Stabilization of Displaced Periodic Orbits in the Solar Sail Restricted Three-body Problem

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Outline

Solar Sails

- Sail Design - Practical Goal

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Earth-Moon Restricted Three-Body Problem

- System Model - Equations of Motion of the Hybrid Sail
- Linearized System
- Conditions for the Existence of Displaced Orbits

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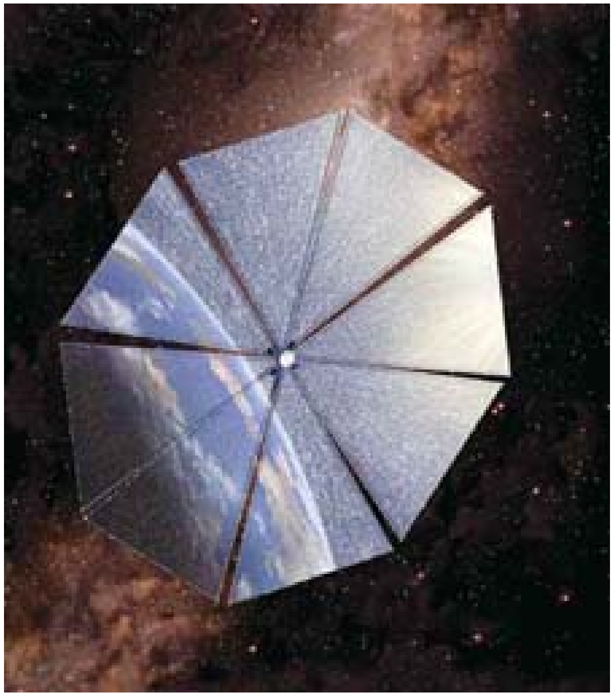
Sail Design

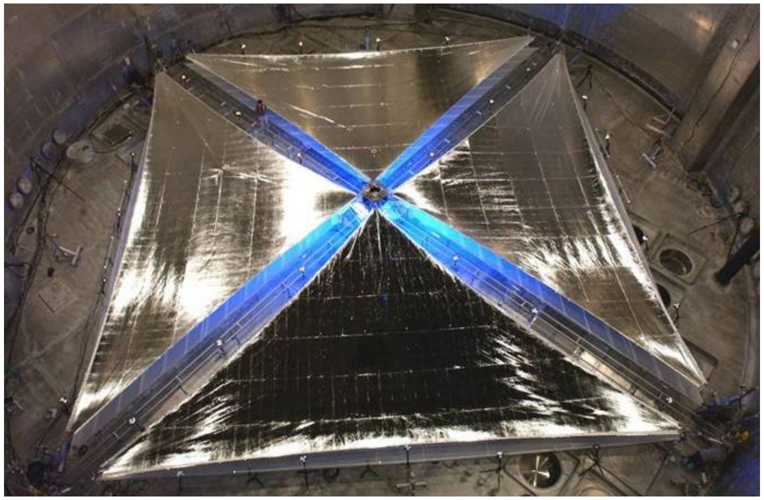


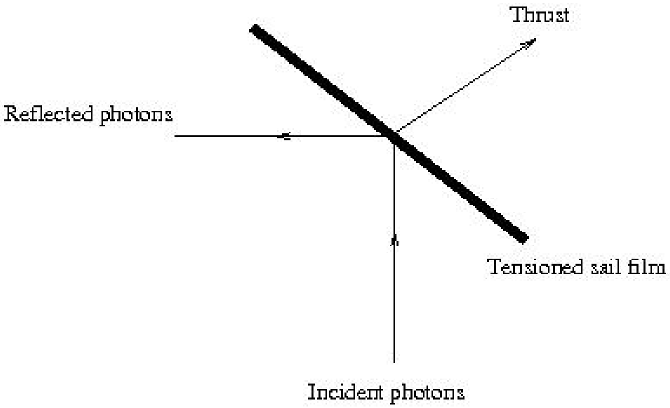
- A solar sail is a spacecraft without an engine, and therefore needs no fuel. It is pushed along by the pressure of photons from the sun hitting the sail.

- Solar sails are typically large square sheets of a highly reflective film supported by booms, although other designs (discs, blades) are popular.

- The material of the sail must be very lightweight and thin, of the order of a couple of microns (one thousandth the width of a sheet of paper), and very large, the order of $(50m) \times (50m)$.

Sail Design





Thrust
Reflected photons
Tensioned sail film
Incident photons

Practical Goal

- Solar sails provide unique families of new orbits (non-Keplerian orbits) with rich properties.

- These new orbits are associated with the artificial libration points.

- The artificial equilibria have potential applications for future space physics and Earth observation missions.

