Singularities of optimal attitude motions

James D. Biggs [*]

[*] *Advanced Space Concepts Laboratory, Department of Mechanical Engineering, University of Strathclyde, Glasgow. (e-mail: james.biggs@strath.ac.uk).*

Abstract: This paper considers the problem of planning optimal attitude motions for spacecraft. The extremal solutions that result from this optimization problem are characterized and their singularities identified. Following this these singularities are solved analytically inferring the form of particular optimal velocities. These particular solutions are then integrated and their corresponding motions derived independently of a local coordinate chart. These motions have the potential to be used as smooth, optimal reference trajectories for performing certain re-orientations for spacecraft.

Keywords: Motion Planning, attitude control, Maximum principle, Singularities.

1. INTRODUCTION

Methodologies for planning and controlling attitude maneuvers of spacecraft based on geometric techniques has a rich history and includes the application of quaternion algebra, Lie group theory and geometric control theory, see Wie (1998); Spindler (1996); Jurdjevic (1997); Leonard et. al. (1995) for a few examples. In this paper we consider the optimal attitude control problem posed in Spindler (1996) from the perspective of using these methods to plan optimal, smooth and practical motions for spacecraft.

Spindler (1996) defines a fixed end point optimal control problem for the attitude control of a rigid body with the angular velocities as the control inputs, where the cost function to be minimized is a quadratic function of the angular velocity components. In particular minimizing such a cost function is desirable during a spacecraft maneuver to keep angular velocity low because high spin rates can cause undesirable tumbling motions. In addition high spin rates make it hard to receive good tracking data to monitor the spacecraft's motion. Spindler (1996) then applies the Maximum Principle (see Jurdjevic (1997); Sussmann (1997)) to this optimal control problem to yield the appropriate Hamiltonian. Following this the corresponding Hamiltonian vector fields are derived to yield the necessary conditions for optimality.

In Spindler (1996) a special case of this optimal control problem is considered where the weights of the cost function are all equal and is representative of the mean square angular velocity. In this case the extremal curves are constant and their projections onto $SO(3)$ can be solved in closed-form. Following this trivial case Spindler (1996) gives a numerical example for the general case (the weights are not equal) and the corresponding extremal curves are solved numerically. In this paper we provide a detailed treatment of singularities that appear in the general solution of the extremal curves and proceed to solve them analytically. This investigation yields an additional set of analytically defined reference trajectories which alongside the constant extremal reference trajectories in Spindler (1996) could potentially be used in practical attitude interpolation problems for spacecraft.

Following the derivation of the extremals at these singularities, techniques based on Lie group theory are used to project them down to $SO(3)$ (see for example Jurdjevic (1997); Biggs (2010)). This procedure enables the derivation of globally defined solutions at the singularities of the extremal functions. These singular motions provide analytic expressions for rotational interpolation in a convenient simplistic form. Furthermore, these rotational motions could be used as practical reference trajectories for a spacecraft to track in order to perform particular manoeuvres.

2. ATTITUDE MOTION PLANNING PROBLEM

The orientation of a spacecraft is represented here by curves in the Special Orthogonal Group $SO(3)$ where $R(t) \in SO(3)$ and where the kinematics are described by the differential equation:

$$\frac{dR(t)}{dt} = R(t)(\Omega_1 A_1 + \Omega_2 A_2 + \Omega_3 A_3) \tag{1}$$

where $\Omega_1, \Omega_2, \Omega_3$ are the angular velocities and where A_1, A_2, A_3 form a basis for the Lie algebra of $SO(3)$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

with the Lie bracket defined by $[X,Y] = XY - YX$ with $X, Y \in \mathfrak{so}(3)$. Therefore, we have $[A_1,A_2] = A_3$, $[A_2,A_3] = A_1$ and $[A_1,A_3] = -A_2$. A_1, A_2, A_3 describe the infinitesimal motion of the spacecraft in the yaw, pitch, roll directions respectively.

The problem addressed by Spindler (1996) is formalized in the following problem statement:

Motion Planning Problem

Compute the optimal velocities and the corresponding rigid body motions $R(t) \in SO(3)$ defined by the kinematic equations (1) that minimizes the cost function:

$$J = \frac{1}{2}\int_0^T c_1\Omega_1^2 + c_2\Omega_2^2 + c_3\Omega_3^2 dt \tag{3}$$

with the given boundary conditions $R(0) = R_0$ and $R(T) = R_T$, where c_1, c_2, c_3 are constant weights and T is the final time.

