Non-Local Dispersal

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Abstract

Equations with non-local dispersal have been used extensively as models in material science, ecology and neurology. We discuss the scalar bistable case

$$u_t(x) = \rho \left\{ \int_\Omega \beta(x,y)u(y)dy - u(x) \right\} + f(u)$$

and contrast it with the corresponding reaction-diffusion equation. We show that for large dispersal rate ρ the asymptotic dynamics is determined by the ODE $\dot{u} = f(u)$. For small ρ and constant kernel β, we prove pointwise convergence to an equilibrium and present some L^∞ stability results.

1 Introduction.

We consider a model of spatial spread that has applications in both material science and biology. The classical models are based upon partial differential equations, in particular reaction-diffusion equations. Here the dispersal term is given in terms of an integral operator and we restrict ourselves to the scalar case. The governing equation is

$$u_t = \rho Du + f(u), \tag{1.1}$$

where

$$(Du)(x) = \int_\Omega \beta(x,y)u(y)\,dy - u(x). \tag{1.2}$$

Here $u : \Omega \times [0,\infty) \to \mathbb{R}$ and the suffix t represents differentiation. Also $\Omega \subset \mathbb{R}^n$ is the spatial region, D the dispersal operator and $\rho > 0$ the dispersal strength. The function f is the reaction term and equation (1.2) may be called a reaction-dispersal equation.

We are interested in the dynamics of this equation. Thus, with regard to the reaction term it is reasonable to assume that f is such that the ordinary differential equation

$$\dot{u} = f(u) \tag{1.3}$$

is dissipative with orbits tending to a bounded global attractor $\mathcal{A}$.

Of greater significance are the restrictions we impose on the dispersal kernel. In particular, we assume the following.

(H1) $\beta : \Omega \times \Omega \to \mathbb{R}$ is continuous, symmetric and strictly positive.

(H2)

$$\int_\Omega \beta(x,y)\,dy \leqslant 1 \ (x \in \Omega). \tag{1.4}$$

For material science or population biology (H2) is quite natural; it essentially implies that material or organisms cannot be created due to dispersal.

As was indicated earlier, models of this type have been used in material science and biology. In the first case, it has been used in the context of phase transition and has been fairly extensively discussed, see [8], [20], [2], [21] and the many references therein. Much of the discussion has been concerned with the case $\Omega = \mathbb{R}^n$ and with travelling waves. Although the focus here is rather different, in view of what follows it is worth noting that for a bistable reaction term, there may be major qualitative differences from the standard reaction-diffusion case. Of particular note is the phenomenon of propagation failure in waves, see [21].

For applications related to population biology the reader is referred to [11, 10, 17]. In [11], Ω is taken to be bounded and the scalar case as well as the case of two competing species is considered. Most of the analysis is concerned with the monostable case, that is when there is a single, stable equilibrium for the reaction system.

There are also neurological models ([16] Ch 12) which have a form similar to that of (1.1). However, in this case β takes on both positive and negative values which violates (H2).

As was mentioned at the beginning of this introduction, classically some of the aforementioned problems have been modeled using the reaction-diffusion equation

$$u_t = \rho \Delta u + f(u) \tag{1.5}$$

on a bounded domain with zero Neumann boundary conditions. Therefore, part of our analysis is aimed at comparing the dynamics of (1.1) and (1.5). In the monostable case they are much the same and some of the similarities may even extend to systems. The bistable case, however, presents a different picture and a much more difficult mathematical problem. This is studied in [20] and [6], but many interesting mathematical questions remain unanswered. Some have important modelling implications but these are not considered here. Since this case is perhaps not very well known, we commence with some informal remarks on the background.

For large ρ the scalar equation (1.1), and indeed a system of dispersal equations, behaves asymptotically exactly like the corresponding reaction-diffusion system (see [5]). In Section 3, we consider in particular the situation, analogous to (1.5) with a zero Neumann condition, where (H1) holds and there is equality in (H2) (so that the population is conserved); for a precise definition see (A1) in Section 3. The orbits then asymptotically approach those for the corresponding reaction system. The results are presented in Theorems 3.1 and 3.2.

The situation is very different when ρ is small. Suppose, for example, that Ω is a finite interval. Consider first the equilibria or stationary states (see [20, 6, 4]). The set of equilibria is 'large' in the sense that it is not compact in L^∞ (see [20]) and, in some circumstances, it is easy to show that this is also true in L^1. This in turn implies the non-existence of a compact global attractor in these spaces, a result that is fundamentally different from the dynamics of (1.5). Associated with this is the interesting result, given in detail later, which shows that for an initial function with small 'wrinkles', these wrinkles are not smoothed out when ρ is small. Again this is in striking contrast to (1.5) in which 'nearly all' initial functions tend to constant solutions of $f(u) = 0$ as $t \to \infty$. For extensive discussion in the case when Ω is an interval see [3, 9]. The lack of compactness, even in this fairly weak sense, is an obstacle to the study of the dynamics and in order to make progress it has been necessary to specialise further. Our main result is as follows; for the details of the restrictions on the initial conditions, see Section 6.

Theorem 1.1 *Suppose that* $\Omega = [0, 1]$, $\beta \equiv 1$ *and* $f(u) = (u - \alpha)(1 - u^2)$ *where* $\alpha \in (-1, 1)$. *Then for a broad class of initial conditions, the positive semi-orbit tends pointwise to an equilibrium of the system.*

We suggest, however, that a much more general result holds.

Conjecture. *Assume* β *satisfies (H1) and (H2) and that* $\dot{u} = f(u)$ *has a compact global attractor* $\mathcal{A}$. *If* $u(x, 0)$ *is measurable, then any solution* $u(x, t)$ *of (1.1) tends pointwise for* $x \in \Omega$ *to an equilibrium as* $t \to \infty$.

