Entanglement of multiparty stabilizer, symmetric, and antisymmetric states

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We study various distance-like entanglement measures of multipartite states under certain symmetries. Using group averaging techniques we provide conditions under which the relative entropy of entanglement, the geometric measure of entanglement and the logarithmic robustness are equivalent. We consider important classes of multiparty states, and in particular show that these measures are equivalent for all stabilizer states, symmetric basis and antisymmetric basis states. We rigorously prove a conjecture that the closest product state of permutation symmetric states can always be chosen to be permutation symmetric. This allows us to calculate the explicit values of various entanglement measures for symmetric and antisymmetric basis states, observing that antisymmetric states are generally more entangled. We use these results to obtain a variety of interesting ensembles of quantum states for which the optimal LOCC discrimination probability may be explicitly determined and achieved. We also discuss applications to the construction of optimal entanglement witnesses.

I. INTRODUCTION

The quantification of the entanglement of multipartite quantum states has attracted a great deal of attention in recent years. Entanglement measures are real valued functions of quantum states that attempt to quantify the amount of entanglement possessed by different quantum states [1, 2, 3, 4]. In the case of multipartite entanglement the quantification of entanglement is complicated by the fact that multipartite entanglement is known to exist in a variety of different inequivalent forms, and it is still not clear what the significance of these different forms is [5]. Nevertheless, a variety of different entanglement measures have been proposed for the multipartite setting, with a variety of different motivations [6]. Computing these measures, and understanding the relationships between them, is usually very difficult as most measures are defined as the solutions to difficult variational problems.

In this paper we will make progress on this problem by considering three multipartite entanglement measures, which attempt to quantify the 'distance' between a quantum state and the set of separable states. The measures that we will consider are the *(Global) Robustness of Entanglement* [7], the *Relative Entropy of Entanglement* [1, 2], and the *Geometric Measure* [8]. Although these quantities do not capture all of the subtleties of entanglement (in particular the variant of the Geometric measure that we will consider is not an entanglement monotone on mixed states, and none of these measures discriminate between the different forms of multiparty entanglement), all these quantities have an operational interpretation as bounds on the information that may be gained by LOCC measurements [9], and the relative entropy of entanglement in particular has applications to the distillation of multipartite entanglement [10].

The three measures that we consider are related by known inequalities [9]. In this paper we will investigate conditions under which those inequalities can be shown to be tight. Our methods rely heavily upon the use of symmetry techniques that have been applied in papers such as [11, 12, 13, 14, 15]. We use these methods, together with some methods from linear algebra, to show that the inequalities in [9] are saturated for stabilizer states (cf. [22]), antisymmetric states, and symmetric states with fixed "Hamming weight" (or fixed "Type" for constituent particles of dimensions greater than a qubit). In the case of the last two families, explicit expressions may be derived for the entanglement - these values are summarized in Table I. Explicit expressions may also be derived for several families of stabilizer state - we refer the reader to [22] for details.

In the next section we introduce these measures and the relationship between them and we discuss the motivation for our investigations in terms of entanglement witnesses and applications to LOCC information gain. In section III we present our general approach and two simple examples (stabilizer states and symmetric states of fixed *type*) where we can use group averaging to prove equivalence of the measures. For symmetric states of

n-party state	$E_R = E_g = \log_2(1+R_g)$
Symmetric state $\|S(n,\overrightarrow{k})\rangle$	$n\log_2 n - \log_2 n!$
	$-\sum_{j=1}^{d}(k_j \log_2 k_j - \log_2 k_j!)$
Antisymmetric state $\|\Psi_a\rangle$	$\log_2(n!)$

TABLE I: Summary of the entanglement values obtained for the symmetric (eq.(32)) and antisymmetric (eq.(47)) states that we consider. In addition the measures are equal for all pure stabilizer states, and equivalent for the (normalised) projector onto the symmetric subspace. The values of these measures for several classes of stabilizer states are obtained in [22]

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fixed type, we require a useful result from linear algebra, the Takagi decomposition, which is discussed in Appendix A. In the final section we apply these methods to certain antisymmetric states, for which the measures can also be calculated exactly. In Appendix B we review some notions of group theory. In Appendix C, we give a more general group theoretic treatment of the problem - the results in the Appendixes B and C are used throughout the rest of this paper.

II. OUTLINE OF PROBLEM AND MOTIVATION

We now go through the definition of the entanglement measures that we will use throughout this paper, along with some of their operational interpretations.

In the following, we assume that our Hilbert space consists of m local Hilbert space, $\mathcal{H} \stackrel{\text{def}}{=} \otimes_{i=1}^{m}\mathcal{H}_i$, each with finite dimensionality. Unless stated otherwise, we treat entanglement and LOCC with respect to the cut $\otimes_{i=1}^{m}\mathcal{H}_i$, i.e. each Hilbert space is assumed to belong entirely to distinct parties.

The *geometric measure of entanglement* [8, 16, 17], is defined as

$$E_g(|\psi\rangle) = \min_{|\Phi\rangle\in\text{Pro}(\mathcal{H})} -\log_2(|\langle\Phi|\psi\rangle|^2), \quad (1)$$

where $\text{Pro}(\mathcal{H})$ is the set of product states on $\mathcal{H}$. This is the distance between state $|\psi\rangle$ and the closest product state $|\Phi\rangle$ in terms of fidelity, and has operational significance, for example in relation to channel capacities [18]. This measure can be extended to the mixed state case in a natural way via the convex roof method [17]. However, here we will define the "geometric measure" $G(\rho)$ for mixed states as,

$$G(\rho) \stackrel{\text{def}}{=} -\log_2 \max_{\sigma\in\text{Sep}} \text{tr}\sigma\rho = -\log_2 \max_{|\Phi\rangle\in\text{Pro}(\mathcal{H})} \langle\Phi|\rho|\Phi\rangle, \quad (2)$$

where "Sep " means the set of all separable states on $\mathcal{H} = \otimes_{i=1}^{m}\mathcal{H}_i$. Note that $G(\rho)$ is no longer an entanglement monotone for general mixed states. Indeed, $G(\rho) > 0$ for many non-pure separable states - it can attain its maximal value, for example, on the maximally mixed state. Nevertheless, in the following discussion, $G(\rho)$ works as a natural extension of $E_g(|\Psi\rangle)$ from a mathematical view point, although $G(\rho)$ has an unusual physical meaning - as it represents a 'distance' to the nearest *pure* product state (not just the nearest separable state) it acts more like a measure of both mixedness and entanglement. In the context of LOCC state discrimination, in which both purity and entanglement have operational significance, it is natural that quantities measuring both entanglement and mixedness should play an important role. $G(\rho)$ is also a useful quantity when constructing entanglement witnesses as we will see in Sec. II B.

The *relative entropy of entanglement* is defined as the "distance" to the closest separable state with respect to the relative entropy [2],

$$E_R(\rho) := \min_{\omega\in\text{Sep}} S(\rho||\omega), \quad (3)$$

where $S(\rho||\omega) = -S(\rho) - \text{tr}\{\rho\log_2\omega\}$ is the relative entropy, $S(\rho)$ is the von Neumann entropy, and Sep is the set of separable states. Note that strictly speaking $S(\rho||\omega)$ is not a distance function. Operationally it tells us, for example, how easy it is to confuse the state ρ for a separable state in the asymptotic setting [1].

The *global robustness of entanglement* $R_g(\rho)$ is defined as [7]:

$$\begin{aligned} R_g(\rho) &:= \min t \\ &\text{such that } \exists \text{ a state } \Delta, \text{ satisfying} \\ \omega &= \tfrac{1}{1+t}(\rho + t\Delta) \in \text{Sep}, \end{aligned} \quad (4)$$

where Sep is the set of separable states. We can understand this as the minimum (arbitrary) noise Δ that we need to add to make the state separable. It can be used also to consider the robustness of operations against noise [19]. In the bipartite setting it gives a bound on how well teleportation can be performed [20]. Recently in the general multipartite setting, it has been shown to be related to optimal entanglement witnesses [29] (as used in Sec. II B). We will often refer to this measure simply as the robustness. For simplicity in expressions, we will sometimes make use of the logarithmic version, the logarithmic robustness [21]:

$$LR_g(\rho) := \log_2(1 + R_g(\rho)). \quad (5)$$

In a sense, these are very broadly defined measures, and do not pick out many of the possible subtleties of entanglement in the multipartite scenario (for example the difference between entanglement arising from multipartite entanglement and that from bipartite entanglement). However, in addition to those applications already mentioned, they have recently found several interesting operational interpretations (e.g. [4]), including as bounds on how much information can be accessed from states under LOCC [9, 22].

In [9] (cf. [17]) it has been shown that the following relation holds between the three different distance-like entanglement measures that we have defined above,

$$r(\rho) \geq E_R(\rho) + S(\rho) \geq G(\rho), \tag{6}$$

where we denote $|A| := \mathrm{tr}(A)$, define P as the projector onto the support of ρ [23], and $r(\rho)$ is defined as:

$$r(\rho) := \log_2 |P| \left(1 + R(\frac{P}{|P|})\right). \tag{7}$$

For pure states the inequalities (6) reduce to

$$LR_g(|\psi\rangle) \geq E_R(|\psi\rangle) \geq E_g(|\psi\rangle). \tag{8}$$

The difficulty in calculating these measures usually increases from right to left as the defining optimization problems get harder. We will see that in certain cases we can show equivalence across (6) and (8).

Before we go into any proofs and examples, we will discuss some motivations for studying this problem. Firstly, showing equivalence across (6) immediately allows the optimization problems of all measures to be reduced to that of the geometric measure, which is easiest amongst the measures. This means that all the possible operational interpretations of all the measures can be studied in terms of the easier, more calculable measure.

In particular we now focus on two applications of these measures - to LOCC state discrimination and the study of entanglement witnesses.

A. Bounds on state discrimination by separable operations

The quantities described in the preceding section all arose naturally in the authors' previous work [9] on LOCC state discrimination. There it was shown that the measures defined above supply *upper* bounds on the effectiveness of orthogonal state discrimination when the measurements are implemented separable or LOCC operations. In this section we will discuss how in situations of high symmetry the above quantities can also give tight *lower* bounds on what may be achieved by separable operations.

Let us consider at first an ensemble of states $\{p_i, \rho_i\}$ (the p_i are probabilities), the ρ_i are states that we must discriminate by separable operations. Then we may derive the following upper bound on the total success probability for discrimination by a separable POVM $\{M_i\}$ [22],

$$\begin{aligned} P_s &= \sum_i p_i \mathrm{tr}\{M_i \rho_i\} \leq \sum_i p_i \mathrm{tr}\{M_i\} 2^{-G(\rho_i)} \\ &\leq \max_i \{p_i 2^{-G(\rho_i)}\} \sum_i \mathrm{tr} M_i = D \max_i \{p_i 2^{-G(\rho_i)}\} \end{aligned} \tag{9}$$

where the first inequality follows from the fact that each M_i is proportional to a separable state, and D is the total dimension of the system.

This upper bound can be achieved by separable operations in cases where the ensemble is generated by a local irreducible unitary group acting on some *fiducial* state ϕ, i.e. $\{\rho_i = U_i \phi U_i^\dagger | i = 1..N\}$, where each state is given to us with uniform prior probability $1/N$. This can be seen as follows. As all the states are local unitarily equivalent to the fiducial state, the upper bound becomes :

$$P_s \leq \frac{D}{N} 2^{-G(\phi)}. \tag{10}$$

This can be achieved by the separable POVM defined by:

$$M_i := \frac{D}{N} U_i \Omega U_i^\dagger \tag{11}$$

where Ω is the optimal product state that achieves the geometric measure of the fiducial state ϕ.

This motivates the question as to whether 'closest separable states' may be used to obtain separable POVMs that give good lower bounds in other cases. Let us suppose that each state in the ensemble has a 'closest' product state ψ_i, i.e. for each i the quantity

$$\mathrm{tr}\{\rho_i \psi_\mathrm{i}\} \tag{12}$$

is as large as it can be for an overlap between ρ_i and a separable state. Our goal in trying to find a good separable measurement to discriminate the ensemble will be to 'pretend' that we are instead trying to discriminate these closest separable states from each other, and use the outcomes to infer information about the original ensemble $\{\rho_i\}$. With this goal in mind we write down the square root measurement for discriminating the ψ_i:

$$M_i := {\psi_m}^{-1/2} p_i \psi_i {\psi_m}^{-1/2} \tag{13}$$

where ψ_m is the mean state:

$$\psi_{mean} := \psi_m := \sum_i p_i \psi_i. \tag{14}$$

For general ensembles $\{\rho_i\}$ with general closest states ψ_i there is no guarantee that the POVM elements M_i defined in equation (13) will themselves be separable. However, under the restriction that *the mean separable state is itself maximally mixed* then the M_i defined in equation (13) will indeed define a separable measurement. In fact the POVM elements will be given by the separable operators:

$$M_i := p_i D \psi_i \tag{15}$$

where D is the total dimension of the system. If we apply this measurement to the original ensemble, then we find that the optimal probability of successful discrimination P_s will be bounded by the following expression:

$$\begin{aligned} P_S \geq & \sum_i p_i \mathrm{tr}\{\rho_i p_i D \psi_i\} = D \sum_i p_i^2 \mathrm{tr}\{\rho_i \psi_i\} \\ = & \; D \sum_i p_i^2 2^{-G(\rho_i)} \geq D \min_i \{p_i 2^{-G(\rho_i)}\} \end{aligned} \tag{16}$$

It is not difficult to construct ensembles for which this lower bound matches the upper bound of equation 9. For example, consider any state multi-qubit state ρ for which a closest product state (under the Geometric measure) is an element of the computational basis, such as $|000...\rangle$. Then because the set of product states $\{X^a \otimes X^b \otimes X^c...|000...\rangle|a,b,c... = 0,1\}$ define a complete product basis, then the ensemble:

$$\{\{X^a \otimes X^b \otimes X^c...\rho X^a \otimes X^b \otimes X^c...|a,b,c... = 0,1\}\} \quad (17)$$

where each state is taken with equal prior probability, will be an example of an ensemble for which the mean closest product state is maximally mixed. Any such ensemble will also be one for which the upper bound (9) and the lower bound (16) match. Note that this example is not contained within the examples involving irreducible representations discussed above, as the group $\{X^a \otimes X^b \otimes X^c...|a,b,c... = 0,1\}$ is not irreducible. Moreover, in such cases the equations (9) and (16) can be achieved by $LOCC$ operations, as the POVM defined by the projectors onto the computational basis may clearly be achieved by LOCC operations. Hence a large number of ensembles may be constructed for which equations (9) and (16) provide the exact optimal discrimination probability for both separable and LOCC operations.

