On the gravity-driven draining of a rivulet of fluid with temperature-dependent viscosity down a uniformly heated or cooled substrate

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Abstract

We use the lubrication approximation to investigate the unsteady gravity-driven draining of a thin rivulet of Newtonian fluid with temperature-dependent viscosity down a substrate that is either uniformly hotter or uniformly colder than the surrounding atmosphere. First we derive the general nonlinear evolution equation for a thin film of fluid with an arbitrary dependence of viscosity on temperature. Then we show that at leading order in the limit of small Biot number the rivulet is isothermal, as expected, but that at leading order in the limit of large Biot number (in which the rivulet is not isothermal) the governing equation can, rather unexpectedly, always be reduced to that in the isothermal case with a suitable rescaling. These results are then used to give a complete description of steady flow of a slender rivulet in the limit of large Biot number in two situations in which the corresponding isothermal problem has previously been solved analytically, namely non-uniform flow down an inclined plane, and locally unidirectional flow down a slowly varying substrate. In particular, we find that if a suitably defined integral measure of the fluidity of the film is a decreasing function of the temperature of the atmosphere (as it is for all three specific viscosity models we consider) then decreasing the temperature of the atmosphere always has the effect of making the rivulet wider and deeper.

1 Introduction

The gravity-driven draining of a rivulet of viscous fluid down an inclined substrate occurs in a number of practical situations ranging from many industrial devices and coating processes to a variety of geophysical flows. In many of these situations heating or cooling effects are significant and so it is of considerable interest to investigate rivulet flows in which non-isothermal effects, such as evaporation or the dependence of one or more of the fluid properties on temperature, play a role. In the present paper we shall investigate one such problem, namely non-isothermal rivulet flow of a fluid with temperature-dependent viscosity.

In recent years there has been considerable work on the gravity-driven draining of an isothermal rivulet down an inclined substrate, much of it building on the pioneering analysis of steady unidirectional flow of Newtonian fluid down an inclined plane undertaken by Towell and Rothfeld [1]. Duffy and Moffatt [2] used the lubrication approximation employed by Allen and Biggin [3] to obtain analytically the leading-order solution in the special case when the cross-sectional profile of the rivulet in the direction transverse to the flow is thin. Duffy and Moffatt [2] calculated the shape of the rivulet (and, in particular, its width and maximum height) as a function of α, the angle of inclination of the substrate to the horizontal, for $0 \leq \alpha \leq \pi$. Duffy and Moffatt [2] also interpreted their results as describing the locally unidirectional flow down a locally planar substrate whose local slope α varies slowly in the flow-wise direction and, in particular, used them to describe the flow in the azimuthal direction round a large horizontal circular cylinder. Duffy and Moffatt's [2] approach has been used by Wilson and Duffy [4] to study the locally unidirectional flow of a rivulet down a slowly varying substrate with variation transverse to the direction of flow, and by Wilson, Duffy and Ross [5] to study the locally unidirectional flow of a rivulet of viscoplastic material down a slowly varying substrate. Taking a somewhat different approach Smith [6] obtained a unique similarity solution of the thin-film equation describing the steady gravity-driven draining of a slender non-uniform rivulet of Newtonian fluid from a point source or to a point sink on an inclined plane in the case when surface-tension effects are weak. Smith's [6] solution predicts that the width of the rivulet increases or decreases like the 3/7th power of the distance measured down the substrate from the source or sink and that the height of the rivulet correspondingly decreases or increases like the $-1/7$th power, and is in good agreement with his own experimental measurements and with the unsteady solutions of the appropriate thin-film equation obtained numerically by Schwartz and Michaelides [7]. Subsequently Duffy and Moffatt [8] performed the corresponding analysis in the case when surface-tension effects are strong. In particular, they found that in this case there is a one-parameter family of solutions representing rivulets whose width increases or decreases like the 3/13th power and whose height correspondingly decreases or increases like the $-1/13$th power of the distance measured down the substrate respectively. All of these similarity solutions predict a varying contact angle at the contact line; Wilson, Duffy and Davis [9] showed that these solutions can be modified to accommodate a fixed-contact-angle condition at the contact line by incorporating sufficiently strong slip at the solid/fluid interface into the model. Recently Wilson, Duffy and Hunt [10] generalised the approach of Smith [6] and Duffy and Moffatt [8] to study rivulet flow of a non-Newtonian power-law fluid driven by either gravity or an imposed constant shear stress at the free surface.

Despite its practical importance there has thus far been surprisingly little work on non-isothermal rivulet flow. In a recent paper Holland, Duffy and Wilson [11] investigated the locally uniform (but not locally unidirectional) flow of a thin rivulet of a fluid with constant viscosity whose surface tension varies linearly with temperature down a slowly varying substrate that is either uniformly hotter or uniformly colder than the surrounding atmosphere. In particular, they found that the variation in surface tension drives a transverse flow that causes the fluid particles to spiral down the rivulet in helical vortices (absent in the corresponding isothermal problem). They also found that a single continuous rivulet can run from the top to the bottom of a large horizontal circular cylinder provided that the cylinder is either warmer or significantly cooler than the surrounding atmosphere, but that if it is only slightly cooler than the surrounding atmosphere then a continuous rivulet is possible only for a sufficiently small volume flux.

