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Singularities of optimal attitude motions

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Abstract: This paper considers the problem of planning optimal attitude motions for spacecraft. The extremal solutions that result from this optimization problem are characterized and their singularities identified. Following this these singularities are solved analytically inferring the form of particular optimal velocities. These particular solutions are then integrated and their corresponding motions derived independently of a local coordinate chart. These motions have the potential to be used as smooth, optimal reference trajectories for performing certain re-orientations for spacecraft.

Keywords: Motion Planning, attitude control, Maximum principle, Singularities.

1. INTRODUCTION

Methodologies for planning and controlling attitude maneuvers of spacecraft based on geometric techniques has a rich history and includes the application of quaternion algebra, Lie group theory and geometric control theory, see Wie (1998); Spindler (1996); Jurdjevic (1997); Leonard et. al. (1995) for a few examples. In this paper we consider the optimal attitude control problem posed in Spindler (1996) from the perspective of using these methods to plan optimal, smooth and practical motions for spacecraft.

Spindler (1996) defines a fixed end point optimal control problem for the attitude control of a rigid body with the angular velocities as the control inputs, where the cost function to be minimized is a quadratic function of the angular velocity components. In particular minimizing such a cost function is desirable during a spacecraft maneuver to keep angular velocity low because high spin rates can cause undesirable tumbling motions. In addition high spin rates make it hard to receive good tracking data to monitor the spacecraft’s motion. Spindler (1996) then applies the Maximum Principle (see Jurdjevic (1997); Biggs (2010)). This procedure enables the derivation of globally defined solutions at the singularities of the extremal functions. These singular motions provide analytic expressions for rotational interpolation in a convenient simplistic form. Furthermore, these rotational motions could be used as practical reference trajectories for a spacecraft to track in order to perform particular manoeuvres.

2. ATTITUDE MOTION PLANNING PROBLEM

The orientation of a spacecraft is represented here by curves in the Special Orthogonal Group $SO(3)$ where $R(t) \in SO(3)$ and where the kinematics are described by the differential equation:

$$\frac{dR(t)}{dt} = R(t)(\Omega_1 A_1 + \Omega_2 A_2 + \Omega_3 A_3)$$

(1)

where $\Omega_1, \Omega_2, \Omega_3$ are the angular velocities and where $A_1, A_2, A_3$ form a basis for the Lie algebra of $SO(3)$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(2)

with the Lie bracket defined by $[X,Y] = XY - YX$ with $X, Y \in so(3)$. Therefore, we have $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$ and $[A_1, A_3] = -A_2$. $A_1, A_2, A_3$ describe the infinitesimal motion of the spacecraft in the yaw, pitch, roll directions respectively.

The problem addressed by Spindler (1996) is formalized in the following problem statement:

Motion Planning Problem

Compute the optimal velocities and the corresponding rigid body motions $R(t) \in SO(3)$ defined by the kinematic equations (1) that minimizes the cost function:

$$J = \frac{1}{2} \int_0^T c_1 \Omega_1^2 + c_2 \Omega_2^2 + c_3 \Omega_3^2 dt$$

(3)
with the given boundary conditions $R(0) = R_0$ and $R(T) = R_T$, where $c_1, c_2, c_3$ are constant weights and $T$ is the final time.

In an optimal control problem where we desire to motion between two fixed orientations, the weights $c_1, c_2, c_3$ can be manipulated to achieve the final desired configuration. This problem can be viewed as an optimal control problem where the angular velocities are the control inputs. The tool used to tackle this Motion planning problem is the coordinate free Maximum Principle of optimal control as described in Spindler (1996); Jurdjevic (1997); Sussmann (1997). The Maximum Principle of optimal control identifies the appropriate left-invariant Hamiltonian $H$ on the dual of the Lie algebra $\mathfrak{so}(3)^*$. The Hamiltonian corresponding to (1) and (3) is written as (see for example Jurdjevic (1997)):

$$H(p,u) = \sum_{i=1}^{3} \Omega_i p(R(t)A_i) - p_0 (\frac{1}{2} \sum_{i=1}^{3} c_i \Omega_i^2)$$  \hspace{1cm} (4)