The process of constructing such ensembles is by working in reverse - we pick a standard product computational basis, and then we find states that have these product states as "closest" separable ones. The ensembles of states that can be identified in this way are ones for which the lower bounds presented above apply.

Following a similar line of reasoning we may also consider the closest separable states for the robustness of entanglement. If the states in the ensemble have the closest states

$$\omega_i := \frac{\rho_i + R_g(\rho_i)\sigma_i}{1 + R_g(\rho_i)}, \quad (18)$$

then as before we may write down the square root measurement for discriminating the ω_i as: $M_i := {\omega_m}^{-1/2} p_i \omega_i {\omega_m}^{-1/2}$ where ω_m is the mean state $\omega_{mean} := \omega_m := \sum_i p_i \omega_i$. Again, if we assume that ω_m *is itself maximally mixed*, then the M_i will indeed define a separable measurement. If we apply this measurement to the original ensemble, then we find that the optimal probability of successful discrimination P_s will be bounded by the following expression:

$$\begin{aligned} P_S \geq & \sum_i p_i \mathrm{tr}\{\rho_i p_i D \omega_i\} = D \sum_i p_i^2 \mathrm{tr}\{\rho_i \omega_i\} \\ = & D \sum_i p_i^2 \mathrm{tr}\{\rho_i \frac{\rho_i + R_g(\rho_i)\sigma_i}{1 + R_g(\rho_i)}\} \\ \geq & D \sum_i p_i^2 \frac{\mathrm{tr}\{\rho_i^2\}}{1 + R_g(\rho_i)} \\ \geq & D \min_i \left(\mathrm{tr}\{\rho_i^2\} \frac{p_i}{1 + R_g(\rho_i)} \right) \end{aligned}$$

Putting the lower and upper bounds together *for ensembles such that the average closest separable state (for the robustness) is maximally mixed* we find that:

$$D \max_i \{p_i 2^{-G(\rho_i)}\} \geq P_s \geq D \min_i \left(\mathrm{tr}\{\rho_i^2\} \frac{p_i}{1 + R_g(\rho_i)} \right)$$

(the upper bound is independent of the nature of the ensemble). We can weaken the lower bound further by using the inequality $\mathrm{tr}\{\rho_i^2\} \geq 1/|P_i|$, where P_i is the projector onto the support of ρ_i, in which case the bounds become:

$$D \max_i \{p_i 2^{-G(\rho_i)}\} \geq P_s \geq D \min_i \left(\frac{p_i}{|P_i|(1 + R(\rho_i))} \right) \quad (19)$$

As a consequence of (6) one might expect that this lower bound is typically not as tight as the one derived in equation (16). However, it is quite possible that the requirement that the mean closest separable state be maximally mixed is not valid for one measure while being valid for the other, hence the two lower bounds (16),(19) may separately prove useful in different cases.

These observations also beg the question as to whether the stringent constraint on the nature of the ensemble - the lower bounds are only valid when the mean closest separable states is maximally mixed - may be relaxed. Some generalisations should be possible - for instance, if the average mean state is sufficiently close to maximally mixed, then a perturbation of the above approach should lead to similar bounds as all the quantities considered above are continuous. However, it would be of more general interest to consider how one can define a separable analogue of the square root measurement in situations where ω_m is not constrained at all. A more general approach, for example, would be to write the global square root measurement, and compute bounds on the minimal noise required to make that global POVM separable. We will not, however, pursue this approach any further here, as we hope to pursue it in future work.

B. Optimal Entanglement Witnesses

We will now see how two of the entanglement measures considered, the robustness and the geometric measure, are naturally related to the concept of entanglement witnesses. The geometric measure can be used to define a particular entanglement witness which we will denote W_G. The robustness of entanglement can be considered as a quantification of the amount a state violates a kind of optimal witness which we denote W_R. As we shall see, if the geometric measure and logarithmic robustness are equal, then both W_G and W_R are optimal in the sense of ρ optimality considered in [29]. Note that this notion of optimality is actually different to the notions of optimality considered both in [25] and [24] - in those papers a witness is only said to be optimal if it is impossible to find another witness that detects a strictly larger set of entangled states. The notion of ρ optimality is likely to be more relevant when considering the statistical significance of violations in experimental implementations.

An entanglement witness W is a Hermitian operator (hence an observable) such that for all separable states ω, $\mathrm{tr}(\omega W) \geq 0$, and for some entangled state ρ, $\mathrm{tr}(\rho W) < 0$. W is said to witness the entanglement of ρ [25].

Similar to those used in, for example Ref. [26], it can easily be seen that the geometric measure $G(\rho)$ naturally defines a normalised entanglement witness associated to state ρ,

$$\begin{aligned} W_G(\rho) : &= \frac{1}{\alpha}(\alpha \mathbb{1} - \rho) \\ \alpha &= \max_{\omega \in SEP} \mathrm{tr}(\rho\omega) = 2^{-G(\rho)}. \end{aligned} \tag{20}$$

Some of these witnesses may be trivial, because if the maximal eigenvalue of ρ corresponds to a product eigenstate, then the witness will not detect any entangled states at all. However, if the maximal eigenvalue of ρ is non-degenerate and corresponds to an entangled eigenstate, then the witness will certainly detect some entangled states.

A so-called *ρ-optimal* entanglement witness (ρ-OEW) relative to a set $\mathcal{M}$ is a witness $W_\rho^{\mathcal{M}}$, that is associated to a state ρ, and which satisfies [27]

$$\mathrm{tr}(W_\rho^{\mathcal{M}}\rho) = \min_{W \in \mathcal{M}} \mathrm{tr}(W\rho), \tag{21}$$

where $\mathcal{M}$ is a compact subset of entanglement witnesses [28]. In this way a ρ-OEW is one which is violated maximally for the state ρ at hand, for a given class of witnesses $\mathcal{M}$. Experimentally we may like to choose such a witness since the violation would then be the most visible.

We will see that equality of the logarithmic robustness and the geometric measure implies that the witnesses W_G are ρ optimal for the set $\mathcal{M}$ of entanglement witnesses satisfying $\mathcal{M} = \{W | W \in \mathcal{W}, W \leq \mathbb{1}\}$. This is the set of witnesses that can be associated in a special way to the robustness of entanglement: In [29] it is shown that the robustness is given by

$$R_g(\rho) = \max\{0, -\min_{W \in \mathcal{M}} \mathrm{tr}(W\rho)\}, \tag{22}$$

where $\mathcal{M} = \{W | W \in \mathcal{W}, W \leq \mathbb{1}\}$. This implies that, for any state ρ, if there exists a witness, we write W_R such that $R_g(\rho) = -\mathrm{tr}(\rho W_R)$, then W_R is ρ-OEW relative to the set $\mathcal{M} = \{W | W \in \mathcal{W}, W \leq \mathbb{1}\}$.

Proposition: For a projection state $\rho = \frac{P}{|P|}$, if we have equivalence of measures $\log_2(|P|(1 + R_g(\rho))) = E_R(\rho) + S(\rho) = G(\rho)$, then the normalised witness $W_G(\rho)$ is a ρ-OEW relative to the set $\mathcal{M} = \{W | W \in \mathcal{W}, W \leq \mathbb{1}\}$.

Proof: If $G(\rho) = \log_2(|P|(1 + R_g(\rho)))$, then

$$R_g(\rho) = \frac{2^{G(\rho)}}{|P|} - 1. \tag{23}$$

The proposition is proved by comparing this to the expectation value of $W_G(\rho)$ for ρ:

$$-\mathrm{tr}(W_G(\rho)\rho) = -1 + \frac{2^{G(\rho)}}{|P|} = R_g(\rho). \tag{24}$$

By (22), $W_G(\rho)$ is also a ρ-OEW relative to the set $\mathcal{M} = \{W | W \in \mathcal{W}, W \leq \mathbb{1}\}$. □

III. OUTLINE OF APPROACH: STABILIZER STATES AND PERMUTATION INVARIANT BASIS STATES

The essence of the argument to prove equivalence of the measures across (6) is to take the product state $|\Phi\rangle$ which achieves the geometric measure (1), and perform a local "twirling" operation (a group averaging), to give a separable mixed state. If the symmetries have a suitable structure, or if the product state $|\Phi\rangle$ has certain properties, then the twirled version of $|\Phi\rangle$ can be a good candidate for the state ω in the optimisation for the global robustness (4). This then gives an upper bound to the robustness which which sits on the left of (6), (8). We will see that for certain states this upper bound matches the geometric measure, hence implying equality across (6), (8). For this to work it is essential that the twirled product state be of the correct form (4). A more formal group theoretical statement of this is given in appendix B. In general these conditions must be checked by knowing the closest product state $|\Phi\rangle$ (see Theorem 1 for projection states and Theorem 2 for pure states in appendix B). In certain cases some group symmetry properties of $|\Phi\rangle$ will suffice. This is the case for the symmetric bases states as we will see. In other cases the conditions may be satisfied simply by the properties of the group averaging and we do not need to know anything about the state $|\Phi\rangle$ (see Theorem 3 for projection states and Theorem 4 for pure states in appendix B). This is the case for the stabilizer states as we will see. In this section we will first give a sketch of the ideas, and two sets of examples which illustrate the methods that we will use.

If $|\Phi\rangle$ is the closest product state to pure state $|\psi\rangle$, the effect of averaging over some group $\{\mathrm{U}\}$ is essentially to project onto the invariant subspaces (see Lemma B3)

$$\omega' = \int \mathrm{U}|\Phi\rangle\langle\Phi|\mathrm{U}^\dagger d\mathrm{U} = \sum_i P_i|\Phi\rangle\langle\Phi|P_i, \tag{25}$$

where P_i are the projectors onto the invariant subspaces. Since the U are local, ω' is separable. In order to be a valid candidate for the robustness state ω in (4), we require that it is possible to reach ω' by adding noise to $|\psi\rangle\langle\psi|$. This is certainly possible if for some i we have

(i) $P_i|\Phi\rangle\langle\Phi|P_i = \lambda|\psi\rangle\langle\psi|$,

hence if $|\psi\rangle$ is invariant under the action of the group. Further, if we also have

(ii) $\lambda = 2^{-E_g(|\psi\rangle)}$,

then it can be shown quite easily that $LG(|\psi\rangle) = E_g(|\psi\rangle)$, hence we have equality across (8) (see Theorem 1 and Theorem 2 in appendix B for a more general group theoretic statement of this fact).

Both (i) and (ii) are immediately satisfied if $|\psi\rangle$ is itself a full invariant subspace, i.e. one of the P_i is itself the projector $|\psi\rangle\langle\psi|$ (see Theorem 3 and Theorem 4 in appendix B). This will be the case for our first set of examples below, the stabilizer states. If this is not the case, we need to find other ways to check that (i) and (ii) are explicitly satisfied (note, $|\psi\rangle$ must still be invariant). We do this by explicitly finding the closest product state and checking. The symmetric basis states provide an example of this case, as we will see below.

A. Stabilizer States

A stabiliser states $|S\rangle$ is defined by the associated group $S = \{G_i\}_{i=1}^{2^n}$, where G_i are made up of local Pauli operators, which stabilize the state in the sense of the eigen-equations [30]

$$G_i|S\rangle = |S\rangle, \quad \forall G_i \in S. \tag{26}$$

The group S is called the *stabilizer group*, and the equations (26) completely characterize the state. In fact, by considering the plus and minus mutual eigen-states of S we define a complete basis. Taking any n generators, we define the 2^n basis states $\{|S_{g_1,g_2..g_n}\rangle\}$, with

$$G_i|S_{g_1,g_2..g_n}\rangle = (-1)^{g_i}|S_{g_1,g_2..g_n}\rangle \tag{27}$$

where $g_i = 1, 0$ corresponding to eigen values $+1$ or -1 respectively, label the basis states. These states are exactly the invariant subspaces of the stabilizer group, i.e. $P_{\bar{g}} = |S_{\bar{g}}\rangle\langle S_{\bar{g}}|$, where $\bar{g}$ is the binary list $g_1, g_2..g_n$. The stabilizer state (26) is then $|S\rangle = |S_{0,0,..0}\rangle$.

