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When is it Energetically Favourable for a Rivulet of Perfectly Wetting Fluid to Split?

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Abstract

We determine when it is energetically favourable for a thin rivulet of perfectly wetting fluid (i.e. a thin rivulet with zero contact angle) on the underside of a planar substrate to split into two or more sub-rivulets.

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Rivulets occur in a wide variety of physical contexts ranging from industrial coating processes to geological flows of mud and lava, and have been of enduring fascination and interest for many years. Specifically, there has been work on various aspects of the stability of rivulet flows by Bankoff [1], Mikielewicz and Moszynski [2], Chung and Bankoff [3], Davis [4], Weiland and Davis [5], Young and Davis [6], Schmuki and Laso [7], Wilson and Duffy [8], Myers, Liang and Wetton [9], and Saber and El-Genk [10]. In particular, Bankoff [1], Mikielewicz and Moszynski [2], Chung and Bankoff [3], and Saber and El-Genk [10] considered the breakup of a fluid film into a periodic array of rivulets by comparing the energy of the film with the energy of the rivulets, while Schmuki and Laso [7] and Myers *et al.* [9] considered the splitting of a rivulet into two or more sub-rivulets by comparing the energies of the different rivulet configurations.

In the present paper we determine when it is energetically favourable for a thin rivulet of perfectly wetting fluid (i.e. a thin rivulet with zero contact angle) to split into two or more sub-rivulets.

Consider the steady unidirectional flow of a thin symmetric rivulet with constant semi-width a and constant volume flux Q down a planar substrate inclined at an angle α ($0 \leq \alpha \leq \pi$) to the horizontal. We assume that the fluid is Newtonian and has constant density ρ, viscosity μ and surface tension γ. We choose Cartesian axes $Oxyz$ as shown in Fig. 1 with the x axis down the slope and the y axis horizontal, both of them being parallel to the substrate $z = 0$, and the z axis being normal to the substrate. The velocity $\mathbf{u} = u(y, z)\mathbf{i}$ and pressure $p = p(x, y, z)$ of the fluid are governed by the familiar mass-conservation and Navier–Stokes equations subject to the usual

normal and tangential stress balances and the kinematic condition at the (unknown) free surface $z = h(y)$, and zero velocity at the substrate $z = 0$. At the contact line $y = a$ where $h = 0$ the contact angle is zero. As Duffy and Wilson [11] and Wilson and Duffy [12] show, if we non-dimensionalise y and a with l, z and h with ϵl, u with $U = \rho g \epsilon^2 l^2 / \mu$, Q with $\epsilon l^2 U = \rho g \epsilon^3 l^4 / \mu$, and $p - p_\infty$ with $\rho g \epsilon l$, where $l = (\gamma / \rho g)^{1/2}$ is the capillary length, g is gravitational acceleration, and p_∞ is the uniform pressure in the surrounding atmosphere, then at leading order in the limit of small transverse aspect ratio $\epsilon \ll 1$ the leading-order versions of the Navier–Stokes equations can be readily solved to yield $u = \sin \alpha (2h - z)z/2$ and $p = (h - z)\cos \alpha - h''$. The transverse profile of the rivulet h satisfies $(h'' - \cos \alpha h)' = 0$ and hence there are no solutions when $0 \leq \alpha \leq \pi/2$ (i.e. no solutions corresponding to a sessile rivulet or a rivulet on a vertical substrate) but there is a solution when $\pi/2 < \alpha \leq \pi$ (i.e. a solution corresponding to a pendent rivulet) [13] given by

$$a = \frac{\pi}{m}, \quad h = \frac{h_{\mathrm{m}}}{2}(1 + \cos my), \tag{1}$$

where we have defined $m = |\cos \alpha|^{1/2}$. Note that, rather unexpectedly, the width of this rivulet is independent of the volume flux down it. The requirement that the flux Q takes a prescribed value determines the maximum thickness of the rivulet, $h_{\mathrm{m}} = h(0)$, to be

$$h_{\mathrm{m}} = \left(\frac{24Qm}{5\pi \sin \alpha}\right)^{1/3}. \tag{2}$$

The total energy of the rivulet is the sum of its kinetic energy and surface energy. If we non-dimensionalise energy (per unit length) with $\rho U^2 \epsilon l^2 = \rho^3 g^2 \epsilon^5 l^6 / \mu^2$ then the