where $p(\cdot) : T_R g \rightarrow \mathbb{R}$ such that $p(R(t)B_i), p(R(t)A_i)$ are scalar components of an element in $T_R g \mathfrak{so}(3)$, where $p_0 \geq 0$ is a fixed positive constant. The curves which satisfy the Hamiltonian (4) with $p_0 = 1$ are called regular extremals and with $p_0 = 0$ abnormal extremals. However, it has been shown in Spindler (1996) that for this optimal control problem no abnormal extremals exist. Following from the Maximum Principle and the fact that (4) is a concave function in $\Omega$, the optimal velocity inputs are given by $\frac{\partial H}{\partial \Omega_i} = 0$ it follows that:

$$\Omega_i = \frac{1}{c_i} p(R(t)A_i)$$  \hspace{1cm} (5)

As the configuration of the spacecraft is the Lie group $SO(3)$, the cotangent bundle $T^*SO(3)$ can be realized as the direct product $SO(3) \times \mathfrak{so}(3)^*$ where $\mathfrak{so}(3)^*$ is the dual of the Lie algebra Jurdjevic (1997). Therefore, the original Hamiltonian defined on $T^*SO(3)$ can be expressed as a reduced Hamiltonian on the dual of the Lie algebra $\mathfrak{so}(3)^*$. We define the linear functions $M_i = p(R(t)A_i) = \rho(A_i)$ for $i = 1, 2, 3$, see Jurdjevic (1997). Therefore, from (16) it follows that the maximizing inputs are:

$$C_i = \Omega_i$$  \hspace{1cm} (6)

Substituting (6) into (4) gives the optimal Hamiltonian

$$H = \frac{1}{2} (\frac{M_1^2}{c_1} + \frac{M_2^2}{c_2} + \frac{M_3^2}{c_3})$$  \hspace{1cm} (7)

The necessary conditions for optimality are then computed by making use of the Poisson bracket defined in terms of the Lie bracket $\{\rho(\cdot), \rho(\cdot)\} = -\dot{\rho}(\cdot, \cdot)$ which yields:

$$M_1 = \frac{c_2^2 - c_3^2}{c_1 c_2^2} M_2 M_3$$

$$M_2 = \frac{c_3^2 - c_1^2}{c_1 c_2} M_1 M_3$$

$$M_3 = \frac{c_1^2 - c_2^2}{c_1 c_2^2} M_1 M_2$$

where $M_1, M_2, M_3 \in \mathfrak{so}^*(3)$ are the extremal curves. It is interesting to note that if the weights (that would be set according to the final desired configuration) are set to correspond to the moments of inertia then the equations are exactly the Euler equations for a free-rigid body (see Whittaker (1999)). However, we note that these are a generalisation of these equations as the weights of the cost function are arbitrary. In order to characterize the extremal curves described by the equations (17) we conveniently express them in the form:

$$M_i = \frac{c_j - c_k}{c_j c_k} M_j M_k$$  \hspace{1cm} (9)

where $i = 1, 2, 3$, $j = 2, 3, 1$ and $k = 3, 1, 2$ respectively. In the next section we observe that by exploiting an additional constant of motion inherent in all left-invariant Hamiltonian systems on $SO(3)$ the evolution of the extremal curves can be reduced to the analysis of a 1 dimensional ordinary differential equation and solved in terms of Jacobi elliptic functions. In Spindler (1996) only a numerical example of the general case is given.