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System Model: Hybrid Sail

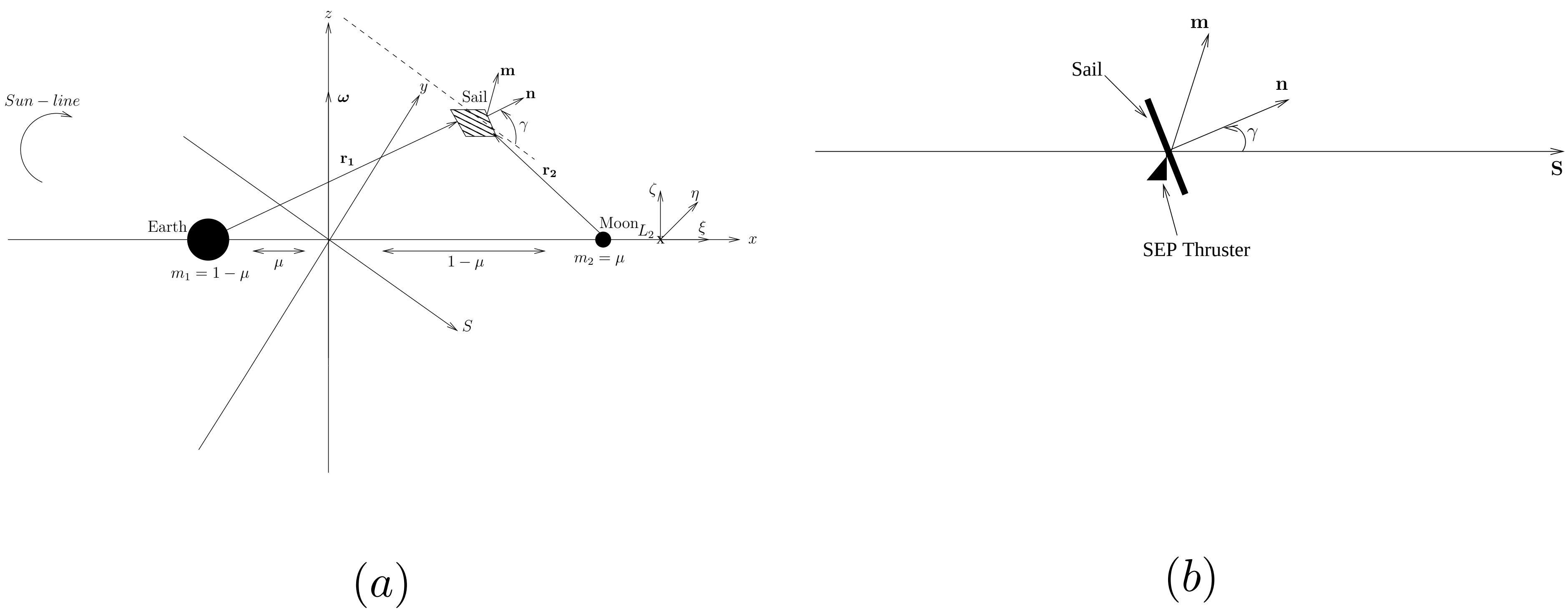

Figure 1: (a) Schematic geometry of the Hybrid Sail in the Earth-Moon restricted three-body problem; (b) Angle γ between the Hybrid Sail surface ***n*** and the Sun-line direction ***S***, and SEP thrust vector direction ***m***.

Equations of Motion of the Hybrid Sail

- The equations of motion of the hybrid sail in a rotating coordinate frame are given by

$$\frac{d^2\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}) = \boldsymbol{a}_S + \boldsymbol{a}_{SEP}, \tag{1}$$

where

$$\begin{aligned}
U(\boldsymbol{r}) &= -\left[\frac{1}{2}|\boldsymbol{\omega} \times \boldsymbol{r}|^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right], \\
\boldsymbol{a}_S &= a_0(\boldsymbol{S} \cdot \boldsymbol{n})^2 \boldsymbol{n}, \\
\boldsymbol{a}_{SEP} &= a_{SEP}\boldsymbol{m}, \\
\boldsymbol{n} &= \begin{bmatrix} \cos(\gamma)\cos(\omega_\star t) & -\cos(\gamma)\sin(\omega_\star t) & \sin(\gamma) \end{bmatrix}^T, \\
\boldsymbol{S} &= \begin{bmatrix} \cos(\omega_\star t) & -\sin(\omega_\star t) & 0 \end{bmatrix}^T,
\end{aligned}$$

- Because the solar radiation pressure force can never be directed sunward, the sail attitude is constrained such that $\boldsymbol{S} \cdot \boldsymbol{n} \geq 0$.

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Linearized System

The equations for the hybrid sail can be written as

$$\frac{d^2\delta \boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta \boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}_L + \delta \boldsymbol{r}) = \boldsymbol{a}_S(\boldsymbol{r}_L + \delta \boldsymbol{r}) + \boldsymbol{a}_{SEP}(\boldsymbol{r}_L + \delta \boldsymbol{r}), \qquad (2)$$

Linearized System

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and retaining only the first-order term in $\delta \boldsymbol{r} = [\delta x, \delta y, \delta y]^T$ in a Taylor-series expansion, the gradient of the potential and the acceleration can be expressed as

$$\nabla U(\boldsymbol{r}_L + \delta \boldsymbol{r}) = \nabla U(\boldsymbol{r}_L) + \left.\frac{\partial \nabla U(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} \delta \boldsymbol{r} + O(\delta \boldsymbol{r}^2), \qquad (3)$$

$$\boldsymbol{a}_S(\boldsymbol{r}_L + \delta \boldsymbol{r}) = \boldsymbol{a}_S(\boldsymbol{r}_L) + \left.\frac{\partial \boldsymbol{a}_S(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} \delta \boldsymbol{r} + O(\delta \boldsymbol{r}^2), \qquad (4)$$

$$\boldsymbol{a}_{SEP}(\boldsymbol{r}_L + \delta \boldsymbol{r}) = \boldsymbol{a}_{SEP}(\boldsymbol{r}_L) + \left.\frac{\partial \boldsymbol{a}_{SEP}(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} \delta \boldsymbol{r} + O(\delta \boldsymbol{r}^2). \qquad (5)$$