In an optimal control problem where we desire to motion between two fixed orientations, the weights c_1, c_2, c_3 can be manipulated to achieve the final desired configuration. This problem can be viewed as an optimal control problem where the angular velocities are the control inputs. The tool used to tackle this **Motion planning problem** is the coordinate free Maximum Principle of optimal control as described in Spindler (1996); Jurdjevic (1997); Sussmann (1997). The Maximum Principle of optimal control identifies the appropriate left-invariant Hamiltonian H on the dual of the Lie algebra $\mathfrak{so}(3)^*$. The Hamiltonian corresponding to (1) and (3) is written as (see for example Jurdjevic (1997)):

$$H(p,u) = \sum_{i=1}^{3} \Omega_i p(R(t)A_i) - p_0\left(\frac{1}{2}\left(\sum_{i=1}^{3} c_i \Omega_i^2\right)\right) \tag{4}$$

where $p(\cdot) : T_{R(t)}SO(3) \mapsto \mathbb{R}$ such that $p(R(t)B_i), p(R(t)A_i)$ are scalar components of an element in $T^*_{R(t)}SO(3)$, where $p_0 \geqslant 0$ is a fixed positive constant. The curves which satisfy the Hamiltonian (4) with $p_0 = 1$ are called regular extremals and with $p_0 = 0$ abnormal extremals. However, it has been shown in Spindler (1996) that for this optimal control problem no abnormal extremals exist. Following from the Maximum Principle and the fact that (4) is a concave function in Ω_i the optimal velocity inputs are given by $\frac{dH}{d\Omega_i} = 0$ it follows that:

$$\Omega_i = \frac{1}{c_i} p(R(t)A_i) \tag{5}$$

As the configuration of the spacecraft is the Lie group $SO(3)$, the cotangent bundle $T^*SO(3)$ can be realized as the direct product $SO(3) \times \mathfrak{so}(3)^*$ where $\mathfrak{so}(3)^*$ is the dual of the Lie algebra Jurdjevic (1997). Therefore, the original Hamiltonian defined on $T^*SO(3)$ can be expressed as a reduced Hamiltonian on the dual of the Lie algebra $\mathfrak{so}(3)^*$. We define the linear functions $M_i = p(R(t)A_i) = \hat{p}(A_i)$ for $i = 1,2,3$, see Jurdjevic (1997). Therefore, from (16) it follows that the maximizing inputs are:

$$\Omega_i = \frac{1}{c_i} M_i \tag{6}$$

Substituting (6) into (4) gives the optimal Hamiltonian

$$H = \frac{1}{2}\left(\frac{M_1^2}{c_1} + \frac{M_2^2}{c_2} + \frac{M_3^2}{c_3}\right) \tag{7}$$

The necessary conditions for optimality are then computed by making use of the Poisson bracket defined in terms of the Lie bracket $\{\hat{p}(\cdot), \hat{p}(\cdot)\} = -\hat{p}([\cdot,\cdot])$ which yields:

$$\begin{aligned} \dot{M}_1 &= \frac{c_2 - c_3}{c_2 c_3} M_2 M_3 \\ \dot{M}_2 &= \frac{c_3 - c_1}{c_1 c_3} M_1 M_3 \\ \dot{M}_3 &= \frac{c_1 - c_2}{c_1 c_2} M_1 M_2 \end{aligned} \tag{8}$$

where $M_1, M_2, M_3 \in \mathfrak{so}^*(3)$ are the extremal curves. It is interesting to note that if the weights (that would be set according to the final desired configuration) are set to correspond to the moments of inertia then the equations are exactly the Euler equations for a free-rigid body (see Whittaker (1999)). However, we note that these are a generalisation of these equations as the weights of the cost function are arbitrary. In order to characterize the extremal curves described by the equations (17) we conveniently express them in the form:

$$\dot{M}_i = \frac{c_j - c_k}{c_j c_k} M_j M_k \tag{9}$$

where $i = 1,2,3,\ j = 2,3,1$ and $k = 3,1,2$ respectively. In the next section we observe that by exploiting an additional constant of motion inherent in all left-invariant Hamiltonian systems on $SO(3)$ the evolution of the extremal curves can be reduced to the analysis of a 1 dimensional ordinary differential equation and solved in terms of Jacobi elliptic functions. In Spindler (1996) only a numerical example of the general case is given.

