flow was responsible for the head rise. He developed an early one-dimensional model and predicted the pump's characteristics. His equations expressed the performance however they relied on experimentally defined coefficients. This fact limits the applicability of Engels model.

Wilson et al.[12] developed a one-dimensional model that is the basis for most publications to date. He suggested an equation for the momentum exchange and identified losses occurring in the pump. However these losses were defined experimentally and do not rely on geometrical or physically derived data. The simplifications used in this model also proved to have a significant effect on the performance prediction capability of the suggested model.

Song et. al. [1] developed a one-dimensional model with a fundamental change. Most publishers succeeding Wilson neglected the change of circulatory velocity along the channel and corrected this by introducing a linear region factor. This lead to an analytical model, but the linear region factor was *a priori* unknown and a source of error. Song et. al. did not simplify the differential equations for the momentum exchange, but used a numerical technique to calculate the circulatory velocity at discrete points to correct the deficiency.

Yoo et. al. [2] proposed equations to calculate the flow geometry namely, the centre of the circulation, and the inlet and outlet radius. Since Wilson et. al., a primary assumption of previous analysis has been that the flow exits the blade only at the impeller tip. Yoo et. al. applied the continuity equation and calculated the circulation and derived inlet and outlet points that are independent of the impeller analysis. Apart from the more accurate description of the flow geometry, the model by Yoo et al., still relies on the linear region factor to correct the error due to the simplification of the differential equation. The linear region factor presented in most published one-dimensional models does not represent the fluid behaviour satisfactory.

In the current study a numerical modelling approach is presented that demonstrates the ability to successfully resolve the flow field in a manner that has not been achieved previously. The one-dimensional model used in this paper incorporates a new model which does not rely on experimental data from conducted test, but can predict the performance only from the geometry and fluid properties.

Six assumptions are made in the development of the mathematical model.

1. Steady flow

2. Incompressible fluid

3. All processes are adiabatic

4. There are no effects due to suction and discharge, except for losses

5. The pump flow is characterised by a tangential and a circulatory velocity

6. Tangential pressure gradient is independant of radius

The head rise for the pump is calculated from the angular momentum exchange or circulatory head rise gH_c and the losses occurring in the pump. These are suction port head loss ΔgH_{sp}, discharge port head loss ΔgH_{dp} and tangential head loss ΔgH_t. The head rise of the pump is expressed by

$$ gH = gH_c - \Delta gH_{sp} - \Delta gH_{dp} - \Delta gH_t \qquad \text{Eq. (1)} $$

where

$$ gH_c = \frac{A_c \overline{V_c}}{Q_s} \left(\sigma U_e^2 - \alpha U_i^2 \right) \qquad \text{Eq. (2)} $$

Ac is the circulatory flow area and Q_s the solid body rotation of the fluid between the impeller vanes. V_c is the mean circulatory velocity evaluated by numerically averaging over the whole pumping region using the composite Simpson's rule [13]. The circulatory velocity is calculated by discretising and linearising of the fist-order non-linear differential equation expressed by

$$ \frac{\partial V_c}{\partial X_{g\rho}} = F\left(X_{g\rho}\right)V_c^2 + G\left(X_{g\rho}\right)V_c + H\left(X_{g\rho}\right) \qquad \text{Eq. (3)} $$

using a predictor-corrector technique. The factors σ and α are the slip factor and the shock factor as detailed by [2].

CFD analysis

Regenerative pumps, as with other kinds of turbomachinery, can be analysed using computational fluid dynamics (CFD). This technique to predict performance and to examine flow characteristics has been successful in standard regenerative pumps recently[7]. Turbomachinery applications for CFD models are proving more robust and have found greater utilisation in the academic and commercial fields [8].

CFD models can offer benefits over experiments in lower costs and in accuracy as the flow features are not disturbed by measuring equipment [8]. This paper presents some of the benefits of the CFD analysis and also considers some of the limitations in this respect.

