

**CONTINUATIONS OF HERMITIAN INDEFINITE
FUNCTIONS AND CORRESPONDING
CANONICAL SYSTEMS: AN EXAMPLE**

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ABSTRACT. M. G. Kreĭn established a close connection between the continuation problem of positive definite functions from a finite interval to the real axis and the inverse spectral problem for differential operators. In this note we study such a connection for the function $f(t) = 1 - |t|$, $t \in \mathbb{R}$, which is not positive definite on \mathbb{R} : its restrictions $f_a := f|_{(-2a, 2a)}$ are positive definite if $a \leq 1$ and have one negative square if $a > 1$. We show that with f a canonical differential equation or a Sturm–Liouville equation can be associated which have a singularity.

1. INTRODUCTION

1.1. Let $0 < a \leq \infty$ and $\kappa \in \mathbb{N} \cup \{0\}$. Recall that a continuous complex valued function (or *kernel*) $F(t, s)$, defined for s, t in some set \mathcal{D} , is said to have κ *negative squares* if $F(t, s) = \overline{F(s, t)}$, $s, t \in \mathcal{D}$, and the maximum of the numbers of negative eigenvalues, counted with multiplicities, of the matrices $(F(t_j, t_k))_{j, k=1}^n$, $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathcal{D}$, is equal to κ ; if $\kappa = 0$, the kernel $F(t, s)$ is called *positive definite*.

Following [12], by $\mathfrak{P}_{\kappa; a}$ ($\mathfrak{G}_{\kappa; a}$, respectively) we denote the set of all functions f (g , respectively) that are defined and continuous on the interval $(-2a, 2a)$ and such that $f(-t) = \overline{f(t)}$ ($g(-t) = \overline{g(t)}$, respectively) and the kernel

$$f(t-s) \quad \left(G_g(t, s) := g(t-s) - \overline{g(t)} + \overline{g(s)} + g(0), \text{ respectively} \right), \quad s, t \in (-a, a),$$

has κ negative squares; functions of the class $\mathfrak{P}_{0; a}$ are also called *positive definite* on $(-2a, 2a)$. If $\kappa = 0$, functions $f \in \mathfrak{P}_{0; a}$ and $g \in \mathfrak{G}_{0; a}$ and their continuations to the whole real axis \mathbb{R} within the class $\mathfrak{P}_{0; \infty}$ and $\mathfrak{G}_{0; \infty}$, respectively, play a decisive role in M. G. Kreĭn’s method for solving inverse spectral problems, see [9], [10], [11].

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In particular, suppose that a given function $g \in \mathfrak{G}_{0;\infty}$ has an *accelerant*, that is it admits a representation

$$(1.1) \quad g(t) = g(0) - \alpha|t| - \int_0^t (t-s)H(s) ds, \quad t \in \mathbb{R},$$

with some $\alpha > 0$ and a function $H \in L^1_{\text{loc}}(\mathbb{R})$, $H(-t) = \overline{H(t)}$, $t \in \mathbb{R}$ (note that $H(t) = -g''(t)$ almost everywhere). Using continuous analogues of orthogonal polynomials, M. G. Kreĭn derived a canonical system, whose spectral measure coincides with the spectral measure of g (which is the measure in the integral representation of $g \in \mathfrak{G}_{0;\infty}$, see [14, (11.16)]).

If $\kappa > 0$ and $0 < a < \infty$, in [14] for a function $g \in \mathfrak{G}_{\kappa;a}$ with accelerant which has infinitely many continuations in $\mathfrak{G}_{\kappa;\infty}$ a description of all these continuations was given, and a canonical system was associated with g . However, this canonical system could be defined only on intervals not containing a singular point of g . We recall the definition of a *singular point* of $g \in \mathfrak{G}_{\kappa;a}$: consider for $0 < a' < a$ the restriction $g_{a'} := g|_{(-2a', 2a')}$ and let $\kappa(a')$ be the number of negative squares of the kernel $G_{g_{a'}}(t, s)$, $s, t \in (-a', a')$. Since g is supposed to have an accelerant, $\kappa(a')$ is zero for sufficiently small a' , and it is piecewise constant and nondecreasing on $(0, a)$. A point $a_0 \in (0, a)$ is *singular* for g if $\kappa(a')$ has a jump at a_0 . Up to now the general form of the ‘canonical system’ in such a singular point is not known.

The canonical system which is defined by a function $g \in \mathfrak{G}_{\kappa;a}$ with accelerant is related to the continuation problem as follows, see [14]. Consider for $x \in (0, a)$ which is not a singular point for g the function $g_x = g|_{(-2x, 2x)}$. Then $g_x \in \mathfrak{G}_{\kappa(x);x}$ with some $\kappa(x)$, $0 \leq \kappa(x) \leq \kappa$, and all its continuations $\tilde{g}_x \in \mathfrak{G}_{\kappa(x);\infty}$ are described by a formula

$$(1.2) \quad i z^2 \int_0^\infty e^{i z t} \overline{\tilde{g}_x(t)} dt = \frac{w_{11}(x; z)\tau(z) + w_{12}(x; z)}{w_{21}(x; z)\tau(z) + w_{22}(x; z)}, \quad \text{Im } z > \gamma,$$

for some $\gamma \geq 0$. In fact, there exist four entire functions $w_{jk}(x; \cdot)$, $j, k = 1, 2$, such that the fractional linear transformation (1.2) establishes a bijective correspondence between all continuations $\tilde{g}_x \in \mathfrak{G}_{\kappa(x);\infty}$ of g_x and all $\tau \in \tilde{N}_0$; by \tilde{N}_0 we denote the class of all Nevanlinna functions (these are the functions which are holomorphic in the upper half-plane and have a non-negative imaginary part there) augmented by the constant ∞ . The matrix function $\mathcal{W}(\cdot; z) = \begin{pmatrix} w_{11}(\cdot; z) & w_{12}(\cdot; z) \\ w_{21}(\cdot; z) & w_{22}(\cdot; z) \end{pmatrix}$, sometimes called the *resolvent matrix* of this continuation problem, is shown to satisfy the canonical system

$$(1.3) \quad \frac{d\mathcal{W}(x; z)}{dx} \mathcal{J} = z\mathcal{W}(x; z)\mathcal{H}(x)$$

on the subintervals of $[0, \infty)$ containing no singular points of g , and the initial condition

$$(1.4) \quad \mathcal{W}(0; z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

here $\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the *Hamiltonian* $\mathcal{H}(x)$ is a continuous 2×2 -matrix function, with values being real symmetric matrices, see [14, Theorem 11.2].