3. REDUCTION AND SOLUTION OF THE EXTREMAL CURVES

The initial stage of the procedure is to derive analytic expressions for the optimal angular velocity history and the corresponding rotation matrix. Note that when $i = 1$ then $j = 2, k = 3$ when $i = 2$ then $j = 3, k = 1$ and when $i = 3$, then $j = 1, k = 2$:

Theorem 1. The optimal angular velocities Ω_i^* that minimize the cost function (3) can be expressed in the form:

$$\Omega_i^* = \frac{\sqrt{s_i}}{c_i} sn\left(\sqrt{\alpha s_j} + C_i, \frac{s_i}{s_j}\right) \tag{10}$$

where $sn(\cdot,\cdot)$ is a Jacobi elliptic function and where the constants C_i are defined by

$$C_i = sn^{-1}\left(\frac{c_i \Omega_i(0)}{\sqrt{s_i}}, \frac{s_i}{s_j}\right) \tag{11}$$

with

$$s_i = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\chi}}{2\alpha} \qquad s_j = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\chi}}{2\alpha} \tag{12}$$

and

$$\begin{aligned} \alpha &= -\frac{(c_i - c_j)(c_i - c_k)}{c_i^2 c_j c_k} \\ \beta &= \frac{4c_j c_k H - 2c_i(c_j + c_k)H + 2c_i K^2 - (c_j + c_k)K^2}{c_i c_j c_k} \\ \chi &= -\frac{\left(2c_j H - K^2\right)\left(2c_k H - K^2\right)}{c_j c_k} \end{aligned} \tag{13}$$

with the conserved quantities H and K defined in terms of the initial angular velocities:

$$\begin{aligned} H &= \frac{1}{2}\left(c_1 \Omega_1^2(0) + c_2 \Omega_2^2(0) + c_3 \Omega_3^2(0)\right) \\ K^2 &= c_1^2 \Omega_1^2(0) + c_2^2 \Omega_2^2(0) + c_3^2 \Omega_3^2(0) \end{aligned} \tag{14}$$

Proof.

the optimal Hamiltonian is given by:

$$H = \frac{1}{2}\left(\frac{M_1^2}{c_1} + \frac{M_2^2}{c_2} + \frac{M_3^2}{c_3}\right) \tag{15}$$

where the extremal curves $M_1, M_2, M_3 \in \mathfrak{so}^*(3)$ are defined in terms of the weights c_i of the cost function and the optimal angular velocities:

$$M_i = c_i \Omega_i^* \tag{16}$$

the corresponding Hamiltonian vector fields are then given by the Poisson bracket (see Jurdjevic (1997) for details):

$$\dot{M}_i = \left(\frac{c_j - c_k}{c_j c_k}\right) M_j M_k \tag{17}$$

it is easily shown that the Casimir function:

$$K^2 = M_1^2 + M_2^2 + M_3^2 \tag{18}$$

is constant along the Hamiltonian flow. Illustrating the solution for M_1 the solutions for M_2 and M_3 follow analogously. From (17) we have:

$$\left(\dot{M}_1\right)^2 = \left(\frac{c_2 - c_3}{c_2 c_3}\right)^2 M_2^2 M_3^2 \tag{19}$$

then using the Hamiltonian (15) and the Casimir function (18) write M_2 and M_3 explicitly in terms of M_1 to yield:

$$\begin{aligned} M_2^2 &= \frac{c_2\left(2c_3 H - K^2 + M_1^2 - (c_3 M_1^2)/c_1\right)}{c_3 - c_2} \\ M_3^2 &= \frac{c_3\left(2c_2 H - K^2 + M_1^2 - (c_2 M_1^2)/c_1\right)}{c_2 - c_3} \end{aligned} \tag{20}$$

then substituting (20) into (19) and simplifying yields:

$$\left(\dot{M}_1\right)^2 = \alpha M_1^4 + \beta M_1^2 + \chi \tag{21}$$

where the constants α, β and χ are defined by equation (13) with $i = 1, j = 2, k = 3$. Defining the constants:

$$s_1 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\chi}}{2\alpha} \quad s_2 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\chi}}{2\alpha} \tag{22}$$

then equation (21) can be expressed as:

$$\dot{M}_1 = \left(s_1 - M_1^2\right)\left(s_2 - M_1^2\right) \tag{23}$$

and therefore

$$\int_0^t dt = \frac{1}{\sqrt{\alpha}} \int_{M_1(0)}^{M_1(t)} \frac{dM_1}{\sqrt{\left(s_1 - M_1^2\right)\left(s_2 - M_1^2\right)}} \tag{24}$$

