

ELECTRODYNAMICS WITH NON-LINEAR CONSTITUTIVE LAWS AND MEMORY EFFECTS

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Key words: Maxwell equations, material with memory, ferro-magnetic material, ferro-electric material

Abstract. *Maxwell's equations governing the propagation of electro-magnetic fields are considered in conjunction with a class of material relations, which are capable of representing memory effects and time delay.*

1 INTRODUCTION

Maxwell's equations are usually given in the form

$$\begin{aligned}\operatorname{curl} H - \partial_0 D &= j, \\ \operatorname{curl} E + \partial_0 B &= 0,\end{aligned}$$

where curl is the usual vector-analytical differential operator and ∂_0 denotes the derivative with respect to time. The electric field is here denoted by E , D denotes the displacement current density, H the magnetic field, B the magnetic induction and j a known current density. In addition a non-linear and non-local material relation

$$\begin{pmatrix} D \\ B \end{pmatrix} = \zeta \begin{pmatrix} E \\ H \end{pmatrix}$$

is assumed. The system can be written in the form

$$\partial_0 \begin{pmatrix} D \\ B \end{pmatrix} - \mathcal{M} \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -j \\ 0 \end{pmatrix}$$

with

$$\mathcal{M} := \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}.$$

A class of material relations of the form

$$\partial_0 \zeta = \zeta_0 \partial_0 + \Phi,$$

completes the system. Here Φ is a non-linear abstract Volterra type operator (defined later) and ζ_0 linear. Under weak assumptions the solution theory of initial boundary value problems for this system are considered in the frame work of extrapolation spaces – see e.g. [7] for the general setting – associated with the time-derivative ∂_0 as a normal operator and a skew-selfadjoint realization A of the formal Maxwell operator $\zeta_0^{-1} \mathcal{M}$ in a suitably weighted space. The approach expands on ideas previously presented in [4].

2 Formulation of the Problem Class

2.1 Abstract Volterra Operators

By $\operatorname{supp}_0 f$ we shall denote the so-called time-support given by

$$\operatorname{supp}_0 f := \overline{\bigcup \{ \operatorname{supp}(\phi \mapsto f(\phi \otimes \eta)) \mid \eta \in H \}}$$

for f in $\left(\overset{\circ}{C}_\infty(\mathbb{R}) \overset{a}{\otimes} H \right)'$, the linear space of complex linear functionals acting on the algebraic tensor product $\overset{\circ}{C}_\infty(\mathbb{R}) \overset{a}{\otimes} H$, H a Hilbert space. Here $\phi \mapsto f(\phi \otimes \eta)$ is the obvious linear functional on $\overset{\circ}{C}_\infty(\mathbb{R})$.

Definition 1 *Let H_0, H_1 be Hilbert spaces and*

$$W : D(W) \subseteq \left(\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a H_0 \right)' \rightarrow \left(\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a H_1 \right)'.$$

If

$$\inf \text{supp}_0 (f - g) \leq \inf \text{supp}_0 (W(f) - W(g)) \tag{1}$$

for all $f, g \in D(W)$ then we shall call W causal.

Here we interpret $\inf \text{supp}_0 f = +\infty$ if $\text{supp}_0 f$ empty and $\inf \text{supp}_0 f = -\infty$ if $\text{supp}_0 f$ is not bounded below, so that (1) is only restrictive if we take f with $\text{supp}_0 f$ bounded below.

For defining abstract Volterra operators we also need a suitable topology, which we shall base on a discussion of properly established operators ∂_0 and A . First we define $\partial_0 := \partial_{0,\nu} + \nu$, where the operator $\partial_{0,\nu}$ is initially given as the closure of $(\partial_0 - \nu) \Big|_{\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a H_0}$ considered as an operator in $L^2(\mathbb{R}, \exp(-2\nu s) ds) \otimes H_0$.

Let $A : D(A) \subseteq H_0 \rightarrow H_0$ be a densely defined, closed, linear operator with non-empty resolvent set - say $\lambda_0 \in \rho(A)$. Then, the appropriate structure for discussing an operator equation of the form

$$(\partial_0 - A) u = g,$$

$g \in H_0$ given, is the lattice of Hilbert spaces

$$(H_{\nu,j,k})_{j,k \in \mathbb{Z}},$$

where $H_{\nu,j,k}$ abbreviates the Hilbert space $H_{(j,k)}(\partial_{0,\nu} + \nu, A - \lambda_0)$, which is given as the completion of $\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a \bigcap_{s \in \mathbb{N}} D(A^s)$ with respect to the norm

$$\phi \mapsto \left| (\partial_{0,\nu} + 1)^j (A - \lambda_0)^k \phi \right|_{\nu,0,0}$$