3. REDUCTION AND SOLUTION OF THE EXTREMAL CURVES

The initial stage of the procedure is to derive analytic expressions for the optimal angular velocity history and the corresponding rotation matrix. Note that when $i = 1$ then $j = 2, k = 3$ when $i = 2$ then $j = 3, k = 1$ and when $i = 3$, then $j = 1, k = 2$:

Theorem 1. The optimal angular velocities $\Omega_i^*$ that minimize the cost function (3) can be expressed in the form:

$$\Omega_i^* = \frac{s_3}{c_i} \sqrt{\alpha \Omega^* + c_i \frac{s_i}{s_j}}$$  \hspace{1cm} (10)

where $sn(\cdot, \cdot)$ is a Jacobi elliptic function and where the constants $C_i$ are defined by:

$$C_i = sn^{-1} \left( \frac{c_i \Omega_i(0)}{\sqrt{s_i} \sqrt{s_j}} \right)$$  \hspace{1cm} (11)

with

$$s_i = -\beta + \frac{\sqrt{\beta^2 - 4 \alpha \chi}}{2 \alpha}$$

$$s_j = -\beta - \frac{\sqrt{\beta^2 - 4 \alpha \chi}}{2 \alpha}$$

and

$$\alpha = \frac{(c_i - c_j)(c_i - c_k)}{c_i c_j c_k}$$

$$\beta = \frac{4 c_i c_j H - 2 c_i (c_j + c_k) H + 2 c_i K^2 - (c_j + c_k) K^2}{c_i c_j c_k}$$

$$\chi = \frac{(2 c_j H - K^2)(2 c_i H - K^2)}{c_i c_j c_k}$$

with the conserved quantities $H$ and $K$ defined in terms of the initial angular velocities:

$$H = \frac{1}{2} (c_i \Omega_i^2(0) + c_j \Omega_i^2(0) + c_j \Omega_i^2(0))$$

$$K^2 = c_i^2 \Omega_i^2(0) + c_j^2 \Omega_i^2(0) + c_k^2 \Omega_i^2(0)$$

(14)

Proof. The optimal Hamiltonian is given by:

$$H = \frac{1}{2} \left( \frac{M_1^2}{c_1} + \frac{M_2^2}{c_2} + \frac{M_3^2}{c_3} \right)$$

(15)

where the extremal curves $M_1, M_2, M_3 \in \mathfrak{so}^*(3)$ are defined in terms of the weights $c_i$ of the cost function and the optimal angular velocities:

$$M_i = c_i \Omega_i^*$$

(16)

the corresponding Hamiltonian vector fields are then given by the Poisson bracket (see Jurdjevic (1997) for details):

$$M_i = \left( \frac{c_j - c_k}{c_j c_k} \right) M_j M_k$$

(17)

it is easily shown that the Casimir function:

$$K^2 = M_1^2 + M_2^2 + M_3^2$$

(18)
is constant along the Hamiltonian flow. Illustrating the solution for \( M_1 \) the solutions for \( M_2 \) and \( M_3 \) follow analogously. From (17) we have:

\[
(M_1)^2 = \left( \frac{c_2 - c_3}{c_2c_3} \right)^2 M_2^2 M_3^2
\]

then using the Hamiltonian (15) and the Casimir function (18) write \( M_2 \) and \( M_3 \) explicitly in terms of \( M_1 \) to yield:

\[
M_2^2 = c_2 \left( 2 c_3 H - K^2 + M_1^2 \right) / (c_3 M_2^2)
\]

\[
M_3^2 = c_3 \left( 2 c_2 H - K^2 + M_1^2 \right) / (c_3 M_3^2)
\]

then substituting (20) into (19) and simplifying yields:

\[
\int dM_1 = \frac{1}{\sqrt{\alpha s_2}} \frac{1}{\sqrt{M_1^2 s_2}} \int dM_1
\]

and therefore

\[
M_1 = \sqrt{\frac{s_1}{s_2}} \text{sn}(\sqrt{\frac{s_1}{s_2}})
\]

where \( s_1 = \sqrt{1 + \beta^2 - 4\alpha\chi} \) and \( s_2 = -\sqrt{1 + \beta^2 - 4\alpha\chi} / \alpha \)

from (28) it can be seen that the singularities of the elliptic integral correspond to \( M_1 = 0 \) in equation (21) and this fact is used to derive analytic expressions for this particular case. In this section we investigate the singularities of the extremal curves, that is where the elliptic integral (27) is not well defined. These singularities correspond to fixed points of the differential equation (21) and we define a root of this by the constant \( M_1 = s_1 \). At this singularity we can solve the extremal curves \( M_j \) and \( M_k \) using the Casimir function such that:

\[
K^2 - s_1^2 = M_j^2 + M_k^2
\]

as the left hand side of (29) is constant this suggests using polar coordinates for \( M_j \) and \( M_k \):

\[
M_j = r \sin \theta, \quad M_k = r \cos \theta
\]

where \( \theta \) is given by substituting (48) into (29):

\[
r = \left( K^2 - s_1^2 \right)^{1/2}
\]

where \( K \) is defined by the equation (13) with \( i = 1, j = 2, k = 3 \). Defining the constants:

\[
\begin{align*}
\alpha &= M_1^2 + \beta M_2^2 = \chi \\
\beta &= -\sqrt{2} \chi \\
\chi &= \sqrt{1 + \beta^2 - 4\alpha}\chi
\end{align*}
\]

then equation (21) can be expressed as:

\[
M_1 = (s_1 - M_1^2) (s_2 - M_1^2)
\]

and setting \( m = \frac{1}{2} \) the equation (24) becomes:

\[
\int_0^\alpha dt = \frac{1}{\alpha s_2} \int_{\alpha}^{\alpha s_2} \frac{dM_1}{\sqrt{M_1^2 s_2}}
\]

and therefore

\[
M_1 = \sqrt{s_1} \text{sn}(\sqrt{\frac{s_1}{s_2}} C_1 + s_1)
\]

where \( C_1 \) is defined in (11) and therefore from (16) the angular velocity is constant.

Note that this analytic solution has been verified against numerical integration.

4. SINGULARITIES OF THE EXTREMAL CURVES

The differential equation (21) can be expressed as an Elliptic integral of the first kind:

\[
\int \frac{M_1(t)}{M_1(0)} \sqrt{\alpha M_1^2 + \beta M_2^2 + \chi} dM_1
\]

In this paper we investigate the case at the singularities of this elliptic integral, that is where

\[
\alpha M_1^2 + \beta M_2^2 + \chi = 0
\]

from (28) it can be seen that the singularities of the elliptic integral correspond to \( M_1 = 0 \) in equation (21) and this fact is used to derive analytic expressions for this particular case. In this section we investigate the singularities of the extremal curves, that is where the elliptic integral (27) is not well defined. These singularities correspond to fixed points of the differential equation (21) and we define a root of this by the constant \( M_1 = s_1 \). At this singularity we can solve the extremal curves \( M_j \) and \( M_k \) using the Casimir function such that:

\[
K^2 - s_1^2 = M_j^2 + M_k^2
\]

5. INTEGRATION PROCEDURE

To obtain the most convenient form for the rotation matrix the differential equations are expressed in Lax Pair form:

\[
L(t) = [L(t), \nabla H]
\]

\[
\frac{dR(t)}{dt} = R(t)\nabla H
\]

where \( R(t) \in SO(3) \) with

\[
L(t) = M_1 A_1 + M_2 A_2 + M_3 A_3
\]

\[
\nabla H = \frac{M_1}{c_1} A_1 + \frac{M_2}{c_2} A_2 + \frac{M_3}{c_3} A_3
\]

this allows us to state the following theorem:

\textbf{Theorem 2.} The projection of the extremal curves \( M_1, M_2, M_3 \) (that satisfy the condition \( M_1^2 + M_2^2 + M_3^2 > 0 \) for all \( t \)) onto \( SO(3) \) are of the form:

\[
\dot{R}(t) = \left[ \begin{array}{ccc} b & ac & ad \\ ae & d - bce - bde - fc \\ -af & de + bcf & bdf - ec \end{array} \right]
\]

where:

\[
\begin{align*}
\alpha &= \sqrt{M_1^2 + M_2^2} \\
b &= \frac{M_1}{K} \\
c &= \frac{M_2}{K} \\
d &= \frac{M_3}{K} \\
e &= \sin \phi_1 \\
f &= \cos \phi_1
\end{align*}
\]

\[
\frac{d\phi_1}{dt} = \frac{K}{M_1} \left( H - \frac{M_1^2}{c_1} \right)
\]

Proof.