Linearized System

It is assumed that $\nabla U(\boldsymbol{r}_L) = 0$, and the accelerations $\boldsymbol{a}_S$ and $\boldsymbol{a}_{SEP}$ are constant with respect to the small displacement $\delta\boldsymbol{r}$, so that

$$\begin{aligned} \left.\frac{\partial \boldsymbol{a}_S(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} &= 0, \\ \left.\frac{\partial \boldsymbol{a}_{SEP}(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} &= 0. \end{aligned}$$

The linear variational system associated with the collinear libration points at $\boldsymbol{r}_L$ can be determined through a Taylor series expansion by substituting Eqs. (3), (4) and (5) into (2)

Linearized System

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The linear variational system associated with the collinear libration points at $\boldsymbol{r}_L$ can be determined through a Taylor series expansion by substituting Eqs. (3), (4) and (5) into (2)

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta\boldsymbol{r}}{dt} - K\delta\boldsymbol{r} = \boldsymbol{a}_S(\boldsymbol{r}_L) + \boldsymbol{a}_{SEP}(\boldsymbol{r}_L), \tag{6}$$

where the matrix K is defined as

$$K = -\left[\left.\frac{\partial \nabla U(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L}\right]. \tag{7}$$

Linearized System

Using matrix notation the linearized equation about the libration point (Eq. (6)) can be represented by the inhomogeneous linear system $\dot{\mathbf{X}} = A\mathbf{X} + \boldsymbol{b}(t)$, where the state vector $\mathbf{X} = (\delta \boldsymbol{r}, \delta \dot{\boldsymbol{r}})^T$, and $\boldsymbol{b}(t)$ is a 6×1 vector, which represents the solar sail acceleration.

Linearized System

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The Jacobian matrix A has the general form

$$A = \begin{pmatrix} 0_3 & I_3 \\ K & \Omega \end{pmatrix}, \qquad (8)$$

where I_3 is a identity matrix, and

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (9)$$

Linearized System

This yields the linearized nondimensional equations of motion in component form of a solar sail near the collinear libration points

$$\ddot{\xi} - 2\dot{\eta} - U^o_{xx}\xi = a_\xi + a_{SEP_\xi}, \quad (10)$$

$$\ddot{\eta} + 2\dot{\xi} - U^o_{yy}\eta = a_\eta + a_{SEP_\eta}, \quad (11)$$

$$\ddot{\zeta} - U^o_{zz}\zeta = a_\zeta + a_{SEP_\zeta}. \quad (12)$$

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$$\ddot{\zeta} - U^o_{zz}\zeta = a_\zeta + a_{SEP_\zeta}. \tag{12}$$

The solar sail acceleration components are given by

$$\begin{aligned} a_\xi &= a_0 \cos(\omega_\star t) \cos^3(\gamma), \\ a_\eta &= -a_0 \sin(\omega_\star t) \cos^3(\gamma), \\ a_\zeta &= a_0 \cos^2(\gamma) \sin(\gamma). \end{aligned}$$

Outline

Conditions for the Existence of Displaced Orbits

By taking $\boldsymbol{a}_{SEP} = 0$ (pure sail at linear order), Eqs. (10) - (12) have a simple periodic solution with a constant out-of-plane displacement of the form

$$
\begin{aligned}
\xi(t) &= \xi_0 \cos(\omega_\star t), && (13)\\
\eta(t) &= \eta_0 \sin(\omega_\star t), && (14)\\
\zeta(t) &= \zeta_0. && (15)
\end{aligned}
$$

By inserting equations (13) and (14) in the differential equations (10) and (11), we obtain the linear system in ξ_0 and η_0,

$$
\begin{cases}
\left(U^o_{xx} - \omega_\star^2\right)\xi_0 - 2\omega_\star\eta_0 = a_0\cos^3(\gamma), \\
-2\omega_\star\xi_0 + \left(U^o_{yy} - \omega_\star^2\right)\eta_0 = -a_0\cos^3(\gamma).
\end{cases}
\qquad (16)
$$

Conditions for the Existence of Displaced Orbits

Then the amplitudes ξ_0 and η_0 are given by

$$\xi_0 = a_0 \frac{\left(U^o_{yy} - \omega_\star^2 - 2\omega_\star\right)\cos^3(\gamma)}{\left(U^o_{xx} - \omega_\star^2\right)\left(U^o_{yy} - \omega_\star^2\right) - 4\omega_\star^2}, \tag{17}$$

$$\eta_0 = a_0 \frac{\left(-U^o_{xx} + \omega_\star^2 + 2\omega_\star\right)\cos^3(\gamma)}{\left(U^o_{xx} - \omega_\star^2\right)\left(U^o_{yy} - \omega_\star^2\right) - 4\omega_\star^2}, \tag{18}$$

and we have the equality

$$\frac{\xi_0}{\eta_0} = \frac{\omega_\star^2 + 2\omega_\star - U^o_{yy}}{-\omega_\star^2 - 2\omega_\star + U^o_{xx}}. \tag{19}$$

Then with the condition given by Eq. (19), Eqs. (13)-(15) will be used as a reference trajectory.