of $H_{\nu,0,0} = L^2(\mathbb{R}, \exp(-2\nu s) ds) \otimes H_0$, $j, k \in \mathbb{Z}$, $\nu \in \mathbb{R}_{>0}$ (see [5, 7] for details of this construction). We shall also use H_k as an abbreviation of $H_k(A - \lambda_0)$, $k \in \mathbb{Z}$, which in turn is the completion of $\bigcap_{s \in \mathbb{N}} D(A^s)$ with respect to the norm

$$\phi \mapsto \left| (A - \lambda_0)^k \phi \right|_0.$$

By construction $\partial_0 = \partial_{0,\nu} + \nu$ and A extend continuously to operators mapping from $H_{\nu,j,k}$ to $H_{\nu,j-1,k}$ and $H_{\nu,j,k-1}$, respectively, $j, k \in \mathbb{Z}$. We shall use the same names for these extensions. In the following, however, only the cases $k = -1, 0, 1$, $j = -1, 0, 1$ will be relevant.

Definition 2 We shall call a family of operators $(\Phi_\nu : H_{\nu,0,0} \rightarrow H_{\nu,0,0})_{\nu \in \mathbb{R}_{\geq \nu_\#}}$, $\nu_\# \in \mathbb{R}_{>0}$, an abstract Volterra operator if there is a causal operator

$$\Phi : \mathring{C}_\infty(\mathbb{R}) \otimes_a H_0 \subseteq \left(\mathring{C}_\infty(\mathbb{R}) \otimes_a H_0 \right)' \rightarrow \left(\mathring{C}_\infty(\mathbb{R}) \otimes_a H_0 \right)'$$

satisfying the following requirements:

1. for the range of Φ we have

$$\Phi \left[\mathring{C}_\infty(\mathbb{R}) \otimes_a H_0 \right] \subseteq \bigcap_{\nu \geq \nu_\#} H_{\nu,0,0},$$

2. there is a constant $C \in \mathbb{R}$ such that

$$|\Phi(u) - \Phi(v)|_{\nu,0,0} \leq C |u - v|_{\nu,0,0}$$

for all $u, v \in \mathring{C}_\infty(\mathbb{R}) \otimes_a H_0$ and all $\nu \in \mathbb{R}_{\geq \nu_\#}$,

3. the mappings $\Phi_\nu : H_{\nu,0,0} \rightarrow H_{\nu,0,0}$ are given as the (*Lipschitz*) continuous extension of Φ the considered as a mapping in $H_{\nu,0,0}$, $\nu \geq \nu_\#$.

Since all the Φ_ν are generated by the same Φ , we shall write Φ instead of Φ_ν , $\nu \in \mathbb{R}_{\geq \nu_\#}$, leaving it again to the context, which particular ν is intended. In this sense we shall speak of Φ as an abstract Volterra operator in $H_{\nu,0,0}$ for $\nu \in \mathbb{R}_{\geq \nu_\#}$.

2.2 On Skew-Selfadjoint Realizations of $\zeta_0^{-1}\mathcal{M}$

In order to turn $\zeta_0^{-1}\mathcal{M}$ into a skew-selfadjoint operator, which in turn will then fill the role of the above operator A (with e.g. $\lambda_0 = -1$), we first have to modify the inner product of the complex Hilbert space $(L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)) \oplus (L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega))$, which we shall denote simply by $L^2(\Omega)$, thus leaving here and in the following the number of component spaces to be determined from the context. Here Ω is an open set in \mathbb{R}^3 to which we wish to confine our considerations. The resulting Hilbert space will serve as H_0 and the appropriate inner product is simply given by

$$\langle \phi | \eta \rangle_0 := \langle \phi | \zeta_0 \eta \rangle_{L^2(\Omega)}$$

for $\phi, \eta \in L^2(\Omega)$, where $\langle \cdot | \cdot \rangle_{L^2(\Omega)}$ denotes the standard inner product in $L^2(\Omega)$ (assumed to be linear in the second factor). For this construction we assume that $\zeta_0 : L^2(\Omega) \rightarrow L^2(\Omega)$ is a positive definite and continuous linear mapping. Thus, in particular, the norm

$|\cdot|_0$ of H_0 and the norm $|\cdot|_{L^2(\Omega)}$ of $L^2(\Omega)$ are equivalent and $\zeta_0^{1/2}$ may be interpreted as a unitary mapping between H_0 and $L^2(\Omega)$, i.e. $H_0 = \zeta_0^{-1/2} L^2(\Omega)$. By a simple integration by parts argument it can now be seen that

$$\zeta_0^{-1} \mathcal{M} \Big|_{\mathring{C}_\infty(\Omega)}$$

is skew-symmetric in H_0 . Here $\mathring{C}_\infty(\Omega)$ abbreviates the subspace of

$$(L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)) \oplus (L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega))$$

consisting of elements with components being infinitely smooth functions having compact support. Noting that this is equivalent to saying that \mathcal{M} being skew-symmetric in $L^2(\Omega)$, we also see that a skew-selfadjoint realization of $\zeta_0^{-1} \mathcal{M}$ in H_0 gives rise to one of \mathcal{M} in $L^2(\Omega)$ and vice versa. To find such a realization amounts to choosing a suitable boundary condition. The standard choice of the boundary condition of vanishing tangential components of the electric field E on smooth boundaries – on which we shall focus here – can be generalized to the general case of boundary points $\mathring{\Omega}$ of the open set Ω in the following simple way. We require