The stability of the equilibria is an important practical issue. Consider $\beta \equiv 1$ and $\Omega = [0, 1]$. For $\rho < 2/3$ there are L^∞-stable non-constant equilibria. Again, this is a remarkable departure from the reaction-diffusion case. In order to present as complete a picture as possible of the dynamics of (1.1), we discuss stability results, most of which were presented in [6], in Section 7.

The contents are as follows. The notation is introduced and the problem is set up as a dynamical system in Section 2. Section 3 gives results for large dispersal. In Section 4 the analysis is restricted to $\beta \equiv 1$ and we introduce a class of initial functions such that the semi-orbits are relatively compact in L^1. In Section 5 some essential background results on equilibria are discussed and in Section 6 our main convergence result is proved. In Section 7 we mention some results on asymptotic stability and review the picture of convergence as it seen at the moment.

2 Preliminaries.

Throughout $\Omega \subset \mathbb{R}^n$ is compact. Let m be the Lebesgue measure. Assume without loss of generality that the volume of Ω is unity (otherwise we rescale the spatial variable). L^p will be the usual Banach space with norm $\|\cdot\|_p$. Define

$$\overline{u}(t) = \int_\Omega u(x, t)\, dx.$$

Let Z be a metric space with distance d. For a semi-flow on Z, the positive semi-orbit through u_0 is denoted by $\gamma^+(u_0)$, the ω-limit set of u_0 by $\omega(u_0)$ and a point on the orbit for $t > 0$ by $u(t)$.

The basic scalar equation is

$$u_t = \rho \left\{ \int_\Omega \beta(x,y)u(y)\,dy - u(x) \right\} + f(u) \tag{2.1}$$

with $u(x,0) = u_0$. The following is assumed henceforth.

(H3) (a) f is C^2 on $J = [-1,1]$.
(b) $f(-1) = 0 = f(1)$.

It is convenient to set

$$(Xu)(x) = \int_\Omega \beta(x,y)u(y)\,dy,$$

so that the dispersal operator is $D = X - I$. A consequence of the maximum principle ([11], [8]) is as follows. Let $B = \{u(x) : -1 \leqslant u(x) \leqslant 1, x \in \Omega\}$. Define the metric space (Z, d) to be the set of measurable functions in B with the metric induced by the L^1 norm. (Note that $L^p\,(p > 1)$ norms give equivalent topologies on Z. The embedding of L^1 in L^∞ is continuous, but not the other way round). We will occasionally want to consider another L^p norm and will then use Z_p.

Lemma 2.1 *For $u_0 \in Z$ the solution $u(t) \in C^1((0,\infty),\ L^p(\Omega))$ for $p \geqslant 1$. Z_p is positively invariant under equation (2.1) and this generates a semi-flow.*

Proof. Since $u \equiv 1$ is a supersolution and $u \equiv -1$ is a subsolution, the the invariance follows from the maximum principle. The positive invariance of B gives us the Lipschitz condition globally and we can then use [18] theorem 6.1.7, p. 190. ☐

3 General Remarks on Asymptotic Behavior.

We compare here some features of the asymptotic behavior with those for the corresponding reaction-diffusion equation. We work in the space (Z_2, d).

We first show that for large dispersal ρ, asymptotically every orbit behaves like the orbit of an ODE. Although we prove this for the scalar case, with which we are dealing in this paper, this result may be very easily extended to a system. It is thus the analogue of a well-known property for diffusion governed by the Laplacian, see [5] for example. In fact the analysis suggests that it is likely that several other results in [5] have an analogue for non-local dispersal. We then consider small ρ and show that in direct contrast with the Laplacian, a non-coarsening result holds, that is in a rather strong sense, initial 'wrinkles' are not smoothed.

We impose the following condition.

(A1) $\int_\Omega \beta(x,y)\,dy = 1$ $(\forall x \in \Omega)$. This corresponds to an interesting biological case ([11] section 2) and gives dispersal analogous to the Laplacian with zero Neumann conditions because of its conservation property.

From (A1), D is obviously a negative semi-definite operator on L^2 with principal eigenvalue zero (1 is the principal eigenfunction of the compact self-adjoint operator X corresponding to the eigenvalue 1). Let λ_2 be the smallest positive eigenvalue of $-D$.

Theorem 3.1 *Assume that (H1), (H2), (H3) and (A1) hold. Let*

$$\sigma = 2(\rho\lambda_2 - M\|D\|_2\lambda_2^{-1}), \tag{3.1}$$

where $\|D\|_2$ is the operator norm of $D : L^2 \to L^2$ and M is the Lipschitz constant of f. Suppose that $\sigma > 0$. Then there are $c_i > 0$ such that the following hold for all $u(0) \in Z_2$.

(a) $0 \leqslant (-Du, u) \leqslant (-Du(0), u(0))e^{-\sigma t}$.

(b) $\|u(t) - \overline{u}(t)\|_2 \leqslant c_1 e^{-\sigma t}$.

(c) $\dfrac{d\overline{u}}{dt} = f(\overline{u}) + g(t),\ \overline{u}(0) = \displaystyle\int_\Omega u_0(x)\,dx$ $(u_0 \in Z_2)$ *where* $|g(t)| \leqslant c_2 e^{-\sigma t}$.

