A Renormalisation Approach to Wave Propagation in 0-3/3-3 Connectivity Composites

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Abstract - Particulate composite materials with variable connectivity are encountered extensively in ultrasonic applications. Examples include a wide range of propagation media, active piezocomposite transducers, matching layers and backing materials. The manner in which mechanical waves propagate in such materials depends strongly on particle distribution and dimensions, in addition to volume fraction. In this paper a commercial finite element package is used to study wave propagation through a realistic model of the complex microstructure of such composites. We also mathematically model a particular class of deterministically generated fractal composites using renormalisation and demonstrate this methodology by examining the propagation of a wave impulse through the Sierpinski Gasket.

INTRODUCTION

In this paper we examine the propagation of ultrasonic waves through dispersive composite materials. We stochastically generate realistic microstructure geometries of such materials and then simulate the propagation of an acoustic wave by means of a finite element analysis. We then examine, using a renormalisation method, this wave propagation in a fractal structure obtained by a deterministic generation process. Sectional micrographs were taken of a PZ-27-epoxy composite and the digitised images obtained processed so that the geometry of each individual particle was identified and the measured. The cluster size distribution then inputs the geometry and material characteristics into a commmercial finite element/wave propagation code PZFlex, which models ultrasonic wave propagation in peizoelectric

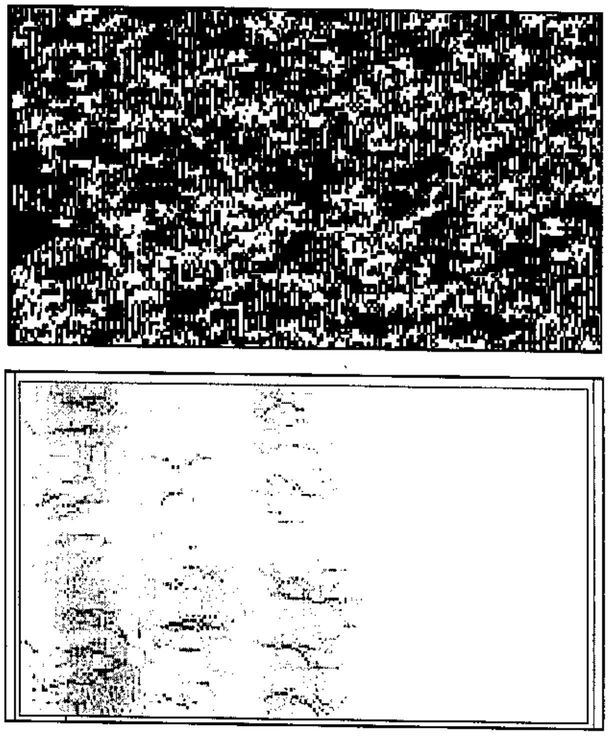

Figure 1. Simulated composite structure (volume fraction 63.3%) and corresponding pressure profile of wave impulse travelling from left to right.

materials [1, 2]. Figure 1 shows a sample of a stochastically simulated composite and the spatial distribution of a travelling pressure wave in the medium. An estimate for the longitudinal velocity of the wave can thus be obtained. It is found that as one increases the volume fraction of the particulate this velocity will change and around the percolation threshold obtains a minimum. These values have been found to compare favourably with those

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obtained experimentally for tungsten-epoxy composites and a fuller discussion of this can be found in [1].

DETERMINISTIC FRACTAL STRUCTURES

Heterogeneous geometries have a dramatic effect on the propagating wave, particularly in fractal media [3, 4]. A recent approach which provides a mathematical framework for the analysis of wave propagation in such media is input/output renormalisation [4, 5]. In particular it is possible to examine the dynamic properties of linear field equations, incorporating boundary conditions, in a class of finitely ramified fractal structures. In this paper we will analyse the propagation of solutions to the wave equation in the Sierpinski Gasket (SG), as an idealised 0-3 composite, to measure the efficacy of the FEM approach. The SG consists of a series of nodes which are initially imbedded in a two-dimensional plane. A temporal vibration is applied to one boundary (a corner or *input* node) and the vibration traverses the structure by causing transverse oscillations in a succession of neighbouring nodes (see Figure 2).