In the present note we derive this Hamiltonian of the canonical system in (1.3) explicitly for the function g :

$$(1.5) \quad g(t) := -|t| + t^2/2, \quad t \in \mathbb{R},$$

i.e., $\alpha = 1$ and $H(s) \equiv -1$ in (1.1). It follows from [14, Theorem 2.1]¹ that

$$g_a = g|_{(-2a, 2a)} \in \begin{cases} \mathfrak{G}_{0;a} & \text{if } 0 < a \leq 1, \\ \mathfrak{G}_{1;a} & \text{if } 1 < a \leq \infty, \end{cases}$$

hence $g \in \mathfrak{G}_{1;\infty}$ and it has one singular point $a_0 = 1$. The Hamiltonian $\mathcal{H}(x)$ of the corresponding canonical system (1.3) is singular at 1, which means that its trace is not summable there. In fact we show in Theorem 7.1 below that for the function g in (1.5) we have

$$(1.6) \quad \mathcal{H}(x) = \begin{pmatrix} (x-1)^2 & 0 \\ 0 & \frac{1}{(x-1)^2} \end{pmatrix}, \quad x \in [0, \infty) \setminus \{1\}.$$

That is, the initial value problem (1.3), (1.4) for the resolvent matrix $\mathcal{W}(x; z)$ from (1.2) becomes

$$(1.7) \quad \frac{d\mathcal{W}(x; z)}{dx} \mathcal{J} = z\mathcal{W}(x; z) \begin{pmatrix} (x-1)^2 & 0 \\ 0 & \frac{1}{(x-1)^2} \end{pmatrix}, \quad \mathcal{W}(0; z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1.2. The formula (1.2) can be derived using the extension theory of symmetric operators in a Hilbert or Pontryagin space, see [14]. In fact, the right-hand side of (1.2) describes the set of all u -resolvents of a symmetric operator in the reproducing kernel Hilbert or Pontryagin space generated by the kernel $G_{g_x}(t, s)$, $s, t \in (-x, x)$, with an improper (that is a generalized) module element u . In order to avoid generalized elements in the present note we use a connection between functions $g \in \mathfrak{G}_{\kappa;a}$ and $f \in \mathfrak{P}_{\kappa;a}$ from [12, Section 5] and [14, Section 5]. In fact, together with g we consider the simpler function f :

$$(1.8) \quad f(t) := 1 - |t|, \quad t \in \mathbb{R}.$$

It is connected with the function g in (1.5) by the integral equation

$$(1.9) \quad - \int_0^t \frac{(t-s)^2}{2} f(s) ds + \int_0^t f(s)g(t-s) ds = \int_0^t g(s) ds, \quad t \in \mathbb{R};$$

this connection between g and f becomes more transparent if one considers the one-sided Fourier transforms, see (6.4) below. The function f has the property (see (i) in Subsection 1.5)

$$f_x = f|_{(-2x, 2x)} \in \begin{cases} \mathfrak{P}_{0;x} & \text{if } 0 < x \leq 1, \\ \mathfrak{P}_{1;x} & \text{if } 1 < x < \infty, \end{cases}$$

¹Observe that in (1.5) $\alpha = 1/2$, whereas in [14, Theorem 2.1] $\alpha = 1$.

and a great part of this note is concerned with the explicit description of the set of all continuations $\tilde{f}_x \in \mathfrak{P}_{0;\infty}$ or $\in \mathfrak{P}_{1;\infty}$ of f_x for $x < 1$ and $x > 1$, respectively (observe (ii)–(iv) in Subsection 1.5). Analogously to (1.2), these continuations are given by a formula

$$(1.10) \quad i \int_0^\infty e^{izt} \overline{\tilde{f}_x(t)} dt = \frac{\widehat{w}_{11}(x; z)\tau(z) + \widehat{w}_{12}(x; z)}{\widehat{w}_{21}(x; z)\tau(z) + \widehat{w}_{22}(x; z)}, \quad \text{Im } z > \gamma,$$

and it is shown that the resolvent matrix $\widehat{\mathcal{W}}(x; z)$ from (1.10) satisfies the same differential equation as $\mathcal{W}(x; z)$ in (1.7) but a different initial condition:

$$(1.11) \quad \widehat{\mathcal{W}}(0; z) = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}.$$

We mention that the z -dependence of the initial condition (1.11) for $\widehat{\mathcal{W}}(x; z)$ could be avoided by extending the canonical system to the interval $[-1, \infty)$ with the Hamiltonian

$$\tilde{\mathcal{H}}(x) := \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } -1 \leq x < 0, \\ \mathcal{H}(x) & \text{if } x \geq 0. \end{cases}$$

1.3. The differential equation in (1.3) corresponds to the initial value problem

$$(1.12) \quad -\mathcal{J} \frac{dy(x)}{dx} = z \mathcal{H}(x) y(x), \quad 0 \leq x \leq a, \quad y_1(0) = 0,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Here ‘corresponds’ means that in case of a non-singular non-negative Hamiltonian (in our example for instance for $a < 1$) the spectral measures for the problem (1.12) are described through a formula (1.2) (with $\mathcal{W}(x; z)$ determined by the problem (1.3)) as the spectral measures of the Nevanlinna function on the right-hand side of (1.2) with $x = a$ if the parameter τ runs through the class $\tilde{\mathcal{N}}_0$, see [5].

For the function f from (1.8) if $a < 1$, the Hilbert space $\Pi(f_a)$ and the symmetric operator S_a , which are the main objects in our construction (see Sections 2 and 3), are in a canonical way isomorphic to the Hilbert space $L^2(\tilde{\mathcal{H}}; a)$ of all 2-vector functions on $[-1, a]$ with inner product

$$\int_{-1}^a \left(\tilde{\mathcal{H}}(x) \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \right) dx,$$

and the minimal symmetric operator in $L^2(\tilde{\mathcal{H}}; a)$ generated by the initial value problem

$$(1.13) \quad -\mathcal{J} \frac{dy(x)}{dx} = z \tilde{\mathcal{H}}(x) y(x), \quad -1 \leq x \leq a, \quad y_1(-1) = 0.$$

If $a > 1$, with the singular (in $x = 1$) Hamiltonian $\tilde{\mathcal{H}}$ a Pontryagin space with negative index one (cf. [4]) and also a minimal symmetric operator can be associated which are again isomorphic to the space $\Pi(f_a)$ and the symmetric operator S_a

from Section 3. The construction of Pontryagin spaces ‘ $L^2(\widetilde{\mathcal{H}}; a)$ ’ and operators associated with a Hamiltonian with singularities will be considered elsewhere.

If the problem (1.12) with \mathcal{H} from (1.6) is considered as a singular problem on the interval $[0, 1)$, the corresponding Titchmarsh–Weyl coefficient in the class N_0 is

$$\frac{\tan z}{z^2} - \frac{1}{z} = i \int_0^\infty e^{izt} \overline{\widetilde{f}_1(t)} dt,$$

where \widetilde{f}_1 is the periodic extension (1.14) of f_1 . If $a > 1$, then all the extensions \widetilde{f}_a of f_a are extensions in $\mathfrak{P}_{1;\infty}$ of the function f_1 . In other examples it happens that for a singular point x_0 of the Hamiltonian there exists a number $\kappa_0 > 1$ such that the positive definite function f_{x_0} has no continuous continuations in the classes $\mathfrak{P}_{\kappa;\infty}$ for $1 \leq \kappa < \kappa_0$ but infinitely many continuations in $\mathfrak{P}_{\kappa_0;\infty}$ (see [15], where this number κ_0 was determined by means of the Titchmarsh–Weyl coefficient for the interval $[0, x_0)$, cf. also [6]).