In this study, a Multiple Reference Frame-modelling (MRF) approach is chosen. Only a stationary performance prediction is desired and the MRF- model is sufficient in this case. For turbomachinery the approach using MRF modelling is also referred to as the "frozen rotor" technique. The inner region close to the impeller is a moving reference frame while the channel region and inlet and outlet ports are treated as a stationary reference frame. The calculation is conducted as steady state and therefore fairly efficient in computational effort. FLUENT application briefs [14-16] report a matching of about 7% between CFD results and experimental data. Unsteady interaction effects like wakes are neglected by this model but this can only be addressed by using the much more expensive sliding mesh modelling. This is considered the remit of future work.

A grid was chosen after conducting a grid independence analysis. Four separate meshes with 0.4 million, 1.3 million, 1.7 million and 2.3 million cells were compared. All of the grids had been created using the same technique but applying different cell sizes. Eventually, the grid with approximately 1.7 million cells was chosen for the comparative analysis. Further refinement did lead to less than one percent change in parameter accuracy in the pump model. Fig. 3 shows the basic shape and refinements of the CFD mesh. The inner region is constructed of mainly hexahedrons with refinements towards wall boundaries and the interface region. The outer frame is meshed with tetrahedrons to model the geometry while obtaining an acceptable mesh quality [17]. The skewness criterion did not exceed 92 per cent at any point.

The geometry for the grid is the same used for manufacturing the pump exported directly from the CAD application.
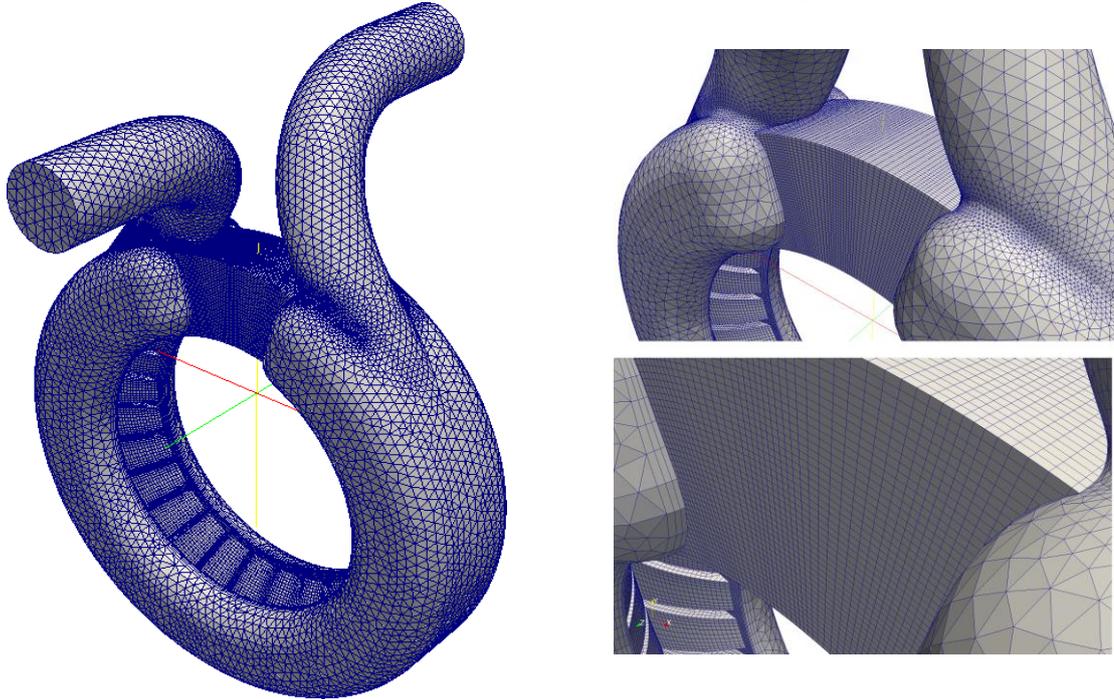
Figure 3: CFD Grid arrangement