We will restrict our study to deterministic fractal graphs which are defined by a recursive process $\wp^{(n+1)} = \Im\left[\wp^{(n)}\right]$. Each graph consists of v copies of itself at level n to form its equivalent at level $n+1$. The adjacency matrix $H^{(n)}$ describes the connectivity properties of the graph with its $(i,j)^{th}$ element giving the number of edges connecting site i to site j. For the family of graphs considered in this paper the total number of edges emanating from a given site is a constant, e. The recursive nature of these graphs can now be expressed $H^{(n+1)} = \overline{H}^{(n)} + V^{(n)}$ where $\overline{H}^{(n)}$ is a block diagonal matrix whose v blocks are equal to $H^{(n)}$ and $V^{(n)}$ is the connection matrix. For the SG,

$$H^{(n)}_{p,q} = \begin{cases} 1 & : H^{(0)}_{\{(p-1)\bmod 3+1,(q-1)\bmod 3+1\}} = 1 \\ & \quad and \left\lceil p/3 \right\rceil = \left\lceil q/3 \right\rceil \\ 0 & : Otherwise \end{cases}$$

where $p,q = 1,\ldots,N_n$ and $H^{(0)}_{i,j} = 0, i = j$, $H^{(0)}_{i,j} = 1, i \neq j$. The connection matrix is given recursively: $V^{(n)}_{p,q} = \{1 \;: \{p,q\} \in L_n; 0 : Otherwise\}$ where $L_n = \bigcup_{i=2}^{n} \left\{W^{(i)}, L_{i-1}, L_{i-1}+3^{i-1}, L_{i-1}+2\times 3^{i-1}\right\}$, $L_2 = W^{(2)}$ and $W^{(n)} = \{\{(3^{n-1}+1)/2, 1+3^{n-1}\}, \{3^{n-1}, 1+2\times 3^{n-1}\}, \{2\times 3^{n-1}, (5\times 3^{n-1}+1)/2\}, \{1+3^{n-1}, (3^{n-1}+1)/2\}, \{1+2\times 3^{n-1}, 3^{n-1}\} \{(5\times 3^{n-1}+1)/2, 2\times 3^{n-1}\}\}$. Let $u^{(n)}$ be the *vertical displacement* vector at each of the lattice sites. The spatially discretised wave equation can be written,

$$\frac{\partial^2 u^{(n)}}{\partial t^2} = A^{(n)} u^{(n)} + f^{n}(t) \qquad (1)$$

where $A^{(n)}$ is a matrix equivalent of the Laplacian operator incorporating the boundary conditions and $f^{(n)}$ is the forcing term used to initiate the travelling impulse. Note that this approach is distinctly different from that in [3, 6] as here the Laplacian is defined solely by the immediate neighbourhood of a given node at a particular generation level rather than with regard to nodes at the previous generation level. By transforming (1) into the Laplace domain the solution takes the form, $\hat{u}^{(n)} = \hat{G}^{(n)}(s)\left(\hat{f}^{(n)}(s) + s\,u_0 + u'_0\right)$ where $\hat{u}^{(n)}(s) = \mathcal{L}\left[u^{(n)}(t)\right]$, $u_0 = u(t=0), u'_0 = u_t(t=0)$ and $\hat{G}^{(n)}(s) = \left[s^2 I - \frac{D}{\Delta x^2} A^{(n)}\right]^{-1}$ is the Green function matrix of the system. For the SG, $A^{(n)}_{p,q} = H^{(n)}_{p,q} + V^{(n)}_{p,q} + B^{(n)}_{p,q} - 3\delta_{p,q}$ where B is a sparse matrix containing the boundary conditions at the input and output nodes $B^{(n)}_{p,q} = \begin{cases} 1 & : p = q \; and \left(p = (3^n+1)/2 \; or \; p = 3^n \right) \\ 0 & : Otherwise \end{cases}$. The set of constituent nodes which comprise the SG fractal are given by,

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$SG^{(n)} = \bigcup_{j=2}^{n} \left\{ \begin{array}{l} SG^{(j-1)}, SG^{(j-1)} + \left\{ 2^{j-1}\eta / \varepsilon(j), 0 \right\}, \\ SG^{(j-1)} + \left\{ 2^{j-2}\eta / \varepsilon(j), 2^{j-1}a / \varepsilon(j) \right\} \end{array} \right\}$ where $SG^{(1)} = \left\{ \{0,0\}, \{\eta,0\}, \{\eta/2, a\} \right\}$, the length scale at generation level n is $\varepsilon(n) = 2^n - 1$, the diameter of the SG is η and $a = \sqrt{3}/2\,\eta$.

FD SOLUTION OF THE WAVE EQUATION

Using (1) the wave equation can be discretised on the SG using finite differences (FD) and the above neighbourhood structure. We start with the unperturbed SG ($u_i^{(n)}(0) = u_i'^{(n)}(0) = 0$, $i = 1, \ldots, N_n = 3^n$) and apply the forcing function $f(t) = \gamma \sin(2\pi\omega t),\ t \le 1/2\omega;\ 0,\ t > 1/2\omega$ to input node 1 with Neumann boundary conditions at the output nodes and Dirichlet at the input node [6]. The explicit discretisation scheme used here is,

$$u_i^{(n),m+1} = 2u_i^{(n),m} - u_i^{(n),m-1} + \left(\frac{\Delta t}{\Delta x}\right)^2 D \left\{ \sum_{j=1}^{N_n} A_{i,j}^{(n)} u_i^{(n),m-1} + \delta_{i1} f((m-1)\Delta t) \right\}$$