Proof. Initially the analysis is in L^2, the inner product being denoted by $(\cdot,\cdot)$. Let $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$ be the complete orthonormal system generated by $-D$ with $\lambda_1 = 0$, $\phi_1 = 1$ and $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ With $u_n = (u, \phi_n)$,

$$(-Du, u) = \sum_{n=2}^{\infty} \lambda_n u_n^2,$$

$$(Du, Du) = \sum_{n=2}^{\infty} \lambda_n^2 u_n^2,$$

$$\|u - \overline{u}\|_2^2 = \sum_{n=2}^{\infty} u_n^2.$$

Hence

$$\begin{aligned} (Du, Du) &\geqslant \lambda_2(-Du, u), && (3.2)\\ \|u - \overline{u}\|_2^2 &\leqslant \lambda_2^{-1}(-Du, u), && (3.3) \end{aligned}$$

the last inequality being an obvious analogue of the Poincaré inequality. With $\chi = (-Du, u)/2$,

$$\begin{aligned} \dot{\chi} &= -(Du, \dot{u}) \\ &= -\rho(Du, Du) - (Du, f(u)) \quad \text{(from (2.1))} \\ &\leqslant \rho\lambda_2(Du, u) - (Du, f(u)) \quad \text{(from (3.2))} \\ &= -2\rho\lambda_2\chi - (Du, f(u)). \end{aligned}$$

To estimate $(Du, f(u))$, put $u = \overline{u} + v$, so that $Du = Dv$. Also, from the invariance of B and the definition of M,

$$\|f(u) - f(\overline{u})\|_2 \leqslant M\|v\|_2. \tag{3.4}$$

Then

$$\begin{aligned} -(Du, f(u)) &= -(Dv, f(u) - f(\overline{u})) - (Dv, f(\overline{u})), \\ &= -(Dv, f(u) - f(\overline{u})), \end{aligned}$$

since Dv is orthogonal to a constant. Therefore, from Schwarz's inequality,

$$\begin{aligned} |(-Du, f(u))| &\leqslant \|Dv\|_2 \|f(u) - f(\overline{u})\|_2, \\ &\leqslant M\|D\|_2\|v\|_2 && \text{(from (3.4))} \\ &\leqslant M\lambda_2^{-1}\|D\|_2(-Du, u) && \text{(from (3.3))} \\ &= 2M\lambda_2^{-1}\|D\|_2\chi. \end{aligned}$$

Thus

$$\dot{\chi} \leqslant -2(\rho\lambda_2 - M\|D\|_2\lambda_2^{-1})\chi,$$

and with σ defined as in (3.1) and with $c_1 = 2\chi(0)/\lambda_2 > 0$, we obtain

$$\begin{aligned} \chi(t) &\leqslant e^{-\sigma t}\chi(0), && (3.5) \\ \|v\|_2 &\leqslant c_1 e^{-\sigma t} && (3.6) \end{aligned}$$

from (3.3) and (3.5), which are (a) and (b). Next, with $w = \overline{u}$, noting that $\int_\Omega Du \, dx = 0$ from (A1), we obtain

$$\begin{aligned} \dot{w} &= \int_\Omega u_t \, dx, \\ &= \rho \textstyle\int_\Omega Du \, dx + \int_\Omega f(u) \, dx, \\ &= \textstyle\int_\Omega [f(u) - f(w)] \, dx + f(w), \end{aligned}$$

A simple estimate for the integral term and (3.6) lead to the ODE equation and (c). □

Using the estimates in Theorem 3.1 we obtain the following

Theorem 3.2 *Suppose $\mathcal{A}$ is a compact attractor for the ODE*

$$u_t = f(u). \tag{3.7}$$

and consider $\mathcal{A}$ as a subset of the constant functions in Z_2. Then if $\sigma > 0$, $\mathcal{A}$ is a compact attractor for (1.1) in Z_2.

Proof. If we consider $\mathcal{A}$ as a subset of constant functions in $L^2 \cap B$, then $\mathcal{A}$ is exactly those constant functions which take on values in the interval $[-1, 1]$. $\mathcal{A}$ is invariant and compact in L^2. Since Z_2 is invariant, the set of possible averages of solutions starting in Z_2 is also the interval $[-1, 1]$. Let U be an L^2 neighborhood of $\mathcal{A}$. The value of c_1 in

(b) is independent of $u_0 \in U$. Therefore U is attracted uniformly by $\mathcal{A}$ and $\omega(U) = \mathcal{A}$. Therefore $\mathcal{A}$ is an attractor for (1.1). $\square$

Remark. For large ρ the only equilibria are the constant solutions of $f(u) = 0$.

For small ρ we have the following non-coarsening theorem (in [6] a more general version of this theorem is given; the proof there only applies under the conditions formulated below).

Theorem 3.3 *Suppose that* $u_0(x) = u(x, 0) \in Z$. *Assume that* $u_0(x)$ *is positive on a set* Ω_+ *and negative on a set* Ω_-, *where* Ω_- *and* Ω_+ *have the property that* $m(\Omega_-)+m(\Omega_+) = 1$. *Assume that there is a number* $\delta > 0$ *such that on each component* $\Omega_-^1, \ldots$ *of* Ω_- *and each component* $\Omega_+^1, \ldots$ *of* Ω_+ *the initial data satisfies* $\operatorname{ess\,sup}_x |u_0(x)| > \delta$.

Then there are sets, $\omega_-^1, \ldots, \omega_+^1, \ldots$ *(where* $\omega_\pm^i \subset \Omega_\pm^i$*), of nonzero measure, such that for* ρ *sufficiently small (depending on* δ*) and all* $t > 0$, $u(x, t) < 0$ *on each of* ω_-^i, $i = 1, \ldots$ *and* $u(x, t) > 0$ *on each of* ω_+^i, $i = 1, \ldots$.

4 The ω-limit Set.

As discussed in the previous section, the asymptotic behavior of orbits is very much like that for classical diffusion if the dispersal rate ρ is large. However, this is far from the case for general ρ (unless the reaction term is monostable, see [11]) and we now commence the task of showing that at least for a fairly broad class of initial conditions, every orbit converges (pointwise) to an equilibrium. In fact for technical reasons, as discussed in the introduction, we henceforth treat the case of a constant kernel. Specifically we assume the following.