Denoting $|\Phi_S\rangle$ as the closest product state, we have

$$E_g(|S\rangle) = -\log_2 |\langle\Phi_S|S\rangle|^2. \tag{28}$$

We construct our candidate for the closest separable state ω in 4, by averaging (or "twirling") over a local group, in this case, the stabilizer group. We thus define

$$\begin{aligned} \omega' &= \sum_{G_i \in S} G_i|\Phi_S\rangle\langle\Phi_S|G_i \\ &= \sum_{\bar{g}} |\langle S_{\bar{g}}|\Phi_S\rangle|^2 |S_{\bar{g}}\rangle\langle S_{\bar{g}}|. \end{aligned} \tag{29}$$

Since the operators G_i are local, the state ω' is a separable state, we can hence consider it as a candidate for closest separable state. For any candidate state $\omega' = \frac{1}{1+t'}(\rho + t'\delta)$, we have that $t' \geq R_g(\rho)$. State (29) is of this form for $|S\rangle$ with $t' = \frac{1}{|\langle S|\Phi_S\rangle|^2} - 1 = 2^{E_g(|S\rangle)} - 1$. Hence we have

$$E_g(|S\rangle) \geq \log_2(1 + R_g(|S\rangle)) \geq E_R(|S\rangle) \geq E_g(|S\rangle), \tag{30}$$

proving equality across all measures, i.e.

$$\log_2(1 + R_g(|S\rangle)) = E_R(|S\rangle) = E_g(|S\rangle). \tag{31}$$

We can now consider what this means in terms of measurements and witnesses from our earlier discussion. Suppose that we are working in a basis where closest product state to $|S_{0,0,0..}\rangle$ is $|000...\rangle$, then the optimal probability of discriminating the ensemble of graph states $\{X^a \otimes X^b \otimes X^c...|S_{0,0,0..}\rangle \| a, b, c... = 0, 1\}$ (all with equal a-priori probability $p = 2^{-n}$), is given exactly by equation (16) - this follows from the discussion in section II. Using the explicit formulae presented in [22] for the entanglement of a variety of classes of stabilizer state, many ensembles of graph states may be constructed whose optimal LOCC discrimination probability may be obtained in this way.

To define the proposed entanglement witness W_G we need the value of the geometric measure. Here we do not have it, however, we can say that for any cases where it is known such witness will also hold as W_R. Examples of where it is known for many important stabilizer states including cluster states is given in [22].

B. Permutation Symmetric States

In the previous case the state itself is an invariant subspace of the group, and this is sufficient for showing the equivalence of the measures (as stated more precisely in Theorem 3 and 4 of Appendix B). If we also know the state $|\Phi\rangle$ which gives the geometric measure we can relax this requirement a little (Theorems 1 and 2 in Appendix B). We will do just that to prove equivalence of these measures for the so called symmetric basis states.

In $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, symmetric basis states $|S(n, \overrightarrow{k})\rangle$, which form a basis of the symmetric subspace $\mathcal{S}_n$, are defined as

$$|S(n, \overrightarrow{k})\rangle := \frac{1}{\sqrt{C_{n,\overrightarrow{k}}}} \sum_{\overrightarrow{i} \in \text{perm}} |\overbrace{0\cdots0}^{k_0}\overbrace{1\cdots1}^{k_1}\cdots\overbrace{d-1\cdots d-1}^{k_{d-1}}\rangle, \tag{32}$$

where the summation is over all permutations of the sequence $(\overbrace{0\cdots0}^{k_0}\cdots\overbrace{d-1\cdots d-1}^{k_{d-1}})$, (that is, a n-length sequence in which "i" appears just k_i times), and $C_{n,\vec{k}} := |\mathrm{perm}| = \frac{\mathrm{n!}}{\Pi_{\mathrm{j=1}}^{\mathrm{d}}\mathrm{k_j!}}$. We also note that $\vec{k} = (k_0, \cdots, k_{d-1})$ satisfies $\sum_{a=0}^{d-1} k_a = n$.

For symmetric basis states, the value of the geometric measure of entanglement is already known [17], under the assumption that the closest product state is also symmetric,

$$E_g(|S(n,\vec{k})\rangle) = n\log_2 n - \log_2 n! - \sum_{j=1}^{d}(k_j \log_2 k_j - \log_2 k_j!), \tag{33}$$

and a closest product state is given by

$$|\Phi\rangle = \left(\sum_{l=1}^{d}\sqrt{\frac{k_l}{n}}|l\rangle\right)^{\otimes n}. \tag{34}$$

Before we show the equivalence of the entanglement measures, we will first prove rigourously the working assumption leading to (33), (34), by using symmetry arguments.

***Lemma* 1** If $|\Psi\rangle \in \mathcal{S}_n$, then, there exist a closest product state $|\Phi\rangle$ in the symmetric Hilbert space, thus,

$$|\Phi\rangle = |\phi\rangle^{\otimes n}, \quad E_g(|\Psi\rangle) = -\log_2 \max_{|\phi\rangle\in\mathcal{H}} |\langle\phi|^{\otimes n}|\Psi\rangle|^2, \tag{35}$$

where $\mathcal{S}_n$ is symmetric subspace of $\mathcal{H}^{\otimes n}$.

Proof of Lemma 1: We prove this in A.

Using this lemma it is possible to show that (34) gives the closest product state [8]. For completeness we give a simplified proof of this:

***Lemma* 2** If $|\Psi\rangle = |S(n,\vec{k})\rangle$, then a closest product state $|\Phi\rangle$ for the geometric measure is given by Eq. (34), i.e.:

$$|\Phi\rangle = \left(\sum_{l=1}^{d}\sqrt{\frac{k_l}{n}}|l\rangle\right)^{\otimes n}. \tag{36}$$

Proof By Lemma 1, $|\langle S(n,\vec{k}||\Phi\rangle|$ attains its maximum when $|\Phi\rangle$ can be written as $|\Phi\rangle = |\phi\rangle^{\otimes n}$ for a local state $|\phi\rangle \in \mathcal{H}$. Moreover, since all coefficients of $|S(n,\vec{k})\rangle$ are positive in the computational basis, $|\langle S(n,\vec{k})||\phi\rangle^{\otimes n}|$ attains its maximum when all coefficients of $|\phi\rangle$ are positive in the computational basis. Thus, we can write down $|\phi\rangle$ as $|\phi\rangle = |\vec{p}\rangle \stackrel{\text{def}}{=} \sum_{l=1}^{d}\sqrt{p_l}|l\rangle$ for some probability distribution $\vec{p}$. Using this we can derive an upper bound as follows,

$$\begin{aligned}\langle S(n,\vec{k})||\vec{p}\rangle^{\otimes n} &= \sqrt{C_{n,\vec{k}}}\,\Pi_{l=1}^{d}\sqrt{p_l}^{k_l} \\ &= \sqrt{C_{n,\vec{k}}}\,2^{\frac{n}{2}(\sum_{l=1}^{d}\frac{\vec{k}}{n}\log_2 p_l)} \\ &= \sqrt{C_{n,\vec{k}}}\,2^{\frac{n}{2}H(\frac{\vec{k}}{n})-D(\frac{\vec{k}}{n}\|\vec{p})} \\ &\leq \sqrt{C_{n,\vec{k}}}\,2^{\frac{n}{2}H(\frac{\vec{k}}{n})},\end{aligned} \tag{37}$$

where $H(\vec{p})$ is the Shannon entropy, $D(\vec{p}\|\vec{q})$ is the Classical relative entropy, and the inequality follows from the positivity of the relative entropy. In (37), equality holds if and only if $\vec{p} = \vec{k}/n$, since a necessary and sufficient condition for $D(\vec{p}\|\vec{q}) = 0$ is $\vec{p} = \vec{q}$. □

We are now ready to show equality of the measures

$$\begin{aligned}&\log_2(1 + R_g(|S(n,\vec{k})\rangle) \\ &= E_r(|S(n,\vec{k})\rangle) = E_g(|S(n,\vec{k})\rangle) \\ &= n\log_2 n - \log_2 n! - \sum_{j=1}^{d}(k_j\log_2 k_j - \log_2 k_j!).\end{aligned} \tag{38}$$

To show this we average over the group $\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$, with representation

$$\mathrm{U}(\theta_1,\theta_2..\theta_{d-1}) = (\sum_{j_1=0}^{d-1}\exp(i\theta_{j_1})|j_1\rangle\langle j_1|)\otimes\cdots\otimes(\sum_{j_n=0}^{d-1}\exp(i\theta_{j_n})|j_n\rangle\langle j_n|) \tag{39}$$

The symmetric states $|S(n,\vec{k})\rangle\langle S(n,\vec{k})|$ are invariant elements of this representation if we choose $\theta_0 = 0$. However, they are not the total invariant subspaces.

At this point to check that the twirled states are of the correct form we could simply apply (39) with $\theta = 0$ to the state (34) and it easily follows that

$$\omega' = \int_0^{2\pi}\cdots\int_0^{2\pi} \mathrm{U}(\theta_1,\theta_2..\theta_{d-1})|\Phi\rangle\langle\Phi|\mathrm{U}(\theta_1,\theta_2..\theta_{d-1})^\dagger d\theta_1\cdots d\theta_{d-1}$$
$$= \sum_{\overrightarrow{k}} \left|\langle\Phi||S(n,\overrightarrow{k})\rangle\right|^2 |S(n,\overrightarrow{k})\rangle\langle S(n,\overrightarrow{k})|, \quad (40)$$

which by construction is separable and is of the appropriate form and hence proves equality of the measures.

In fact, however, this can also be seen without knowing the exact state $|\Phi\rangle$ itself, but using only the fact that it must be symmetric (Lemma 1).

It can easily be seen that the invariant subspace of this unitary group consists of the subspace of the fixed "Type" (or fixed "Hamming weight") $\mathcal{A}_{\overrightarrow{k}}$; by means of the d-dimensional vector $\overrightarrow{k} = (k_0, k_1, \cdots, k_{d-1})$ satisfying $k_i \geq 0$ and $\sum_{i=0}^{d-1} k_i = n$, the subspace $\mathcal{A}_{\overrightarrow{k}}$ is defined as $\mathcal{A}_{\overrightarrow{k}} = span\{|a\rangle\langle b| \mid a,b \in \mathrm{Type}(\overrightarrow{k})\}$, where $\mathrm{Type}(\overrightarrow{k})$ is the set of sequences derived by permutations of $\{\overbrace{0,\cdots,0}^{k_0},\overbrace{1,\cdots,1}^{k_1},\cdots,\overbrace{d-1,\cdots,d-1}^{k_{d-1}}\}$ (sequences of "Type $\overrightarrow{k}$"). Thus, the projection operator corresponding to the total invariant subspaces $\mathcal{A}_{\overrightarrow{k}}$ can be written down as $P_{\mathcal{A}_{\overrightarrow{k}}} = \sum_{a\in\mathrm{Type}(\overrightarrow{k})} |a\rangle\langle a|$. We thus need to check that the twirled $|\Phi\rangle$ is of the correct form.

We choose a closest product state from the symmetric Hilbert space (Lemma 1), and average (34) over $\mathrm{U}(\theta_1,\theta_2..\theta_{d-1})$ to get

$$\omega' = \int_0^{2\pi}\cdots\int_0^{2\pi} \mathrm{U}(\theta_1,\theta_2..\theta_{d-1})|\Phi\rangle\langle\Phi|\mathrm{U}(\theta_1,\theta_2..\theta_{d-1})^\dagger d\theta_1\cdots d\theta_{d-1}$$
$$= \sum_{\overrightarrow{k}} P_{\mathcal{A}_{\overrightarrow{k}}}|\Phi\rangle\langle\Phi|P_{\mathcal{A}_{\overrightarrow{k}}}$$
$$= \sum_{\overrightarrow{k}} P_{\mathcal{A}_{\overrightarrow{k}}}(\sum_{\overrightarrow{l}} |S(n,\overrightarrow{l})\rangle\langle S(n,\overrightarrow{l})|)|\Phi\rangle\langle\Phi|(\sum_{\overrightarrow{m}} |S(n,\overrightarrow{m})\rangle\langle S(n,\overrightarrow{m})|)P_{\mathcal{A}_{\overrightarrow{k}}}$$
$$= \sum_{\overrightarrow{k},\overrightarrow{l},\overrightarrow{m}} \delta_{\overrightarrow{k}\overrightarrow{l}}\delta_{\overrightarrow{k}\overrightarrow{m}}\langle S(n,\overrightarrow{l})|\Phi\rangle\langle\Phi|S(n,\overrightarrow{m})\rangle|S(n,\overrightarrow{l})\rangle\langle S(n,\overrightarrow{m})|$$
$$= \sum_{\overrightarrow{k}} \left|\langle\Phi||S(n,\overrightarrow{k})\rangle\right|^2 |S(n,\overrightarrow{k})\rangle\langle S(n,\overrightarrow{k})|, \quad (41)$$

where we use Lemma 5 in appendix B in the second part, the fact that a closest product state $|\Phi\rangle$ is in the symmetric Hilbert space and the equation $\left(\sum_{\overrightarrow{k}} |S(n,\overrightarrow{k})\rangle\langle S(n,\overrightarrow{k})|\right)|\Phi\rangle = |\Phi\rangle$ in the third part, and the fact $P_{\mathcal{A}_{\overrightarrow{k}}}|S(n,\overrightarrow{l})\rangle = \delta_{\overrightarrow{k}\overrightarrow{l}}|S(n,\overrightarrow{l})\rangle$ in the fourth part. Since the original state $|\Phi\rangle$ is separable, and only local unitaries are used, the final state ω' is separable. We see that the state ω' is now a candidate state for the closest separable state for the robustness, and we again get equivalence of the measures Eq.(38) in the same way as the stabilizer states. Note that, in comparison with the case of the stabilizer states, we must use additional information about the nearest product state $|\Phi\rangle$ in the proof of Eq.(38); that is, in Eq. (41), we use the fact that a closest product state can be chosen from the symmetric Hilbert space (Lemma 1). This shows that we generally cannot conclude the equivalence of the entanglement measures only by invariance of a state under local unitary group actions (see Theorem 1 in appendix B).