Conditions for the Existence of Displaced Orbits

By applying a Laplace transform, the uncoupled out-of-plane ζ-motion defined by the equation (12) can be written as

$$\begin{aligned}
\zeta(t) &= \zeta_0 \cos(\omega_\zeta t) + \dot{\zeta}_0 |U^o_{zz}|^{-1/2} \sin(\omega_\zeta t) \\
&\quad + a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1} [U(t) - \cos(\omega_\zeta t)], \\
&= U(t) a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1} + \dot{\zeta}_0 |U^o_{zz}|^{-1/2} \sin(\omega_\zeta t) \\
&\quad + \cos(\omega_\zeta t)[\zeta_0 - a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1}],
\end{aligned} \tag{20}$$

where the nondimensional frequency is defined as $\omega_\zeta = |U^o_{zz}|^{1/2}$ and $U(t)$ is the unit step function.

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&= U(t) a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1} + \dot{\zeta}_0 |U^o_{zz}|^{-1/2} \sin(\omega_\zeta t) \\
&\quad + \cos(\omega_\zeta t)[\zeta_0 - a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1}],
\end{aligned} \tag{20}$$

where the nondimensional frequency is defined as $\omega_\zeta = |U^o_{zz}|^{1/2}$ and $U(t)$ is the unit step function.

Specifically for the choice of the initial data $\dot{\zeta}_0 = 0$, the equation (20) can be more conveniently expressed as

$$\begin{aligned}
\zeta(t) &= U(t) a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1} \\
&\quad + \cos(\omega_\zeta t)[\zeta_0 - a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1}].
\end{aligned} \tag{21}$$

Conditions for the Existence of Displaced Orbits

The solution can be made to contain only the periodic oscillatory modes at an out-of-plane distance

$$\zeta_0 = a_0 \cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}. \tag{22}$$

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The solution can be made to contain only the periodic oscillatory modes at an out-of-plane distance

$$\zeta_0 = a_0 \cos^2(\gamma) \sin(\gamma) |U^o_{zz}|^{-1}. \tag{22}$$

Furthermore, the out-of-plane distance can be maximized by an optimal choice of the sail pitch angle determined by

$$\begin{aligned} \frac{d}{d\gamma} \cos^2(\gamma) \sin(\gamma) \Big|_{\gamma=\gamma^\star} &= 0, \\ \gamma^\star &= 35.264^\circ. \end{aligned} \tag{23}$$

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Evaluation of Hybrid Sail Performance - Propellant Usage

Objectives

To develop a feedback linearization scheme, the motion of the hybrid solar sail moving in the CRTBP is separated into linear and nonlinear components, such that

$$\ddot{\xi} = f^{\xi}_{Non-Linear} + f^{\xi}_{Linear} + a_{\xi} + u_{\xi}, \tag{24}$$

$$\ddot{\eta} = f^{\eta}_{Non-Linear} + f^{\eta}_{Linear} + a_{\eta} + u_{\eta}, \tag{25}$$

$$\ddot{\zeta} = f^{\zeta}_{Non-Linear} + f^{\zeta}_{Linear} + a_{\zeta} + u_{\zeta}, \tag{26}$$

where the f functions are defined as the linear and the nonlinear terms

Objectives

$$
\begin{aligned}
f^{\xi}_{Non-Linear} &= -(1-\mu)\frac{(x_{L_i}+\xi)+\mu}{r_1^3} - \mu\frac{(x_{L_i}+\xi)-1+\mu}{r_2^3}, \\
f^{\xi}_{Linear} &= 2\dot{\eta} + (x_{L_i}+\xi), \\
f^{\eta}_{Non-Linear} &= -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)\eta, \\
f^{\eta}_{Linear} &= -2\dot{\xi} + \eta, \\
f^{\zeta}_{Non-Linear} &= -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)\zeta, \\
f^{\zeta}_{Linear} &= 0,
\end{aligned}
$$

with

$$
\begin{aligned}
r_1 &= \sqrt{((x_{L_i}+\xi)+\mu)^2+\eta^2+\zeta^2}, \\
r_2 &= \sqrt{((x_{L_i}+\xi)-1+\mu)^2+\eta^2+\zeta^2}.
\end{aligned}
$$

Objectives

We then select the SEP control $\boldsymbol{u}(t)$ such that

$$\boldsymbol{u}(t) = \begin{bmatrix} u_\xi \\ u_\eta \\ u_\zeta \end{bmatrix} = \boldsymbol{U}(t) + \tilde{\boldsymbol{u}}(t), \tag{27}$$

where

$$\boldsymbol{U}(t) = - \begin{bmatrix} (x_{L_2} + \xi) - (1-\mu)\frac{(x_{L_2}+\xi)+\mu}{r_1^3} - \mu\frac{(x_{L_2}+\xi)-1+\mu}{r_2^3} - U_{xx}^o \xi \\ -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)\eta - U_{yy}^o \eta \\ -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)\zeta - U_{zz}^o \zeta \end{bmatrix} \tag{28}$$

is the cancelling term and $\tilde{\boldsymbol{u}}(t)$ the stabilizing term.