Varying volumetric flowrates of the regenerative pump was assessed for multiple differential pressures using the commercial CFD code FLUENT Version 12.0.16. The Reynolds Averaged Navier Stokes Equations were solved on the domain using a pressure based solver and the SIMPLE algorithm [18] for pressure correction. Turbulence was modelled using the k-ε-Realizable model, a highly stable and well developed two-equation approach [19]. This turbulence model is suitable for complex shear flows involving rapid strain, swirl, vortices and locally transitional flows. The applied discretisation scheme obtained second order accuracy. The boundary layer was not resolved, but modelled using a continuous wall function (Enhanced Wall Treatment). Therefore the y^+ -value was kept below 300.

Experimental procedure

Until now regenerative pump port geometry have retained a fairly basic geometric configuration with simple radial inlet and outlet ports machined into the casings [6]. For the pump used in this paper with a rapid prototyping technique robust enough to produce, for the first time, more complex regenerative pump axial inlet and outlet ports were manufactured. The axial port arrangement differs to the classical radial ports. The inflow into the pump and the outflow are parallel to the pump's axis (fig. 2) in contrast to the 90° angle of the radial pump standard. Rapid Prototyping using `Objet Polyjet™ ,[20], (photopolymer jetting technology) techniques , builds components in thin layers (0.016mm) enabling smooth, accurate and highly detailed parts to be produced in a range of materials directly from CAD models. Quail et. al. first demonstrated the ability to produce regenerative pump components in an accurate and robust enough manner to permit experimental testing for a radial standard pump [21].

The experimental rig arrangement, Fig 4, incorporated a reservoir tank which stored and ultimately received the working fluid, in this case water. The fluid was drawn to the pump from the tank and the fluid flow was adjusted via a flow control valve in the return line to the tank.
The fluid flowrate was measured using a Hall Effect turbine flow meter, situated upstream of the flow control valve and upstream of the pump. The pump itself was driven by an induction motor operating at a constant speed of 3000rpm (50 Hz). The motor housing was coupled to a dynamometer containing a load cell to measure strain and hence indicate input torque to be used in the pump efficiency calculations.
The test impeller had 30 blades of width 11mm and diameter 74.5mm. The pump was of double suction shape designed with alignment of the blades to balance axial thrust. In the current study a radial impeller was used with teeth or vanes machined into each side at its periphery. The measurements were collected via a data acquisition unit.
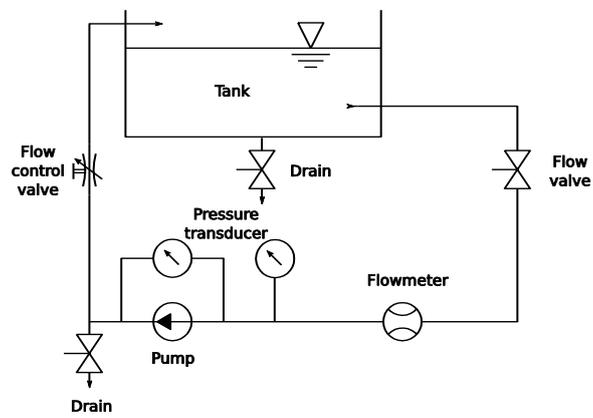
**Figure 4: Schematic of the experimental rig arrangement**

Results

The results of the one-dimensional model and the CFD are compared to the experimental results for two pumps. Pump 1 is the pump developed by Quail et. al. , [7,8,21] incorporating a radial suction and discharge port arrangement . Here the flow enters the pump radially and joins with the flow in the pump with a 90° angle to the peripheral velocity. Pump 2 is the pump designed by the authors (fig 1). The suction and discharge ports are axially aligned to support a multistage arrangement. In addition the flow enters aligned to the peripheral velocity in the channel.