kinetic energy (per unit length) is given by

$$\frac{1}{2} \int_{-a}^{+a} \int_{0}^{h} u^2(y, z) \, \mathrm{d}z \, \mathrm{d}y \tag{3}$$

and the surface energy, or, more precisely, the difference between the surface energy of the rivulet and the surface energy of the dry substrate (per unit length) is given by

$$\frac{1}{\epsilon^2 W} \left[\int_{-a}^{+a} \left(1 + \epsilon^2 h'^2(y) \right)^{1/2} \mathrm{d}y - 2a \right], \tag{4}$$

where

$$W = \frac{\rho l U^2}{\gamma \epsilon} = \frac{\gamma^2 \epsilon^3}{g l \mu^2} \tag{5}$$

is an appropriately defined Weber number (a non-dimensional measure of the relative importance of surface energy and kinetic energy). Thus the leading-order expression for the energy of a thin rivulet of perfectly wetting fluid, E, is given by

$$E = \frac{\sin^2 \alpha}{15} \int_{-a}^{+a} h(y)^5 \, \mathrm{d}y + \frac{1}{2W} \int_{-a}^{+a} h'^2(y) \, \mathrm{d}y. \tag{6}$$

Using the present solution (1) we can evaluate (6) to yield

$$E = \frac{21\pi \sin^2 \alpha}{640m} h_{\mathrm{m}}^5 + \frac{\pi m}{8W} h_{\mathrm{m}}^2. \tag{7}$$

We can now determine when it is energetically favourable for a rivulet of perfectly wetting fluid to split into two or more sub-rivulets.

The natural first question is whether it is ever energetically favourable for a rivulet to split into two sub-rivulets. Specifically, it is energetically favourable for a rivulet with flux Q to split into two sub-rivulets, one with flux λQ and the other with flux $(1 - \lambda)Q$, where $0 < \lambda \leq 1/2$, if the difference between the energies of the two states, denoted by

ΔE, is positive. Using (2) and (7) yields

$$\Delta E = \frac{21\pi \sin^2 \alpha}{640m} \left(\frac{24Qm}{5\pi \sin \alpha}\right)^{5/3} \left[1 - \lambda^{5/3} - (1-\lambda)^{5/3} + \frac{1}{\hat{W}}\left(1 - \lambda^{2/3} - (1-\lambda)^{2/3}\right)\right],$$

(8)

where

$$\hat{W} = \frac{63QW \sin \alpha}{50\pi m}$$

(9)

is an appropriately redefined Weber number. Inspection of (8) reveals that $\Delta E < 0$ for all $0 < \lambda \le 1/2$ when $\hat{W} < \hat{W}_{\mathrm{c}}$, $\Delta E = 0$ at $\lambda = 1/2$ when $\hat{W} = \hat{W}_{\mathrm{c}}$, and ΔE has a positive global maximum at $\lambda = 1/2$ when $\hat{W} > \hat{W}_{\mathrm{c}}$, where

$$\hat{W}_{\mathrm{c}} = \frac{2^{1/3} - 1}{1 - 2^{-2/3}} \simeq 0.7024.$$

(10)

Thus it is energetically favourable for a rivulet to split into two equal sub-rivulets when $\hat{W} > \hat{W}_{\mathrm{c}}$.

This result begs the natural second question of whether it is ever energetically favourable for a rivulet to split into *more* than two sub-rivulets. Specifically, it is energetically favourable for a rivulet with flux Q to split into n equal sub-rivulets each with flux Q/n if the difference between the energies of the two states, denoted by ΔE_n, is positive. Again using (2) and (7) yields

$$\Delta E_n = \frac{21\pi \sin^2 \alpha}{640m} \left(\frac{24Qm}{5\pi \sin \alpha}\right)^{5/3} \left[1 - n^{-2/3} + \frac{1}{\hat{W}}\left(1 - n^{1/3}\right)\right],$$

(11)

where $\hat{W}$ is again given by (9). The most energetically favourable state is the one with the largest value of ΔE_n for $n = 1, 2, 3, \ldots$, and inspection of (11) reveals that this is the one-rivulet state for $0 = \hat{W}_{\mathrm{c}0} < \hat{W} < \hat{W}_{\mathrm{c}1}$, the two-rivulet state for $\hat{W}_{\mathrm{c}1} < \hat{W} < \hat{W}_{\mathrm{c}2}$,

the three-rivulet state for $\hat{W}_{c2} < \hat{W} < \hat{W}_{c3}$, the four-rivulet state for $\hat{W}_{c3} < \hat{W} < \hat{W}_{c4}$, and so on, where

$$\hat{W}_{cn} = \frac{(n+1)^{1/3} - n^{1/3}}{n^{-2/3} - (n+1)^{-2/3}}. \tag{12}$$