Objectives

The motion of the hybrid solar sail in the CRTBP is then described by the equations

$$\ddot{\xi} = 2\dot{\eta} + U^o_{xx}\xi + a_0 \cos(\omega_\star t) \cos^3(\gamma) + \tilde{u}_\xi, \quad (29)$$

$$\ddot{\eta} = -2\dot{\xi} + U^o_{yy}\eta - a_0 \sin(\omega_\star t) \cos^3(\gamma) + \tilde{u}_\eta, \quad (30)$$

$$\ddot{\zeta} = U^o_{zz}\zeta + a_0 \cos^2(\gamma) \sin(\gamma) + \tilde{u}_\zeta. \quad (31)$$

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Stabilization

Let us consider nonlinear system described by

$$\ddot{\boldsymbol{x}} = f(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{u}, \tag{32}$$

where $\boldsymbol{x} \in \mathbb{R}^3$ is the position. Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{ref}(t)$ denote the position error relative to some reference solution, where the reference trajectory

$$\boldsymbol{x}_{ref}(t) = \begin{bmatrix} \xi_{ref} & \eta_{ref} & \zeta_{ref} \end{bmatrix}^T \tag{33}$$

is given by the analytical solution

$$\begin{aligned} \xi_{ref}(t) &= \xi_0 \cos(\omega_\star t), \\ \eta_{ref}(t) &= \eta_0 \sin(\omega_\star t), \\ \zeta_{ref}(t) &= \zeta_0. \end{aligned}$$

Stabilization

We then differentiate $\boldsymbol{e}(t)$ until the control appears so that

$$\begin{aligned}
\boldsymbol{e}(t) &= \boldsymbol{x}(t) - \boldsymbol{x}_{ref}(t), & (34)\\
\dot{\boldsymbol{e}}(t) &= \dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{x}}_{ref}(t), & (35)\\
\ddot{\boldsymbol{e}}(t) &= \ddot{\boldsymbol{x}}(t) - \ddot{\boldsymbol{x}}_{ref}(t), & (36)\\
&= f(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{u} - \ddot{\boldsymbol{x}}_{ref}(t),\\
&= -\lambda_1 \dot{\boldsymbol{e}} - \lambda_2 \boldsymbol{e},
\end{aligned}$$

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\end{aligned}$$

and so, we have

$$\boldsymbol{u}(t) = -f(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \ddot{\boldsymbol{x}}_{ref}(t) - \lambda_1 \dot{\boldsymbol{e}} - \lambda_2 \boldsymbol{e}. \quad (37)$$

Trajectory Tracking

- Consider the system given by Eq. (32), where our objective is to make the output $\boldsymbol{x} \in \mathbb{R}^3$ track a desired trajectory given by the reference trajectory $\boldsymbol{x}_{ref} \in \mathbb{R}^3$ while keeping the whole position bounded.

- Therefore, we want to find a control law for the input $\tilde{\boldsymbol{u}} \in \mathbb{R}^3$ such that, starting from any initial position in a domain $D \subset \mathbb{R}^3$, the tracking error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{ref}(t)$ goes to zero, while the whole position $\boldsymbol{x} \in \mathbb{R}^3$ remains bounded.

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- Hence, asymtotic tracking will be achieved if we design a state feedback control law to ensure that $e(t)$ is bounded and converges to zero as t tends to infinity.

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- Hence, asymtotic tracking will be achieved if we design a state feedback control law to ensure that $e(t)$ is bounded and converges to zero as t tends to infinity.

Thus, the control law

$$\tilde{\boldsymbol{u}} = -\lambda_1 \dot{\boldsymbol{e}} - \lambda_2 \boldsymbol{e} \tag{38}$$

yields the tracking error equation

$$\ddot{\boldsymbol{e}} + \lambda_1 \dot{\boldsymbol{e}} + \lambda_2 \boldsymbol{e} = 0. \tag{39}$$

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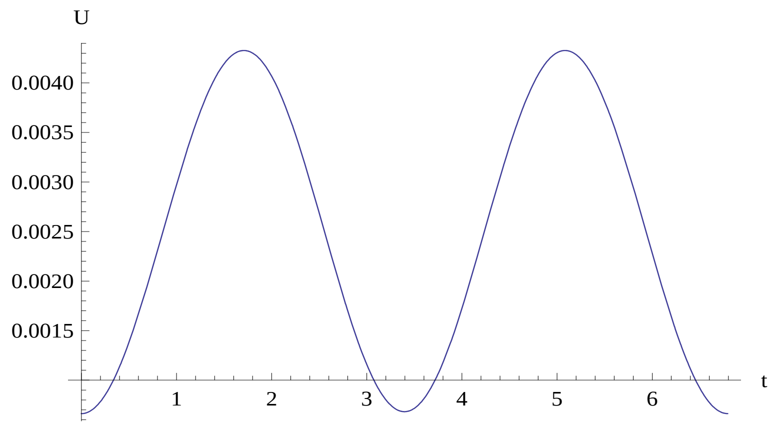
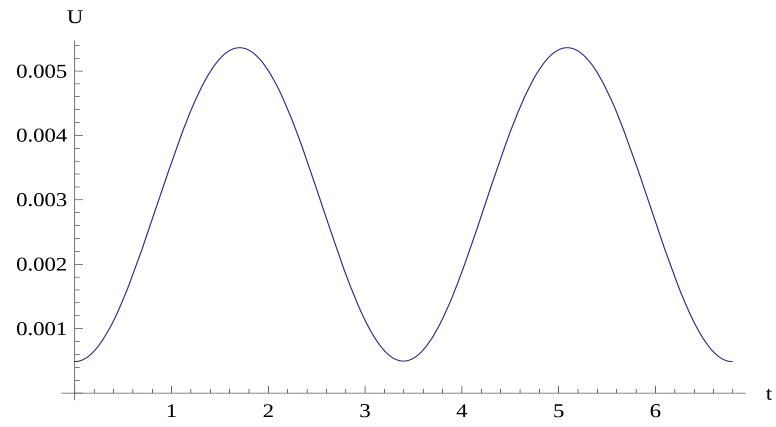
Evaluation of Hybrid Sail Performance