$$E \in H(\mathring{\text{curl}}),$$

where $H(\mathring{\text{curl}})$ denotes the domain of the closure $\mathring{\text{curl}}$ of the operator

$$\begin{aligned} \text{curl} \Big|_{\mathring{C}_\infty(\Omega)} : \mathring{C}_\infty(\Omega) \subseteq L^2(\Omega) &\rightarrow L^2(\Omega), \\ \phi &\mapsto \text{curl } \phi. \end{aligned}$$

Recall that $H(\mathring{\text{curl}})$ is a Hilbert space with respect to the graph norm $\phi \mapsto \sqrt{|\phi|_0^2 + |\mathring{\text{curl}} \phi|_0^2}$.

The symmetry of $\text{curl} \Big|_{\mathring{C}_\infty(\Omega)}$ in $L^2(\Omega)$ motivates the definition of the operator

$$\text{curl} := \left(\text{curl} \Big|_{\mathring{C}_\infty(\Omega)} \right)^*$$

with domain denoted by $H(\text{curl})$. With these definitions we have that $\mathcal{M} \Big|_{H(\mathring{\text{curl}}) \oplus H(\text{curl})}$ as a skew-selfadjoint realization of \mathcal{M} in $L^2(\Omega)$. Consequently, this yields

$$A := \zeta_0^{-1} \mathcal{M} \Big|_{H(\mathring{\text{curl}}) \oplus H(\text{curl})}$$

as a skew-selfadjoint realization of $\zeta_0^{-1} \mathcal{M}$ in H_0 .

2.3 The Evolution Problem

We are now able to formulate the law governing electro-magnetic field in the the particular class of media considered here as

$$(\partial_0 - \mathbf{A}) U = F + \delta \otimes U_0 + \Phi(U), \quad (2)$$

The term $\delta \otimes U_0 \in H_{\nu, -1, 0}$ represents here initial data U_0 at time zero and is defined by

$$(\delta \otimes U_0)(\varphi \otimes h) := \varphi(0) \langle U_0 | h \rangle_0$$

for all $h \in H_0$, $\varphi \in \mathring{C}_\infty(\mathbb{R})$. In 2 it is assumed that Φ is an abstract Volterra operator in $H_{\nu, 0, 0}$ and

$$F \in H_{\nu, 0, 0}, \text{ supp}_0 F \subseteq \mathbb{R}_{\geq 0}, U_0 \in H_0. \quad (3)$$

for all $\nu \in \mathbb{R}_{\geq \nu_\#}$.

We recall now from [7] the following – slightly adapted – version of the linear solution theory in the frame work of Sobolev lattices.

Theorem 3 *Let $\Re\sigma(\mathbf{A})$ be bounded above by $\nu_\# \in \mathbb{R} \setminus \Re\sigma(\mathbf{A})$, $\nu_\# \in \mathbb{R}_{> 0}$, and for $\nu \geq \nu_\#$*

$$(\lambda + \nu - \mathbf{A})^{-1} : H_0 \rightarrow H_0$$

is uniformly bounded for all $\lambda \in \mathbb{R}_{\geq 0} + \mathfrak{B}\mathbb{R}$, and moreover

$$\sup \{ \|(\lambda + \nu - \mathbf{A})^{-1}\| \mid \lambda \in \mathbb{R}_{\geq 0} + \mathfrak{B}\mathbb{R} \} = o(1) \text{ for } \nu \rightarrow \infty. \quad (4)$$

Then $(\partial_0 - \mathbf{A})$ is continuously invertible in $H_{\nu, j, k}$,

$$\|(\partial_0 - \mathbf{A})^{-1} : H_{\nu, j, k} \rightarrow H_{\nu, j, k}\| = o(1) \text{ for } \nu \rightarrow \infty$$

and

$$(\partial_0 - \mathbf{A})^{-1} : H_{\nu, j, k} \rightarrow H_{\nu, j, k}$$

is (forward) causal for every $\nu \geq \nu_\#$, $j, k \in \mathbb{Z}$.

3 The Solution Theory

Theorem 4 *Under the stated general assumptions problem (2) has for all sufficiently large $\nu \in \mathbb{R}_{\geq \nu_\#}$ a unique solution¹ $V \in H_{\nu, 0, 0}$, which depends continuously on the data F and V_0 in the sense that there is a constant $C \in \mathbb{R}_{> 0}$ such that*

$$|V_1 - V_2|_{\nu, 0, 0} \leq C \left(|F_1 - F_2|_{\nu, 0, 0} + |V_{0,1} - V_{0,2}|_0 \right),$$

where V_i denotes the solution associated with the forcing term F_i and the initial data $V_{0,i}$, $i = 1, 2$.