The characteristics are displayed in a dimensionless manner using a head coefficient and a flow coefficient. They are expressed by

$$\psi = \frac{gH}{\omega^2 r_{g,o}} \qquad\qquad \text{Eq. (4)}$$

$$\phi = \frac{Q}{\omega A_o r_{g,o}} \qquad\qquad \text{Eq. (5)}$$

Fig. 5 shows the comparison between both pumps and the tools for performance prediction.
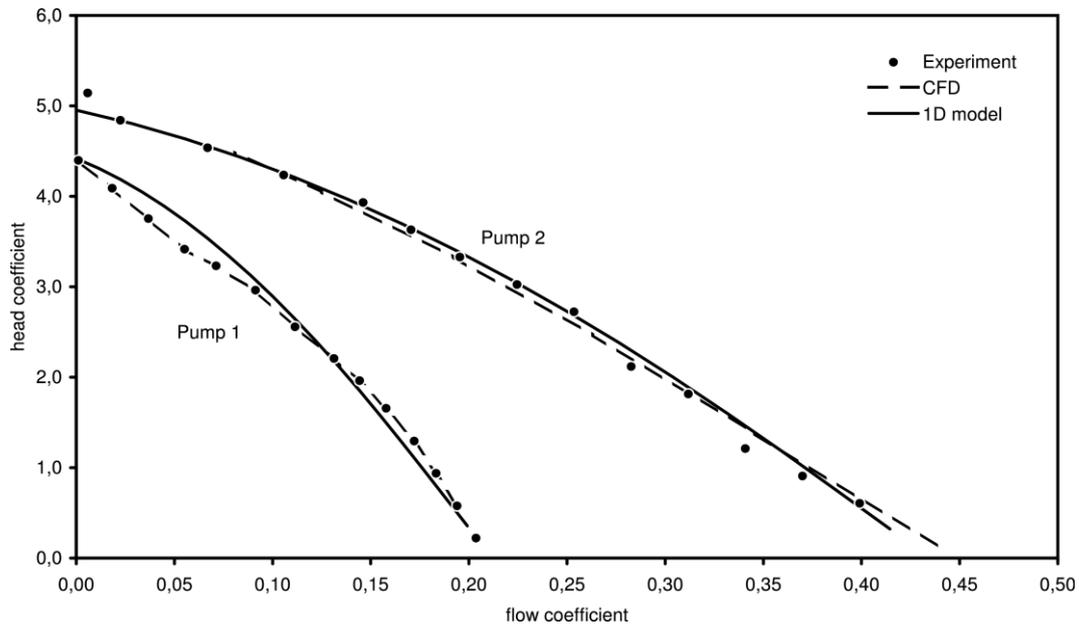
Figure 5: CFD / Experimental Results Comparison

The CFD is capable of much more than just predict the overall performance of the pump.

As previously shown by Quail et al. [7] a powerful advantage of CFD is the full folwfield resolved. This approach of using generally valid equations also includes all flow characteristics and is only restricted by the numerical technique.

The one dimensional models can be defined in two main categories, the ones following the circulation or liquid filament theory and the ones following the turbulence and mixing theory [22]. Neither model is able to fully represent the flow in a regenerative pump, but both have their strengths as it was shown by Senoo et al. [23]. Nevertheless no existing one- dimensional model includes both theories of operation. The CFD on the other hand does not rely on one restricted model with simplifying assumptions, but captures properties of both theories in a more abstract way.

As Figure 5 demonstrates, both the CFD and the one –dimensional analysis match the experimental results quite well.