(H4) (a) $\Omega = [0, 1]$,

(b) $\beta(x, y) = 1 \quad (x, y \in \Omega)$,

(c)
$$f(u) = (u - \alpha)(1 - u^2), \text{where} \alpha \in (-1, 1). \tag{4.1}$$

It is not hard to see that the argument could be extended to a much wider class of reaction terms, but the extra technicalities are tedious rather than illuminating, so we shall concentrate on equation (4.1) here.

The governing equation is now

$$u_t = \rho(\overline{u} - u) + f(u) \quad (x \in \Omega), \tag{4.2}$$

with f as given by (4.1) and initial function $u_0 \in Z$, where Z has the metric induced by the L^1 norm.

One major difficulty is in establishing some compactness for semi-orbits and hence the existence of an ω-limit set. We start with a sufficient condition on u_0 for compactness of $cl\gamma^+(u_0)$; we remark that the special form of β, that is β is constant, is crucial to the proof. This is seen in the following preliminary lemma. Here terms involving 'monotone' do not carry the implication of 'strictness' unless it is explicitly added.

The broad idea is as follows. We assume that a subdivision of Ω may be chosen with an infinite, but 'not too large', number of intervals such that u_0 is monotone increasing or decreasing on each interval except for a 'small part' of Ω. Lemma 4.1 shows that this property persists along an orbit. A standard L^1 theorem is then used to prove relative compactness of $\gamma^+(u_0)$.

Lemma 4.1 *If $u_0(x) = u(x,0)$ is monotone (strictly monotone) on an interval then so is $u(x,t)$ for $t > 0$.*

Proof. Take any $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$. Put $u(x_i, t) = u_i$. Substituting x_1, x_2 into equation (4.2) and subtracting, we obtain the equation

$$(u_1 - u_2)_t = (u_1 - u_2)[1 - \rho + \alpha(u_1 + u_2) - u_1^2 - u_1 u_2 - u_2^2].$$

The result follows from a standard ODE argument. □

Definition 4.2 *A function $u_0 \in Z$ is said to be admissible if there is a subdivision of $[0,1]$ with the following properties.*

- *(i) $[0,1]$ is the union of a finite number n of closed intervals $I_j = [y_j, y_{j+1}]$ with $y_0 = 0$, $y_n = 1$.*
- *(ii) For each j there is a (possibly finite) increasing sequence $\{y_k^{(j)}\}$ with $y_0^{(j)} = y_j$ and $\lim_{k\to\infty} y_k^{(j)} = y_{j+1}$. (We also allow decreasing sequences.)*
- *(iii) On each interval $I_{jk} = (y_k^{(j)}, y_{k+1}^{(j)})$ u_0 is monotone.*

Remark. The idea is to include functions which have a finite number of cluster points of intervals on which they are monotone, a simple example being $\sin(1/x)$.

In the following, functions $u : \Omega \to [-1,1]$ are extended to $\mathbb{R}$ by setting them zero outside Ω.

Theorem 4.3 (See [7] IV.8.20) *The set $W \subset Z$ is relatively compact if and only if for each $\epsilon > 0$ and all $u \in W$, there is a $\delta > 0$ such that*

$$\int_{\mathbb{R}} |u(x+h) - u(x)|\, dx < \epsilon \quad (|h| < \delta). \tag{4.3}$$

Lemma 4.4 *If u_0 is admissible, then $\gamma^+(u_0)$ is relatively compact.*

Proof. Consider a typical interval I_j and let $S(\eta)$, $\tilde{S}(\eta)$ be the intervals

$$S(\eta) = [y_{j+1} - \eta, y_{j+1}], \quad \tilde{S}(\eta) = [y_j, y_{j+1} - \eta].$$

Let $[x_1, x_2]$ be a representative subinterval I_{jk} as in Definition 4.2. We may suppose without loss of generality that u_0 is monotone non-decreasing and $h > 0$. Recall in the following that since $u_0 \in Z$, $|u_0(x)| \leqslant 1$ $(x \in \Omega)$.

From Lemma 4.1 the monotonicity on an interval persists along an orbit. Suppose then that u is monotone non-decreasing on $[x_1, x_2]$. Then for each t (we drop the t from the notation for clarity),

$$\begin{aligned}
\int_{x_1}^{x_2} |u(x+h) - u(x)|\,dx &= \int_{x_1}^{x_2-h} |u(x+h) - u(x)|\,dx + \int_{x_2-h}^{x_2} |u(x+h) - u(x)|\,dx \\
&\leqslant 2h + \int_{x_1}^{x_2-h} |u(x+h) - u(x)|\,dx \\
&= 2h + \int_{x_1}^{x_2-h} [u(x+h) - u(x)]\,dx \\
&= 2h + \int_{x_1+h}^{x_2} u(x)\,dx - \int_{x_1}^{x_2-h} u(x)\,dx \\
&= 2h + \int_{x_2-h}^{x_2} u(x)\,dx - \int_{x_1}^{x_1+h} u(x)\,dx \\
&\leqslant 4h.
\end{aligned}$$

If $N(\eta)$ is the number of intervals of monotonicity of $\tilde{S}(\eta)$, we thus have

$$\int_{\tilde{S}(\eta)} |u(x+h) - u(x)|\,dx \leqslant 4hN(\eta). \tag{4.4}$$

Also, from the definition of $S(\eta)$,

$$\int_{S(\eta)} |u(x+h) - u(x)|\,dx \leqslant m(S(\eta)). \tag{4.5}$$

Hence, from equations (4.4) and (4.5),

$$\int_{\mathbb{R}} |u(x+h) - u(x)|\,dx \leqslant 4hN(\eta) + m(S(\eta)).$$

Choose η such that $m(S(\eta)) < \epsilon/4$. Then take δ such that

$$\delta . 4N(\eta) < \epsilon/4.$$

The inequality (4.3) follows for $|h| < \delta$. $\square$

It is presumably the case that $\gamma^+(u_0)$ is relatively compact for a more general class of functions and kernels β, possibly even for Z itself, but it is not apparent how this may be proved.