¹Note that the solution V does not depend on ν as long as ν is sufficiently large.

Proof: The result follows by the obvious contraction mapping argument. Indeed, for sufficiently large $\nu \in \mathbb{R}_{\geq \nu\#}$ the mapping $T_\nu : H_{\nu,0,0} \rightarrow H_{\nu,0,0}$ (note that $H_0 \rightarrow H_{\nu,0,0}$, $V_0 \mapsto (\partial_0 - \mathbf{A})^{-1} \delta \otimes V_0$) given by

$$T_\nu \phi := (\partial_0 - \mathbf{A})^{-1} (F + \delta \otimes V_0 + \Phi(\phi))$$

is Lipschitz continuous with

$$|T_\nu|_{Lip} \leq \sup \left\{ \|(\lambda + \nu - \mathbf{A})^{-1}\| \mid \lambda \in \mathbb{R}_{\geq 0} + \beta \mathbb{R} \right\} \sup \left\{ |\Phi_\mu|_{Lip} \mid \mu \geq \nu \right\}.$$

Since assumption (4) is clearly satisfied in our case, indeed we have

$$\sup \left\{ \|(\lambda + \nu - \mathbf{A})^{-1}\| \mid \lambda \in \mathbb{R}_{\geq 0} + \beta \mathbb{R} \right\} \leq \frac{1}{\nu},$$

the right-hand side can be made smaller than 1 for all sufficiently large ν . For such a choice of the parameter $\nu \in \mathbb{R}_{\geq \nu\#}$ the mapping T_ν becomes a contraction and the unique existence of a fixed point $V \in H_{\nu,0,0}$ follows together with the continuous dependence estimate. The fixed point V satisfies

$$V = T_\nu V = (\partial_0 - \mathbf{A})^{-1} (F + \delta \otimes V_0 + \Phi(V)).$$

Applying now $(\partial_0 - \mathbf{A})|_{H_{\nu,0,0}} : H_{\nu,0,0} \subseteq H_{\nu,-1,-1} \rightarrow H_{\nu,-1,-1}$ (in the sense of the Sobolev lattice construction above) to both sides we obtain (2). It remains to determine uniqueness of a solution to (2). But if

$$(\partial_0 - \mathbf{A}) V = F + \delta \otimes V_0 + \Phi(V)$$

then we have by applying $(\partial_0 - \mathbf{A})^{-1} : H_{\nu,-1,-1} \rightarrow H_{\nu,-1,-1}$

$$V = (\partial_0 - \mathbf{A})^{-1} (F + \delta \otimes V_0 + \Phi(V)).$$

By the assumptions on V we see that the right-hand side is however just T_ν and by the uniqueness of a fixed point we obtain that the solution of (2) is unique in $H_{\nu,0,0}$ for sufficiently large $\nu \in \mathbb{R}_{\geq \nu\#}$. \square

It is frequently desired that the data prior to the initial time zero are assumed to be known, so that the actual solution of interest is of the form

$$U = \begin{cases} V & \text{on } \mathbb{R}_{>0} \\ V_{-\infty} & \text{on } \mathbb{R}_{\leq 0} \end{cases}$$

with $V_{-\infty} = \chi_{\mathbb{R}_{<0}}(m_0) V_{-\infty}$ the given pre-history. The initial data are then induced by assuming U to be continuous at zero, i.e.

$$V_0 = V_{-\infty}(0-).$$

This requires suitable assumptions on the pre-history to ensure the existence of this limit and a match with the initial data, such as

$$\varphi(m_0) \left(V_{-\infty} + \chi_{\mathbb{R}_{>0}}(m_0) \otimes V_0 \right) \in H_{\nu,1,0}, \quad (5)$$

where $\varphi = \partial_0^{-1}\psi$ for some $\psi \in \mathring{C}_\infty(\mathbb{R}_{<0})$, which we shall assume to hold throughout. Also the other terms on the right-hand side usually contain some form of dependence on the pre-history encoded as a modification of F . In the next section we shall illustrate this in more detail.

4 Applications

To illustrate the utility of the developed frame work let us look at several more specific forms of material laws.

4.1 The Nimitzky Operator Case

Φ is induced by a Lipschitz continuous mapping

$$\Gamma : H_0 \rightarrow H_0,$$

i.e.

$$\Phi(V) := t \mapsto \Gamma(V(t))$$

for $V \in \mathring{C}_\infty(\mathbb{R}) \otimes_a H_0$. In this case there is clearly no dependence on the pre-history other than the information encoded in the initial data.