- The magnitude of the total control effort appears in Figure 2. Thus, the control acceleration effort ***U*** required to track the reference orbit while rejecting the nonlinearities varies up to 0.004 ($0.012\ mm/s^2$) for the orbit about L_1 and 0.005 ($0.014\ mm/s^2$) for the orbit about the L_2 point.

- The control accelerations are continous smooth signals.



(a)



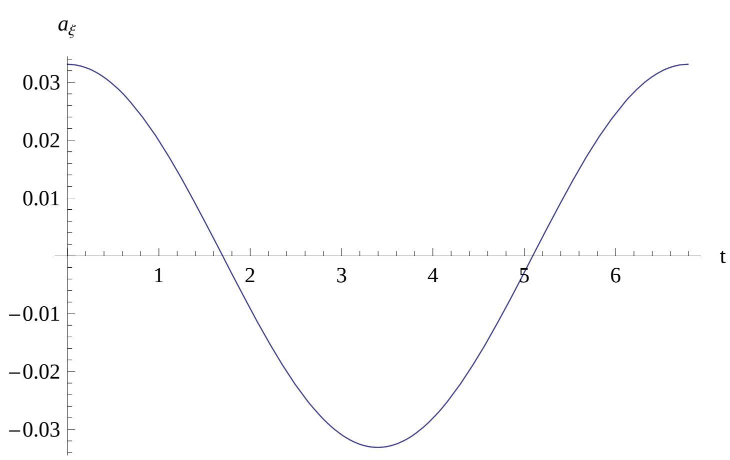
(b)

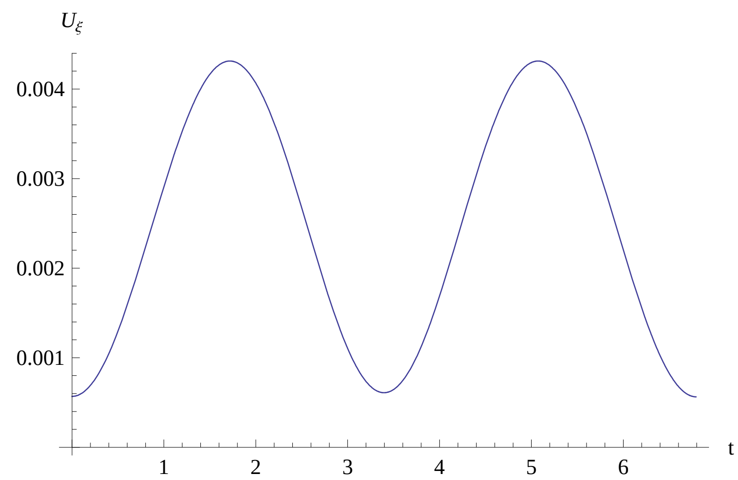
Figure 2: (a) Magnitude of the total control effort about the L_1 point; (b) Magnitude of the total control effort about the L_2 point.

Evaluation of Hybrid Sail Performance

- The acceleration derived from the solar sail (denoted by a_ξ, a_η, a_ζ) is plotted in terms of components for one-month orbits in Figure 3 (a) about L_1, Figure 4 (a) about L_2, and the SEP acceleration components appears in Figure 3 (b) about L_1, Figure 4 (b) about L_2

- The control acceleration effort derived from the thruster (denoted by U_ξ, U_η, U_ζ) is order of 10^{-3} - 10^{-4}, while the acceleration derived from the solar sail is over approximately 10^{-2}.