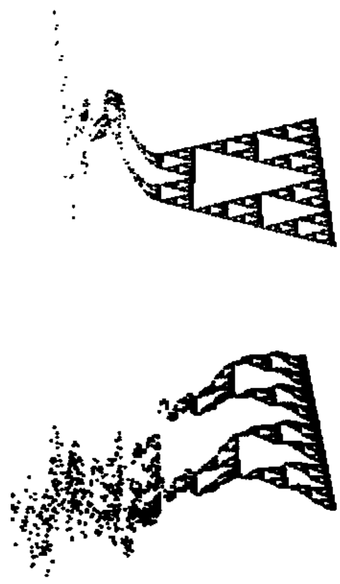
In Figure 2 the propagation of this impulse in the SG is shown at two instants of time.

RENORMALISATION

To solve (1) using renormalisation we initially ignore the boundary conditions and then reintroduce it later by means of a suitable matrix transformation. Given this geometrical framework it is possible to derive the following renormalisation relationship [5],

$$\hat{G}^{(n+1)} = \overline{G}^{(n)} + \overline{G}^{(n)} V^{(n)} \hat{G}^{(n+1)} \qquad (2)$$

where $\overline{G}^{(n)}$ is a block diagonal matrix of dimensions $N_{n+1} \times N_{n+1}$ whose blocks equal $\hat{G}^{(n)}$. This system of linear equations in $\hat{G}_{ij}^{(n)}$ will give rise to the renormalisation recursions for the bare (i.e. without boundary conditions) pivotal Green functions. To reintroduce the boundary conditions



(a)

(b)

Figure 2. Wave impulse profiles in the Sierpinski Gasket (n=7, Δt=10^{-5}, η = 1, ω = 2.5, (a) t = 0.4s (b) t=2.5s).

we use the matrix transformation,

$$\hat{G}^{*} = \hat{G}^{(n)} + \hat{G}^{(n)} B\ \hat{G}^{*} \qquad (3)$$

where $\hat{G}^{*}$ is the Green function matrix which incorporates the boundary conditions. It is then possible to use (2) and (3) to derive a set of coupled recursion relationships,

$$X = x + \frac{2y^2}{\Delta_1} - 2Y, \quad Y = y^2 \frac{1 - x + y}{\Delta_1 \Delta_2},$$

$$x^{*} = x + \frac{2y^2}{1 - x - y}, \quad y^{*} = \frac{y}{1 - x - y}$$

where $\Delta_1 = 1 - x + y - 2y$, $\Delta_2 = 1 + y - (x - y)(x + y)$ and $x(s) \equiv \hat{G}_{11}^{(n)}(s)$, $y(s) \equiv \hat{G}_{15}^{(n)}(s)$ are the two bare pivotal Green functions at the forced (*input*) node and the *transport* between the input node and the output nodes respectively. Capital letters denote the two pivotal Green functions at generation $n+1$ and * denotes the inclusion of boundary conditions. The effective properties of the composite can then be derived from these recursion relationships or specific

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information on the temporal evolution of a particular node or set of nodes by applying an inverse Laplace transform to obtain, $u_i^{(n)}(t) = \mathcal{L}^{-1}\left\{\hat{G}_{i1}^{(n)}(s)\hat{f}(s)\right\}$. In Figure 3 we show the time evolution of the vertical displacement for one of the output nodes for the 7th generation Sierpinski Gasket. Essentially the time step Δt must be so drastically reduced to ensure stability that the number of iterations required to realise a propagating wave grows prohibitively large and/or rounding error dominates the solution.

Figure 3. Time evolution for an output boundary node of the 7th level SG with $\eta = 1$ and $\omega = 2.5$.

CONCLUSIONS

An investigation of the dispersive effect of particle volume fraction in random 0-3 composites has been investigated using a finite element approach. The results suggest that at intermediate volume fractions there is a *percolation effect* which slows the wave to a velocity below the effective velocity that would arise in a homogeneous matrix of either of the two constituent materials. This result concurs with experimental findings for tungsten-epoxy composites. To further our understanding of this phenomenon and as a test on the efficacy of the FEM approach a deterministically generated fractal composite structure, the Sierpinski Gasket (SG) was examined using a renormalisation approach applied to the wave equation. The results for the perturbations at the output nodes agree with those obtained numerically using a finite difference approximation to the Laplacian. Of course the renormalisation approach provides explicit details of the wave structure as it traverses the composite. This provides a concrete link between the internal geometry of the structure and the wave profile. This will act as a guide in utilising the wealth of information contained in the waveform emerging from the random composites and an investigation into this inverse problem will be the subject of future investigation.

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