(H5) The initial function u_0 is such that $\gamma^+(u_0)$ is relatively compact in Z.

Lemma 4.5 *Suppose that (H4) and (H5) hold. Then $\omega(u_0)$ is non-empty, closed, compact, invariant, connected and attracts the orbit. It consists of equilibria. Also, if*

$$F'(u) = f(u) \quad \textit{and} \quad V(t) = \rho\overline{(u - \overline{u})^2} - 2\overline{F},$$

then V is a constant on $\omega(u_0)$.

Proof. The first statement is standard. To prove that $\omega(u_0)$ consists of equilibria, note first that by [20] (Prop 3.11), $\|u_t\|_2 \to 0$ as $t \to \infty$. However, from equation (4.2), $\|u_t\|_\infty$ is bounded and it follows that

$$\lim_{t\to\infty} \|u_t\|_1 = 0. \tag{4.6}$$

Let $\tilde{u} \in \omega(u_0)$. Then $\exists\{t_n\} \nearrow \infty$ such that $u(t_n) \to \tilde{u}$. From continuity, since $D : L^1 \to L^1$ is a bounded linear operator,

$$\rho Du(t_n) + f(u(t_n)) \to \rho D\tilde{u} + f(\tilde{u}). \tag{4.7}$$

But from equation (4.6), the LHS of (4.7) tends to zero in L^1. Therefore

$$\rho D\tilde{u} + f(\tilde{u}) = 0,$$

that is, $\tilde{u}$ is an equilibrium.

Following [6], $V(t)$ is defined by

$$V(t) = \rho\overline{(u - \overline{u})^2} - 2\overline{F}$$

and then, using equation (4.2), it follows that

$$\dot{V} = -2\overline{u_t^2} = -2\|u_t\|_2^2.$$

Thus V is a non-increasing function of t which, for the f of interest here, is bounded below. Thus $V_\infty = \lim_{t\to\infty} V(t)$ exists and so V is constant on $\omega(u_0)$. $\square$

Let $\tilde{u} \in \omega(u_0)$. Then the last lemma shows that $\tilde{u}$ is an equilibrium and so

$$\rho(\tilde{u} - \overline{\tilde{u}}) = f(\tilde{u}).$$

Thus

$$V_\infty = \rho\overline{(\tilde{u} - \overline{\tilde{u}})^2} - 2\overline{F(\tilde{u})}$$

$$= \overline{(\tilde{u} - \overline{\tilde{u}})f(\tilde{u})} - 2\overline{F(\tilde{u})}$$

$$= \overline{\tilde{u}f(\tilde{u})} - 2\overline{F(\tilde{u})}.$$

With $f(u) = (u - \alpha)(1 - u^2)$, we find that

$$V_\infty = -\frac{1}{2}\overline{\tilde{u}^4} + \alpha\overline{\tilde{u}} + \frac{1}{3}\alpha\overline{\tilde{u}^3},$$

and, with the aid of

$$\rho(\tilde{u} - \overline{\tilde{u}}) = (\tilde{u} - \alpha)(1 - \tilde{u}^2)$$

and some algebra, we obtain

$$\overline{\tilde{u}^4} = \frac{(8\alpha\overline{\tilde{u}} - 6V_\infty)(1 - \rho + \alpha^2) - 2\alpha^2(1 - \rho + \rho\overline{\tilde{u}}^2)}{3 - 3\rho + \alpha^2}. \tag{4.8}$$

This result is essential in the characterization of equilibria in the next section.

5 The Equilibria.

It is convenient to gather together in Lemma 5.1 below certain elementary facts about the equilibria in our special case (4.2) where (H4) holds. With $\gamma \in [-1, 1]$ a parameter, define the family of cubic polynomials in z as follows:

$$G(z, \gamma) = (z - \alpha)(1 - z^2) + \rho(\gamma - z). \tag{5.1}$$

In order to find the equilibria it is natural to consider first the cubic equation $G(u, \gamma) = 0$ for the real function $u(x) \in [-1, 1]$ and then to impose the consistency condition $\overline{u} = \gamma$.

The location of the equilibria is in principle straightforward, but in practice can be confusing. Therefore we make the following preliminary remarks with the hope that they may help the reader. Because 5.1 is a cubic equation, it may have 1, 2 or 3 real solutions (for fixed γ). Thus an equilibrium has the potential to have 1, 2 or 3 values (it is not claimed that the number of values corresponds to the number of real roots of equation (5.1)). We shall refer to the equilibrium as a one, two or three phased if it takes the values of one, two or three roots respectively on a set of measure greater than zero. If equation (5.1) has only one real solution, then it is easy to see that the consistency condition $\overline{u} = \gamma$ implies that $\overline{u}$ is -1, α or 1. Thus the equilibrium is necessarily one phased. Therefore most interest centers on 2 or 3 phase solutions. Suppose that equation (5.1) has 3 real roots (2 real roots is a non-generic case which for these preliminary remarks we ignore), and consider 2 or 3 phase solutions. The consistency condition $\overline{u} = \gamma$ imposes a restriction upon the measures of the sets where the equilibrium takes the value of the roots. However, we do not need a detailed calculation here. The proof of the following lemma is elementary and we make only a few remarks on it. It is perhaps helpful to look at the solutions of equation (5.1) as the intersection of the graphs $(z-\alpha)(1-z^2)$ and $\rho(z-\gamma)$.