1.4. We outline the contents of the paper. For $a \neq 1$ in Section 2 we introduce the Hilbert or Pontryagin space $\Pi(f_a)$ which is the reproducing kernel space generated by the Hermitian kernel $f_a(t-s)$, $s, t \in (-a, a)$. A corresponding symmetric operator S_a with defect index $(1, 1)$, which is, roughly, the differentiation operator, is introduced in Section 3, and the self-adjoint extensions of S_a are described in Section 4. An application of M. G. Kreĭn’s formula for the generalized resolvents yields in Section 5 an explicit formula of the form (1.10) for the continuations $\widetilde{f}_a \in \mathfrak{P}_{0;\infty}$ or $\in \mathfrak{P}_{1;\infty}$, depending on $a < 1$ or $a > 1$, of f_a . In Section 6 the continuation problem for the function g is solved using the above mentioned connection between f and g .

It is shown in Section 7 that the resolvent matrices $\mathcal{W}(x; z)$ and $\widehat{\mathcal{W}}(x; z)$ satisfy the same canonical differential equation (1.7), but different initial conditions. Finally, corresponding scalar differential equations of second order are derived which have singular coefficients. In particular, here appear close connections with the study of the Bessel differential operator in an indefinite setting in [3].

1.5. We conclude this introduction with some statements about the function

$$f(t) = 1 - |t|, \quad t \in \mathbb{R},$$

and its restrictions $f_a = f|_{(-2a, 2a)}$. They will also follow from the considerations in Sections 2–5, but the proofs indicated here are more elementary.

- (i) $f_a \in \mathfrak{P}_{0;a}$ if $0 < a \leq 1$ and $f_a \in \mathfrak{P}_{1;a}$ if $a > 1$.
- (ii) If $a < 1$, then f_a has infinitely many continuations in $\mathfrak{P}_{0;\infty}$.²
- (iii) If $a > 1$, then f_a has infinitely many continuations in $\mathfrak{P}_{1;\infty}$.
- (iv) If $a = 1$, then f_a has exactly one continuation in $\mathfrak{P}_{0;\infty}$.

The statement (i) is a consequence of (ii)–(iv).

If $a \leq 1$, then the periodic extension \widetilde{f}_a of f_a is positive definite, which can be seen from its Fourier series

$$(1.14) \quad \widetilde{f}_a(t) = 1 - a + \sum_{k=-\infty}^{\infty} \frac{a}{(k + \frac{1}{2})^2 \pi^2} e^{it(k + \frac{1}{2})\frac{\pi}{a}}.$$

²M. G. Kreĭn notes already in [8] that f_a has infinitely many continuations in $\mathfrak{P}_{0;\infty}$ if and only if $a < 1$.

If $a < 1$, then for each $b \in (a, 1)$, the periodic extension \tilde{f}_b in (1.14) is an extension of f_a .

If $a = 1$ and \tilde{f}_1 is a positive definite extension of f_1 , then $\tilde{f}_1(2) = -1$ and therefore \tilde{f}_1 is periodic with period 4 in view of [16, Corollary 1.4.2].

If $a > 1$, then f_a is not positive definite since its modulus is not majorized by $f(0)$. On the other hand, for each $b \geq a$ the function $h(t) = \max(-|t| + 2b, 0)$, $t \in \mathbb{R}$, is positive definite (since it is non-negative, non-decreasing, and convex on $(0, \infty)$, Theorem of Pólya, see [16, Exercise 1.10.12]) and hence $\tilde{f}_a(t) = h(t) + 1 - 2b$, which is a continuation of f_a , has (at most) one negative square. This also implies that $f_a \in \mathfrak{P}_{1;a}$.

2. THE MODEL SPACE $\Pi^m(f_a)$

Recall that, for $a > 0$,

$$f_a(t) = 1 - |t|, \quad -2a < t < 2a.$$

In the space $C_a := C([-a, a])$ of continuous functions on $[-a, a]$ the following inner product is defined by

$$(2.1) \quad \begin{aligned} [u, v] &:= \int_{-a}^a \int_{-a}^a f_a(t-s) u(t) \overline{v(s)} dt ds \\ &= \int_{-a}^a \int_{-a}^a (1 - |t-s|) u(t) \overline{v(s)} dt ds, \quad u, v \in C_a. \end{aligned}$$

Integration by parts yields the relation

$$(2.2) \quad [u, v] = 2 \int_{-a}^a \hat{u}(s) \overline{\hat{v}(s)} ds + \hat{u}(a) \overline{\hat{v}(a)} - \int_{-a}^a \hat{u}(s) ds \cdot \overline{\hat{v}(a)} - \hat{u}(a) \int_{-a}^a \overline{\hat{v}(s)} ds,$$

where

$$\hat{u}(s) := \int_{-a}^s u(t) dt.$$

The inner product in (2.1) has at most one negative square. In order to see this we consider the Hilbert space $L^2([-a, a]) \oplus \mathbb{C}$, and equip it with the inner product

$$(2.3) \quad \begin{aligned} \langle \tilde{u}, \tilde{v} \rangle &:= 2(\hat{u}, \hat{v}) + \xi \bar{\eta} - (\hat{u}, \mathbf{1}_a) \bar{\eta} - \xi (\mathbf{1}_a, \hat{v}), \\ \tilde{u} &= \begin{pmatrix} \hat{u} \\ \xi \end{pmatrix}, \quad \tilde{v} = \begin{pmatrix} \hat{v} \\ \eta \end{pmatrix} \in L^2([-a, a]) \oplus \mathbb{C}, \end{aligned}$$

where (\cdot, \cdot) denotes the inner product of $L^2([-a, a])$ and $\mathbf{1}_a$ is the function identically equal to 1 on $[-a, a]$. This inner product in $L^2([-a, a]) \oplus \mathbb{C}$ is generated by the Gram operator

$$\mathcal{G}_a := \begin{pmatrix} 2I & -\mathbf{1}_a \\ -(\cdot, \mathbf{1}_a) & 1 \end{pmatrix}.$$

It is easy to see that the operator \mathcal{G}_a is uniformly positive if $a < 1$, that it is non-negative with the isolated and simple eigenvalue zero if $a = 1$, and that its non-positive spectrum consists of one simple negative eigenvalue if $a > 1$. It follows that the space $(L^2([-a, a]) \oplus \mathbb{C}, \langle \cdot, \cdot \rangle)$ is a Hilbert space if $a < 1$,

a Pontryagin space with negative index 1 if $a > 1$, and that the factor space $(L^2([-a, a]) \oplus \mathbb{C} / \ker \mathcal{G}_a, \langle \cdot, \cdot \rangle)$ is a Hilbert space if $a = 1$; in all three cases we denote this space by $\Pi^m(f_a)$, where m stands for model.