In practice the variation of viscosity with temperature can be more significant than the variation of surface tension with temperature considered by Holland, Duffy and Wilson [11]. As far as the present authors are aware, there has thus far been no work on non-isothermal rivulet flow of a fluid with temperature-dependent viscosity. There has, however, been some work on other non-isothermal thin-film flows of such fluids. For example, Goussis and Kelly [12] and Hwang and Weng [13] investigated the linear and non-linear stability of a two-dimensional thin film of fluid with temperature-dependent viscosity draining down a uniformly heated or cooled inclined plane in the limit of large Biot number (in which the free surface is at a uniform temperature). Both works considered the particular case when viscosity depends on temperature according to an exponential model, and both concluded that heating the substrate destabilises the flow while cooling the substrate stabilises it. Non-linear evolution equations for two-dimensional thin films of fluid with temperature-dependent viscosity with surface-tension and van-der-Waals effects have been derived by Reisfeld and Bankoff [14] (who considered a linear model for the dependence of viscosity on temperature) and Wu and Hwang [15] (who considered an exponential model). Both works treated the limit of large Biot number and found that in this limit the resulting evolution equations can (rather surprisingly) be reduced to the corresponding equation for an isothermal film with an appropriate choice of timescale. In particular, both Reisfeld and Bankoff [14] and Wu and Hwang [15] concluded that heating the substrate decreases the time for the film to rupture, while cooling the substrate increases it. Oron, Davis and Bankoff [16, pp. 945–946] derived the corresponding general evolution equation for a two-dimensional thin film of fluid with an arbitrary dependence of viscosity on temperature. In addition there has been considerable work on thin-film flows with a variety of other non-isothermal effects (see, for example, the excellent review by Oron, Davis and Bankoff [16] for further details).

In the present paper we shall use the lubrication approximation to investigate the unsteady gravity-driven draining of a thin rivulet of Newtonian fluid with temperature-dependent viscosity down a substrate that is either uniformly hotter or uniformly colder than the surrounding atmosphere. First we derive the general nonlinear evolution equation for a thin film of fluid with an arbitrary dependence of viscosity on temperature. Then we obtain the leading-order versions of this equation in the limits of small and large Biot number respectively. These results are then used to give a complete description of steady flow of a slender rivulet in the limit of large Biot number in two situations in which the corresponding isothermal problem has previously been solved analytically, namely non-uniform flow down an inclined plane, and locally unidirectional flow down a slowly varying substrate.

2 Problem Formulation

Consider the unsteady gravity-driven draining of a rivulet of Newtonian fluid with temperature-dependent viscosity down a planar substrate inclined at an angle α ($0 \leq \alpha \leq \pi$) to the horizontal, shown in Fig. 1. We choose Cartesian axes $Oxyz$ as indicated in Fig. 1, with the x axis down the line of greatest slope and the z axis normal to the substrate $z = 0$, and denote time by t. The velocity $\mathbf{u} = (u, v, w) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$, pressure $p = p(x, y, z, t)$ and temperature $T = T(x, y, z, t)$ of the fluid are governed by the familiar mass-conservation, Navier–Stokes and energy equations. The semi-width of the rivulet is denoted by $a = a(x, t)$, and the volume flux of fluid down the rivulet by $Q = Q(x, t)$. The fluid is assumed to have constant density ρ, surface tension γ and thermal conductivity k_{th}, but a non-constant viscosity $\mu = \mu(T)$ that depends on temperature. On the solid substrate $z = 0$ the fluid velocity is zero and the uniform temperature is prescribed to be $T = T_0$. On the free surface $z = h(x, y, t)$ the usual normal and tangential stress balances, the energy balance

$$-k_{\mathrm{th}} \nabla T \cdot \mathbf{n} = \alpha_{\mathrm{th}}(T - T_\infty) \tag{1}$$

(where T_∞ denotes the uniform temperature of the surrounding atmosphere, α_{th} the surface heat-transfer coefficient and $\mathbf{n}$ the unit normal to the free surface) and the kinematic condition apply. At the edges of the rivulet $y = \pm a(x, t)$ where $h = 0$ a condition must be specified concerning the contact angle $\beta = \beta(x, y, t)$. For example, β may be assumed to take a prescribed constant value, or to satisfy a "Tanner law" relating it to the velocity of the contact line; however for our present purposes it is not necessary to be specific about this condition at this stage. The thickness of the rivulet at $y = 0$ (not necessarily the maximum thickness) is denoted by $h_{\mathrm{m}} = h_{\mathrm{m}}(x, t) = h(x, 0, t)$.

Most of the results in the present work will be valid for an arbitrary dependence of viscosity on temperature satisfying $\mu = \mu_0$ and $\mathrm{d}\mu/\mathrm{d}T = -\lambda$ at $T = T_0$, where λ is a prescribed positive constant. Some examples will be presented for three specific models, namely the *linear* model

$$\mu(T) = \mu_0 - \lambda(T - T_0) \tag{2}$$

(valid only for $\mu > 0$, i.e. $\mu_0 > \lambda(T - T_0)$), the *exponential* model

$$\mu(T) = \mu_0 \exp\left[-\frac{\lambda(T - T_0)}{\mu_0}\right], \tag{3}$$

and the *Eyring* model

$$\mu(T) = \mu_0 \exp\left[\frac{\lambda T_0^2}{\mu_0}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right]. \tag{4}$$