Evaluation of Hybrid Sail Performance

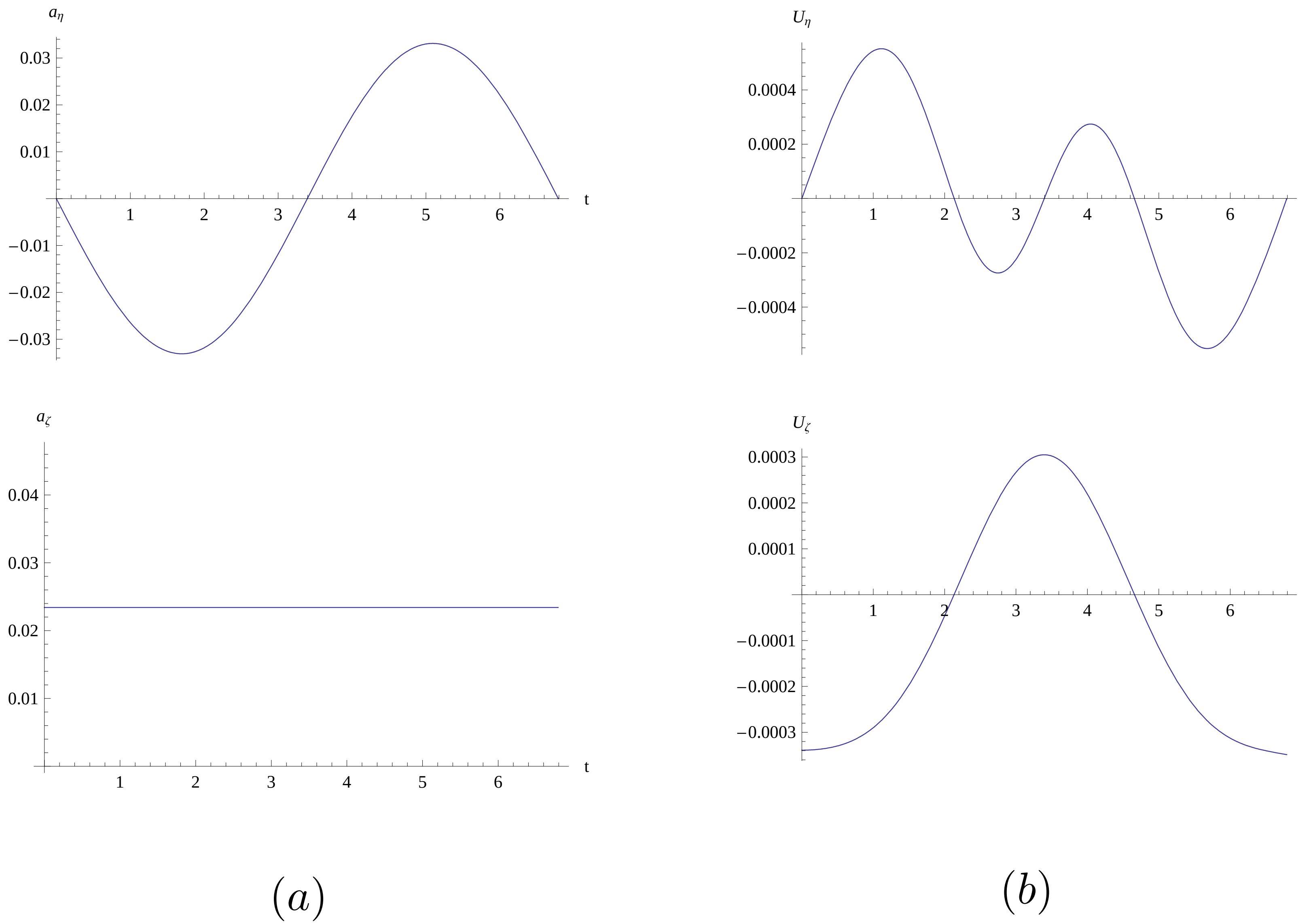
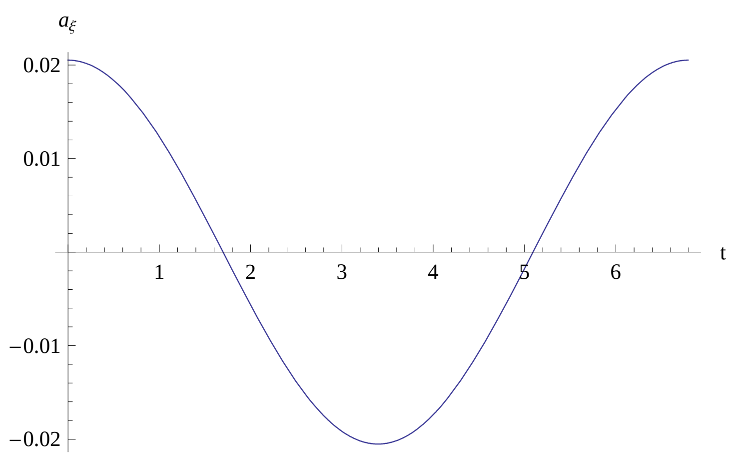
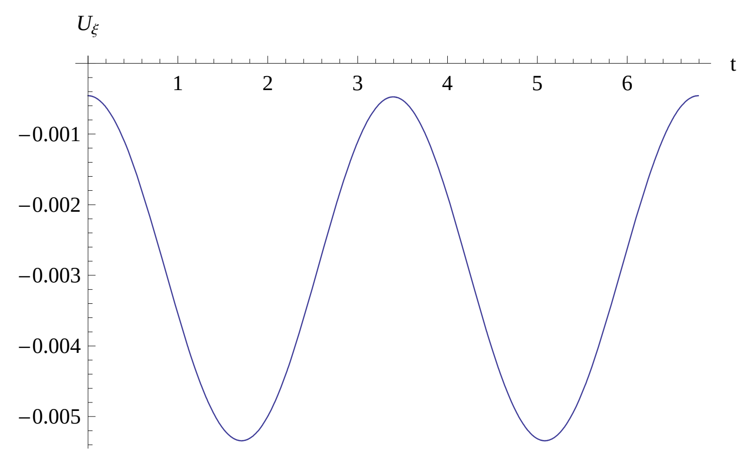


Figure 3: (a) Acceleration derived from the solar sail about the L_1 point; (b) Acceleration derived from the thruster about the L_1 point.