Lemma 5.1 *Let (H4) hold and $v \in \omega(u_0)$.*

1. *If the cubic equation $G(z,\overline{v}) = 0$ has only one real solution, then the equilibrium v is one of the constant functions $-1,\ \alpha,\ 1$.*
2. *If $v =$ constant a.e., then the value is -1, α or 1.*
3. *If the cubic $G(z,\overline{v}) = 0$ has more than one real solution and v is not a constant then $|\overline{v}| < 1$ and the following hold.*
   - *(a) The roots r_1, r_2, r_3 of $G(z,\overline{v}) = 0$ (with $r_3 \leqslant r_2 \leqslant r_1$) are determined only by $\overline{v}$.*
   - *(b) There is no loss of generality in assuming that $\alpha > r_2$ since if $\alpha < r_2$ we may replace u by $-u$ and α by $-\alpha$.*
   - *(c) $\overline{v} > \alpha \Rightarrow -1 \leqslant r_3 \leqslant r_2 < \alpha < r_1 \leqslant 1$.*
   - *(d) We have $\rho < 1 + 2\alpha^2$. Let*

$$v(x) = r_i \ (x \in S_i,\ i = 1,2,3) \tag{5.2}$$

where S_1, S_2, S_3 are disjoint sets with union $[0,1]$. Let $s_i = m(S_i)$. With V_∞ a constant determined by u_0,

$$\sum_i s_i = 1, \quad \sum_i s_i r_i = \overline{v} \tag{5.3}$$

and

$$\sum_i s_i r_i^4 = \frac{(8\alpha\overline{v} - 6V_\infty)(1-\rho+\alpha^2) - 2\alpha^2(1-\rho+\rho\overline{v}^2)}{3-3\rho+\alpha^2}. \tag{5.4}$$

The r_i and the s_i are determined by $\overline{v}$ (the r_i via $G(u,\overline{u}) = 0$ and the s_i via (5.3) and (5.4)).

   - *(e) If s_1 is chosen, then there are only a finite number of values of $\overline{v}$ for $v \in \omega(u_0)$.*
   - *(f) The set $I' = \{\overline{v} : v \in \omega(u_0)\}$ is a closed interval. Let $I' = [a', b']$ and suppose that $a' \neq b'$. We may assume that $b' > \alpha$ for otherwise the signs of u and α may be changed. Choose $a'' \in I'$ with $\alpha < a'' < b'$ and define $I = [a,b] \subset (a'', b')$ such that $s_1(a) \neq s_1(b)$. By the construction of I, we can choose δ so that $a - \alpha > \delta > 0$,*

$$I'' = [a-\delta, b+\delta] \ \text{ and } \ I \subset I'' \subset I'.$$

Then s_1 is a C^1 function of $\overline{v} \in I''$ and the $r_i(\overline{v})$ are bounded away from α. Thus there is an $\eta > 0$ such that

$$|s_1(w_1) - s_1(w_2)| < \eta |w_1 - w_2| \ \ (w_1, w_2 \in I''). \tag{5.5}$$

Remarks on proof.

1. If the cubic has only one real solution, say θ, then $u(x) = \theta$ $(0 \leqslant x \leqslant 1)$ and so $\overline{u} = \theta$. The cubic equation is now $(\theta - \alpha)(1 - \theta^2) = 0$.

2. The argument is similar to that in 1.

3.(d) If $\rho > 1 + 2\alpha^2$ then the slope of $\rho(z - \gamma)$ is so large that $G(z, \gamma)$ cannot have more than one solution. Note that (5.4) follows from (4.8). From an elementary property of determinants, the s_i are given uniquely as the solution to the linear equations (5.3) and (5.4).

3.(e) We are dealing with rational functions.

3.(f) The set $\omega(u_0)$ is connected in L^1, from which it follows (since $\overline{v}$ is a continuous function on a compact, connected set) that I' is a closed interval (which may be one point). The smoothness of $s_1(\cdot)$ follows from being able to write it as an algebraic expression.
It is possible to ensure that $s_1(a) \neq s_1(b)$ because of 3.(e).

6 Convergence to an Equilibrium.

We consider the equation

$$u_t = \rho(\overline{u} - u) + (u - \alpha)(1 - u^2) \tag{6.1}$$

where $\alpha \in (-1, 1)$, with $u(x, 0) = u_0(x)$. Throughout, u_0 is taken as a fixed function satisfying (H5). We shall prove that there is a $u_\infty \in Z$ such that $u \to u_\infty$ pointwise as $t \to \infty$. The first step is to show that $\overline{u}(t)$ tends to a limit, l say, as $t \to \infty$. Recall that as u_0 is fixed, so is $\omega(u_0)$ and therefore also V for $v \in \omega(u_0)$ (by Lemma 4.5).

Below, the interval I is defined in Lemma 5.1 3.(f). It is a closed interval contained in the interior of I', the range of $\overline{v}$ for $v \in \omega(u_0)$ and corresponds to points where $\overline{v} > 0$. The idea is that asymptotically in time the orbit is close to $\omega(u_0)$ and hence for a range of large t, points u on the orbit are such that $\overline{u} > 0$.

In the following proofs, convergence, norms etc. are relative to the metric space Z. Note that u_0, $\omega(u_0)$, I, I'', δ and η of Lemma 5.1 are fixed in the following argument.