Now we return to the space C_a , equipped with the inner product (2.1), and consider the mapping

$$(2.4) \quad u \mapsto \tilde{u} := \begin{pmatrix} \hat{u} \\ \hat{u}(a) \end{pmatrix} \quad \text{from } (C_a, [\cdot, \cdot]) \text{ into } \Pi^m(f_a).$$

Since the linear functional $\varphi \mapsto \varphi(a)$, defined for all functions $\varphi \in C_a$, is not continuous on $L^2([-a, a])$, the image of the mapping (2.4) is a dense subset of $\Pi^m(f_a)$, and because of the relations (2.2) and (2.3) the mapping (2.4) is an isometry. Therefore, if $a < 1$, the space $(C_a, [\cdot, \cdot])$ can be completed to a Hilbert space, if $a > 1$, it can be completed to a Pontryagin space with negative index 1, and for $a = 1$ the corresponding factor space can be completed to a Hilbert space. In all three cases, we denote this space by $\Pi(f_a)$. It is isomorphic to the space $\Pi^m(f_a)$, the isomorphism given by (2.4).

For $t \in (-a, a)$, consider a sequence $(u_{t,n})_1^\infty \subset C_a$ which converges in the sense of distributions to the Dirac distribution δ_t concentrated at t . Because of the continuity of f the sequence $(u_{t,n})_1^\infty$ is a Cauchy sequence in $\Pi(f_a)$, hence we find that $\delta_t \in \Pi(f_a)$:

$$\lim_{n \rightarrow \infty} \tilde{u}_{t,n} = \tilde{\delta}_t = \begin{pmatrix} \chi_{[t,a]} \\ 1 \end{pmatrix} \in \Pi^m(f_a),$$

where $\chi_{[t,a]}$ is the characteristic function of the interval $[t, a]$, and

$$f(t-s) = \langle \tilde{\delta}_t, \tilde{\delta}_s \rangle, \quad s, t \in (-a, a).$$

3. THE SYMMETRIC OPERATORS S_a AND $S_{a,m}$

In the space $\Pi(f_a)$ we define a symmetric operator S_a as follows:

$$\begin{aligned} \text{dom } S_a &:= \{u \in C_a : u \text{ absolutely continuous, } u' \in C_a, u(-a) = u(a) = 0\}, \\ S_a u &:= -i u'. \end{aligned}$$

Under the isomorphism between $\Pi(f_a)$ and $\Pi^m(f_a)$, given by (2.4), the closure of the operator S_a in $\Pi(f_a)$ becomes the operator $S_{a,m}$ in $\Pi^m(f_a)$ defined as follows:

$$\begin{aligned} \text{dom } S_{a,m} &:= \left\{ \tilde{u} = \begin{pmatrix} u \\ u(a) \end{pmatrix} : u \text{ abs. continuous, } u' \in L^2([-a, a]), u(-a) = 0 \right\}, \\ S_{a,m} \tilde{u} &:= \begin{pmatrix} -i u' \\ 0 \end{pmatrix}. \end{aligned}$$

The simple proof of this fact is left to the reader.

Next we determine the adjoint $S_{a,m}^*$ of $S_{a,m}$. To this end we consider the relation

$$\langle S_{a,m} \tilde{u}, \tilde{v} \rangle = \langle \tilde{u}, \tilde{w} \rangle, \quad \tilde{u} \in \text{dom } S_{a,m},$$

where $\tilde{u} = \begin{pmatrix} u \\ \xi \end{pmatrix}$, $\tilde{v} = \begin{pmatrix} v \\ \eta \end{pmatrix}$ and $\tilde{w} = \begin{pmatrix} w \\ \zeta \end{pmatrix}$. Using (2.3) and the definition of $S_{a,m}$ this relation becomes

$$-2i(u', v) + i(u', \mathbf{1}_a) \bar{\eta} = 2(u, w) + u(a) \bar{\zeta} - (u, \mathbf{1}_a) \bar{\zeta} - u(a)(\mathbf{1}_a, w).$$

Further, $i(u', \mathbf{1}_a) = i u(a)$ and $2(u, w) - (u, \mathbf{1}_a) \bar{\zeta} = (u, 2w - \zeta)$, and we obtain

$$(3.1) \quad -2i(u', v) = (u, 2w - \zeta) + u(a) \left(-i\bar{\eta} + \bar{\zeta} - (\mathbf{1}_a, w) \right).$$

Since

$$\{u \in L^2(0, 1) : u \text{ abs. continuous, } u' \in L^2, u(-a) = u(a) = 0\} \subset \text{dom } S_{a,m},$$

for all such u the relation $-i(u', v) = (u, w - \frac{1}{2}\zeta)$ holds. Consequently, v is absolutely continuous and $-i v' = w - \frac{1}{2}\zeta$. The relation

$$-i(u', v) = -i u(a) \overline{v(a)} + (u, -i v') = -i u(a) \overline{v(a)} + \left(u, w - \frac{1}{2}\zeta \right)$$

and (3.1) imply

$$-i u(a) \overline{v(a)} = \frac{1}{2} u(a) (-i\bar{\eta} + \bar{\zeta} - (\mathbf{1}_a, w)).$$

Using this and the relation

$$(\mathbf{1}_a, w) = \left(\mathbf{1}_a, -i v' + \frac{1}{2}\zeta \right) = i \left(\overline{v(a)} - \overline{v(-a)} \right) + a \bar{\zeta},$$

we obtain $\zeta = \frac{i}{(a-1)}(\eta - v(a) - v(-a))$ if $a \neq 1$, and $\eta = v(a) + v(-a)$ if $a = 1$.

Summing up, if $a \neq 1$, then

$$(3.2) \quad \begin{aligned} \text{dom } S_{a,m}^* &= \left\{ \begin{pmatrix} v \\ \eta \end{pmatrix} \in \Pi^m(f_a) : v \text{ absolutely continuous, } v' \in L^2([-a, a]) \right\}, \\ S_{a,m}^* \begin{pmatrix} v \\ \eta \end{pmatrix} &= \begin{pmatrix} -i v' + \frac{i}{2(a-1)}(\eta - v(a) - v(-a)) \\ \frac{i}{(a-1)}(\eta - v(a) - v(-a)) \end{pmatrix}. \end{aligned}$$

If $a = 1$, it can be shown that $S_{a,m}$ is self-adjoint.