We consider the gravity-driven flow of a thin rivulet (with, in particular, $\beta \ll 1$) for which the length scale in the z direction is much smaller than the length scales in both the x and y

directions. Hence we scale the system appropriately by writing

$$
\begin{gathered}
x = lx^*, \quad y = \epsilon l y^*, \quad z = \epsilon\delta l z^*, \quad h = \epsilon\delta l h^*, \quad \beta = \delta\beta^*, \\
u = \frac{\epsilon^2\delta^2\rho g l^2}{\mu_0} u^*, \quad v = \frac{\epsilon^3\delta^2\rho g l^2}{\mu_0} v^*, \quad w = \frac{\epsilon^3\delta^3\rho g l^2}{\mu_0} w^*, \quad t = \frac{\mu_0}{\epsilon^2\delta^2\rho g l} t^*, \\
p = p_\infty + \epsilon\delta\rho g l p^*, \quad Q = \frac{\epsilon^4\delta^3\rho g l^4}{\mu_0} Q^*, \quad \mu = \mu_0\mu^*, \quad T = T_\infty + (T_0 - T_\infty)T^*,
\end{gathered} \tag{5}
$$

where $\delta \ll 1$ is the (small) aspect ratio in the transverse direction, ϵ (which may be either small or of unit order, depending on the specific situation being considered) is the aspect ratio in the longitudinal direction, l is the length scale in the x direction, g denotes acceleration due to gravity, and p_∞ is the uniform pressure of the surrounding atmosphere. Note that for simplicity we have chosen to use μ_0, the viscosity of the fluid at the substrate temperature $T = T_0$, as the characteristic viscosity scale. The star subscripts will be dropped immediately for clarity, and hereafter all quantities are non-dimensional unless it is stated otherwise.

Provided that the suitably defined Reynolds number is sufficiently small (specifically, $\epsilon^4\delta^4\rho^2 g l^3/\mu_0^2 \ll 1$) the leading-order versions of the governing mass-conservation, Navier–Stokes and energy equations are

$$u_x + u_y + w_z = 0 \tag{6}$$

$$0 = \sin\alpha + (\mu u_z)_z, \tag{7}$$

$$0 = -\delta p_y + \epsilon(\mu v_z)_z, \tag{8}$$

$$0 = -p_z - \cos\alpha, \tag{9}$$

$$T_{zz} = 0, \tag{10}$$

to be integrated subject to the boundary conditions of no slip and prescribed temperature at the substrate $z = 0$,

$$u = v = w = 0, \quad T = 1, \tag{11}$$

and balances of normal and tangential stress, an energy balance and the kinematic condition at the free surface $z = h$,

$$p = -C^{-1}(\epsilon^2 h_{xx} + h_{yy}), \tag{12}$$

$$u_z = v_z = 0, \tag{13}$$

$$T_z + BT = 0, \tag{14}$$

$$h_t + \bar{u}_x + \bar{v}_y = 0, \tag{15}$$

where

$$C = \frac{\rho g \epsilon^2 l^2}{\gamma}, \quad B = \frac{\epsilon \delta l \alpha_{\mathrm{th}}}{k_{\mathrm{th}}} \tag{16}$$

denote the non-dimensional capillary and Biot numbers respectively, and the local fluxes $\bar{u} = \bar{u}(x, y, t)$ and $\bar{v} = \bar{v}(x, y, t)$ are defined by

$$\bar{u} = \int_0^h u \, \mathrm{d}z, \quad \bar{v} = \int_0^h v \, \mathrm{d}z; \tag{17}$$

at the edges of the rivulet $y = \pm a$ we have

$$h = 0, \tag{18}$$

$$h_y = \mp \beta. \tag{19}$$

Integrating (9) subject to (12) at $z = h$ yields the pressure distribution

$$p = (h - z)\cos\alpha - C^{-1}(\epsilon^2 h_{xx} + h_{yy}), \tag{20}$$

while integrating (10) twice subject to (11) at $z = 0$ and (14) at $z = h$ yields the temperature distribution

$$T = 1 - \frac{Bz}{1 + Bh}. \tag{21}$$

Thus if we define a *thermoviscosity number* V, a non-dimensional measure of the variation of viscosity with temperature, by

$$V = \frac{\lambda(T_0 - T_\infty)}{\mu_0} \tag{22}$$

then the linear model (2) yields

$$\mu = 1 - V(T-1) = 1 + \frac{BVz}{1+Bh} \tag{23}$$

(valid only for $\mu > 0$, i.e. $B(1+V)h_{\mathrm{m}} > -1$), while the exponential model (3) yields

$$\mu = \exp\left(-V(T-1)\right) = \exp\left(\frac{BVz}{1+Bh}\right), \tag{24}$$

and the Eyring model (4) yields

$$\mu = \exp\left(-\frac{V_{\mathrm{m}}V(T-1)}{V_{\mathrm{m}} + V(T-1)}\right) = \exp\left(\frac{BV_{\mathrm{m}}Vz}{V_{\mathrm{m}}(1+Bh) - BVz}\right), \tag{25}$$

where we have defined $V_{\mathrm{m}} = \lambda T_0/\mu_0$. Note that by definition $V_{\mathrm{m}} > 0$ and $V_{\mathrm{m}} > V$. Positive (negative) values of V correspond to $T_0 > T_\infty$ ($T_0 < T_\infty$), i.e. to situations in which the atmosphere is colder (hotter) than the substrate and in which the viscosity at the free surface is accordingly greater than (less than) the viscosity at the substrate. The case $V = 0$ corresponds to the isothermal problem $T_0 = T_\infty$ in which the viscosity is constant, $\mu \equiv 1$.