4.2 Fixed Time Delay

If Φ is given in terms of $s \mapsto \eta(\tau_{-t_0}\phi(s))$ with a fixed delay time $t_0 > 0$ then the needed uniform Lipschitz continuity is also easily seen:

$$\begin{aligned} |\eta(\tau_{-t_0}\phi_0) - \eta(\tau_{-t_0}\phi_1)|_{\nu,0,0}^2 &\leq |\eta|_{Lip}^2 \int_{\mathbb{R}} |\tau_{-t_0}\phi_0(s) - \tau_{-t_0}\phi_1(s)|_0^2 \exp(-2\nu s) ds, \\ &= |\eta|_{Lip}^2 e^{-2\nu t_0} \int_{\mathbb{R}} |\phi_0(s) - \phi_1(s)|_0^2 e^{-2\nu s} ds, \\ &\leq |\eta|_{Lip}^2 |\phi_0 - \phi_1|_{\nu,0,0}^2, \end{aligned}$$

for all $\phi_0, \phi_1 \in H_{\nu,0,0}$. Here $|\eta|_{Lip}^2$ is the best Lipschitz constant of $\eta : H_0 \rightarrow H_0$. In this case the pre-history can be taken into account by letting

$$\Phi(V) := \chi_{\mathbb{R}_{>t_0}}(m_0) \eta(\tau_{-t_0}V) + \chi_{[0,t_0]}(m_0) \eta(\tau_{-t_0}V_{-\infty})$$

which can again be interpreted as a modification of the source term F . Clearly, here only a pre-history with time support in $[0, t_0]$ is relevant. To ensure that $\chi_{[0, t_0]}(m_0) \eta(\tau_{-t_0} V_{-\infty}) \in H_{\nu, 0, 0}$ we observe that

$$\begin{aligned}
 & \left| \chi_{[0, t_0]}(m_0) \eta(\tau_{-t_0} \phi_0) - \chi_{[0, t_0]}(m_0) \eta(\tau_{-t_0} \phi_1) \right|_{\nu, 0, 0}^2 \leq \\
 & \leq |\eta|_{Lip}^2 \int_0^{t_0} |\phi_0(t-t_0) - \phi_1(t-t_0)|_0^2 e^{-2\nu t} dt \\
 & \leq |\eta|_{Lip}^2 \int_{-t_0}^0 |\phi_0(s) - \phi_1(s)|_0^2 e^{-2\nu(s+t_0)} ds \\
 & = e^{-2\nu t_0} |\eta|_{Lip}^2 \int_{-t_0}^0 |\phi_0(s) - \phi_1(s)|_0^2 e^{-2\nu s} ds \\
 & = e^{-2\nu t_0} |\eta|_{Lip}^2 \int_{-t_0}^0 |\phi_0(s) - \phi_1(s)|_0^2 e^{-2\nu_0 s} e^{-2(\nu-\nu_0)s} ds \\
 & \leq e^{-2\nu_0 t_0} |\eta|_{Lip}^2 \int_{-t_0}^0 |\phi_0(s) - \phi_1(s)|_0^2 e^{-2\nu_0 s} ds
 \end{aligned}$$

for all $\phi_0, \phi_1 \in H_{\nu_0, 0, 0}$, $\nu_0 \in \mathbb{R}_{\leq \nu}$. Thus, it suffices to ensure that $V_{-\infty} \in H_{\nu_0, 0, 0}$ for some $\nu_0 \in \mathbb{R}$ to obtain unique solvability and continuous dependence on the data (including the pre-history) for all sufficiently large $\nu \geq \nu_{\#}$.

4.3 The Volterra Integral Operator Case

This case has been dealt with in generality already in [4]. However, for illustration purposes we recall the main arguments here. We take $\eta : H_0 \rightarrow H_0$ to be Lipschitz continuous. Then let Φ be given by

$$V \mapsto \int_{\mathbb{R}_{\geq 0}} \mathbf{K}(\cdot - s) \eta(V(s)) ds$$

such that

$$\mathbf{K}(t) = 0 \text{ for } t < 0 \quad (\mathbf{K} \text{ causal})$$

and

$$\int_{\mathbb{R}} \|\mathbf{K}(t)\| e^{-\nu_K t} dt < \infty$$

for some $\nu_K \in \mathbb{R}_{> 0}$.

Here

$$|\eta|_{Lip} \int_{\mathbb{R}} \|\mathbf{K}(t)\| e^{-\nu_K t} dt < \infty \quad (6)$$

features the desired Lipschitz constant. Indeed, noting that $e^{-2(\nu-\nu_K)(t-s)} \leq 1$ for $\nu \geq \nu_K$ and $t \geq s$, we may estimate