then with a change of variable $M_1 = \sqrt{s_1} sn(u,m)$ and setting $m = \frac{s_1}{s_2}$ the equation (24) becomes:

$$t = \frac{1}{\sqrt{\alpha s_2}} \int_{sn^{-1}(M_1(0)/\sqrt{s_1}, \frac{s_1}{s_2})}^{sn^{-1}(M_1(t)/\sqrt{s_1}, \frac{s_1}{s_2})} du \tag{25}$$

and therefore:

$$M_1 = \sqrt{s_1} sn(\sqrt{\alpha s_2} t + C_1, \frac{s_1}{s_2}) \tag{26}$$

where C_1 is defined in (11) and therefore from (16) the angular velocity is (10) $\square$.

Note that this analytic solution has been verified against numerical integration.

4. SINGULARITIES OF THE EXTREMAL CURVES

The differential equation (21) can be expressed as an Elliptic integral of the first kind:

$$t = \int_{M_i(0)}^{M_i(T)} \frac{1}{\sqrt{\alpha M_i^4 + \beta M_i^2 + \chi}} dM_i \tag{27}$$

In this paper we investigate the case at the singularities of this elliptic integral, that is where

$$\alpha M_i^4 + \beta M_i^2 + \chi = 0 \tag{28}$$

from (28) it can be seen that the singularities of the elliptic integral correspond to $\dot{M}_i = 0$ in equation (21) and this fact is used to derive analytic expressions for this particular case. In this section we investigate the singularities of the extremal curves, that is where the elliptic integral (27) is not well defined. These singularities correspond to fixed points of the differential equation (21) and we define a root of this by the constant $M_i = s_1$. At this singularity we can solve the extremal curves M_j and M_k using the Casimir function such that:

$$K^2 - s_1^2 = M_j^2 + M_k^2 \tag{29}$$

as the left hand side of (29) is constant this suggests using polar coordinates for M_j and M_k:

$$M_j = r\sin\theta, \quad M_k = r\cos\theta \tag{30}$$

r is given by substituting (48) into (29):

$$r = \left(K^2 - s_1^2\right)^{1/2} \tag{31}$$

and θ is given as follows:

$$\theta = \arctan\left(\frac{M_j}{M_k}\right) \tag{32}$$

$$\dot{\theta} = \frac{M_k \dot{M}_j - M_j \dot{M}_k}{M_j^2 + M_k^2} \tag{33}$$

substituting in the values for $\dot{M}_j$ and $\dot{M}_k$ from (17) and simplifying gives:

$$\dot{\theta} = s_1\left(\frac{K^2 - c_1 H}{c_1\left(K^2 - s_1^2\right)}\right) \tag{34}$$

therefore $\dot{\theta}$ is constant and it is convenient to write $\dot{\theta} = C$. Assuming a particular solution such that the constant of integration is zero we have $\theta = Ct$. Therefore, we can express the singular extremal curves explicitly as:

$$M_i = s_1 \quad M_j = r\sin Ct, \quad M_k = r\cos Ct \tag{35}$$

where s_1, r, C are constants. Therefore, at the singularities the extremal M_i is constant and M_j and M_k are sinusoids. In the following section we illustrate a method for projecting the extremal curves onto $SO(3)$.

5. INTEGRATION PROCEDURE

To obtain the most convenient form for the rotation matrix the differential equations are expressed in Lax Pair form:

$$\begin{aligned} \dot{L}(t) &= [L(t), \nabla H] \\ \frac{dR(t)}{dt} &= R(t)\nabla H \end{aligned} \tag{36}$$

where $R(t) \in SO(3)$ with

$$\begin{aligned} L(t) &= M_1 A_1 + M_2 A_2 + M_3 A_3 \\ \nabla H &= \frac{M_1}{c_1} A_1 + \frac{M_2}{c_2} A_2 + \frac{M_3}{c_3} A_3 \end{aligned} \tag{37}$$

this allows us to state the following theorem:

Theorem 2. The projection of the extremal curves M_1, M_2, M_3 (that satisfy the condition $M_2^2 + M_3^2 > 0$ for all t) onto $SO(3)$ are of the form:

$$R(t) = \begin{pmatrix} b & ac & ad \\ ae & df - bce & -bde - fc \\ -af & de + bcf & bdf - ec \end{pmatrix} \tag{38}$$

where:

$$\begin{aligned} a &= \frac{\sqrt{M_2^2 + M_3^2}}{K}, \quad b = \frac{M_1}{K}, \\ c &= \frac{M_2}{\sqrt{M_2^2 + M_3^2}}, \quad d = \frac{M_3}{\sqrt{M_2^2 + M_3^2}} \\ e &= \sin\phi_1 \quad f = \cos\phi_1 \end{aligned} \tag{39}$$

where

$$\frac{d\phi_1}{dt} = K\frac{H - \left(M_1^2/c_1\right)}{K^2 - M_1^2} \tag{40}$$

Proof.