Integrating (7) and (8) once each subject to (11) at $z = 0$ yields the velocity distributions

$$u = \sin\alpha \int_0^z \frac{h - \tilde{z}}{\mu(T)}\,\mathrm{d}\tilde{z}, \quad \epsilon v = -\delta p_y \int_0^z \frac{h-\tilde{z}}{\mu(T)}\,\mathrm{d}\tilde{z}, \tag{26}$$

and hence from (17) the local fluxes are given by

$$\bar{u} = \frac{\sin\alpha}{3} f h^3, \quad \epsilon\bar{v} = -\frac{\delta p_y}{3} f h^3, \tag{27}$$

where the function $f = f(x, y, t)$, defined by

$$f = \frac{3}{h^3}\int_0^h \int_0^z \frac{h-\tilde{z}}{\mu(T)}\,\mathrm{d}\tilde{z}\,\mathrm{d}z, \tag{28}$$

is an integral measure of the fluidity of the film (hereafter simply referred to as the *fluidity*); therefore the flux of fluid down the rivulet $Q \geq 0$ is given by

$$Q = \int_{-a}^{+a} \bar{u}\,\mathrm{d}y = \frac{\sin\alpha}{3}\int_{-a}^{+a} f h^3\,\mathrm{d}y. \tag{29}$$

In the special case of an isothermal rivulet $V = 0$ we have $f = 1$ and so $\bar{u} = \bar{u}_0$, $\bar{v} = \bar{v}_0$ and $Q = Q_0$, where

$$\bar{u}_0 = \frac{\sin\alpha}{3} h^3, \quad \epsilon\bar{v}_0 = -\frac{\delta p_y}{3} h^3, \quad Q_0 = \frac{\sin\alpha}{3} \int_{-a}^{+a} h^3 \,\mathrm{d}y. \tag{30}$$

Comparing (27) and (30) shows that *for all values of* B the local fluxes $\bar{u}$ and $\bar{v}$ are related to their values in the isothermal case by $\bar{u} = f\bar{u}_0$ and $\bar{v} = f\bar{v}_0$, so that (15) becomes

$$h_t + (f\bar{u}_0)_x + (f\bar{v}_0)_y = 0; \tag{31}$$

however, comparing (29) and (30) shows that, in general, the flux Q is *not* simply equal to fQ_0. Thus the general nonlinear evolution equation for the unsteady flow of a thin rivulet of a Newtonian fluid with an arbitrary dependence of viscosity on temperature with prescribed flux $\bar{Q} = \bar{Q}(t) \geq 0$ at $x = 0$ down an inclined plane that is either uniformly hotter or uniformly colder than the surrounding atmosphere is given by (31) with f given by (28) and with $\bar{u}_0$ and $\bar{v}_0$ given by (30), to be integrated subject to the flux condition $Q = \bar{Q}$ at $x = 0$ where Q is given by (29), equations (18) and (19) at $y = \pm a$ with some additional condition on β, and appropriate initial conditions. In the special case of steady flow the flux Q is independent of both x and t, and hence Q takes the prescribed constant value $\bar{Q} > 0$ thoughout the flow.

The Limit $B \to 0$

At leading order in the limit $B \to 0$ the free surface is thermally insulated (adiabatic), and from (21) we have $T \equiv 1$, i.e. the rivulet is isothermal. Hence at leading order $f = 1$ and so $\bar{u} = \bar{u}_0$, $\bar{v} = \bar{v}_0$ and $Q = Q_0$, where $\bar{u}_0$, $\bar{v}_0$ and Q_0 are the values in the isothermal case given by (30), and therefore (29) and (31) reduce to the familiar equations and boundary conditions describing an isothermal rivulet, as expected.

Note that the same (effectively isothermal) behaviour occurs in the corresponding rivulet flow with thermocapillary effects but constant viscosity in the limit $B \to 0$. This is in contrast to the situation studied by Holland, Duffy and Wilson [11] in which the variation of surface tension with temperature is sufficiently strong that thermocapillary effects appear at leading order in the limit $B \to 0$.

The Limit $B \to \infty$

On the other hand, at leading order in the limit $B \to \infty$ the free surface is at the same uniform temperature as the atmosphere, and from (21) we have $T = 1 - z/h$. Thus at leading order we can write $\mu(T) = g(z/h)$, where $g(\cdot)$ is a known function, and hence

$$f = 3\int_0^1 \int_0^Z \frac{1-\tilde{Z}}{g(\tilde{Z})}\,\mathrm{d}\tilde{Z}\,\mathrm{d}Z = 3\int_0^1 \frac{(1-Z)^2}{g(Z)}\,\mathrm{d}Z = 3\int_0^1 \frac{T^2}{\mu(T)}\,\mathrm{d}T. \tag{32}$$

The key observation here is that *the fluidity f is independent of x, y and t.* Therefore (31) simplifies to

$$h_t + f(\bar{u}_{0x} + \bar{v}_{0y}) = 0, \tag{33}$$

and with (30) equation (29) simplifies to

$$Q = fQ_0, \tag{34}$$

and hence the behaviour in this limit for an *arbitrary* dependence of viscosity on temperature can be obtained directly from the corresponding behaviour in the isothermal case simply by rescaling time t with f and flux Q with $1/f$, i.e. by replacing t by ft and Q_0 by Q/f. This result is entirely consistent with the corresponding results of Reisfeld and Bankoff [14] and Wu and Hwang [15] for unsteady flow of two-dimensional thin films in the limit $B \to \infty$ for the particular viscosity models they considered. In particular, in the special case of steady flow equation (31) is identically the same as in the isothermal case, and only the expression for the flux Q in (29) differs from that in the isothermal case by a factor of f.