Two simple, preliminary lemmas are needed. In both we assume that q is measurable and use the notation

$$Q^+ = \{x : q(x) \geqslant \alpha\}.$$

Lemma 6.1 *There exists $\zeta > 0$ such that, given $q \in B$, $v \in \omega(u_0)$ (with $\overline{v} \in I$) and $\epsilon > 0$ with*

$$\int_0^1 |q(x) - v(x)|\, dx < \epsilon, \tag{6.2}$$

then

$$|m(Q^+) - s_1(\overline{v})| \leqslant \epsilon\zeta. \tag{6.3}$$

Proof. By Lemma 5.1 (3c, 3f), for $\overline{v} \in I$ there is an $r > 0$ such that $r_1 > \alpha + r$ and $r_2 < \alpha - r$. We shall show that (6.3) holds with $\zeta = 1/r$. The argument is by contradiction: we rule out in turn the possibilities

$$m(Q^+) < s_1(\overline{v}) - \epsilon/r, \tag{6.4}$$

$$m(Q^+) > s_1(\overline{v}) + \epsilon/r. \tag{6.5}$$

Suppose first that (6.4) holds. With X the subset of S_1 where $q(x) < \alpha$, we have

$$m(X) \geqslant \epsilon/r, \quad q(x) < \alpha \ \text{ and } \ v(x) = r_1 > \alpha \quad (x \in X).$$

Then from (6.2),

$$\begin{aligned} \epsilon &> \int_0^1 |v(x) - q(x)|\,dx \\ &\geqslant \int_X [v(x) - q(x)]\,dx \\ &\geqslant m(X)r_1 \geqslant \epsilon r_1/r > \epsilon. \end{aligned}$$

This contradiction proves that (6.4) is false.

If (6.5) holds, then with X the subset of $S_2 \cup S_3$ where $q(x) > \alpha$, we have

$$m(X) \geqslant \epsilon/r, \quad q(x) > \alpha \ \text{ and } \ v(x) \leqslant r_2 < \alpha \quad (x \in X).$$

Using (6.2) as before, we get a contradiction. □

Lemma 6.2 *With δ as in Lemma 5.1 3.(f), choose $\epsilon \in (0, \delta)$ and suppose that for some $q \in B$ with $\overline{q} \in I$,*

$$d(q, \omega(u_0)) \leqslant \epsilon. \tag{6.6}$$

Then for any $v \in \omega(u_0)$ with $\overline{v} \in I$ and $\overline{q} = \overline{v}$,

$$|m(Q^+) - s_1(\overline{v})| < \epsilon(\eta + \zeta), \tag{6.7}$$

where η is the Lipschitz constant in (5.5).

Proof. From (6.6), there is a $p \in \omega(u_0)$ such that

$$\begin{aligned} \epsilon &\geqslant \int_0^1 |p(x) - q(x)|\,dx \\ &\geqslant \left| \int [p(x) - q(x)]\,dx \right| \\ &= |\overline{p} - \overline{q}|. \end{aligned}$$

Hence, since $\overline{v} = \overline{q}$ we have $|\overline{v} - \overline{p}| \leqslant \epsilon$. It follows that $\overline{p} \in [a - \epsilon, b + \epsilon] \subset I''$. From (5.5) we now have

$$|s_1(\overline{v}) - s_1(\overline{p})| \leqslant \epsilon\eta \tag{6.8}$$

and from Lemma 6.1

$$|m(Q^+) - s_1(\overline{p})| \leqslant \epsilon\zeta \tag{6.9}$$

Thus (6.7) follows from the triangle inequality applied to (6.8) and (6.9). □

Proposition 6.3 *Let u_0 satisfy (H5). Then there is an $l \in [-1, 1]$ such that*

$$\lim_{t \to \infty} \overline{u}(t) = l.$$

Proof. Obviously if in Lemma 5.1 3.(f) $a' = b'$ then I is a single point and the result follows. So we assume that $a' \neq b'$ and choose a, b such that $s_1(a) \neq s_1(b)$. By Lemma 5.1 (2), if $v \in \omega(u_0)$ and $\overline{v} \in I$ then v cannot be a constant solution and so we may assume that the cubic $G(v, \overline{v}) = 0$ has three real roots.

With δ as in Lemma 5.1 3.(f), choose $\epsilon \in (0, \delta)$ such that

$$2(\eta + \zeta)\epsilon < |s_1(a) - s_1(b)|. \tag{6.10}$$

We emphasize that η and ζ depend only upon u_0 and I. From the definition of ω-limit sets, there exists T such that

$$d(u(\cdot, t), \omega(u_0)) < \epsilon \quad (t > T).$$

Furthermore, from the choice of a and b, $\overline{u}(t)$ will take each of these values an infinite number of times. Choose t^* so that $\overline{u}(t^*) = b$. Let t_1 be the last time before t^* that $\overline{u}(t_1) = a$ and t_4 be the first time after t^* that $\overline{u}(t_4) = a$. Now let t_2 be the first time after t_1 that $\overline{u}(t_2) = b$ and let t_3 be the last time before t_4 that $\overline{u}(t_3) = b$. Thus

$$\overline{u}(t) \in (a, b) \quad (t \in (t_1, t_2) \cup (t_3, t_4)).$$

Let now

$$U^+(t) = \{x : u(x, t) \geqslant \alpha\}$$

and set $g(t) = m(U^+(t))$. From the governing equation (4.2), when $u = \alpha$,

$$u_t = \rho(\overline{u} - u) > 0$$

since $\overline{u} \geqslant a > \alpha$. Hence $g(t)$ is non-decreasing for $t_1 \leqslant t \leqslant t_4$. From Lemma 6.2, with $U^+(t)$ replacing Q^+,

$$|g(t_1) - s_1(a)| < \epsilon(\eta + \zeta),$$
$$|g(t_2) - s_1(b)| < \epsilon(\eta + \zeta),$$

whence

$$-\epsilon(\eta + \zeta) + s_1(a) < g(t_1) \leqslant g(t_2) < s_1(b) + \epsilon(\eta + \zeta).$$

Combining this with a similar argument on $[t_3, t_4]$ one finds that

$$|s_1(a) - s_1(b)| < 2\epsilon(\eta + \zeta),$$

which contradicts (6.10). Thus $a = b$. □

Theorem 6.4 *Assume that (H4) holds and let u_0 satisfy (H5). Then the solution $u(\cdot,t)$ of equation (4.2) converges pointwise on $[0,1]$ to some equilibrium $u_\infty \in Z$.*

Proof. By Proposition 6.3, $\lim_{t\to\infty} \overline{u}(t) = l$. Hence, for any fixed x, equation (4.2) may be written as the non-autonomous ODE

$$\dot{u} = (u-\alpha)(1-u^2) + \rho(l-u) + h(t),$$

where $h(t) \to 0$ as $t \to \infty$. The result follows from [1, 14] or [15].