4. THE SELF-ADJOINT EXTENSIONS OF $S_{a,m}$

In this section we describe the self-adjoint extensions of the operator $S_{a,m}$ if $a \neq 1$. To do this we use boundary mappings, see for instance [1]. Recall that for a symmetric operator S with defect index $(1, 1)$, two linear functionals Γ_1, Γ_2 on $\text{dom } S^*$ are called a *pair of boundary mappings* of S if

$$\langle S^* \tilde{u}, \tilde{v} \rangle - \langle \tilde{u}, S^* \tilde{v} \rangle = \Gamma_1 \tilde{u} \cdot \overline{\Gamma_2 \tilde{v}} - \Gamma_2 \tilde{u} \cdot \overline{\Gamma_1 \tilde{v}} \quad \text{for all } \tilde{u}, \tilde{v} \in \text{dom } S^*.$$

For the operator $S_{a,m}$ and the elements $\tilde{u} = \begin{pmatrix} u \\ \xi \end{pmatrix}$, $\tilde{v} = \begin{pmatrix} v \\ \eta \end{pmatrix}$ we obtain

$$\begin{aligned}
& \langle S_{a,m}^* \tilde{u}, \tilde{v} \rangle - \langle \tilde{u}, S_{a,m}^* \tilde{v} \rangle \\
&= -i \left\langle \begin{pmatrix} u' - \frac{\xi - u(a) - u(-a)}{2(a-1)} \\ -\frac{\xi - u(a) - u(-a)}{(a-1)} \end{pmatrix}, \begin{pmatrix} v \\ \eta \end{pmatrix} \right\rangle - i \left\langle \begin{pmatrix} u \\ \xi \end{pmatrix}, \begin{pmatrix} v' - \frac{\eta - v(a) - v(-a)}{2(a-1)} \\ -\frac{\eta - v(a) - v(-a)}{(a-1)} \end{pmatrix} \right\rangle \\
&= -i \left[2(u', v) - (u', \mathbf{1}_a)\bar{\eta} + (\xi - u(a) - u(-a))\bar{\eta} \right. \\
&\quad \left. + 2(u, v') - \xi(\mathbf{1}_a, v') + \xi \overline{(\eta - v(a) - v(-a))} \right] \\
&= -i \left[2 \left(u(a)\overline{v(a)} - u(-a)\overline{v(-a)} \right) - (u(a) - u(-a))\bar{\eta} - \xi \overline{(v(a) - v(-a))} \right. \\
&\quad \left. - (u(a) + u(-a) - \xi)\bar{\eta} - \xi \overline{(v(a) + v(-a) - \eta)} \right] \\
&= -i(u(a) - u(-a) - \xi) \overline{(v(a) + v(-a) - \eta)} \\
&\quad - i(u(a) + u(-a) - \xi) \overline{(v(a) - v(-a) - \eta)}.
\end{aligned}$$

Therefore a possible choice for Γ_1, Γ_2 is

$$\Gamma_1 \tilde{u} = (a-1) \begin{pmatrix} u(a) - u(-a) - \xi \\ \xi \end{pmatrix}, \quad \Gamma_2 \tilde{u} = \frac{i}{(a-1)} \begin{pmatrix} u(a) + u(-a) - \xi \\ \xi \end{pmatrix}.$$

The factor $(a-1)$ is chosen in order to get a resolvent matrix $\widehat{\mathcal{W}}$ with $\widehat{\mathcal{W}}(a; 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, see (5.8). According to [1, Proposition 2.1] the canonical self-adjoint extensions of $S_{a,m}$ are the restrictions of $S_{a,m}^*$ to $\ker(\Gamma_1 + \tau\Gamma_2)$, where $\tau \in \mathbb{R} \cup \{\infty\}$, i.e., restrictions to those elements $\tilde{u} = \begin{pmatrix} u \\ \xi \end{pmatrix} \in \text{dom } S_{a,m}^*$ for which

$$(4.1) \quad (a-1)^2 (u(a) - u(-a) - \xi) + \tau i (u(a) + u(-a) - \xi) = 0.$$

For a given $\tau \in \mathbb{R} \cup \{\infty\}$, we denote this self-adjoint extension by $A_{a,m}^{(\tau)}$.

The relation (4.1) can be written as

$$\xi = u(a) + \theta u(-a), \quad \text{where} \quad \theta = \frac{\tau + i(a-1)^2}{\tau - i(a-1)^2}.$$

If τ runs through $\mathbb{R} \cup \{\infty\}$, then θ runs through the unit circle \mathbb{T} , and we denote the self-adjoint extension $A_{a,m}^{(\tau)}$ also by $A_{a,m}^{(\theta)}$. Thus we have proved the following theorem.

Theorem 4.1. *If $a \neq 1$, then the self-adjoint extensions of $S_{a,m}$ in $\Pi^m(f_a)$ are the operators $A_{a,m}^{(\theta)}$, $\theta \in \mathbb{T}$, given by the relations*

$$\begin{aligned}
\text{dom } A_{a,m}^{(\theta)} &= \left\{ \begin{pmatrix} u \\ \xi \end{pmatrix} : u \text{ is abs. continuous, } u' \in L^2([-a, a]), \xi = u(a) + \theta u(-a) \right\}, \\
A_{a,m}^{(\theta)} \begin{pmatrix} u \\ \xi \end{pmatrix} &= \begin{pmatrix} -i u' + \frac{i(\theta-1)}{2(a-1)} u(-a) \\ \frac{i(\theta-1)}{(a-1)} u(-a) \end{pmatrix}.
\end{aligned}$$

In the next section we describe the $\tilde{\delta}_0$ -resolvents of $S_{a,m}$ by M. G. Kreĭn's resolvent formula. To this end we calculate the Q -function corresponding to the fixed self-adjoint extension $A_{a,m}^{(1)} \equiv A_{a,m}^{(\infty)}$:

$$A_{a,m}^{(1)} \begin{pmatrix} u \\ u(a) + u(-a) \end{pmatrix} = \begin{pmatrix} -i u' \\ 0 \end{pmatrix}.$$

As defect elements $\varphi(z)$, which are solutions of the equation $(S_{a,m}^* - z)\varphi(z) = 0$, we can choose

$$(4.2) \quad \varphi(z) = \begin{pmatrix} -\frac{(a-1)ie^{itz}}{2 \cos az} - \frac{1}{2z} \\ \frac{1}{-z} \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

which fulfill $\Gamma_2 \varphi(z) \equiv 1$. According to [1, Proposition 2.2] we have

$$\varphi(z) = (A_{a,m}^{(1)} - z_0)(A_{a,m}^{(1)} - z)^{-1} \varphi(z_0).$$

Due to [1, Definition 2.2 and Proposition 2.3] a corresponding Q -function is given by

$$(4.3) \quad Q_a(z) = \Gamma_1 \varphi(z) = (a-1)^2 \tan az + \frac{a-1}{z}.$$

In order to describe the generalized resolvents we use the parameter τ . The following theorem, which is an immediate consequence of M. G. Kreĭn's resolvent formula, follows directly from [1, Theorem 2.2], see also [13, Behauptung 1.1]. By N_κ we denote the set of all complex functions Q that are meromorphic in the upper half-plane and such that the kernel $N_Q(z, \zeta) := \frac{Q(z) - \overline{Q(\zeta)}}{z - \zeta}$ for z, ζ in the upper half-plane and in the domain of holomorphy of Q has κ negative squares; and we set as in the introduction $\tilde{N}_0 := N_0 \cup \{\infty\}$.