For all three models considered in the present paper $\partial\mu/\partial V > 0$ and hence from (32) $\partial f/\partial V < 0$, i.e. the fluidity decreases as the temperature of the atmosphere is decreased, as expected. All three models coincide up to and including $O(V)$ in the limit $V \to 0$, and satisfy

$$f = 1 - \frac{V}{4} + O(V^2) \tag{35}$$

as $V \to 0$. For the linear model (2) we have $g(\tilde{Z}) = 1 + V\tilde{Z}$ and so f is given by

$$f = \frac{3}{2V^3}\left[-V(2+3V) + 2(1+V)^2 \log(1+V)\right], \tag{36}$$

and satisfies

$$f = \frac{3}{2} - \frac{3}{2}(1+V) + O((1+V)^2 \log(1+V)) \quad \text{as} \quad V \to -1, \tag{37}$$

$$f = \frac{3(2\log V - 3)}{2V} + O\left(\frac{\log V}{V^2}\right) \quad \text{as} \quad V \to \infty. \tag{38}$$

For the exponential model (3) we have $g(\tilde{Z}) = \exp(V\tilde{Z})$ and so f is given by

$$f = \frac{3}{V^3}\left[(V-1)^2 + 1 - 2\exp(-V)\right], \tag{39}$$

and satisfies

$$f = \frac{6\exp(-V)}{(-V)^3} + O\left(\frac{1}{V}\right) \quad \text{as} \quad V \to -\infty, \tag{40}$$

$$f = \frac{3}{V} + O\left(\frac{1}{V^2}\right) \quad \text{as} \quad V \to \infty. \tag{41}$$

For the Eyring model (4) we have

$$g(\tilde{Z}) = \exp\left(\frac{V V_{\mathrm{m}} \tilde{Z}}{V_{\mathrm{m}} - V\tilde{Z}}\right) \tag{42}$$

and so f is given by

$$\begin{aligned} f = 3\exp(V_{\mathrm{m}})\left(\frac{V_{\mathrm{m}}}{V}\right)^3 \Bigg[& \mathrm{E}_4(V_{\mathrm{m}}) - 2\left(\frac{V_{\mathrm{m}} - V}{V_{\mathrm{m}}}\right)\mathrm{E}_3(V_{\mathrm{m}}) + \left(\frac{V_{\mathrm{m}} - V}{V_{\mathrm{m}}}\right)^2 \mathrm{E}_2(V_{\mathrm{m}}) \\ & - \left(\frac{V_{\mathrm{m}} - V}{V_{\mathrm{m}}}\right)^3 \left\{\mathrm{E}_4\left(\frac{V_{\mathrm{m}}^2}{V_{\mathrm{m}} - V}\right) - 2\mathrm{E}_3\left(\frac{V_{\mathrm{m}}^2}{V_{\mathrm{m}} - V}\right) + \mathrm{E}_2\left(\frac{V_{\mathrm{m}}^2}{V_{\mathrm{m}} - V}\right)\right\}\Bigg], \end{aligned} \tag{43}$$

where $\mathrm{E}_n(X)$ denotes the usual exponential integral

$$\mathrm{E}_n(X) = \int_1^\infty \frac{\exp(-Xs)}{s^n}\,\mathrm{d}s. \tag{44}$$

Thus for the Eyring model f satisfies

$$f = \exp(V_{\mathrm{m}}) + O\left(\frac{\log V}{V}\right) \quad \text{as} \quad V \to -\infty, \tag{45}$$

$$f \sim 3\exp(V_{\mathrm{m}})\mathrm{E}_4(V_{\mathrm{m}}) \quad \text{as} \quad V \to V_{\mathrm{m}}^-; \tag{46}$$

moreover if $V > 0$ then at leading order in the limit $V_{\mathrm{m}} \to V^+$ we have $f \sim 3\exp(V)\mathrm{E}_4(V)$, and if $V < 0$ then at leading order in the limit $V_{\mathrm{m}} \to 0^+$ we have $f \sim 1$, while at leading order in the limit $V_{\mathrm{m}} \to \infty$ the Eyring model reduces to the exponential model and so f is given by (39). The fluidity f for the linear model (given by (36)), the exponential model (given by (39)) and the Eyring model (given by (43)) (the last-named for a range of values of V_{m}) are plotted as functions of V in Figure 2. In particular, Figure 2 confirms that all three models coincide up to and including $O(V)$ in the limit $V \to 0$ and that in each case f is a monotonically decreasing function of V.

Note that, unlike in the limit $B \to 0$ discussed above, the same behaviour does *not* occur in the corresponding rivulet flow with thermocapillary effects but constant viscosity in the limit $B \to \infty$. In fact, since the temperature of the free surface is uniform at leading order, the leading-order problem is isothermal in that case, and thermocapillary effects appear only as higher-order corrections to the isothermal results.

In the remainder of this paper we indicate how the above results can be used give a complete description of steady flow of a slender rivulet in the limit of large Biot number in two situations in which the isothermal problem has previously been solved analytically, namely non-uniform flow down an inclined plane, and locally unidirectional flow down a slowly varying substrate.