Equation (12) shows that $\hat{W}_{cn}$ is a monotonically increasing function of n satisfying $\hat{W}_{c0} = 0$, $\hat{W}_{c1} = \hat{W}_c \simeq 0.7024$, $\hat{W}_{c2} \simeq 1.2220$, $\hat{W}_{c3} \simeq 1.7301$, $\hat{W}_{c4} \simeq 2.2345$, and $\hat{W}_{cn} = n/2 + 1/4 + O(1/n) \to \infty$ as $n \to \infty$. Note that, as expected, in the special case $n = 2$ we recover the critical value $\hat{W}_{c1} = \hat{W}_c$ for one rivulet to split into two equal sub-rivulets obtained previously.

In summary, the results of the present calculations reveal that it is most energetically favourable for a rivulet to split into n equal sub-rivulets when Q satisfies $0 \leq Q_{cn-1} < Q < Q_{cn}$, where

$$Q_{cn} = \frac{50\pi\hat{W}_{cn}(-\cos\alpha)^{1/2}}{63W\sin\alpha}, \tag{13}$$

or, equivalently, when α satisfies $\pi/2 < \alpha_{cn} < \alpha < \alpha_{cn-1} \leq \pi$, where

$$\alpha_{cn} = \cos^{-1}\left[\frac{1}{2}\left(\frac{50\pi\hat{W}_{cn}}{63QW}\right)^2 - \left\{\frac{1}{4}\left(\frac{50\pi\hat{W}_{cn}}{63QW}\right)^4 + 1\right\}^{1/2}\right]. \tag{14}$$

In particular, Q_{cn} is a monotonically increasing function of α satisfying $Q_{cn} = O(\alpha - \pi/2)^{1/2} \to 0^+$ as $\alpha \to \pi/2^+$ and $Q_{cn} = O(\pi - \alpha)^{-1} \to \infty$ as $\alpha \to \pi^-$, while α_{cn} is a monotonically increasing function of QW satisfying $\alpha_{cn} = \pi/2 + O(QW)^2 \to \pi/2^+$ as $QW \to 0^+$ and $\alpha_{cn} = \pi + O(QW)^{-1} \to \pi^-$ as $QW \to \infty$. (Note that, when expressed in terms of dimensional quantities, the non-dimensional parameter QW is equal to $\rho Q/\mu l$.)

Figure 2 shows how $QW - \alpha$ parameter space is divided into a region in which it is energetically unfavourable for a rivulet to split (i.e. in which the one-rivulet state is

energetically favourable) and regions in which it is most energetically favourable for a rivulet to split into two or more equal sub-rivulets. In particular, Fig. 2 shows that when the flux Q is sufficiently small or when the angle of the inclination of the substrate to the horizontal α is sufficiently close to π then the one-rivulet state is always energetically favourable, but that increasing Q or decreasing α eventually causes the two-rivulet state (namely, two shallower sub-rivulets each of the same width as the original one, but with $2^{-1/3}$ times the depth and half the flux) to become the most energetically favourable. Increasing Q or decreasing α further eventually causes states with three or more rivulets each to become, in turn, the most energetically favourable.

The authors know of no experiments involving the splitting of a rivulet of perfectly wetting fluid against which the current theoretical results can be tested. However, we hope that such experiments will be conducted in the future.

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[13] As described by Duffy and Wilson (Ref. [11]) and Wilson and Duffy (Ref. [12]) there are actually infinitely many other such solutions, but since the additional solutions merely represent "arrays" of identical contiguous rivulets each of which is simply a suitably re-scaled copy of the simplest solution we shall not consider them here.

Figure Captions

Figure 1: The geometry of the problem.

Figure 2: Plot of $QW - \alpha$ parameter space showing how it is divided into a region in which it is energtically unfavourable for a rivulet to split (i.e. in which the one-rivulet state is energetically favourable) and regions in which it is most energetically favourable for a rivulet to split into two or more equal sub-rivulets. The labels indicate the number of rivulets in the most energetically favourable state.