$$\begin{aligned}
 & \left| \int_{\mathbb{R}_{\geq 0}} \mathbf{K}(\cdot - s) \eta(\phi(s)) ds - \int_{\mathbb{R}_{\geq 0}} \mathbf{K}(\cdot - s) \eta(\psi(s)) ds \right|_{\nu, 0, 0} \leq \\
 & \leq |\eta|_{Lip} \sqrt{\int_{\mathbb{R}_{\geq 0}} \left(\int_{\mathbb{R}_{\geq 0}} \|\mathbf{K}(t-s)\| |\phi(s) - \psi(s)|_0 ds \right)^2 e^{-2\nu t} dt} \leq \\
 & \leq |\eta|_{Lip} \sqrt{\int_{\mathbb{R}} \|\mathbf{K}(t)\| e^{-\nu_K t} dt} \sqrt{\int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}_{\geq 0}} \|\mathbf{K}(t-s)\| e^{-\nu_K(t-s)} e^{-2(\nu-\nu_K)(t-s)} dt |\phi(s) - \psi(s)|_0^2 e^{-2\nu s} ds} \leq \\
 & \leq |\eta|_{Lip} \int_{\mathbb{R}} \|\mathbf{K}(t)\| e^{-\nu_K t} dt |\phi - \psi|_{\nu, 0, 0}
 \end{aligned}$$

for all $\phi, \psi \in H_{\nu, 0, 0}$ uniformly in ν for $\nu \geq \nu_K$. Assumption (K causal) yields causality. If we would want to consider a dependence on the pre-history of the same form, i.e.

$$\Phi(V) = \chi_{\mathbb{R}_{> 0}}(m_0) \left(\int_{\mathbb{R}_{\geq 0}} \mathbf{K}(\cdot - s) \eta(V(s)) ds + \int_{\mathbb{R}_{< 0}} \mathbf{K}(\cdot - s) \eta(V_{-\infty}(s)) ds \right)$$

then the desired uniform Lipschitz continuity is seen in the same way. Alternatively, the term $\chi_{\mathbb{R}_{> 0}}(m_0) \int_{\mathbb{R}_{< 0}} \mathbf{K}(\cdot - s) \eta(V_{-\infty}(s)) ds$ could be considered merely as a modification of F . In any case we need the pre-history to be such that this term is in $H_{\nu, 0, 0}$. By an analogous calculation we obtain

$$\begin{aligned}
 & \left| \chi_{\mathbb{R}_{> 0}}(m_0) \int_{\mathbb{R}_{< 0}} \mathbf{K}(\cdot - s) \eta(\phi(s)) ds - \chi_{\mathbb{R}_{> 0}}(m_0) \int_{\mathbb{R}_{< 0}} \mathbf{K}(\cdot - s) \eta(\psi(s)) ds \right|_{\nu, 0, 0} \leq \\
 & \leq |\eta|_{Lip} \sqrt{\int_{\mathbb{R}_{\geq 0}} \left(\int_{\mathbb{R}_{< 0}} \|\mathbf{K}(t-s)\| |\phi(s) - \psi(s)|_0 ds \right)^2 e^{-2\nu t} dt} \leq \\
 & \leq |\eta|_{Lip} \sqrt{\int_{\mathbb{R}} \|\mathbf{K}(t)\| e^{-\nu_K t} dt} \sqrt{\int_{\mathbb{R}_{< 0}} \int_{\mathbb{R}_{\geq 0}} \|\mathbf{K}(t-s)\| e^{-\nu_K(t-s)} e^{-2(\nu-\nu_K)(t-s)} e^{-2(\nu-\nu_0)s} dt |\phi(s) - \psi(s)|_0^2 e^{-2\nu_0 s} ds} \leq \\
 & \leq |\eta|_{Lip} \int_{\mathbb{R}} \|\mathbf{K}(t)\| e^{-\nu_K t} dt \sqrt{\int_{\mathbb{R}_{< 0}} |\phi(s) - \psi(s)|_0^2 e^{-2\nu_0 s} ds}
 \end{aligned}$$

for every $\nu_0 \geq \nu_K$ as a Lipschitz type estimate for the pre-history. Thus we only have to ensure that e.g. $\chi_{\mathbb{R}_{> 0}}(m_0) \int_{\mathbb{R}_{< 0}} \mathbf{K}(\cdot - s) \eta(0) ds \in H_{\nu, 0, 0}$, which can be obtained if we additionally assume $\eta(0) = 0$. Thus, the above general solution result applies. However, here the additional question arises if the solution also depends continuously on the pre-history. But since we may consider the term $\chi_{\mathbb{R}_{> 0}}(m_0) \int_{\mathbb{R}_{< 0}} \mathbf{K}(\cdot - s) \eta(V_{-\infty}(s)) ds$ as part of F also this question has a positive answer according to the above Lipschitz type estimate for the pre-history.