3 Steady Non-Uniform Flow Down an Inclined Plane

3.1 Weak Surface-Tension Effects

In the case $C^{-1} \ll \cos\alpha$ if we choose $\delta = \epsilon$ then at leading order equations (29) and (31) reduce to the equations describing steady flow of a slender rivulet with weak surface-tension effects, namely

$$\cos\alpha(h^3 h_y)_y - \sin\alpha(h^3)_x = 0, \quad \frac{Q}{f} = \frac{\sin\alpha}{3}\int_{-a}^{+a} h^3\,\mathrm{d}y. \tag{47}$$

Smith [6] obtained the unique similarity solution to (47) for a rivulet with prescribed flux $Q = \bar{Q}$ in the isothermal case ($f = 1$), from which we can readily deduce the solution to the present non-isothermal problem, namely

$$h = h_{\mathrm{m}}\left(1 - \frac{y^2}{a^2}\right), \quad a = (cx)^{3/7}, \tag{48}$$

where

$$h_{\mathrm{m}} = \frac{3c\tan\alpha}{14(cx)^{1/7}}, \quad c = \frac{7\cot\alpha}{3}\left(\frac{105\bar{Q}}{4f\sin\alpha}\right)^{1/3}. \tag{49}$$

For $\cos\alpha > 0$ (so that $c > 0$) this solution represents a widening and shallowing pendant rivulet in $x > 0$, while for $\cos\alpha < 0$ (so that $c < 0$) it represents a narrowing and deepening sessile rivulet in $x < 0$. Note that in this case the transverse profile always has a single global maximum $h = h_{\mathrm{m}}$ at $y = 0$, and that, in general, the contact angle β given by (19) varies with x like $x^{-4/7}$.

In this case a and h_{m} vary with $\bar{Q}/f$ like $(\bar{Q}/f)^{1/7}$ and $(\bar{Q}/f)^{2/7}$ respectively, and so, since f is a monotonic decreasing function of V, decreasing the temperature of the atmosphere (i.e. increasing V) always has the effect of increasing a and h, i.e. of making the rivulet wider and deeper.

3.2 Strong Surface-Tension Effects

On the other hand, in the case $C^{-1} \gg \cos\alpha$ if we choose $\delta\ell^2 = \epsilon^3 l^2$ (where $\ell = (\gamma/\rho g)^{1/2}$ is the dimensional capillary length) then at leading order equations (29) and (31) reduce to the equations describing steady flow of a slender rivulet with strong surface-tension effects, namely

$$(h^3 h_{yyy})_y + \sin\alpha(h^3)_x = 0, \quad \frac{Q}{f} = \frac{\sin\alpha}{3}\int_{-a}^{+a} h^3\,\mathrm{d}y. \tag{50}$$

Duffy and Moffatt [8] obtained the one-parameter family of similarity solutions (parameterised by $G_0 \geq 0$) to (50) for a rivulet with prescribed flux $Q = \bar{Q}$ in the isothermal case ($f = 1$), from which we can readily deduce the solution to the present non-isothermal problem, namely

$$h = h_{\mathrm{m}} \left(1 - \frac{y^2}{a^2}\right)\left(G_0 - \frac{Sy^2}{24a^2}\right), \quad a = (cx)^{3/13}, \tag{51}$$

where

$$h_{\mathrm{m}} = \frac{3|c| \sin\alpha}{13(cx)^{1/13}}, \quad |c| = \frac{13}{3\sin\alpha}\left(\frac{3\bar{Q}}{If\sin\alpha}\right)^{1/3}, \tag{52}$$

in which we have defined $I = I(G_0)$ by

$$I = \int_{-1}^{+1} \left[(1-\eta^2)\left(G_0 - \frac{S\eta^2}{24}\right)\right]^3 \mathrm{d}\eta = \frac{32}{35}G_0^3 - \frac{4S}{315}G_0^2 + \frac{1}{6930}G_0 - \frac{S}{1297296} \tag{53}$$

and introduced the notation $S = \mathrm{sgn}(c)$. For $S = 1$ (so that $c > 0$) this solution represents a widening and shallowing rivulet in $x > 0$, while for $S = -1$ (so that $c < 0$) it represents a narrowing and deepening rivulet in $x < 0$. Note that in this case physically realisable solutions are obtained only for $G_0 \geq 1/24$ when $S = 1$, in which case the transverse profile always has a single global maximum $h = h_{\mathrm{m}}$ at $y = 0$, but are obtained for all values of $G_0 \geq 0$ when $S = -1$, in which case the transverse profile has a single global maximum $h = h_{\mathrm{m}}$ at $y = 0$ for $G_0 \geq 1/24$ but two equal global maxima $h/h_{\mathrm{m}} = (1 + 24G_0)^2/96$ at $y/a = \pm((1 - 24G_0)/2)^{1/2}$ and a local minimum $h = h_{\mathrm{m}}$ at $y = 0$ for $0 \leq G_0 < 1/24$. As in the case of weak surface tension, in general, the contact angle β given by (19) varies with x like $x^{-4/13}$.

In this case a and h_{m} vary with $\bar{Q}/f$ like $(\bar{Q}/f)^{1/13}$ and $(\bar{Q}/f)^{4/13}$ respectively, and so, as in the case of weak surface tension, decreasing the temperature of the atmosphere (i.e. increasing V) always has the effect of increasing a and h, i.e. of making the rivulet wider and deeper.