We now turn again to the topics of separable measurements and witnesses. Again the methods of Sec. II A can be applied to obtain ensembles of states that are local unitarily equivalent to the symmetric basis states, and for which the optimal LOCC discrimination procedure is given by a simple product measurement. We may also easily apply the discussion concerning optimal entanglement witnesses. Since in this case we know the value of E_G we can define the entanglement witness as in section

II B,

$$W_G(|S(n,k)\rangle) = \frac{1}{\alpha}(\alpha\mathbb{1} - |S(n,k)\rangle\langle S(n,k)|)$$
$$\alpha = C_{n,\vec{k}} \prod_{l=1}^{d} \left(\frac{k_l}{n}\right)^{k_l} \quad (42)$$

which by the equality of the measures will be ρ-OEW.

IV. FURTHER EXAMPLES: MULTI-PARTITE STATES RELATED TO THE TENSOR PRODUCT REPRESENTATION OF $\mathrm{U}(n)$

We now consider a set of further examples. Suppose our Hilbert space is $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. We consider the tensor product representation of $\mathrm{U}(d)$, that is, $\pi : \mathrm{U} \in \mathrm{U}(d) \mapsto \overbrace{\mathrm{U}\otimes\cdots\otimes\mathrm{U}}^{n} \in \mathfrak{B}_2(\mathcal{H})$. This representation clearly only involves local unitary operations. It is well known that, by means of "*Weyl's unitary trick*", there exists a natural bijection between all irreducible representations derived from the above representation of $\mathrm{U}(d)$ and all irreducible representations which are derived from the tensor product representation of $GL(d)$, that is, $A \in GL(d) \mapsto \overbrace{A\otimes\cdots\otimes A}^{n} \in \mathfrak{B}_2(\mathcal{H})$ [31, 32, 33]. Moreover, by "*Schur duality*", the tensor product representation of $GL(d)$ can be decomposed as follows[31, 32],

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda\in \mathrm{Par}(n,d)} G^\lambda \otimes F_d^\lambda, \quad (43)$$

where $\mathrm{Par}(n,d)$ is a partition of n with depth $d \leq n$, that is, a set of $\lambda \in \mathbb{N}^d$ satisfying $\lambda_1 \geq \cdots \geq \lambda_d$ and $\sum_{i=1}^d \lambda_i = n$, G^λ is the space of an irreducible representation of the symmetric group of degree n (which we denote $\mathfrak{S}_n$) defined by partition λ, and F_d^λ is the representation space of the irreducible representation of $GL(d)$ with the highest weight λ [31, 32]. Using Young tableaux terminology, $\lambda \in \mathrm{Par}(n,d)$ corresponds to a Young tableau which has λ_k boxes in the kth row. Since this representation $(\pi,\mathcal{H})$ can be decomposed by

$$(\pi,\mathcal{H}) \cong (\bigoplus_{\lambda\in\mathrm{Par}(n,d)} G^\lambda\otimes F_d^\lambda, \bigoplus_{\lambda\in\mathrm{Par}(n,d)} I_{G^\lambda}\otimes\pi_\lambda), \quad (44)$$

where π_λ is an irreducible representation with highest weight λ, we can apply Theorem 3 for this representation of $\mathrm{U}(n)$. In order to apply Theorem 3 for the projection states corresponding to subspace F_d^λ, the dimension of G_λ must be one. Since the dimension of G_λ is given by the number of *standard Young tableaux* (that is, a Young tableau in which the numbers form an increasing sequence along each line and along each column) corresponding to the partition λ, a necessary and sufficient condition for $\dim G_\lambda = 1$ is $\lambda = (n,0,\cdots,0)$, or $(\overbrace{1,\cdots,1}^{n})$. It is also well known that the representation space F_d^λ of partition $\lambda = (n,0,\cdots,0)$ corresponds to the symmetric Hilbert space $\mathcal{S}_n$, and the representation space of partition $\lambda = (1,\cdots,1)$ corresponds to the anti-symmetric Hilbert space $\mathcal{A}_n$, which only exists under the condition $n \leq d$. Hence we have proven the following Corollary,

***Corollary* 1** In $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, the projection states $\frac{P}{\mathrm{tr}P}$ corresponding to the symmetric ($\mathrm{Ran}P = \mathcal{S}_n$) and anti-symmetric ($\mathrm{Ran}P = \mathcal{A}_n$) Hilbert spaces satisfy

$$\log_2(1+R_g(\frac{P}{\mathrm{tr}P})) = E_R(\frac{P}{\mathrm{tr}P}) = G(\frac{P}{\mathrm{tr}P}) - \log_2 \mathrm{tr}P. \quad (45)$$

As we will see in the following part, an anti-symmetric basis state is an example to which this corollary may be applied.

Anti-symmetric basis states. Suppose $\mathcal{H} = (\mathbb{C}^n)^{\otimes n}$, $n \leq d$, and $|\Psi_a\rangle \stackrel{\mathrm{def}}{=} |1\rangle\wedge\cdots\wedge|n\rangle$, (we call $|\Psi_a\rangle$ an anti-symmetric basis state), where $\{|i\rangle\}_{i=1}^n$ is an orthonormal basis of $\mathbb{C}^n$, and $\wedge$ is the wedge product ($|a\rangle\wedge|b\rangle = \frac{1}{\sqrt{2}}(|a\rangle\otimes|b\rangle - |b\rangle\otimes|a\rangle)$). Since for the irreducible representation $(\pi_{(1,\cdots,1)}, F_d^{(1,\cdots,1)})$, $F_d^{(1,\cdots,1)} = \mathcal{A}_d = \mathbb{C}|\Psi_a\rangle$, by means of Theorem 1, we have equivalence of distance like measures

$$\log_2(1+R_g(|\Psi_a\rangle)) = E_R(|\Psi_a\rangle) = E_g(|\Psi_a\rangle). \quad (46)$$

Moreover, the value of the geometric measure of entanglement is known in this case as follows [34]:

***Lemma* 3** In $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, anti-symmetric basis states

$$|\Psi_a\rangle \stackrel{\mathrm{def}}{=} |l\rangle\wedge\cdots\wedge|l+n\rangle$$
$$= \frac{1}{N!}\sum_{\{k_l\}} \epsilon_{k_1,k_2,..,k_N}|\alpha_{k_1},..,\alpha_{k_N}\rangle \quad (47)$$

satisfy

$$E_g(|\Psi_a\rangle) = \log_2 n!, \quad (48)$$

where $\epsilon_{k_1,k_2,..,k_N}$ is the Levi-Civita symbol, $n \leq d$, $1 \leq l \leq d-n$, and $\{|i\rangle\}_{i=1}^d$ is an orthonormal basis on $\mathbb{C}^d$.

Proof Firstly, the entanglement of $|l\rangle\wedge\cdots\wedge|l+n\rangle$ in $(\mathbb{C}^d)^{\otimes n}$, and the entanglement of $|1\rangle\wedge\cdots\wedge|n\rangle$ in $(\mathbb{C}^n)^{\otimes n}$ are equivalent, because they can be interconverted by LOCC. Thus, we only consider the case $|\Psi_a\rangle = |1\rangle\wedge\cdots\wedge|n\rangle$. Therefore, all we have to do is to calculate the value of the geometric measure of entanglement for $|1\rangle\wedge\cdots\wedge|n\rangle$. From the definition of the wedge product we can easily see that

$$\langle\phi_1|\otimes\langle\phi_1|\otimes\langle\phi_3|\otimes\cdots\otimes\langle\phi_n||\Psi_a\rangle$$
$$= \langle\phi_1|\otimes\langle\phi_1|\otimes\langle\phi_3|\otimes\cdots\otimes\langle\phi_n|\mathrm{U}_{12}^\dagger\mathrm{U}_{12}|\Psi_a\rangle$$
$$= -\langle\phi_1|\otimes\langle\phi_1|\otimes\langle\phi_3|\otimes\cdots\otimes\langle\phi_n||\Psi_a\rangle$$
$$= 0,$$

where U_{ij} is the swap operation between the ith and jth particle. Extending this observation by induction we can easily show the following fact: We can always assume that a state $|\Phi_0\rangle \stackrel{\mathrm{def}}{=} |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ which attains the maximum of $\max_{|\Phi\rangle \in \mathrm{Pro}(\mathcal{H})} |\langle\Phi||\Psi_a\rangle|$ satisfies $|\phi_1\rangle \perp \cdots \perp |\phi_n\rangle$. Then, under the condition of the orthogonality of the $\{|\phi_i\rangle\}_{i=1}^n$, we can calculate $|\langle\Phi||\Psi_a\rangle|$ as follows,

$$\begin{aligned} & |\langle\phi_1| \otimes \cdots \otimes \langle\phi_n||\Psi_a\rangle| \\ = & \ |\langle\phi_1| \otimes \cdots \otimes \langle\phi_n||1\rangle \wedge \cdots \wedge |n\rangle| \\ = & \ |\frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) \langle\phi_1||\sigma(1)\rangle \cdots \langle\phi_n||\sigma(n)\rangle| \\ = & \ \frac{1}{\sqrt{n!}} |\det\{\langle\phi_i||j\rangle\}_{ij}| \\ = & \ \frac{1}{\sqrt{n!}}, \end{aligned}$$

where $\{\langle\phi_i||j\rangle\}_{ij}$ is a matrix with $\langle\phi_i||j\rangle$ as its (i,j)th element, and we used the unitarity of $\{\langle\phi_i||j\rangle\}_{ij}$ in the last equality. Therefore,

$$\begin{aligned} E_g(|\Psi_a\rangle) &= -\log_2 \max_{|\Phi\rangle \in \mathrm{Pro}(\mathcal{H})} |\langle\Phi||\Psi_a\rangle|^2 \\ &= \log_2 n!. \end{aligned}$$

□

Thus in the case of antisymmetric states we can derive the values of the other measures from the value of geometric measure. That is, by Eq.(48) and Eq.(46), we derive the following corollary.

***Corollary* 2** In $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, anti-symmetric basis states $|\Psi_a\rangle \stackrel{\mathrm{def}}{=} |l\rangle \wedge \cdots \wedge |l+n\rangle$ satisfy

$$\log_2(1 + R_g(|\Psi_a\rangle)) = E_R(|\Psi_a\rangle) = E_g(|\Psi_a\rangle) = \log_2 n!, \quad (49)$$

where $n \leq d$, $1 \leq l \leq d-n$, and $\{|i\rangle\}_{i=1}^d$ is an orthonormal basis on $\mathbb{C}^d$.

By Eq.(49) and Eq.(38), we can compare the entanglement of anti-symmetric basis states $|\Psi_a\rangle = |1\rangle \wedge \cdots \wedge |n\rangle$ with that of the symmetric basis states $|\Psi_s\rangle = |S(n,(1,\cdots,1))\rangle = \sum_{\sigma \in \mathfrak{S}_n} |\sigma(1)\rangle \otimes \cdots \otimes |\sigma(n)\rangle$ on a given Hilbert space $(\mathbb{C}^d)^{\otimes n}$, $(n \leq d)$. Since $-\log_2 |\langle 1| \otimes \cdots \otimes \langle n||\Psi_s\rangle|^2 = \log_2 n! = E_g(|\Psi_a\rangle)$, we can easily see $E_g(|\Psi_a\rangle) \geq E_g(|\Psi_s\rangle)$, where equality holds if and only if $n = 2$. Moreover, when n is large enough, by means of Eq.(49), Eq.(38) and the Stirling formula, we derive

$$\begin{aligned} \frac{E_g(|\Psi_a\rangle)}{E_g(|\Psi_s\rangle)} &\approx \frac{n \log_2 n - n + 1}{n+1} \\ &\approx \log_2 n. \end{aligned} \quad (50)$$

Although the differences between anti-symmetric $|\Psi_a\rangle$ and symmetric $|\Psi_s\rangle$ basis states correspond only to phase factors $\mathrm{sign}(n)$, these two states have very different entanglement, and actually an anti-symmetric basis state is more entangled than symmetric basis states. Furthermore, since the symmetric basis states $|\Psi_s\rangle = |S(n,(1,\cdots,1))\rangle$ have the largest values of of entanglement among all symmetric basis states $|S(\overrightarrow{k})\rangle$ (under the condition $n \leq d$), the anti-symmetric basis states $|\Psi_a\rangle$ have larger values of the distance like measures than all symmetric basis states in $(\mathbb{C}^d)^{\otimes n}$.