Theorem 4.2. *Let $a \neq 1$. There is a bijective correspondence between all minimal self-adjoint extensions $A_{a,m}^{(\tau)}$ of $S_{a,m}$ and all functions $\tau \in \tilde{N}_0$, given by*

$$(4.4) \quad P(A_{a,m}^{(\tau)} - z)^{-1} \Big|_{\Pi^m(f_a)} = (A_{a,m}^{(\infty)} - z)^{-1} - \frac{\langle \cdot, \varphi(\bar{z}) \rangle}{\tau(z) + Q_a(z)} \varphi(z),$$

where P is the orthogonal projection onto $\Pi^m(f_a)$ in the extending space, and $\varphi(z)$, $Q_a(z)$ are defined by (4.2) and (4.3), respectively. The Štraus extension of $S_{a,m}$ corresponding to the self-adjoint extension $A_{a,m}^{(\tau)}$ is described by the eigenvalue problem

$$(S_{a,m}^* - z) \begin{pmatrix} u \\ \xi \end{pmatrix} = 0, \quad \xi = u(a) + \frac{\tau(z) + i(a-1)^2}{\tau(z) - i(a-1)^2} u(-a).$$

The expressions on the left-hand side of (4.4) are the *generalized resolvents* of $S_{a,m}$; for the definition of the Štraus extension we refer to [2].

For later use we note that the spectrum of the operator $A_{a,m}^{(\infty)}$ consists of the simple eigenvalues 0 and $(k + \frac{1}{2})\frac{\pi}{a}$, $k \in \mathbb{Z}$, and the resolvent of $A_{a,m}^{(\infty)}$ is given by

$$(4.5) \quad (A_{a,m}^{(\infty)} - z)^{-1} \begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} e^{izt} \left(c + i \int_{-a}^t u(s) e^{-isz} ds \right) \\ -\frac{\xi}{z} \end{pmatrix},$$

where

$$c = \frac{-1}{2 \cos az} \left(\frac{\xi}{z} + i e^{iaz} \int_{-a}^a u(s) e^{-isz} ds \right).$$

5. DESCRIPTION OF THE CONTINUATIONS OF f_a

In order to find a description of the set of all $\tilde{\delta}_0$ -resolvents $\langle (A_{a,m}^{(\tau)} - z)^{-1} \tilde{\delta}_0, \tilde{\delta}_0 \rangle$, $\tau \in \tilde{N}_0$, of $S_{a,m}$, we use the relation (4.4). Recall that $\tilde{\delta}_0 = \begin{pmatrix} \chi_{[0,a]} \\ 1 \end{pmatrix}$, and it remains to calculate $\langle (A_{a,m}^{(\infty)} - z)^{-1} \tilde{\delta}_0, \tilde{\delta}_0 \rangle$ and $\langle \varphi(z), \tilde{\delta}_0 \rangle$.

For the latter we get

$$\langle \varphi(z), \tilde{\delta}_0 \rangle = \left(-\frac{(a-1)i e^{itz}}{2 \cos az} - \frac{1}{2z}, 2\chi_{[0,a]} - \mathbf{1}_a \right) - \frac{1}{z} (-\langle \mathbf{1}_a, \chi_{[0,a]} \rangle + 1) = \frac{a-1}{z \cos az},$$

and with (4.5) we find

$$(A_{a,m}^{(\infty)} - z)^{-1} \tilde{\delta}_0 = \begin{pmatrix} -\frac{e^{iz(t+a)}}{2z \cos az} + \chi_{[0,a]}(t) \frac{e^{izt} - 1}{z} \\ -\frac{1}{z} \end{pmatrix},$$

hence

$$r(z) := \langle (A_{a,m}^{(\infty)} - z)^{-1} \tilde{\delta}_0, \tilde{\delta}_0 \rangle = \frac{\tan az}{z^2} - \frac{1}{z}.$$

By continuity, this relation holds also if $a = 1$. Since $\text{res}(r; 0) = a - 1$, it follows easily that $r \in N_0$ if $a \leq 1$ and $r \in N_1$ if $a > 1$. Putting it all together, it follows from (4.4) that

$$(5.1) \quad \langle (A_{a,m}^{(\tau)} - z)^{-1} \tilde{\delta}_0, \tilde{\delta}_0 \rangle = \frac{\tan az}{z^2} - \frac{1}{z} - \frac{(a-1)^2}{z^2 \cos^2 az (\tau(z) + (a-1)^2 \tan az + \frac{a-1}{z})},$$

where $\tau \in \tilde{N}_0$ and $z \in \rho(A_{a,m}^{(\tau)})$.

Now we observe (see, e.g., [14, Theorem 3.1]) that with $\gamma = 0$ for $a < 1$ and some $\gamma > 0$ for $a > 1$ a bijective correspondence between all extensions $\tilde{f}_{a,\tau}$ of f_a , in the class $\mathfrak{P}_{0;\infty}$ if $a < 1$ and in the class $\mathfrak{P}_{1;\infty}$ if $a > 1$, and the minimal self-adjoint extensions $A_{a,m}^{(\tau)}$ of $S_{a,m}$ is given by the formula

$$i \int_0^\infty e^{izt} \overline{\tilde{f}_{a,\tau}(t)} dt = \langle (A_{a,m}^{(\tau)} - z)^{-1} \tilde{\delta}_0, \tilde{\delta}_0 \rangle, \quad \text{Im } z > \gamma.$$

Therefore the following theorem is proved.

Theorem 5.1. *If $a < 1$, then the relation*

$$(5.2) \quad \begin{aligned} & i \int_0^\infty e^{izt} \overline{\tilde{f}_{a,\tau}(t)} dt \\ &= \frac{\tan az}{z^2} - \frac{1}{z} - \frac{(a-1)^2}{z^2 \cos^2 az (\tau(z) + (a-1)^2 \tan az + \frac{a-1}{z})}, \quad \text{Im } z > \gamma, \end{aligned}$$

with $\gamma = 0$ establishes a bijective correspondence between all extensions $\tilde{f}_{a,\tau} \in \mathfrak{P}_{0;\infty}$ of f_a and all $\tau \in \tilde{N}_0$; if $a > 1$, then the relation (5.2) with some $\gamma > 0$ establishes a bijective correspondence between all extensions $\tilde{f}_{a,\tau} \in \mathfrak{P}_{1;\infty}$ of f_a and all $\tau \in \tilde{N}_0$.

The expression in (5.2) as a function of z , which we denote by $p_{a,\tau}(z)$, can be meromorphically extended to the upper half-plane and by symmetry (i.e. $p_{a,\tau}(\bar{z}) = \overline{p_{a,\tau}(z)}$) also to the lower half-plane. This function $p_{a,\tau}$ belongs to $\tilde{N}_0 \cup N_1$.