4 Steady Locally Unidirectional Flow Down a Slowly Varying Substrate

In the case $\epsilon \to 0$ with $C = O(1)$ equations (29) and (31) reduce to the equations describing steady locally unidirectional flow of a rivulet, namely

$$\left[h^3(h\cos\alpha - h_{yy})_y\right]_y = 0, \quad \frac{Q}{f} = \frac{\sin\alpha}{3}\int_{-a}^{+a} h^3\,\mathrm{d}y, \tag{54}$$

where we have chosen $C = 1$ and (with the additional assumption that the contact angle takes the prescribed constant value β) $\delta = \beta$. Duffy and Moffatt [2] obtained the solution to (54) for a rivulet with prescribed flux $Q = \bar{Q}$ in the isothermal case ($f = 1$), from which we can readily deduce the solution to the present non-isothermal problem, namely

$$h(y) = \begin{cases} \dfrac{\cosh ma - \cosh my}{m\sinh ma} & \text{if} \quad 0 \le \alpha < \pi/2, \\ \dfrac{a^2 - y^2}{2a} & \text{if} \quad \alpha = \pi/2, \\ \dfrac{\cos my - \cos ma}{m\sin ma} & \text{if} \quad \pi/2 < \alpha \le \pi, \end{cases} \tag{55}$$

where we have introduced the notation $m = |\cos\alpha|^{1/2}$, and hence obtain an implicit algebraic equation for a, namely

$$\frac{\bar{Q}}{f} = \frac{\sin\alpha}{9m^4}F(ma), \tag{56}$$

where we have defined

$$F(ma) = \begin{cases} 15ma\coth^3 ma - 15\coth^2 ma - 9ma\coth ma + 4 & \text{if} \quad 0 \le \alpha < \pi/2, \\ \dfrac{12}{35}(ma)^4 & \text{if} \quad \alpha = \pi/2, \\ -15ma\cot^3 ma + 15\cot^2 ma - 9ma\cot ma + 4 & \text{if} \quad \pi/2 < \alpha \le \pi. \end{cases} \tag{57}$$

(Note that in the case $\alpha = \pi/2$ we have $m = 0$ and the factors of m^4 must be cancelled before setting $\alpha = \pi/2$.) Note that in this case the transverse profile always has a single global maximum $h = h_{\mathrm{m}}$ at $y = 0$, where $h_{\mathrm{m}} = h(0)$ in (55).

Duffy and Moffatt [2] interpreted their results as describing locally unidirectional flow down a locally planar substrate whose local slope α varies slowly in the flow-wise direction and, in particular, used them to describe the flow in the azimuthal direction round a large horizontal circular cylinder. For $0 \leq \alpha \leq \pi/2$ there is a unique solution for a for each value of α, while for $\pi/2 < \alpha \leq \pi$ there are multiple branches of solutions for a, but of these only the one that connects smoothly with the solution in $0 \leq \alpha \leq \pi/2$ is physically realisable.

As $\alpha \to 0$ the rivulet becomes infinitely wide according to

$$a \sim \frac{3\bar{Q}}{2f\alpha}, \quad h_{\mathrm{m}} \sim 1 + \frac{\alpha^2}{4}, \tag{58}$$

while as $\alpha \to \pi$ the rivulet becomes infinitely deep according to

$$a \sim \pi - \left[\frac{5f\pi(\pi-\alpha)}{3\bar{Q}}\right]^{1/3}, \quad h_{\mathrm{m}} \sim \left[\frac{24\bar{Q}}{5f\pi(\pi-\alpha)}\right]^{1/3}, \tag{59}$$

Holland, Duffy and Wilson [11] showed that in the limit of small flux $\bar{Q}/f \to 0$ the rivulet becomes narrow and shallow according to

$$a \sim \left(\frac{105\bar{Q}}{4f\sin\alpha}\right)^{1/4}, \quad h_{\mathrm{m}} \sim \frac{1}{2}\left(\frac{105\bar{Q}}{4f\sin\alpha}\right)^{1/4}, \tag{60}$$

while in the limit of large flux $\bar{Q}/f \to \infty$ the rivulet becomes infinitely wide for $0 \leq \alpha \leq \pi/2$, and infinitely deep for $\pi/2 \leq \alpha \leq \pi$ according to

$$a \sim \frac{3\bar{Q}m^3}{2f\sin\alpha}, \quad h_{\mathrm{m}} \sim \frac{1}{m} \tag{61}$$

if $0 \leq \alpha < \pi/2$,

$$a \sim \left(\frac{105\bar{Q}}{4f}\right)^{1/4}, \quad h_{\mathrm{m}} \sim \frac{1}{2}\left(\frac{105\bar{Q}}{4f}\right)^{1/4} \tag{62}$$

if $\alpha = \pi/2$, and

$$a \sim \frac{\pi}{m} - \frac{1}{m^2}\left(\frac{5f\pi\sin\alpha}{3\bar{Q}m}\right)^{1/3}, \quad h_{\mathrm{m}} \sim \left(\frac{24\bar{Q}m}{5f\pi\sin\alpha}\right)^{1/3} \tag{63}$$

if $\pi/2 < \alpha \leq \pi$. Note that (60) fails near $\alpha = 0$ and $\alpha = \pi$ where there are boundary layers of width $O(\bar{Q}/f)$, while (61) and (63) fail near $\alpha = \pi/2$ where there are boundary layers of thickness $O(\bar{Q}/f)^{-1/2}$ across which a and h_{m} change from their $O(\bar{Q}/f)$ and $O(1)$ values in

$0 \le \alpha < \pi/2$ to their $O(1)$ and $O(\bar{Q}/f)^{1/3}$ values in $\pi/2 < \alpha \le \pi$ via their $O(\bar{Q}/f)^{1/4}$ values at $\alpha = \pi/2$.