It can be seen that the representation of the CFD is slightly better than the one-dimensional modelling. The new 1D model does not rely on any data from experiments conducted on regenerative pumps and is therefore a cheap and easy-to-use way to calculate the performance. Figure 6 – 8 show captured images of the CFD results. The streamlines plotted in figure 6 clearly shows the helical flow path of the fluid in the pump. The pathlines are coloured by velocity, where a darker colour represents higher velocity. Here it can be seen that the velocity rises from pivot to perimeter as it is assumed in the one-dimensional models. This again shows the advantage of the CFD simulations over one-dimensional modelling.
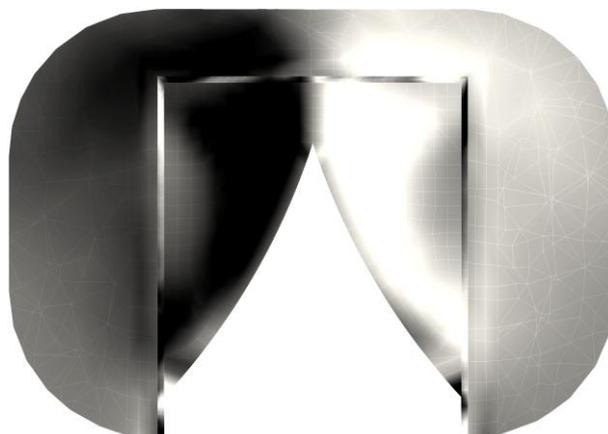

Figure 6: Regenerative Pump helicity pathlines plot

The helical flow can be quantified by the helicity. Helicity is the vorticity or circulation of the flow times the flow normal to the circulation. Formally it is expressed by

$$H = \int u(\nabla \times u) d^3 r$$

 Fig 7. indicates a cross-section though the channel where the flow is fully developed. Black indicates positive and white negative helicity. The flow is directed into the shown plane and the right channel side shows clockwise, the left side counter-clockwise circulation.

Figure 7: Regenerative Pump helicity contours plot.

The helical flowpath or for that matter streamlines, until now, could only be visualised by complex flow visualisation experiments as conducted by Bartels [24], Lazo et al. [25] and Lutz [26]. CFD is able to capture this crucial fact of the regenerative pump in a way that was not possible before. This advantage can be used, not only to determine the overall performance, but to assess and refine certain aspects of the flow. The key to a more efficient regenerative pump lies in the understanding of the complex flow and in the knowledge of how to effect the flow.

Figure 8 indicates the pressure distribution in the regenerative pump from blue (light) to red (dark). Unlike in centrifugal pumps, where the pressure rise occurs in radial direction, the head is developed in a peripheral direction in regenerative pumps. The aim of designing regenerative pumps is to avoid local pressure drops or peaks and to assure constant pressure rises with a high gradient in peripheral direction. The CFD analysis allows examination of the pressure very accurately, without the influence of pressure probes etc. It is also possible to view the pressure distribution at any point. Figure 8 additionally shows the pressure distribution at the improved inlet and outlet ports. The aligned inflow and outflow of the pump lead to much smaller pressure drops than in the version with radial inlet and outlet ports [27].