If $p_{a,\tau} \in N_0$ (i.e., if $a < 1$ and $\tau \in \tilde{N}_0$), then

$$(5.3) \quad p_{a,\tau}(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\mu_{a,\tau}(x)$$

with a finite measure $\mu_{a,\tau}$, and the function $\tilde{f}_{a,\tau}$ has the representation

$$(5.4) \quad \tilde{f}_{a,\tau}(t) = \int_{-\infty}^{\infty} e^{ixt} d\mu_{a,\tau}(x).$$

We consider some special extensions.

(a) If $\tau(z) \equiv \infty$, then

$$(5.5) \quad i \int_0^{\infty} e^{izt} \overline{\tilde{f}_{a,\infty}(t)} dt = \frac{\tan(az)}{z^2} - \frac{1}{z} = \int_{-\infty}^{\infty} \frac{1}{t-z} d\mu_{a,\infty}(t),$$

where $\mu_{a,\infty}$ is the measure concentrated on $\sigma(A_{a,m}^{(\infty)}) = \{0\} \cup \{(k + \frac{1}{2})\frac{\pi}{a} : k \in \mathbb{Z}\}$, given by

$$\mu_{a,\infty}\left(\left\{\left(k + \frac{1}{2}\right)\frac{\pi}{a}\right\}\right) = \frac{a}{(k + \frac{1}{2})^2\pi^2}, \quad \mu_{a,\infty}(\{0\}) = 1 - a.$$

The corresponding extension $\tilde{f}_{a,\infty}$ is the $4a$ -periodic extension of f_a ,

$$\tilde{f}_{a,\infty}(t) = 1 - a + \sum_{k=-\infty}^{\infty} \frac{a}{(k + \frac{1}{2})^2\pi^2} e^{it(k + \frac{1}{2})\frac{\pi}{a}}.$$

(b) For $\tau(z) = (a-1)^2 i - \frac{a-1}{z}$, $\text{Im } z > 0$, the corresponding extension is $\tilde{f}_{a,\tau}(t) = 1 - |t|$, $t \in \mathbb{R}$, which follows from (5.2). Observe that $\tilde{f}_{a,\tau} \in \mathfrak{P}_{1;\infty}$ and if $a > 1$, then $f_a \in \mathfrak{P}_{1;a}$ and $\tau \in N_0$, if $a < 1$, then $f_a \in \mathfrak{P}_{0;a}$ and $\tau \in N_1$, which is in accordance with [13, Behauptung 1.1]. The function $p_{a,\tau}$ in this case is equal to

$$(5.6) \quad p_{a,\tau}(z) = \frac{i}{z^2} - \frac{1}{z} \quad \text{for } \text{Im } z > 0.$$

We can give an interpretation of formulae (5.3) and (5.4) in this case using the distribution

$$\int_{-\infty}^{\infty} \varphi(x) d\mu_{a,\tau}(x) := \varphi(0) + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(0)}{x^2} dx, \\ \varphi \in C^\infty(\mathbb{R}), \quad \varphi \text{ bounded,}$$

where p.v. denotes the principal value at 0. This follows since

$$\frac{i}{z^2} - \frac{1}{z} = \frac{1}{-z} + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \left(\frac{1}{x-z} + \frac{1}{z}\right) \frac{1}{x^2} dx, \quad \text{Im } z > 0, \\ 1 - |t| = 1 + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{itx} - 1}{x^2} dx, \quad t \in \mathbb{R},$$

or equivalently,

$$1 - |t| = 1 + \frac{2}{\pi} \int_0^\infty \frac{\cos(tx) - 1}{x^2} dx, \quad t \in \mathbb{R}.$$

(c) If $\tau(z) = (a-1)^2 i$, $\text{Im } z > 0$, then for the corresponding measure $\mu_{a,\tau}$ we obtain

$$\begin{aligned} d\mu_{a,\tau}(x) &= \frac{1}{\pi} \lim_{y \downarrow 0} \text{Im } p_{a,\tau}(x + iy) dx \\ &= \frac{(a-1)^2}{\pi((a-1)^2 x^2 + (a-1)x \sin(2ax) + \cos^2(ax))} dx. \end{aligned}$$

This implies the relation³

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \frac{(a-1)^2 \cos(tx)}{(a-1)^2 x^2 + (a-1)x \sin(2ax) + \cos^2(ax)} dx &= 1 - |t|, \\ 0 < a < 1, \quad -2a < t < 2a. \end{aligned}$$

Setting $\tau(z) = (a-1)^2 b i$, $b > 0$, we obtain the more complicated formula

$$\frac{2}{\pi} \int_0^\infty \frac{(a-1)^2 b \cos(tx)}{(a-1)^2 x^2 + (a-1)x \sin(2ax) + [1 + x^2(b^2 - 1)(a-1)^2] \cos^2(ax)} dx = 1 - |t|,$$

where again $0 < a < 1$, $-2a < t < 2a$.

The right-hand side of (5.2) can be written as a fractional linear transformation of $\tau(z)$:

$$(5.7) \quad p_{a,\tau}(z) = \frac{\widehat{w}_{11}(a; z)\tau(z) + \widehat{w}_{12}(a; z)}{\widehat{w}_{21}(a; z)\tau(z) + \widehat{w}_{22}(a; z)}, \quad \tau \in \widetilde{N}_0,$$

with the matrix

$$(5.8) \quad \begin{aligned} \widehat{\mathcal{W}}(a; z) &= (\widehat{w}_{jk}(a; z))_{j,k=1}^2 \\ &= \begin{pmatrix} \frac{\sin az - z \cos az}{(a-1)z} & \left(\frac{1}{z^2} - (a-1) \right) \sin az - \frac{a \cos az}{z} \\ \frac{z \cos az}{a-1} & (a-1)z \sin az + \cos az \end{pmatrix}. \end{aligned}$$

It is entire in z and has the properties $\widehat{\mathcal{W}}(a; 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\widehat{\mathcal{W}}(0; z) = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$.