4.4 Delayed Reaction

In this case, which has also been briefly considered in [4], we let Φ be based on a mapping of the form $f \circ \tau_{(\cdot)}$, where

$$(\tau_t \psi)(x) = \psi(x + t)$$

for $t, x \in \mathbb{R}$ for e.g. $\psi \in \mathring{C}_\infty(\mathbb{R}) \otimes H_0$, and

$$f : H_{\nu_0,0,0} \rightarrow H_0$$

Lipschitz continuous for some $\nu_0 \in \mathbb{R}$ and

$$f = f \circ \chi_{\mathbb{R}_{\leq 0}}(m_0).$$

To take a non-trivial pre-history $V_{-\infty}$ into account, we let

$$\Phi(\psi) = \chi_{\mathbb{R}_{> 0}}(m_0) f \left(\tau_{(\cdot)} \left(\chi_{\mathbb{R}_{> 0}}(m_0) V + V_{-\infty} \right) \right).$$

The Lipschitz continuity of Φ follows now in the following way

$$\begin{aligned} & \left| \chi_{\mathbb{R}_{> 0}}(m_0) f(\tau_{(\cdot)} \varphi_0) - \chi_{\mathbb{R}_{> 0}}(m_0) f(\tau_{(\cdot)} \varphi_1) \right|_{\nu,0,0}^2 = \\ & = \int_{\mathbb{R}_{> 0}} |f(\tau_t \varphi_0) - f(\tau_t \varphi_1)|_0^2 e^{-2\nu t} dt \tag{7} \\ & \leq |f|_{Lip}^2 \int_{\mathbb{R}_{> 0}} \int_{\mathbb{R}_{\leq 0}} |\varphi_0(s+t) - \varphi_1(s+t)|_0^2 e^{-2\nu_0 s} ds e^{-2\nu t} dt \\ & = |f|_{Lip}^2 \int_{\mathbb{R}_{> 0}} \int_{\mathbb{R}_{\leq -t}} |\varphi_0(s+t) - \varphi_1(s+t)|_0^2 e^{-2\nu_0 s} ds e^{-2\nu t} dt + \\ & \quad + |f|_{Lip}^2 \int_{\mathbb{R}_{> 0}} \int_{[-t,0]} |\varphi_0(s+t) - \varphi_1(s+t)|_0^2 e^{-2\nu_0 s} ds e^{-2\nu t} dt \\ & = |f|_{Lip}^2 \int_{\mathbb{R}_{> 0}} \int_{\mathbb{R}_{\leq 0}} |\varphi_0(u) - \varphi_1(u)|_0^2 e^{-2\nu_0 u} du e^{-2(\nu-\nu_0)t} dt + \\ & \quad + |f|_{Lip}^2 \int_{\mathbb{R}_{< 0}} \int_{[-s,\infty 0]} |\varphi_0(s+t) - \varphi_1(s+t)|_0^2 e^{-2\nu(t+s)} dt e^{-2(\nu_0-\nu)s} ds \\ & \leq |f|_{Lip}^2 \frac{1}{2(\nu-\nu_0)} \left(\left| \chi_{\mathbb{R}_{> 0}}(m_0) (\varphi_1 - \varphi_2) \right|_{\nu,0,0}^2 + \left| \chi_{\mathbb{R}_{\leq 0}}(m_0) (\varphi_1 - \varphi_2) \right|_{\nu,0,0}^2 \right) \end{aligned}$$

for all $\varphi_0, \varphi_1 \in \mathring{C}_\infty(\mathbb{R}) \otimes H_0$ and uniformly for $-\nu \geq \nu_0 + 1$. As a consequence is $\Phi : \chi_{\mathbb{R}_{\geq 0}}(m_0) H_{\nu,0,0} \rightarrow \chi_{\mathbb{R}_{\geq 0}}(m_0) H_{\nu,0,0}$ for a fixed pre-history $V_{-\infty}$ indeed (uniformly) Lipschitz continuous

$$|\Phi(V_1) - \Phi(V_2)|_{\nu,0,0} \leq \frac{|f|_{Lip}}{\sqrt{2}} |V_1 - V_2|_{\nu,0,0}$$

for all $V_i \in H_{\nu,0,0}$, $i = 1, 2$, uniformly for all $\nu \geq \nu_0 + 1 > \nu_0$. Moreover, we also obtain continuous dependence on the pre-history, which follows with (7) by subtracting corresponding fixed point equations and estimating.

4.5 Material Laws given by Systems of Ordinary Differential Equations

Frequently, the material relation ζ is given via a separate system of differential equations, see e.g. [1] for the case of Maxwell equations in ferro-magnetic media. More specifically $\zeta(U) = U - W(U)$, where $U \mapsto W(U)$ is given as the solution of the system²:

$$\begin{aligned}\partial_0 W &= G_0(U, W, I), \\ \partial_0 I &= G_1(U, W, I).\end{aligned}\tag{8}$$

Thus, with a known pre-history $V_{-\infty}$ this³ turns into

$$\begin{aligned}\partial_0 W &= \chi_{\mathbb{R}_{\geq 0}}(m_0) G_0(V_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) V, W_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) W, I_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) I) + \delta \otimes W_{-\infty}(0-), \\ \partial_0 I &= \chi_{\mathbb{R}_{\geq 0}}(m_0) G_1(V_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) V, W_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) W, I_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) I) + \delta \otimes I_{-\infty}(0-),\end{aligned}$$

where $W_{-\infty}, I_{-\infty}$ are the causal solutions⁴ – say – in $H_{\nu_0,0,0}$, $\nu_0 \in \mathbb{R}_{>0}$, of the system