Figure 8: Tangential pressure rise through axial regenerative pump

Comparison with a centrifugal pump

Pumps are the single largest user of electricity in industry in the European Union, and of those pumps, centrifugal pumps represent some 73% of all pump types [28]. The regenerative pump like the centrifugal pump is a kinetic pump however the regenerative pump can in many applications offer a more efficient alternative [5].
If high pressures are needed, the pump can be assembled to multiple stages. They are the best choice for many applications. Small centrifugal pumps have a low ratio between the contained volume and the interior surface resulting in a high level of friction between the moving fluid and the impeller and the pump casing walls. Large centrifugal pumps with impellers of greater diameter and width impart high velocity to the fluids they transport and have a higher ratio of between the contained volume and interior surface thereby reducing friction and improving efficiency [8].
Fig. 9 compares the characteristics of a centrifugal pump to a regenerative pump. Considering a the use of both pumps at a flowrate of 16.6 l/min (2.77 x 10^{-3} m^3/s) the regenerative pump is capable of developing 450 kPa while the centrifugal pump only delivers 167 kPa.
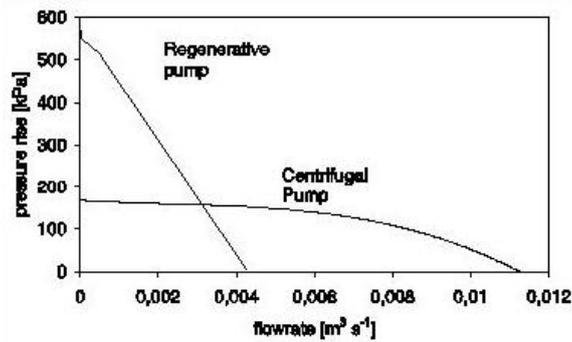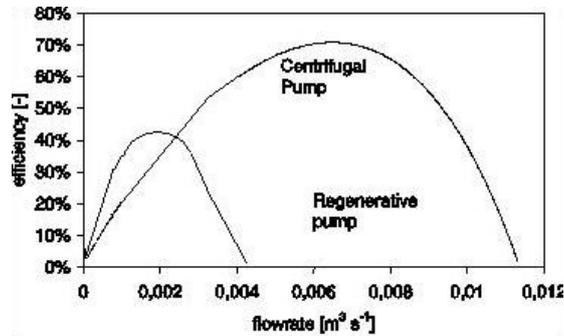
Figure 9: Pump comparison plot.


Figure10: Pump comparison plot.

The efficiency of the centrifugal pump at 16.6 l/min is about 5%. The regenerative pump at this flowrate is able to perform with an efficiency of 13%. Figure 10 shows the efficiency over the flowrate range of the pumps.

The delivered head of the regenerative pump is 2.7 times higher than the capabilities of the centrifugal pump while the efficiency is 2.6 times better. If three stages of the centrifugal pump are used, to gain a similar head rise, the efficiency of the arrangement is less than two per cent.

Proposed Design Procedure

Considering the fact that there is little published data about design criteria a design procedure, incorporating the findings of this work, is proposed. As a first step, after defining the performance specifications, the one-dimensional model is used to match the performance. An optimisation algorithm can be used to automatically fit required head and flowrate while optimising efficiency or pump size. The pump geometry is quick to implement and a solution can be reached in very little time. The second step is to load the results of the one-dimensional optimisation into a parameterised CAD model to create a fully three-dimensional geometry. With this geometry a CFD simulations are conducted. The pump's features can be refined locally and so adjusted to yield best flow through the pump further refining the design. True performance can however only be sufficiently proved by experiments the creation of a prototype is therefore inevitably. In the third step of the design procedure, this can be done by rapid prototyping. For regenerative pumps are usually rather small machines, this can be done with little financial effort. The specimens produced by today's rapid prototyping machines proved to be sufficiently durable to conduct tests without resorting to slow and expensive traditional manufacturing techniques. Mechanical data can easily be obtained from similar centrifugal pumps.