V. CONCLUSION

In this paper, we have discussed sufficient conditions under which the values of the distance like measures of entanglement, (*i.e.* the robustness of entanglement, the relative entropy of entanglement, and the geometric measure of entanglement), are equivalent by means of the representation theory of compact topological groups (Theorem 9 and Theorem 3). As applications of these theorems, we have seen that such distance like measures of entanglement are equivalent for stabilizer states, projection states defined by the symmetric and anti-symmetric subspaces (which include anti-symmetric basis states), and also for symmetric basis states. Moreover, by calculating the value of the geometric measure of entanglement, we derived the values of all the measures for anti-symmetric basis states and symmetric basis states. By comparing these values, we conclude that anti-symmetric basis states are more entangled than any symmetric basis states on $(\mathbb{C}^d)^{\otimes n}$ with $n \leq d$. The results have applications as lower and upper bounds, which can often be tight, on the optimal probability of discrimination by separable or LOCC operations for certain classes of ensemble.

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APPENDIX A: PROOF OF LEMMA 1

Proof First, for an arbitrary $|\Phi\rangle \in \mathrm{Pro}(\mathcal{H}^{\otimes \mathrm{n}})$, suppose that $\langle\Phi||\Psi\rangle = re^{i\theta}$, where $r \geq 0$ and θ is real. By choosing $|\Phi'\rangle \stackrel{\mathrm{def}}{=} e^{i\theta}|\Phi\rangle \in \mathrm{Pro}(\mathcal{H}^{\otimes \mathrm{n}})$, we can always find a state $|\Phi'\rangle$ such that $\langle\Phi'||\Psi\rangle = |\langle\Phi'||\Psi\rangle| = |\langle\Phi||\Psi\rangle| = r$. Thus, when we consider $\max_{|\Phi\rangle\in \mathrm{Pro}(\mathcal{H}^{\otimes n})} |\langle\Phi|\Psi\rangle|$, we can always assume that $|\Phi\rangle$ gives a non-negative real $\langle\Phi|\Psi\rangle$. In the following discussion, we always assume $|\Phi\rangle$ satisfies this condition.

We prove this lemma in two steps; first for the case $n = 2$ and later for the case $n \geq 3$:

i) In the case $|\Psi\rangle \in \mathcal{H}^{\otimes 2}$.

First, we note that the following proof is valid for non-normalized $|\Psi\rangle$.

We define $|\Phi\rangle \stackrel{\mathrm{def}}{=} |a\rangle \otimes |b\rangle$. A diagonalisation theorem known as *Takagi's factorization* [35] states: "*If* Ψ *is a complex symmetric matrix, then there exists a unitary* U *and a real nonnegative diagonal matrix* $\Sigma = \mathrm{diag}(r_1, \cdots, r_n)$ *such that* $\Psi = \mathrm{U}\Sigma\mathrm{U}^T$". By means of this theorem, for any $|\Psi\rangle$ in the symmetric subspace of $\mathcal{H}^2$, we can calculate

$$\begin{aligned} \langle\Psi||a\rangle \otimes |b\rangle &= b^T \Psi a \\ &= (\mathrm{U}^T b)^T \Sigma \mathrm{U}^T a, \end{aligned} \tag{A1}$$

where in the first equality we used the natural correspondence between a bipartite Hilbert space and the space of matrices with respect to a fixed product basis: Ψ is the $\dim\mathcal{H} \times \dim\mathcal{H}$ matrix corresponding to $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, and a and b are the column vectors corresponding to $|a\rangle$ and $|b\rangle$, respectively. In Eq.(A1), we also note that U is a unitary matrix, and Σ is a nonnegative diagonal matrix, both of which are derived from Takagi's factorization. We can assume $r_1 \geq r_2 \geq \cdots \geq r_n$ for Σ. Then, from Eq.(A1), we can observe that the maximum of $\langle\Psi||a\rangle \otimes |b\rangle$ is attained if and only if $\mathrm{U}^T b = \mathrm{U}^T a = e_1 \stackrel{\mathrm{def}}{=} (1, 0, \cdots, 0)^T$. Therefore, $\max_{|a\rangle\otimes|b\rangle} \langle\Psi||a\rangle \otimes |b\rangle = r_1$ and the maximum is attained if and only if $|a\rangle = |b\rangle = |\overline{\mathrm{U}}e_1\rangle$, where $|\overline{\mathrm{U}}e_1\rangle$ is a state on $\mathcal{H}$ corresponding to the column vector $\overline{\mathrm{U}}e_1$, ($\overline{\mathrm{U}}$ is a complex conjugate of U). Hence we have proven that Eq. (35) is valid for bipartite states.

ii) In the case $|\Psi\rangle \in \mathcal{H}^{\otimes n}, n \geq 3$.

Suppose $|\Psi\rangle$ is in the symmetric subspace of $\mathcal{H}^{\otimes n}$, and assume that the state $|a_1\rangle\otimes\cdots\otimes|a_i\rangle\otimes\cdots\otimes|a_j\rangle\otimes\cdots\otimes|a_n\rangle$ attains $\max_{|\Phi\rangle\in\mathrm{Pro}(\mathcal{H}^{\otimes n})}\langle\Psi||\Phi\rangle$ where $|a_i\rangle \neq |a_j\rangle$. Then, $\mathrm{U}_{ij}|\Psi\rangle = |\Psi\rangle$ for all ij, where U_{ij} is the swap operation between the ith and jth Hilbert spaces, i.e. the unitary defined as $\mathrm{U}_{ij}|a_1\rangle \otimes \cdots \otimes |a_i\rangle \otimes \cdots \otimes |a_j\rangle \otimes \cdots \otimes |a_n\rangle = |a_1\rangle \otimes \cdots \otimes |a_j\rangle \otimes \cdots \otimes |a_i\rangle \otimes \cdots \otimes |a_n\rangle$. Suppose $P_{ij} \stackrel{\mathrm{def}}{=} \langle a_1|\otimes\cdots\otimes\langle a_{i-1}|\otimes I_{\mathcal{H}}\otimes\langle a_{i+1}|\otimes\cdots\otimes\langle a_{j-1}|\otimes I_{\mathcal{H}}\otimes\langle a_{j+1}|\otimes\cdots\otimes\langle a_n|$ is projection onto $|a_1\rangle\otimes\cdots\otimes|a_{i-1}\rangle\otimes\mathcal{H}\otimes|a_{i+1}\rangle\otimes\cdots\otimes|a_{j-1}\rangle\otimes\mathcal{H}\otimes|a_{j+1}\rangle\otimes\cdots\otimes|a_n\rangle \cong \mathcal{H}\otimes\mathcal{H}$, where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Since $\mathrm{U}_{ij}P_{ij}|\Psi\rangle = P_{ij}\mathrm{U}_{ij}|\Psi\rangle = P_{ij}|\Psi\rangle$, $|\Psi'\rangle \stackrel{\mathrm{def}}{=} P_{ij}|\Psi\rangle$ is a non-normalized symmetric bipartite state. By the definition of $|a_1\rangle\otimes\cdots\otimes|a_i\rangle\otimes\cdots\otimes|a_j\rangle\otimes\cdots\otimes|a_n\rangle$ and $|\Psi'\rangle$, we can obviously see that $\max_{|a\rangle\otimes|b\rangle}\langle\Psi'||a\rangle\otimes|b\rangle = \max_{|\Phi\rangle\in\mathrm{Pro}(\mathcal{H}^{\otimes n})}\langle\Psi||\Phi\rangle$, and $|a_i\rangle\otimes|a_j\rangle$ attains $\max_{|a\rangle\otimes|b\rangle}\langle\Psi'||a\rangle\otimes|b\rangle$.

Then, from i), we can choose $|a'_i\rangle$ and $|a'_j\rangle$ such that $|a'_i\rangle = |a'_j\rangle$, and $|a'_i\rangle\otimes|a'_j\rangle$ attains $\max_{|a\rangle\otimes|b\rangle}\langle\Psi'||a\rangle\otimes|b\rangle$. That is, $\langle\Psi'||a'_i\rangle\otimes|a'_j\rangle = \langle\Psi'||a_i\rangle\otimes|a_j\rangle = \max_{|a\rangle\otimes|b\rangle}\langle\Psi'||a\rangle\otimes|b\rangle$. Then, $\langle\Psi||a_1\rangle\otimes\cdots\otimes|a'_i\rangle\otimes\cdots\otimes|a'_j\rangle\otimes\cdots\otimes|a_n\rangle = \langle\Psi||a_1\rangle\otimes\cdots\otimes|a_i\rangle\otimes\cdots\otimes|a_j\rangle\otimes\cdots\otimes|a_n\rangle$, and $|a_1\rangle\otimes\cdots\otimes|a'_i\rangle\otimes\cdots\otimes|a'_j\rangle\otimes\cdots\otimes|a_n\rangle$, which is symmetric for (i,j), attains $\max_{|\Phi\rangle\in\mathrm{Pro}(\otimes\mathrm{n})}\langle\Psi||\Phi\rangle$. Therefore, by repeating the above symmetrization process for all (i,j), we can conclude that there always exists a $|a_1\rangle\otimes\cdots\otimes|a_i\rangle\otimes\cdots\otimes|a_j\rangle\otimes\cdots\otimes|a_n\rangle$ which attains $\max_{|\Phi\rangle\in\mathrm{Pro}(\mathcal{H}^{\otimes n})}\langle\Psi||\Phi\rangle$, and also satisfies $|a_i\rangle = |a_j\rangle$ for all (i,j). □

APPENDIX B: ELEMENTS OF GROUP REPRESENTATION THEORY

We first list the definitions and theorems that we use in the proof of this paper.

***Definition* 1** (Intertwining operator)
Suppose $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are both representations of a group G. A linear operator T from $\mathcal{H}$ onto $\mathcal{H}'$ is called an intertwining operator if T satisfies

$$\pi'(g)T = T\pi(g)\ (\forall g \in G). \tag{B1}$$

We write the set of all intertwining operators from $(\pi, \mathcal{H})$ onto $(\pi', \mathcal{H}')$ as $\mathrm{Hom}_G(\mathcal{H}, \mathcal{H}')$.

$\mathrm{Hom}_G(\mathcal{H}, \mathcal{H}')$ is a linear space.

***Definition* 2** (Equivalence of group representations)
We say that two group representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of a group G are equivalent, $(\pi, \mathcal{H}) \cong (\pi', \mathcal{H}')$, if there exists a bijective linear map $A \in \mathrm{Hom}_G(\mathcal{H}, \mathcal{H}')$.

In this case, A gives an isomorphism between the group representations $\pi(G)$ and $\pi'(G)$.

***Definition* 3** (Multiplicity of irreducible representations)
Suppose a finite dimensional representation $(\pi, \mathcal{H})$ of a group G is decomposed into a direct sum of irreducible representations as $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$. Then, for an irreducible representation (τ, W) of G, it can be shown that $\dim \mathrm{Hom}_G(W, \mathcal{H}) = \sharp\{i|(\tau, W) \cong (\pi|_{\mathcal{H}_i}, \mathcal{H}_i)\}$. This dimension is called the *multiplicity* of τ in π.

***Lemma* 4** (Schur's lemma)
Consider two given representations of a group G on finite dimensional complex Hilbert spaces, $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$. If a linear map $A : \mathcal{H} \to \mathcal{H}'$ satisfies,

$$A\pi(g) = \pi'(g)A,\ \forall g \in G, \tag{B2}$$

then, we have following.

1. If $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are not equivalent, $A = 0$
2. If $(\pi, \mathcal{H}) \cong (\pi', \mathcal{H}')$ and $T : \mathcal{H} \to \mathcal{H}'$ gives an isomorphism, then, there exists $\lambda \in \mathbb{C}$ such that $A = \lambda T$. In particular, in the case $(\pi, \mathcal{H}) = (\pi', \mathcal{H}')$, $A = \lambda I$, where I is the identity on $\mathcal{H}$.

Proof See [31, 33]

***Lemma* 5** For a representation $(\pi, \mathcal{H})$ of a group G, Suppose $\mathcal{H}$ can be decomposed as $\mathcal{H} = \oplus_{i=k}^{K} \mathcal{H}_k$, and each $\mathcal{H}_k$ is invariant under the action of G. Then, $\Psi = \oplus_k \Psi_k \in \mathcal{H}$ is an invariant element of $(\pi, \mathcal{H})$ if and only if Ψ_k is an invariant element for all k.

Proof The "if part" is trivial.
"only if part": Suppose there exist k_0 such that $\pi(g)\Psi_{k_0} \neq \Psi_{k_0}$. Then, from the uniqueness of the direct sum decomposition, $\Psi = \oplus_{k=1}^{K} \Psi_k \neq \oplus_{k=1}^{K} \pi(g)\Psi_k = \pi(g)\Psi$. This contradicts the invariance of Ψ. □

The following lemma, which concerns 'averaging over' the Haar measure of a compact topological group, is the key to deriving the sufficient conditions under which the inequalities (6) become equalities,

***Lemma* 6** (See [33].) Let G be a compact topological group, $(\pi, \mathcal{H})$ a finite dimensional unitary representation of the group G, and dg a normalized Haar measure on G. Then, the linear map on the Hilbert-Schmidt space $\mathfrak{B}_2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^\dagger$, (that is, the "*super-operator* "),

$$\rho \mapsto \int_G \pi(g)\rho\pi(g)^\dagger dg, \tag{B3}$$

is the projection (as a map on $\mathfrak{B}_2(\mathcal{H})$) onto $\mathfrak{B}_2(\mathcal{H})^G$, where $\mathfrak{B}_2(\mathcal{H})^G$ is the linear subspace of all G-invariant elements on $\mathfrak{B}_2(\mathcal{H})$;

$$\mathfrak{B}_2(\mathcal{H})^G \stackrel{\text{def}}{=} \{\rho \in \mathfrak{B}_2(\mathcal{H}) | \forall g \in G,\ \pi(g)\rho\pi(g)^\dagger = \rho\}. \tag{B4}$$

All this lemma represents is that the integration (B3) projects a state to the subspace of G-invariant elements on $\mathfrak{B}_2(\mathcal{H})$.