$$\begin{aligned}\partial_0 W_{-\infty} &= G_0(V_{-\infty}, W_{-\infty}, I_{-\infty}), \\ \partial_0 I_{-\infty} &= G_1(V_{-\infty}, W_{-\infty}, I_{-\infty})\end{aligned}$$

restricted to $\mathbb{R}_{<0}$. The evolution problem now becomes

$$\partial_0 V - AV = F + \delta \otimes V_0 + \Phi(V)$$

where

$$\Phi(V) := \chi_{\mathbb{R}_{\geq 0}}(m_0) G_0(V_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) V, \pi_0 L(V_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) V), \pi_1 L(V_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) V))$$

with $U \mapsto L(U)$ denoting the solution operator of the system (8) and π_0, π_1 the canonical projectors on the components (i.e. $\pi_0 \begin{pmatrix} W \\ I \end{pmatrix} = W$, $\pi_1 \begin{pmatrix} W \\ I \end{pmatrix} = I$). Causality and a uniform Lipschitz continuity of

$$(V, W, I) \mapsto \chi_{\mathbb{R}_{\geq 0}}(m_0) G_k(V_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) V, W_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) W, I_{-\infty} + \chi_{\mathbb{R}_{\geq 0}}(m_0) I),$$

in $H_{\nu,0,0}$ for $k = 0, 1$, $\nu \geq \nu_{\#}$, is sufficient to ensure a suitable abstract Volterra operator Φ . Note also that $W_{-\infty}(0-), I_{-\infty}(0-) \in H_0$ are well-defined, since $W_{-\infty}, I_{-\infty} \in H_{\nu_0,1,0}$.

²The component functions of I are referred to as interior variables.

³If the right-hand sides G_0, G_1 are e.g. of Nimitzky type and the initial state (W_0, I_0) is known (e.g. directly from $V_{-\infty}$) the system simplifies to solving

$$\begin{aligned}\partial_0 W &= \chi_{\mathbb{R}_{\geq 0}}(m_0) G_0(V, W, I) + \delta \otimes W_0, \\ \partial_0 I &= \chi_{\mathbb{R}_{\geq 0}}(m_0) G_1(V, W, I) + \delta \otimes I_0.\end{aligned}$$

⁴In the more simple situation mentioned in the previous footnote only the initial data W_0 and I_0 are relevant to represent the pre-history of W and I before time zero.

There is an alternative view on this problem, which may be even more natural. We may look for the solution V directly as the solution of the system

$$\begin{pmatrix} (\partial_0 - A) & -\partial_0 & 0 \\ 0 & \partial_0 & 0 \\ 0 & 0 & \partial_0 \end{pmatrix} \begin{pmatrix} V \\ W \\ I \end{pmatrix} = \begin{pmatrix} F + \delta \otimes V_{-\infty}(0-) \\ G_0(V_{-\infty} + V, W_{-\infty} + W, I_{-\infty} + I) + \delta \otimes W_{-\infty}(0-) \\ G_1(V_{-\infty} + V, W_{-\infty} + W, I_{-\infty} + I) + \delta \otimes I_{-\infty}(0-) \end{pmatrix}.$$

The solution can now be given by a similar contraction argument involving the inverse of the linear part

$$\begin{pmatrix} (\partial_0 - A) & -\partial_0 & 0 \\ 0 & \partial_0 & 0 \\ 0 & 0 & \partial_0 \end{pmatrix}^{-1} = \begin{pmatrix} (\partial_0 - A)^{-1} & (\partial_0 - A)^{-1} & 0 \\ 0 & \partial_0^{-1} & 0 \\ 0 & 0 & \partial_0^{-1} \end{pmatrix},$$

which can be estimated

$$\begin{aligned} \left| \begin{pmatrix} (\partial_0 - A)^{-1} & (\partial_0 - A)^{-1} & 0 \\ 0 & \partial_0^{-1} & 0 \\ 0 & 0 & \partial_0^{-1} \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \end{pmatrix} \right|_{\nu,0,0} &\leq \frac{1}{\nu} |F_0|_{\nu,0,0} + \frac{2}{\nu} |F_1|_{\nu,0,0} + \frac{1}{\nu} |F_2|_{\nu,0,0} \\ &\leq \frac{2\sqrt{3}}{\nu} \left| \begin{pmatrix} F_0 \\ F_1 \\ F_2 \end{pmatrix} \right|_{\nu,0,0} \end{aligned}$$

for all $F_0, F_1, F_2 \in H_{\nu,0,0}$.

Even partial differential equations are used to describe media with memory. As a prominent example we refer to the discussion of ferroelectric media, see e.g. [3, 2]. Such a system is – after suitable adaptation – also covered by the above approach. With this last observation we conclude our investigation into the described class of material relations. Although, the results have been illustrated for electromagnetic fields, which indeed has been the main motivation for inspecting this type of material laws, it is clear from the setup of the problem that these results transfer to a larger class of problems covering also other evolution problems of mathematical physics.

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