In these three steps of the design process, the advantages of all three procedures, one-dimensional analysis, CFD simulation and conducting experiments are used. The one-dimensional approach is fast and able to process a great number of different geometries in short time. The flow features are introduced into a CFD analysis which needs more time, but is also quite cheap. Expensive experiments are minimised, because only optimised pump geometries will be tested. Using rapid prototyping to create specimen, cost and time can be reduced here as well [21]. The use of rapid prototyping techniques becomes even more important considering complex port geometry arrangement defined above which can only be cast or machined via five-axis-milling.
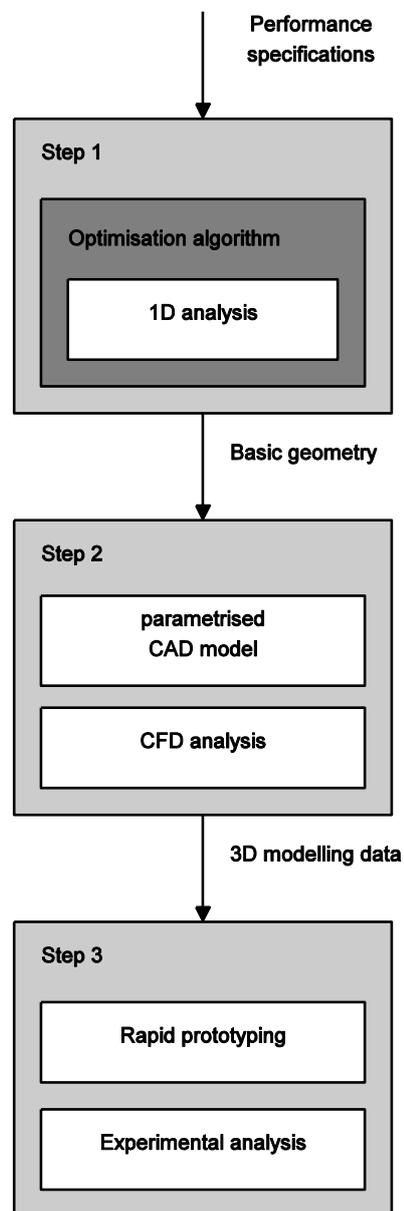
Conclusions

The CFD analysis and the one-dimensional tool are alternatives to experimental testing. They offer potential as advanced design tools without the need for expensive tests. The one-dimensional approach is fast and can be used to evaluate different geometry parameters very quickly. The CFD method takes more time to develop and to calculate the performance, but is still faster and less costly than conducting experimental test. The advantages offered by CFD analysis over all previous methods of calculation and flow visualisation are significant.

The advances in rapid manufacturing offer an opportunity to quickly corroborate the analytical models. The benefit of a corroborated three-dimensional model offers further capabilities for resolving the flow's properties (flow visualisation) and making it possible to focus on occurring local effects within the pump.

To design a regenerative pump a procedure is proposed. After evaluating the needs and given restrictions, a three-step design procedure leads to good results in very little time at low cost.

Future work

The Multiple Reference Frame approach for the CFD analysis gives a good approximation, but neglects unsteady effects. A sliding mesh method (SMM) approach is currently being undertaken and the results will be published in the coming months.

The one-dimensional calculation algorithm will be expanded to include a Simplex Optimisation Algorithm [29]. It will be possible to fit the pumps geometry to the intended purpose automatically by the variation of specified variables. The Simplex algorithm proved to be very stable for other applications and with the fast one-dimensional model a very precise examination of multiple variants of pump geometries will be possible.

The regenerative pump design presented in this paper will be extended to consider multiple stages. This will make the regenerative pump an even more attractive alternative to centrifugal pumps, considering fewer stages and therefore smaller dimensions and less weight and better efficiency in the regenerative pumps performance range.

The new inlet and outlet port arrangement allows a multistage arrangement. The carryover effects from stage to stage will be examined by CDF analysis and finally a true multistage version of the regenerative pump will be built to conduct experiments.

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Appendix

$$F\left(X_{g,\rho}\right) = \frac{A_c}{2Q_0 \theta r_{g,\rho}} \left[K_c + \left(\frac{A_c \tan\beta}{(r_0 - r_c)r_i} \right)^2 \right]$$

$$G\left(X_{g,\rho}\right) = \frac{A_c}{Q_0 \theta r_{g,\rho}} \left(\frac{A_c \tan\beta}{(r_0 - r_c)r_i} \right)(1-\alpha)U_i$$

$$H\left(X_{g,\rho}\right) = \frac{Ac}{2Q_0 \theta r_{g,\rho}} \left[2 \left(1-\varphi\right)\left(\sigma U_e^2 - \alpha U_i^2\right) - \left(1-\alpha\right)^2 U_i^2 \right]$$