In cases where we know the irreducible decomposition of the group representation $(\pi, \mathcal{H})$, we can derive a concrete description of the subspace of G-invariant elements $\mathfrak{B}_2(\mathcal{H})^G$ as follows. Since all compact topological groups are completely reducible, $(\pi, \mathcal{H})$ can be decomposed as

$$(\pi, \mathcal{H}) = (\bigoplus_{k=1}^{K} (I_{\mathcal{A}_k} \otimes \pi_k), \bigoplus_{k=1}^{K} (\mathcal{A}_k \otimes \mathcal{B}_k)), \tag{B5}$$

where $(\pi_k, \mathcal{B}_k)$ is an irreducible representation of the compact topological group G, and $(\pi_k, \mathcal{B}_k)$ and $(\pi_{k'}, \mathcal{B}_{k'})$ are inequivalent group representations for all $k \neq k'$, i.e. there is no bijective intertwining operator (see Definition 1 in this appendix) between the representation subspaces corresponding to different k. In the above decomposition into irreducible subspaces, we used a tensor product to write down equivalent representations. By using this irreducible representation, we can write down $\mathfrak{B}_2(\mathcal{H})^G$ explicitly as follows. Note that the tensor product in Eq (B5) is not related to the tensor product of $\mathcal{H} = \otimes_{i=1}^{m} \mathcal{H}_i$, which is the "*cut*" across which we discuss the entanglement.

***Lemma* 7** For a given compact topological group G and a unitary representation on a finite dimensional complex Hilbert space $\mathcal{H}$, $(\pi, \mathcal{H})$, we can write $\mathfrak{B}_2(\mathcal{H})^G$ as follows:

$$\mathfrak{B}_2(\mathcal{H})^G = \{\bigoplus_{k=1}^{K} (N_k \otimes I_{\mathcal{B}_k}) \in \mathfrak{B}_2(\mathcal{H}) | \forall k, M_k \in \mathfrak{B}_2(\mathcal{A}_k)\}, \tag{B6}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are defined by the irreducible decomposition $(\pi, \mathcal{H}) = (\bigoplus_{k=1}^{K} (I_{\mathcal{A}_k} \otimes \pi_k), \bigoplus_{k=1}^{K} (\mathcal{A}_k \otimes \mathcal{B}_k))$.

Proof As with Theorem 9, we consider the unitary representation of G on $\mathfrak{B}_2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^\dagger$ via the map $\rho \mapsto \pi(g)g\pi(g)^\dagger$. We denote this representation by $(\pi \otimes \pi^\dagger, \mathcal{H} \otimes \mathcal{H}^\dagger)$. Since a compact topological group is completely reducible, this representation can be decomposed as $(\pi, \mathcal{H}) = (\bigoplus_{k=1}^{K} (I_{\mathcal{A}_k} \otimes \pi_k), \bigoplus_{k=1}^{K} (\mathcal{A}_k \otimes \mathcal{B}_k))$, where $(\pi_k, \mathcal{A}_k \otimes \mathcal{B}_k)$ is irreducible for all k, and $(\pi_k, \mathcal{A}_k \otimes \mathcal{B}_k)$ and $(\pi_{k'}, \mathcal{A}_{k'} \otimes \mathcal{B}_{k'})$ are not equivalent for all $k \neq k'$. Then the representation on the Hilbert Schmidt space $\mathcal{H} \otimes \mathcal{H}^\dagger$ also decomposes as

$$\begin{aligned}(\pi \otimes \pi^\dagger, \mathcal{H} \otimes \mathcal{H}^\dagger) &= (\bigoplus_{k,l} (I_{\mathcal{A}_k} \otimes \pi_k \otimes I_{\mathcal{A}_l}^\dagger \otimes \pi_l^\dagger), \bigoplus_{k,l} (\mathcal{A}_k \otimes \mathcal{B}_k \otimes \mathcal{A}_l^\dagger \otimes \mathcal{B}_l^\dagger)) \\ &\cong (\bigoplus_{k,l} (I_{\mathcal{A}_k} \otimes I_{\mathcal{A}_l}^\dagger \otimes \pi_k \otimes \pi_l^\dagger), \bigoplus_{k,l} (\mathcal{A}_k \otimes \mathcal{A}_l^\dagger \otimes \mathcal{B}_k \otimes \mathcal{B}_l^\dagger)),\end{aligned}$$

where in the second line we have reordered the tensor spaces for convenience in later discussions.

That is, each $(\mathcal{A}_k \otimes \mathcal{A}_l^\dagger \otimes \mathcal{B}_k \otimes \mathcal{B}_l^\dagger)$, (that is, the Hilbert Schmidt space of operators between $\mathcal{A}_k \otimes \mathcal{B}_k$ and $\mathcal{A}_l \otimes \mathcal{B}_l$), is a invariant subspace of $\pi \otimes \pi^\dagger$ for any k and l. Then, from Lemma 5 in appendix B, in order to derive the description of an invariant element of $(\pi \otimes \pi^\dagger, \mathcal{H} \otimes \mathcal{H}^\dagger)$, we only need to consider the invariant element of $(I_{\mathcal{A}_k} \otimes I_{\mathcal{A}_l}^\dagger \otimes \pi_k \otimes \pi_l^\dagger, (\mathcal{A}_k \otimes \mathcal{A}_l^\dagger \otimes \mathcal{B}_k \otimes \mathcal{B}_l^\dagger))$, and the invariant element of the whole space is only the direct product of such invariant elements of subspace. That is, $M \in \mathcal{H} \otimes \mathcal{H}^\dagger$ is an invariant element, if and only if $M = \oplus_{k,l} M_{kl}$ and all $M_{kl} \in \mathcal{A}_k \otimes \mathcal{A}_l^\dagger \otimes \mathcal{B}_k \otimes \mathcal{B}_l^\dagger$ are invariant elements.

Suppose $M_{kl} \in \mathcal{A}_k \otimes \mathcal{A}_l^\dagger \otimes \mathcal{B}_k \otimes \mathcal{B}_l^\dagger$ is an invariant element for all k, l. Then, from $\pi(g) M_{kl} \pi(g)^\dagger = I_{\mathcal{A}_k} \otimes \pi_k(g) M_{kl} I_{\mathcal{A}_l} \otimes \pi_l^\dagger(g)$, we derive

$$\pi_k(g) \langle \alpha_p^k | M_{kl} | \alpha_q^l \rangle = \langle \alpha_p^k | M_{kl} | \alpha_q^l \rangle \pi_l(g) \qquad \text{(B7)}$$

for all kl and $g \in G$, where $\{|\alpha_p^k\rangle\}_{p=1}^{\dim \mathcal{A}_k}$ and $\{|\alpha_q^l\rangle\}_{q=1}^{\dim \mathcal{A}_l}$ are orthonormal basis of $\mathcal{A}_k$ and $\mathcal{A}_l$, respectively. Here, we should note $\langle \alpha_p^k | M_{kl} | \alpha_q^l \rangle \in \mathcal{B}_k \otimes \mathcal{B}_l^\dagger$ for all p, q. Next, we use Schur's lemma (Lemma 4 in this appendix) for the representation $(\pi_k, \mathcal{B}_k)$ and $(\pi_l, \mathcal{B}_l)$. Then, since $(\pi_k, \mathcal{B}_k)$ and $(\pi_l, \mathcal{B}_l)$ are not equivalent for all $k \neq l$, by means of Eq.(B7) and Schur's lemma, we derive

$$\langle \alpha_p^k | M_{kl} | \alpha_q^l \rangle = 0 \qquad (\forall k \neq l, \text{and } \forall p, q) \qquad \text{(B8)}$$

$$\langle \alpha_p^k | M_{kl} | \alpha_q^k \rangle = C_{pq}^k I_{\mathcal{B}_k} \qquad (\forall k, \text{and } \forall p, q), \qquad \text{(B9)}$$

where $C_{pq} \in \mathbb{C}$ is a complex number coefficient. By using matrix element m_{pqrs}, M_{kl} can be written as $M_{kl} = \sum_{pqrs} m_{pqrs}^{kl} |\alpha_p^k\rangle\langle\alpha_q^l| \otimes |\beta_r^k\rangle\langle\beta_s^l|$, where $\{|\beta_r^k\rangle\}_{r=1}^{\dim \mathcal{B}_k}$ and $\{|\beta_r^l\rangle\}_{r=1}^{\dim \mathcal{B}_l}$ are orthonormal bases of $\mathcal{B}_k$ and $\mathcal{B}_l$, respectively. Then, Eq. (B8) and Eq. (B9) can be written down as

$$m_{pqrs}^{kl} = 0 \qquad (\forall k \neq l, \text{and } \forall p, q, r, s) \qquad \text{(B10)}$$

$$m_{pqrs}^{kk} = C_{pq} \delta_{rs} \qquad (\forall k, \text{and } \forall p, q), . \qquad \text{(B11)}$$

From Eq. (B10), we derive $M_{kl} = 0$ for all $k \neq l$. From Eq. (B11), we derive

$$\begin{aligned} M_{kk} &= \Big(\sum_{pq} C_{pq}^k |\alpha_p^k\rangle\langle\alpha_q^k|\Big) \otimes \Big(\sum_r |\beta_r^k\rangle\langle\beta_r^k|\Big) \\ &= N_k \otimes I_{\mathcal{B}_k}, \end{aligned}$$

where $N_k \stackrel{\text{def}}{=} \sum_{pq} C_{pq}^k |\alpha_p^k\rangle\langle\alpha_q^k| \in \mathcal{A}_k \otimes \mathcal{A}_k^\dagger$.

Therefore, finally by means of Lemma 5, we can conclude that $M \in \mathcal{H} \otimes \mathcal{H}^\dagger$ is an invariant element of $(\pi \otimes \pi^\dagger, \mathcal{H} \otimes \mathcal{H}^\dagger)$ if M can be written down as $M = \bigoplus_{k=1}^K (N_k \otimes I_{\mathcal{B}_k})$ by using $N_k \in \mathcal{A}_k \otimes \mathcal{A}_k^\dagger$. Conversely, suppose $M \in \mathcal{H} \otimes \mathcal{H}^\dagger$ can be written as $M = \bigoplus_{k=1}^K (N_k \otimes I_{\mathcal{B}_k})$. Then, since all $N_k \otimes I_{\mathcal{B}_k}$ are clearly invariant elements of $(\pi \otimes \pi^\dagger, \mathcal{H} \otimes \mathcal{H}^\dagger)$, by Lemma 5 in this appendix, M is also an invariant element. □

By means of the previous lemma, we can derive an expression for the state that results from averaging over a compact topological group as follows.

***Lemma* 8** For a given compact topological group G and a corresponding unitary representation on a finite dimensional complex Hilbert space $\mathcal{H}$, $(\pi, \mathcal{H})$, a projection (super-) operator $\mathcal{P}_{\mathfrak{B}_2(\mathcal{H})^G}$ onto the Hilbert-Schmidt subspace of G-invariant elements $\mathfrak{B}_2(\mathcal{H})^G$ maps a Hilbert Schmidt class operator $\rho \in \mathfrak{B}_2(\mathcal{H})$ as follows,

$$\mathcal{P}_{\mathfrak{B}_2(\mathcal{H})^G}(\rho) = \sum_{k=1}^K \frac{1}{\dim \mathcal{B}_k} \mathrm{tr}_{\mathcal{B}_k}(P_{\mathcal{A}_k \otimes \mathcal{B}_k} \rho P_{\mathcal{A}_k \otimes \mathcal{B}_k}) \otimes I_{\mathcal{B}_k} \qquad \text{(B12)}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ is defined by the irreducible decomposition $(\pi, \mathcal{H}) = (\bigoplus_{k=1}^K (I_{\mathcal{A}_k} \otimes \pi_k), \bigoplus_{k=1}^K (\mathcal{A}_k \otimes \mathcal{B}_k))$, and $P_{\mathcal{A}_k \otimes \mathcal{B}_k}$ is a projection onto $\mathcal{A}_k \otimes \mathcal{B}_k$.

Proof From Lemma 7, $\mathfrak{B}_2(\mathcal{H})^G$ can be written down as $B_2(\mathcal{H})^G = \bigoplus_{k=1}^K (\mathfrak{B}_2(\mathcal{A}_k) \otimes \{\alpha I_{\mathcal{B}_k}\}_{\alpha \in \mathbb{C}})$, where $I_{\mathcal{B}_k}$ is the identity operator on subspace $\mathcal{B}_k$.