Typical numerically calculated values of a and $h_{\rm m}$ are plotted as functions of α for a range of values of $\bar{Q}/f$ in Figure 3. In particular, the results in Figure 3 illustrate that both a and $h_{\rm m}$ are monotonically increasing functions of $\bar{Q}/f$, and hence decreasing the temperature of the atmosphere (i.e. increasing V) always has the effect of increasing a and h, i.e. of making the rivulet wider and deeper.

5 Conclusions

In the present paper we used the lubrication approximation to investigate the unsteady gravity-driven draining of a thin rivulet of Newtonian fluid with temperature-dependent viscosity down a substrate that is either uniformly hotter or uniformly colder than the surrounding atmosphere. First we derived the general nonlinear evolution equation for a thin film of fluid with an arbitrary dependence of viscosity on temperature. Then we showed that at leading order in the limit of small Biot number the rivulet is isothermal, as expected, but that at leading order in the limit of large Biot number (in which the rivulet is *not* isothermal) the governing equation can, rather unexpectedly, always be reduced to that in the isothermal case with a suitable rescaling. These results were then used to give a complete description of steady flow of a slender rivulet in the limit of large Biot number in two situations in which the corresponding isothermal problem has previously been solved analytically, namely non-uniform flow down an inclined plane, and locally unidirectional flow down a slowly varying substrate. In particular, we found that if the fluidity is a decreasing function of the temperature of the atmosphere (as it is for all three specific viscosity models we considered) then decreasing the temperature of the atmosphere always has the effect of making the rivulet wider and deeper.

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References

[1] G. D. Towell and L. B. Rothfeld, Hydrodynamics of rivulet flow. AIChE. J. **12** (1966) 972–980.

[2] B. R. Duffy and H. K. Moffatt, Flow of a viscous trickle on a slowly varying incline. *Chem. Eng. J.* **60** (1995) 141–146.

[3] R. F. Allen and C. M. Biggin, Longitudinal flow of a lenticular liquid filament down an inclined plane. *Phys. Fluids* **17** (1974) 287–291.

[4] S. K. Wilson and B. R. Duffy, On the gravity-driven draining of a rivulet of viscous fluid down a slowly varying substrate with variation transverse to the direction of flow. *Phys. Fluids* **10** (1998) 13–22.

[5] S. K. Wilson, B. R. Duffy and A. B. Ross, On the gravity-driven draining of a rivulet of viscoplastic material down a slowly varying substrate. *Phys. Fluids* **14** (2002) 555–571.

[6] P. C. Smith, A similarity solution for slow viscous flow down an inclined plane. *J. Fluid Mech.* **58** (1973) 275–288.

[7] L. W. Schwartz and E. E. Michaelides, Gravity flow of a viscous liquid down a slope with injection. *Phys. Fluids*, **31** (1988) 2739–2741.

[8] B. R. Duffy and H. K. Moffatt, A similarity solution for viscous source flow on a vertical plane. *Euro. J. Appl. Math.* **8** (1997) 37–47.

[9] S. K. Wilson, B. R. Duffy and S. H. Davis, On a slender dry patch in a liquid film draining under gravity down an inclined plane. *Euro. J. Appl. Math.*, **12** (2001) 233–252.

[10] S. K. Wilson, B. R. Duffy and R. Hunt, A slender rivulet of a power-law fluid driven by either gravity or a constant shear stress at the free surface. To appear in *Q. J. Mech. Appl. Math.*

[11] D. Holland, B. R. Duffy and S. K. Wilson, Thermocapillary effects on a thin viscous rivulet draining steadily down a uniformly heated or cooled slowly varying substrate. *J. Fluid Mech.* **441** (2001) 195–221.

[12] D. A. Goussis and R. E. Kelly, Effects of viscosity variation on the stability of film flow down heated or cooled inclined surfaces: Long-wavelength analysis. *Phys. Fluids* **28** (1985) 3207–3214.

[13] C.-C. Hwang and C.-I. Weng, Non-linear stability analysis of film flow down a heated or cooled inclined plane with viscosity variation. *Int. J. Heat Mass Transfer* **31** (1988) 1775–1784.

[14] B. Reisfeld and S. G. Bankoff, Nonlinear stability of a heated thin liquid film with variable viscosity. *Phys. Fluids A* **2** (1990) 2066–2067.

[15] M.-C. Wu and C.-C. Hwang, Nonlinear theory of film rupture with viscosity variation. *Int. Comm. Heat Mass Transfer* **18** (1991) 705–713.

[16] A. Oron, S. H. Davis and S. G. Bankoff, Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69** (1997) 931–980.

Figure Captions

Fig. 1 : The geometry of the problem.

Fig. 2 : The fluidity f given by (36) for the linear viscosity model (2), by (39) for the exponential viscosity model (3), and by (43) for the Eyring viscosity model (4) (the last-named for a range of values of V_{m}) plotted as functions of V. Note that for the linear model f is defined only for $V > -1$, while for the Eyring model it is defined only for $V < V_{\mathrm{m}}$.

Fig. 3 : (a) The semi-width a and (b) the maximum thickness h_{m} of a steady locally unidirectional rivulet flow with prescribed flux $\bar{Q}$ plotted as functions of α for a range of values of $\bar{Q}/f$.