6. DESCRIPTION OF THE CONTINUATIONS OF g_a

We consider now the function

$$g(t) = -|t| + \frac{t^2}{2}, \quad t \in \mathbb{R},$$

³See [17] for a different proof.

which belongs to the class $\mathfrak{G}_{1;\infty}$. As mentioned in Subsection 1.2 it is connected with f from (1.8) by the equation

$$(6.1) \quad -\int_0^t \frac{(t-s)^2}{2} f(s) ds + \int_0^t f(s)g(t-s) ds = \int_0^t g(s) ds, \quad t \in \mathbb{R}.$$

The functions

$$(6.2) \quad q(z) := i z^2 \int_0^\infty e^{itz} \overline{g(t)} dt = i + \frac{1}{z}, \quad \text{Im } z > 0,$$

$$(6.3) \quad p(z) := i \int_0^\infty e^{itz} \overline{f(t)} dt = \frac{i}{z^2} - \frac{1}{z}, \quad \text{Im } z > 0,$$

are related by

$$(6.4) \quad q(z) = \frac{p(z)}{zp(z) + 1},$$

cf. [12, (5.7)]. Note that p is equal to $p_{a,\tau}$ in (5.6). Because of relation (6.4) the resolvent matrix $\mathcal{W}(x; z)$ of g can be calculated from $\widehat{\mathcal{W}}(x; z)$ in (5.8):

$$(6.5) \quad \begin{aligned} \mathcal{W}(x; z) &= \begin{pmatrix} w_{11}(x; z) & w_{12}(x; z) \\ w_{21}(x; z) & w_{22}(x; z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \widehat{\mathcal{W}}(x; z) \\ &= \begin{pmatrix} \frac{\sin zx - z \cos zx}{z(x-1)} & \left(\frac{1}{z^2} - (x-1)\right) \sin zx - \frac{x \cos zx}{z} \\ \frac{\sin zx}{x-1} & \frac{\sin zx}{z} - (x-1) \cos zx \end{pmatrix}. \end{aligned}$$

If

$$q_{x,\tau}(z) = \frac{w_{11}(x; z)\tau(z) + w_{12}(x; z)}{w_{21}(x; z)\tau(z) + w_{22}(x; z)} \quad \text{and} \quad p_{x,\tau}(z) = \frac{\widehat{w}_{11}(x; z)\tau(z) + \widehat{w}_{12}(x; z)}{\widehat{w}_{21}(x; z)\tau(z) + \widehat{w}_{22}(x; z)},$$

then

$$(6.6) \quad q_{x,\tau}(z) = \frac{p_{x,\tau}(z)}{zp_{x,\tau}(z) + 1}.$$

With the resolvent matrix \mathcal{W} , according to [14, Theorem 5.2] the solutions $\widetilde{g}_{a,\tau}(t)$ of the continuation problem for g_a are described as follows.

Theorem 6.1. *If $a < 1$, then the relation*

$$(6.7) \quad i z^2 \int_0^\infty e^{izt} \overline{\widetilde{g}_{a,\tau}(t)} dt = \frac{w_{11}(a; z)\tau(z) + w_{12}(a; z)}{w_{21}(a; z)\tau(z) + w_{22}(a; z)}, \quad \text{Im } z > \gamma,$$

with $\gamma = 0$ establishes a bijective correspondence between all extensions $\widetilde{g}_{a,\tau} \in \mathfrak{G}_{0;\infty}$ of g_a and all $\tau \in \widetilde{N}_0$; if $a > 1$, then the relation (6.7) with some $\gamma > 0$ establishes a bijective correspondence between all extensions $\widetilde{g}_{a,\tau} \in \mathfrak{G}_{1;\infty}$ of f_a and all $\tau \in \widetilde{N}_0$.

The function $q(z)$ can be written as

$$q(z) = i + \frac{1}{z} = \int_{-\infty}^\infty \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x)$$

with the signed measure $\mu = \frac{1}{\pi}\lambda - \delta_0$, where λ is the Lebesgue measure on \mathbb{R} and δ_0 is the Dirac measure concentrated in 0. With this measure μ we can write the kernel G_g as

$$\begin{aligned} G_g(t, s) &= \int_{-\infty}^{\infty} \frac{(e^{itx} - 1)(e^{-isx} - 1)}{x^2} d\mu(x) \\ &= -ts + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(e^{itx} - 1)(e^{-isx} - 1)}{x^2} dx, \quad s, t \in \mathbb{R}. \end{aligned}$$

7. THE DIFFERENTIAL EQUATIONS

7.1. Now we consider $\mathcal{W}(x; z)$ and $\widehat{\mathcal{W}}(x; z)$ in (6.5) and (5.8), respectively, as functions of x on $[0, \infty) \setminus \{1\}$. We set $\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$(7.1) \quad \mathcal{H}(x) := \begin{pmatrix} (x-1)^2 & 0 \\ 0 & \frac{1}{(x-1)^2} \end{pmatrix}.$$

By a straightforward calculation the following theorem can be proved.

Theorem 7.1. *The matrix functions $\mathcal{W}(\cdot; z)$ and $\widehat{\mathcal{W}}(\cdot; z)$ satisfy the same differential equation*

$$(7.2) \quad \frac{d\mathcal{V}(x; z)}{dx} \mathcal{J} = z \mathcal{V}(x; z) \mathcal{H}(x), \quad x \in [0, \infty) \setminus \{1\},$$

and the initial conditions

$$\mathcal{W}(0; z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{\mathcal{W}}(0; z) = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}.$$

In relations (5.2) and (6.7) we can let a tend to infinity, which yields the relations (6.3) and (6.2). This means that limit-point case prevails at infinity for the canonical systems (1.12), (1.13) and that p and q are the corresponding Titchmarsh–Weyl coefficients; for the definition of a Titchmarsh–Weyl coefficient associated with a resolvent matrix with singularities see also [7].

7.2. Recall that with the Hamiltonian (7.1) we have associated in (1.12) the differential system

$$(7.3) \quad -\mathcal{J} \frac{dy(x)}{dx} = z \begin{pmatrix} (x-1)^2 & 0 \\ 0 & \frac{1}{(x-1)^2} \end{pmatrix} y(x), \quad 0 \leq x \leq a, \quad y_1(0) = 0.$$

Since the Hamiltonian $\mathcal{H}(x)$ has diagonal form (which is a consequence of the reality of the functions f and g), the system in (7.3) can be transformed into one second order differential equation, e.g., for the component y_2 we obtain

$$(7.4) \quad -\left(\frac{1}{(x-1)^2} y_2' \right)' = \lambda \frac{1}{(x-1)^2} y_2.$$

This equation leads to a function space of L^2 -type with respect to the singular weight $1/(x-1)^2$, similar to those mentioned in Subsection 1.3, and which will be considered elsewhere.

According to [14, (4.9)] the canonical system (7.3) can be transformed into a canonical system with a potential: for

$$v(x) := \mathcal{H}(x)^{1/2}y(x) = \begin{pmatrix} |x-1| & 0 \\ 0 & \frac{1}{|x-1|} \end{pmatrix} y(x)$$

we get the differential equation

$$-\mathcal{J} \frac{dv(x)}{dx} + \begin{pmatrix} 0 & \frac{1}{x-1} \\ \frac{1}{x-1} & 0 \end{pmatrix} v(x) = zv(x).$$

This first order system can again be transformed into one second order equation. For the second component $v_2(x)$ of $v(x)$ we obtain the following Sturm–Liouville equation with a potential which has a singularity at $x = 1$:

$$-v_2'' + \frac{2}{(x-1)^2}v_2 = \lambda v_2,$$

where again $\lambda = z^2$. The Bessel differential equation, where the same type of singularity of the potential appears at $x = 0$, was studied in an indefinite setting in [3]. We hope to return to this connection in future publications.

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