Suppose $\mathcal{P}_{I_{\mathcal{B}_k}}$ is a projection onto a one-dimensional Hilbert-Schmidt subspace $\{\alpha I_{\mathcal{B}_k}\}_{\alpha \in \mathbb{C}}$. Then, for $\rho \in \mathfrak{B}_2(\mathcal{B}_k)$, $\mathcal{P}_{I_{\mathcal{B}_k}}(\rho) = \frac{\mathrm{tr}(\rho)}{\dim \mathcal{B}_k} \cdot I_{\mathcal{B}_k}$ Thus,

$$\begin{aligned} &\mathcal{P}_{\mathfrak{B}_2(\mathcal{H})^G}(\rho) \\ &= \sum_{k=1}^K \mathcal{P}_{\mathfrak{B}_2(\mathcal{A}_k) \otimes \{\alpha I_{\mathcal{B}_k}\}_{\alpha \in \mathbb{C}}} \circ \mathcal{P}_{\mathfrak{B}_2(\mathcal{A}_k \otimes \mathcal{B}_k)}(\rho) \\ &= \sum_{k=1}^K (\mathcal{P}_{\mathfrak{B}_2(\mathcal{A}_k)} \otimes \mathcal{P}_{\{\alpha I_{\mathcal{B}_k}\}_{\alpha \in \mathbb{C}}}) \cdot (P_{\mathcal{A}_k \otimes \mathcal{B}_k} \rho P_{\mathcal{A}_k \otimes \mathcal{B}_k}) \\ &= \sum_{k=1}^K \frac{1}{\dim \mathcal{B}_k} \mathrm{tr}_{\mathcal{B}}(P_{\mathcal{A}_k \otimes \mathcal{B}_k} \rho P_{\mathcal{A}_k \otimes \mathcal{B}_k}) \otimes I_{\mathcal{B}_k} \end{aligned}$$

□

In the next appendix we will attempt to apply the above definitions and lemmas, with the intention of deriving a fairly general sufficient conditions under which equality of the measures may be proven. However, it turns out that these sufficient conditions are equivalent to a more obvious sufficient condition - that the state under consideration is the invariant state of an irreducible subspace of multiplicity one in the local unitary representation. Nevertheless, we present the full arguments below, in order that the origin of this condition be clear.

APPENDIX C: SUFFICIENT CONDITIONS UNDER WHICH DISTANCE LIKE MEASURES OF ENTANGLEMENT COINCIDE

In this appendix, we present the sufficient conditions under which the measures of entanglement that we consider coincide by means of the group theoretical tools reviewed above.

As stated previously, these conditions collapse to the more elementary condition that multiplicity of the irrep upon which the invariant state resides is 1. That this is a sufficient condition may be seen more directly by an intuitive argument - the off-diagonal elements representing coherences between inequivalent irreps must vanish as a consequence of Schur's lemma, hence if the invariant state is proportional to the projector onto an irrep of multiplicity 1, the closest product state under the geometric measure automatically averages under twirling to give a state of the form required for the robustness measure. Hence a similar discussion holds in such examples as with the stabilizer states. However, we present the full sequence of lemmas as the conditions that we develop at first seem to be more general, and so it is of interest to understand why this is not the case.

First, by means of Lemma 6, we derive the following sufficient condition under which the distance like measures have equal value.

***Lemma* 9** A projection state $\frac{P}{\mathrm{tr}P}$ on $\mathcal{H}=\otimes_{i=1}^{m}\mathcal{H}_i$ satisfies,

$$\log_2(1+R_g(\frac{P}{\mathrm{tr}P}))=E_R(\frac{P}{\mathrm{tr}P})=G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P, \tag{C1}$$

if there exists a compact topological group G and a finite dimensional unitary representation $(\pi,\mathcal{H})$ such that P is an invariant element of the representation of G as defined by Eq.(B3), $\pi(g)$ is a local unitary transformation for all $g\in G$, and the following inequality is satisfied,

$$\int_G \pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg \geq \frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P, \tag{C2}$$

where $|\Phi_0\rangle$ attains $\max_{|\Phi\rangle\in Pro(\mathcal{H})}\langle\Phi|\frac{P}{\mathrm{tr}P}|\Phi\rangle$.

Proof Suppose $\frac{P}{\mathrm{tr}P}$ is an invariant element of the representation of G, $\pi(g)$ is a local unitary for all $g\in G$, and the inequality (C2) holds. Then, from Lemma 6,

$$\begin{aligned}
&\int_G \pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg \\
&= \mathcal{P}_{\mathfrak{B}_2(\mathcal{H})^G}(|\Phi_0\rangle\langle\Phi_0|) \\
&= \mathcal{P}_P(|\Phi_0\rangle\langle\Phi_0|)+\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|) \\
&= \langle\Phi_0|\frac{P}{\sqrt{\mathrm{tr}P}}|\Phi_0\rangle\frac{P}{\sqrt{\mathrm{tr}P}}+\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)\in\mathrm{Sep}(\mathcal{H}) \\
&= (\mathrm{tr}P)\langle\Phi_0|\frac{P}{\mathrm{tr}P}|\Phi_0\rangle\frac{P}{\mathrm{tr}P} \\
&\quad +\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)\in\mathrm{Sep}(\mathcal{H}),
\end{aligned} \tag{C3}$$

where $\mathcal{P}_{\mathfrak{B}_2(\mathcal{H})^G}$, $\mathcal{P}_P$, and $\mathcal{P}_{P^\perp}$ are the projections of Hilbert-Schmidt space (super-operator) $\mathfrak{B}_2(\mathcal{H})$ onto $\mathfrak{B}_2(\mathcal{H})^G$, P, and the orthogonal complement of P as a subspace of $\mathfrak{B}_2(\mathcal{H})^G$, respectively.

By definition, $\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)$ satisfies $\mathrm{tr}P\ \ \mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)=0$, and from (C2), $\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)\geq 0$. Hence, from Eq.(4) and Eq.(C3), by corresponding ω and $\frac{1}{1+t}$ to $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$ and $\langle\Phi_0|P|\Phi_0\rangle$, we see that $\frac{1}{\langle\Phi_0|P|\Phi_0\rangle}-1$ satisfies the all condition of t in the definition (4) of $R_g(\rho)$. Thus, we derive $\frac{1}{\langle\Phi_0|P|\Phi_0\rangle}\geq 1+R_g(\frac{P}{\mathrm{tr}P})$. Moreover, by the definition of $|\Phi_0\rangle$, $\frac{1}{\langle\Phi_0|P|\Phi_0\rangle}=2^{G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P}$. That is,

$$\begin{aligned}
1+R_g(\frac{P}{\mathrm{tr}P})\leq\frac{1}{\langle\Phi_0|P|\Phi_0\rangle} &= 2^{G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P} \\
&\leq 1+R_g(\frac{P}{\mathrm{tr}P}),
\end{aligned} \tag{C4}$$

where we use the inequalities (6) in the second inequality. Therefore, $G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P=E_r(\frac{P}{\mathrm{tr}P})=\log_2(1+R_g(\frac{P}{\mathrm{tr}P}))$. □

In the above Theorem, the inequality (C2) corresponds to the condition that $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$ should be in the form $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg=\lambda(\mathrm{tr}P)\frac{P}{\mathrm{tr}P}+\{1-\lambda(\mathrm{tr}P)\}\Delta$ with a positive $\lambda\mathrm{tr}P\leq 1$ and a state Δ, which we used in the definition of $R_g(\frac{P}{\mathrm{tr}P})$ (4). This condition is necessary for this upper bound (the first inequality in Eq.(C4)) to be valid.

From the proof of the above lemma, we can easily see that $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$ is a closest separable state of a projection state $\frac{P}{\mathrm{tr}P}$ in terms of the robustness of entanglement in the case where the projection state satisfies all of the assumption in the above lemma. Moreover, we can also show that this state is a closest separable state in terms of the relative entropy of entanglement. Hence, roughly speaking, if a given multipartite projection state has enough group symmetry, we can derive a closest separable state from a closest product state by just averaging it over a group action. We can check the optimality for the relative entropy of entanglement by the following argument, which was also used in [9].

$$\begin{aligned}
&E_R(\frac{P}{\mathrm{tr}P}) \\
&= \min_{\omega\in\mathrm{Sep}}D(\rho\|\omega) \\
&\leq D(\rho\|\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg) \\
&= -\log_2\mathrm{tr}P \\
&\quad -\mathrm{tr}\left(\frac{P}{\mathrm{tr}P}\log_2\left(\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P+\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)\right)\right) \\
&\leq -\log_2\mathrm{tr}P-\mathrm{tr}\left(\frac{P}{\mathrm{tr}P}\log_2\left(\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P\right)\right) \\
&= -\log_2|\Phi_0\rangle P\langle\Phi_0| \\
&= G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P,
\end{aligned}$$

where we used the operator-monotonicity of the logarithmic function in the second inequality. Since $E_R(\frac{P}{\mathrm{tr}P})=G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P$, all of the above inequalities should be equalities. Therefore, from the second equality, we derive $\min_{\omega\in\mathrm{Sep}}D(\rho\|\omega)=D(\rho\|\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg)$; that is, $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$ is a closest separable state in terms of the relative entropy.

By means of Lemma 8, we can rewrite Lemma 9 as follows.

***Theorem* 1** A projection state $\frac{P}{\mathrm{tr}P}$ on $\mathcal{H}=\otimes_{i=1}^{m}\mathcal{H}_i$ satisfies,

$$\log_2(1+R_g(\frac{P}{\mathrm{tr}P}))=E_R(\frac{P}{\mathrm{tr}P})=G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P, \tag{C5}$$

if there exist a compact topological group G and its finite dimensional unitary representation $(\pi,\mathcal{H})$ such that P is a G-invariant element, $\pi(g)$ is a local unitary transformation for all $g\in G$, and the following inequality is satisfied for all k such that $\mathrm{Ran}(P)\cap(\mathcal{A}_k\otimes\mathcal{B}_k)\neq\{0\}$,

$$\begin{aligned}&\mathrm{tr}_{\mathcal{B}_k}(P_{\mathcal{A}_k\otimes\mathcal{B}_k}|\Phi_0\rangle\langle\Phi_0|P_{\mathcal{A}_k\otimes\mathcal{B}_k})\\&-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}\mathrm{tr}_{\mathcal{B}_k}(P_{\mathcal{A}_k\otimes\mathcal{B}_k}PP_{\mathcal{A}_k\otimes\mathcal{B}_k})\geq 0,\end{aligned} \tag{C6}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are define by the irreducible decomposition $(\pi,\mathcal{H})=(\bigoplus_{k=1}^{K}(I_{\mathcal{A}_k}\otimes\pi_k),\bigoplus_{k=1}^{K}(\mathcal{A}_k\otimes\mathcal{B}_k))$, $P_{\mathcal{A}_k\otimes\mathcal{B}_k}$ is a projection onto $\mathcal{A}_k\otimes\mathcal{B}_k$, and $|\Phi_0\rangle$ attains $\max_{|\Phi\rangle\in Pro(\mathcal{H})}\langle\Phi|\frac{P}{\mathrm{tr}P}|\Phi\rangle$.

Proof Suppose all assumptions in this theorem are valid. Then,

$$\begin{aligned}&\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P\\=&\int_G\pi(g)\left(|\Phi_0\rangle\langle\Phi_0|-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P\right)\pi(g)^\dagger dg\\=&\sum_{k=1}^{K}\mathrm{tr}_{\mathcal{B}_k}\Big(P_{\mathcal{A}_k\otimes\mathcal{B}_k}\big(|\Phi_0\rangle\langle\Phi_0|\\&\qquad-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P\big)P_{\mathcal{A}_k\otimes\mathcal{B}_k}\Big)\otimes I_{\mathcal{B}_k}\\=&\sum_{k=1}^{K}\Big(\mathrm{tr}_{\mathcal{B}_k}(P_{\mathcal{A}_k\otimes\mathcal{B}_k}|\Phi_0\rangle\langle\Phi_0|P_{\mathcal{A}_k\otimes\mathcal{B}_k})\\&\qquad-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}\mathrm{tr}_{\mathcal{B}_k}\left(P_{\mathcal{A}_k\otimes\mathcal{B}_k}PP_{\mathcal{A}_k\otimes\mathcal{B}_k}\right)\Big)\otimes I_{\mathcal{B}_k},\end{aligned}$$

where we used Lemma 8 in the second equality. Thus, $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P\geq 0$, if and only if $\mathrm{tr}_{\mathcal{B}_k}(P_{\mathcal{A}_k\otimes\mathcal{B}_k}|\Phi_0\rangle\langle\Phi_0|P_{\mathcal{A}_k\otimes\mathcal{B}_k})-\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}\mathrm{tr}_{\mathcal{B}_k}\left(P_{\mathcal{A}_k\otimes\mathcal{B}_k}PP_{\mathcal{A}_k\otimes\mathcal{B}_k}\right)\geq 0$ for all k such that $\mathrm{Ran}(P)\cap(\mathcal{A}_k\otimes\mathcal{B}_k)\neq\{0\}$. Therefore, from Lemma 9, we can derive this theorem. □

For a pure state, the sufficient condition in the above theorem can be simplified to the following.