For details, similar proofs can be found in Jurdjevic (1997);

Biggs (2010) where it is shown that for a particular $R(0) \in SO(3)$ that:

$$L(t) = KR(t)^{-1}A_1R(t) \tag{41}$$

Noting that the stabilizer of KA_1 under the adjoint action of $SO(3)$ is the one parameter group $G = \exp(\phi_1 A_1)$, which describes rotations about the yaw axis, it is convenient to write:

$$R(t) = \exp(\phi_1 A_1)\exp(\phi_2 A_2)\exp(\phi_3 A_1) \tag{42}$$

with the appropriate ranges of the angles defined by $\phi_1, \phi_3 \in (-\pi, \pi]$ and $\phi_2 \in [0, \pi]$. Then substituting (42) into (41) and equating the left and right hand side of equation (41) it is shown that:

$$L(t) = K\begin{pmatrix} 0 & -\cos\phi_3 \sin\phi_2 & \sin\phi_2 \sin\phi_3 \\ \cos\phi_3 \sin\phi_2 & 0 & -\cos\phi_2 \\ -\sin\phi_2 \sin\phi_3 & \cos\phi_2 & 0 \end{pmatrix} \tag{43}$$

which gives

$$M_1 = K\cos\phi_2 \tag{44}$$

furthermore

$$\begin{aligned} M_2 &= K\sin\phi_2 \sin\phi_3 \\ M_3 &= K\sin\phi_2 \cos\phi_3 \end{aligned} \tag{45}$$

dividing M_2 by M_3 in (45) gives ϕ_3 in terms of the extremal solutions:

$$\frac{M_2}{M_3} = \tan\phi_3 \tag{46}$$

then for the Euler angle $\phi_3 \in (-\pi, \pi]$ it follows that

$$\sin\phi_3 = \frac{M_2}{\sqrt{M_2^2 + M_3^2}}, \cos\phi_3 = \frac{M_3}{\sqrt{M_2^2 + M_3^2}} \tag{47}$$

to obtain the expression for ϕ_1 substitute (42) into $R(t)^{-1}\frac{dR(t)}{dt} = \nabla H$ and equate the right and left hand side of this equation to yield (38). $\square$

The rotation matrix (38) is expressed in terms of the extremal curves and the Euler parameter ϕ_1. In addition the derivative of ϕ_1 is expressed in terms of the extremal curves. However, as ϕ_1 itself is not expressed analytically the problem is not in a form suitable for parametric optimization. However, we proceed to use these expressions to derive an analytic expression for the rotation matrix corresponding to the optimal angular velocities at the singularity. A singular solution of the extremal curves has been shown to be:

$$M_1 = s_1 \quad M_2 = r\sin Ct, \quad M_3 = r\cos Ct \tag{48}$$

substituting (48) into (40) yields:

$$\frac{d\phi_1}{dt} = K\frac{H - \left(s_1^2/c_1\right)}{K^2 - s_1^2} \tag{49}$$

as the right hand side is constant this yields:

$$\phi_1 = \Gamma t + \gamma \tag{50}$$

where $\Gamma = K\frac{H-\left(s_1^2/c_1\right)}{K^2-s_1^2}$ and γ is a constant of integration. Then substituting (50) and (48) into (39) yields:

$$\begin{gathered} a = \frac{r}{K}, \quad b = \frac{s_1}{K}, \quad c = \sin Ct, \quad d = r\cos Ct \\ e = \sin(\Gamma t + \gamma) \quad f = \cos(\Gamma t + \gamma) \end{gathered} \tag{51}$$

which defines the attitude motion at the singularity of the extremal curves completely analytically.

6. CONCLUSION

This paper has derived the analytic form of an optimal attitude motion at a singularity. This provides an analytic expression for an optimal motion in terms of the angular velocity history and the corresponding rotation matrix. This expression can then be used in a parametric optimization problem, where by the available parameters (the initial angular velocities) can be optimized to match boundary conditions such as initial and final pointing directions.

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