7 Stability of Equilibria.

We have shown that for a constant kernel β and $\Omega = [0,1]$ orbits of admissible initial data converge in L^1 to an equilibrium. However, from the point of view of applications, a question of importance is whether equilibria are (locally asymptotically) stable. Observe first that one cannot expect an equilibrium to be L^1-stable. This is because an L^1-neighborhood of an equilibrium u_0 may contain other equilibria that are a perturbation of u_0 by a small 'spike'. However, the situation with respect to L^∞ stability is much more satisfactory, and probably more relevant in applications. Thus, below 'stable' will mean 'locally asymptotically L^∞-stable' In this section we confine ourselves to the case $f(u) = u(1-u^2)$.

If the analysis is restricted to small ρ then we may extend the theory to a larger class of kernels than has been allowed thus far. Indeed, for any continuous kernel β (not necessarily positive) and bounded domain $\Omega \subset \mathbb{R}^n$ we have the following theorem.

Theorem 7.1 *Suppose u_0 is an equilibrium of the ODE (1.3) such that $u_0(x) = \pm 1$ on sets of non-zero measure $\Omega_{\pm 1}$ and $u(x) = 0$ on Ω_0. Then for ρ sufficiently small:*

1. *u_0 continues to a locally unique (in L^∞) stationary solution u_ρ of (2.1).*
2. *The spectrum of the linearization around u_ρ is concentrated in two intervals of length $O(\rho)$, J_1 around 1 and J_{-2} around -2.*
3. *If $\Omega_0 = \emptyset$, the spectrum of the linearization is concentrated in J_{-2} and u_ρ is stable.*

Proof.

1. Locally unique continuation follows from the implicit function theorem [6, 13].
2. The operator L, defined by $L\phi = f'(u_0)\phi$, has two eigenvalues -2 and 1 if $m(\Omega_0) \neq 0$ and $m(\Omega_{-1}) \neq 0$ or $m(\Omega_1) \neq 0$. Hence the result follows from the general perturbation theory of linear operators ([12, Ch. IV, § 3], see also [13]). Estimates on the length of J_{-2} and J_1 are obtained by a standard regular perturbation argument.

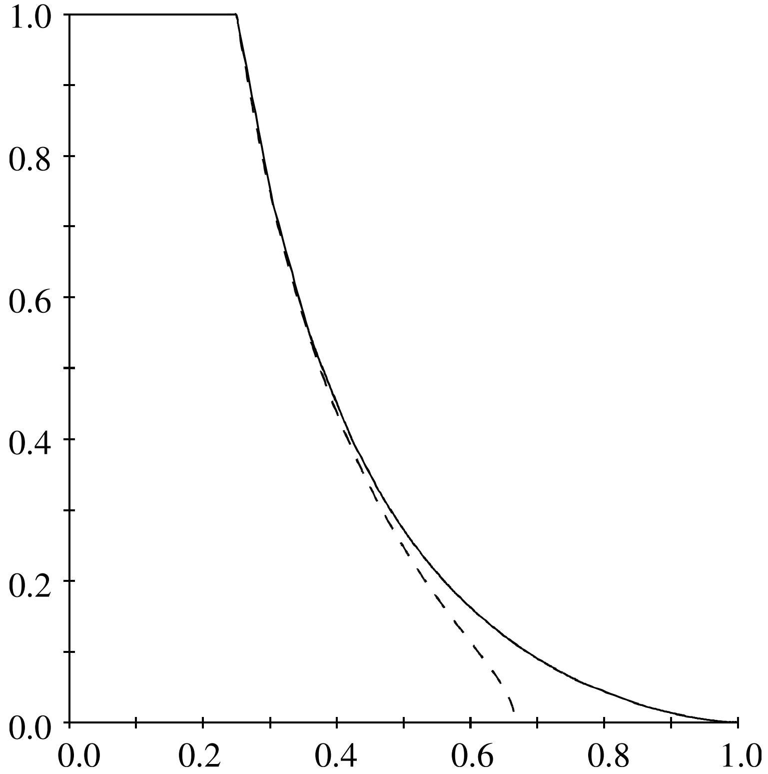

Figure 1: For every point $(\rho, \overline{u})$ below the solid line there are non-constant equilibria. Below the broken line there are stable non-constant equilibria.

3. To deduce local asymptotic stability in L^∞ from spectral stability, we first note that from Lemma 2.1 (2.1) generates a semiflow in L^∞. Then the result follows by a linearized stability theorem, e.g. [19, Theorem 11.22].

We note that, as discussed in [13], there is a value $\rho^* > 0$ such that all equilibria of (1.3) can be continued for $\rho < \rho^*$.

When (H4) holds, more can be said. The constants ±1 are always stable and 0 is unstable, both for the case $\rho > 1$ and the case $\rho < 1$, when these are the only equilibria. One can show that only equilibria with $s_2 = 0$ (see Lemma 5.1 for the notation) can be stable and that indeed they are stable if and only if

$$|r_1 r_3| > \sqrt{\frac{1-\rho}{3}}.$$

The situation concerning existence and stability in this case is summarized in Figure 1.

The conclusion is interesting from a biological point of view. In contrast with the bistable diffusion case, there are a great many patterns (stable non-constant solutions). This may fit better with the general situation in ecology.

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