***Theorem* 2** a state $|\Psi\rangle\in\mathcal{H}=\otimes_{i=1}^{m}\mathcal{H}_i$ satisfies,

$$\log_2(1+R_g(|\Psi\rangle))=E_R(|\Psi\rangle)=E_g(|\Psi\rangle), \tag{C7}$$

if there exists a compact topological group G and its finite dimensional unitary representation $(\pi,\mathcal{H})$ such that $|\Psi\rangle\langle\Psi|$ is a G-invariant element, $\pi(g)$ is a local unitary transformation for all $g\in G$, and the following inequality is satisfied,

$$P_{\mathcal{A}_{k_0}\otimes\mathcal{B}_{k_0}}|\Phi_0\rangle\langle\Phi_0|P_{\mathcal{A}_{k_0}\otimes\mathcal{B}_{k_0}}-|\langle\Psi|\Phi_0\rangle|^2|\Psi\rangle\langle\Psi|\geq 0, \tag{C8}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are defined by the irreducible decomposition $(\pi,\mathcal{H})=(\bigoplus_{k=1}^{K}(I_{\mathcal{A}_k}\otimes\pi_k),\bigoplus_{k=1}^{K}(\mathcal{A}_k\otimes\mathcal{B}_k))$, $P_{\mathcal{A}_k\otimes\mathcal{B}_k}$ is a projection onto $\mathcal{A}_k\otimes\mathcal{B}_k$, k_0 satisfies $\mathcal{A}_{k_0}\otimes\mathcal{B}_{k_0}\ni|\Psi\rangle$, and $|\Phi_0\rangle$ attains $\max_{|\Phi\rangle\in Pro(\mathcal{H})}\langle\Phi|\frac{P}{\mathrm{tr}P}|\Phi\rangle$.

In the above theorem 9, in order to check whether a projection state $\frac{P}{\mathrm{tr}P}$ satisfies Eq.(C2), or not, we need to know the closest product state $|\Phi_0\rangle$, that is, the state which attain $\max_{\Phi\in\mathrm{Pro}(\mathcal{H})}\langle\Phi|\frac{P}{\mathrm{tr}P}|\Phi\rangle$. However, if $\frac{P}{\mathrm{tr}P}$ and a group representation $(\pi,\mathcal{H})$ of a topological group G satisfy an additional condition, we can derive Eq.(C1) without needing to know the closest product state $|\Phi_0\rangle$. We can write down this fact as following lemma.

***Lemma* 10** a projection state $\frac{P}{\mathrm{tr}P}$ on $\mathcal{H}=\otimes_{i=1}^{m}\mathcal{H}_i$ satisfies,

$$\log_2(1+R_g(\frac{P}{\mathrm{tr}P}))=E_R(\frac{P}{\mathrm{tr}P})=G(\frac{P}{\mathrm{tr}P})-\log_2\mathrm{tr}P, \tag{C9}$$

if there exists a compact topological group G and its finite dimensional unitary representation $(\pi,\mathcal{H})$ such that $P\in\mathfrak{B}_2(\mathcal{H})^G$, $\pi(g)$ is a local unitary transformation for all $g\in G$, and $\sigma|\xi\rangle=0$ for all $|\xi\rangle\in\mathrm{Ran}(P)$ and $\sigma\in\mathfrak{B}^G_{P^\perp}$, where $\mathrm{Ran}(P)$ is the range (the image of the domain) of the projection P, and $\mathfrak{B}^G_{P\perp}$ is defined as an orthogonal complement of P in the Hilbert-Schmidt subspace $\mathfrak{B}_2(\mathcal{H})^G$,

$$\mathfrak{B}^G_{P^\perp}=\{\sigma\in\mathfrak{B}_2(\mathcal{H})^G|\mathrm{tr}P\sigma=0\}. \tag{C10}$$

Proof We will see that, if $\sigma|\xi\rangle=0$ for all $|\xi\rangle\in\mathrm{Ran}(P)$ and $\sigma\in\mathfrak{B}^G_{P^\perp}$, the inequality (C2) is satisfied. Suppose all conditions of this lemma are satisfied. Then, by using the same discussion as that of Theorem 9, $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$ can be written down as,

$$\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg=\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P+\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|). \tag{C11}$$

Since $\mathrm{tr}P\mathcal{P}_{|\xi\rangle\langle\xi|^\perp}(|\Phi_0\rangle\langle\Phi_0|)=0$, $\mathcal{P}_{|\xi\rangle\langle\xi|^\perp}(|\Phi_0\rangle\langle\Phi_0|)\in\mathfrak{B}^G_{\Psi^\perp}$. Then, by the assumption of this lemma, $\mathcal{P}_{|\xi\rangle\langle\xi|^\perp}(|\Phi_0\rangle\langle\Phi_0|)|\xi\rangle=0$ for all $|\xi\rangle\in\mathrm{Ran}(P)$. Therefore, for all $|\xi\rangle\in\mathrm{Ran}(P)$,

$$\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg|\xi\rangle=\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}P|\xi\rangle, \tag{C12}$$

that is, $\mathrm{Ran}(P)$ is included by the eigenspace of $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$ with an eigenvalue $\frac{\langle\Phi_0|P|\Phi_0\rangle}{\mathrm{tr}P}$. Thus, since we can see Eq. (C11) as a part of spectral decomposition of a positive operator $\int_G\pi(g)|\Phi_0\rangle\langle\Phi_0|\pi(g)^\dagger dg$, we can conclude $\mathcal{P}_{P^\perp}(|\Phi_0\rangle\langle\Phi_0|)\geq 0$. Therefore, by the lemma 9, we derive Eq.(C9). □

The sufficiency condition of Lemma 10 now depends only on $\mathfrak{B}_2(\mathcal{H})^G$ and a state $\frac{P}{\mathrm{tr}P}$. That is, if we know the structure of the representation $(\pi,\mathcal{H})$, we can check

Lemma 10 without knowing a closest product state $|\Phi_0\rangle$. Actually, by means of Lemma 7, Lemma 10 can be rewritten in a simpler form which is described only in terms of properties of the group representation $(\pi, \mathcal{H})$ of G as follows.

***Theorem* 3** A projection state $\frac{P}{\mathrm{tr}P}$ on $\mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$ satisfies,

$$\log_2(1 + R_g(\frac{P}{\mathrm{tr}P})) = E_R(\frac{P}{\mathrm{tr}P}) = G(\frac{P}{\mathrm{tr}P}) - \log_2 \mathrm{tr}P, \quad \text{(C13)}$$

if there exists a compact topological group G and a finite dimensional unitary representation $(\pi, \mathcal{H})$ such that $\pi(g)$ is a local unitary transformation for all $g \in G$, and $(\pi|_{\mathrm{Ran}P}, \mathrm{Ran}P)$ is an irreducible representation of G whose multiplicity (Definition 3 in Appendix. A) is one on $(\pi, \mathcal{H})$.

Proof Suppose the assumption in the statement of the theorem is valid. Similar to the proof of Lemma 7, we can write $(\pi, \mathcal{H})$ in the form of an irreducible representation as $(\pi, \mathcal{H}) = (\bigoplus_{k=1}^K (I_{\mathcal{A}_k} \otimes \pi_k), \bigoplus_{k=1}^K (\mathcal{A}_k \otimes \mathcal{B}_k))$, where $(\pi_k, \mathcal{A}_k \otimes \mathcal{B}_k)$ is irreducible for all k, and $(\pi_k, \mathcal{A}_k \otimes \mathcal{B}_k)$ and $(\pi_{k'}, \mathcal{A}_{k'} \otimes \mathcal{B}_{k'})$ are not equivalent for all $k \neq k'$. Without losing generality, we can assume $\mathcal{B}_1 = \mathrm{Ran}P$ and $\mathcal{A}_1 = \mathbb{C}$ by the assumption of the theorem. Then, from Lemma 7, by defining $\{|\alpha_p^k\rangle\}_{p=1}^{\dim \mathcal{A}_k}$ as an orthonormal basis of $\mathcal{A}_k$, we can choose $\{\frac{|\alpha_p^k\rangle\langle\alpha_q^k| \otimes I_{\mathcal{B}_k}}{\sqrt{\dim \mathcal{B}_k}}\}_{p,q,k}$ as an orthonormal basis of the Hilbert Schmidt subspace $\mathfrak{B}_2(\mathcal{H})^G$. Since $P = |\alpha_1^1\rangle\langle\alpha_1^1| \otimes I_{\mathcal{B}_1}$, (note that this $\otimes$ is not the tensor product related to the entanglement of P, that is, P is not a "separable state"), $\mathfrak{B}_{P^\perp}^G = \{\rho \in \mathfrak{B}_2(\mathcal{H})^G | \mathrm{tr}P\rho = 0\}$ can be spanned by $\{\frac{|\alpha_p^k\rangle\langle\alpha_q^k| \otimes I_{\mathcal{B}_k}}{\sqrt{\dim \mathcal{B}_k}}\}_{p\geq1,q\geq1,k\geq2}$. Thus, suppose $\rho \in \mathcal{H} \otimes \mathcal{H}^\dagger$ is in $\mathfrak{B}_{P^\perp}^G$. Then, ρ can be decomposed only by $\{\frac{|\alpha_p^k\rangle\langle\alpha_q^k| \otimes I_{\mathcal{B}_k}}{\sqrt{\dim \mathcal{B}_k}}\}_{p\geq1,q\geq1,k\geq2}$. Since $\mathcal{B}_k \perp \mathcal{B}_1$ for all $k \geq 2$, $\rho|\Psi\rangle = 0$ for all $|\Psi\rangle \in \mathrm{Ran}P = \mathcal{B}_1$. Therefore from Lemma 10, Eq.(C9) holds. □

In the above proof, we derived Theorem 3 from Lemma 10. However, in this process we lost no generality; that is, the sufficient conditions of Lemma 10 and Theorem 3 are equivalent. This fact can be seen as follows. Suppose the sufficient condition of Lemma 10 is valid. Since $P \in \mathfrak{B}_2(\mathcal{H})^G$, from Lemma 8 we can see that without losing generality, P can be written down as $P = \sum_{k=1}^{K_0} P_k \otimes I_{\mathcal{B}_k}$, where $P_k \in \mathfrak{B}_2(\mathcal{A}_k)$ is a non-zero projection, and $K_0 \leq K$. First let us assume $K_0 > 1$. Then, by defining $\sigma \in \mathfrak{B}_2(\mathcal{H})$ and $|\xi\rangle \in \mathcal{H}$ as

$$\sigma = -\left(\frac{\sum_{k=2}^{K_0} \dim P_k \dim \mathcal{B}_k}{\dim P_1 \dim \mathcal{B}_1}\right) P_1 \otimes I_{\mathcal{B}_1} + \sum_{k=2}^{K_0} P_k \otimes I_{\mathcal{B}_k}$$

$$|\xi\rangle = |\alpha\rangle \otimes |\beta\rangle,$$

where $|\alpha\rangle \in \mathrm{Ran}(P_1)$ and $|\beta\rangle \in \mathcal{B}_1$, we derive $\mathrm{tr}P\sigma = 0$ and $\sigma|\xi\rangle = -\frac{\sum_{k=2}^{K_0} \dim P_k \dim \mathcal{B}_k}{\dim P}|\xi\rangle \neq 0$; This contradicts the sufficient condition in Lemma 10. Thus, $K_0 = 1$ and $\mathrm{Ran}(P) \in \mathcal{A}_1 \otimes \mathcal{B}_1$. Let us now assume $\dim \mathcal{A}_1 \geq 2$; that is, there exists another equivalent representation with $(\pi|_{\mathrm{Ran}P}, \mathrm{Ran}P)$ in $(\pi, \mathcal{H})$. In this case, we can write down $P = |\alpha_1^1\rangle\langle\alpha_1^1| \otimes I_{\mathcal{B}_1}$ by using $\{|\alpha_p^k\rangle\}_{p=1}^{\dim \mathcal{A}}$ as an orthonormal basis of $\mathcal{A}_k$. However, in this case, $\mathcal{A}_1$ is spanned by $\{|\alpha_p^1\rangle\}_{p=1}^{d_1}$ for $d_1 \geq 2$. We define $\sigma \stackrel{\text{def}}{=} \sum_{pq} a_{pq} |\alpha_p^1\rangle\langle\alpha_q^1| \otimes I_{\mathcal{B}_1}$ with $a_{11} = 0$ and $a_{pq} \neq 0$ $(\forall(p,q) \neq (1,1))$. Then, although $\sigma \in \mathfrak{B}_2(\mathcal{H})^G$, $\sigma|\alpha_1^1\rangle \otimes |\psi\rangle = \sum_{p\geq2} a_{p1} |\alpha_p^1\rangle\langle\alpha_1^1| \otimes I_{\mathcal{B}_1} \neq 0$, where $|\psi\rangle \in \mathcal{B}_1$. This also contradicts the sufficient condition of Lemma 10. Thus, if the sufficient condition of Lemma 10 is valid, then, $(\pi|_{\mathrm{Ran}P}, \mathrm{Ran}P)$ is an irreducible representation of G with multiplicity one on $(\pi, \mathcal{H})$. That is, the sufficient condition in Lemma 10 is equivalent to the sufficient condition in Lemma 3.

Finally, we rewrite the above theorem for pure states.

***Theorem* 4** A pure state $|\Psi\rangle$ on $\mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$ satisfies,

$$\log_2(1 + R_g(|\Psi\rangle)) = E_R(|\Psi\rangle) = E_g(|\Psi\rangle), \quad \text{(C14)}$$

if there exists a compact topological group G and a finite dimensional unitary representation $(\pi, \mathcal{H})$ such that $\pi(g)$ is a local unitary transformation for all $g \in G$, and $(\pi|_{\mathrm{Ran}|\Psi\rangle\langle\Psi|}, \mathrm{Ran}|\Psi\rangle\langle\Psi|)$ is an irreducible representation of G whose multiplicity is one on $(\pi, \mathcal{H})$.

Thus, if a pure state possesses an enough group symmetry, the values of all the three distance like measures of entanglement coincide. Note that, by means of Theorem 2 and Theorem 4, the results of stabilizer states and symmetric basis states in Section III can be easily recovered.

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