For details, similar proofs can be found in Jurdjevic (1997);
Biggs (2010) where it is shown that for a particular $R(0) \in SO(3)$ that:

$$L(t) = KR(t)^{-1}A_1R(t)$$  \hspace{1cm} (41)

Noting that the stabilizer of $KA_1$ under the adjoint action of $SO(3)$ is the one parameter group $G = \exp(\phi A_1)$, which describes rotations about the yaw axis, it is convenient to write:

$$R(t) = \exp(\phi A_1)\exp(\phi A_2)\exp(\phi A_1)$$  \hspace{1cm} (42)

with the appropriate ranges of the angles defined by $\phi_1, \phi_2 \in (-\pi, \pi]$ and $\phi_1 \in [0,\pi]$. Then substituting (42) into (41) and equating the left and right hand side of equation (41) it is shown that:

$$L(t) = K \begin{pmatrix} 0 & -\cos \phi_1 \sin \phi_2 & \sin \phi_2 \sin \phi_3 \\ \cos \phi_1 \sin \phi_2 & 0 & -\cos \phi_2 \\ -\sin \phi_2 \sin \phi_3 & \cos \phi_2 & 0 \end{pmatrix}$$  \hspace{1cm} (43)

which gives

$$M_1 = K \cos \phi_2$$  \hspace{1cm} (44)

$$M_2 = K \sin \phi_2 \sin \phi_3$$  \hspace{1cm} (45)

$$M_3 = K \sin \phi_2 \cos \phi_3$$

dividing $M_2$ by $M_3$ in (45) gives $\phi_1$ in terms of the extremal solutions:

$$\frac{M_2}{M_3} = \tan \phi_3$$  \hspace{1cm} (46)

then for the Euler angle $\phi_3 \in (-\pi, \pi]$ it follows that

$$\sin \phi_3 = \frac{M_2}{\sqrt{M_2^2 + M_3^2}}, \cos \phi_3 = \frac{M_3}{\sqrt{M_2^2 + M_3^2}}$$  \hspace{1cm} (47)

to obtain the expression for $\phi_1$ substitute (42) into $R(t)^{-1}dR(t) = VH$ and equate the right and left hand side of this equation to yield (38). □

The rotation matrix (38) is expressed in terms of the extremal curves and the Euler parameter $\phi_1$. In addition the derivative of $\phi_1$ is expressed in terms of the extremal curves. However, as $\phi_1$ itself is not expressed analytically the problem is not in a form suitable for parametric optimization. However, we proceed to use these expressions to derive an analytic expression for the rotation matrix corresponding to the optimal angular velocities at the singularity. A singular solution of the extremal curves has been shown to be:

$$M_1 = s_1, \quad M_2 = r \sin \Gamma t, \quad M_3 = r \cos \Gamma t$$  \hspace{1cm} (48)

substituting (48) into (40) yields:

$$\frac{d\phi_1}{dt} = K \frac{H - (s_1^2/c_1)}{H^2 - s_1^2}$$  \hspace{1cm} (49)

as the right hand side is constant this yields:

$$\phi_1 = \Gamma t + \gamma$$  \hspace{1cm} (50)

where $\Gamma = K \frac{H - (s_1^2/c_1)}{H^2 - s_1^2}$ and $\gamma$ is a constant of integration. Then substituting (50) and (48) into (39) yields:

$$a = \frac{r}{K}, \quad b = \frac{s_1}{K}, \quad c = \sin \Gamma t, \quad d = \cos \Gamma t$$

$$e = \sin(\Gamma t + \gamma), \quad f = \cos(\Gamma t + \gamma)$$

which defines the attitude motion at the singularity of the extremal curves completely analytically.

6. CONCLUSION

This paper has derived the analytic form of an optimal attitude motion at a singularity. This provides an analytic expression for an optimal motion in terms of the angular velocity history and the corresponding rotation matrix. This expression can then be used in a parametric optimization problem, where by the available parameters (the initial angular velocities) can be optimized to match boundary conditions such as initial and final pointing directions.

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