Evaluation of Hybrid Sail Performance

- The small control acceleration from the SEP thruster is then applied to ensure that the displacement of the periodic orbit is constant. The solar sail provides a constant out-of-plane force.





Evaluation of Hybrid Sail Performance

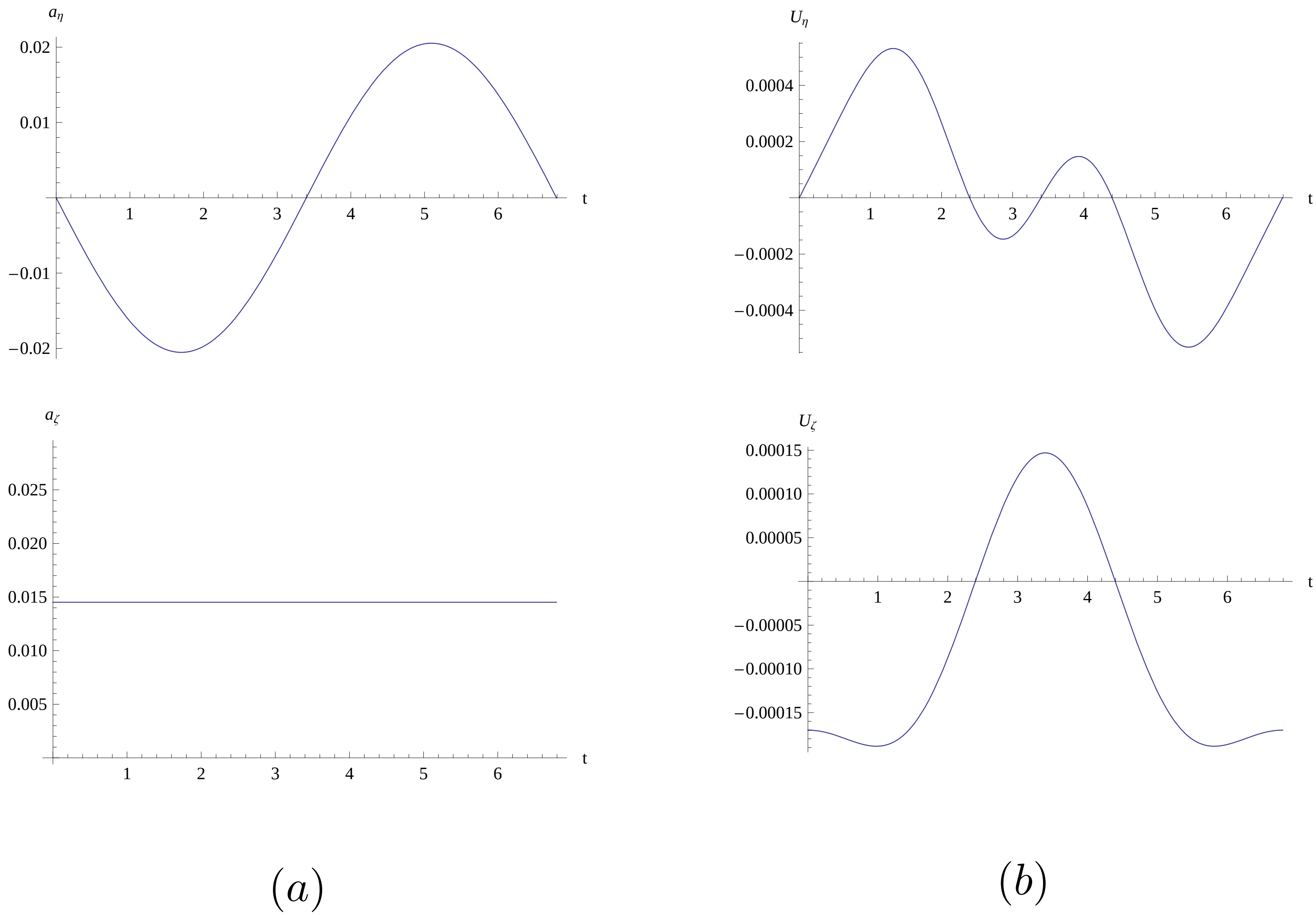
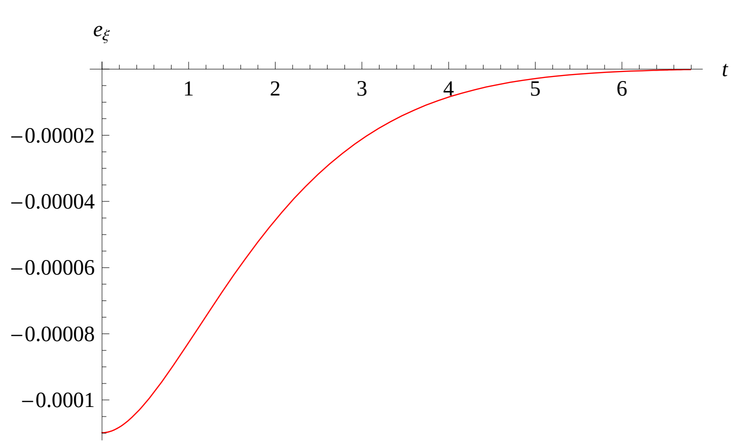


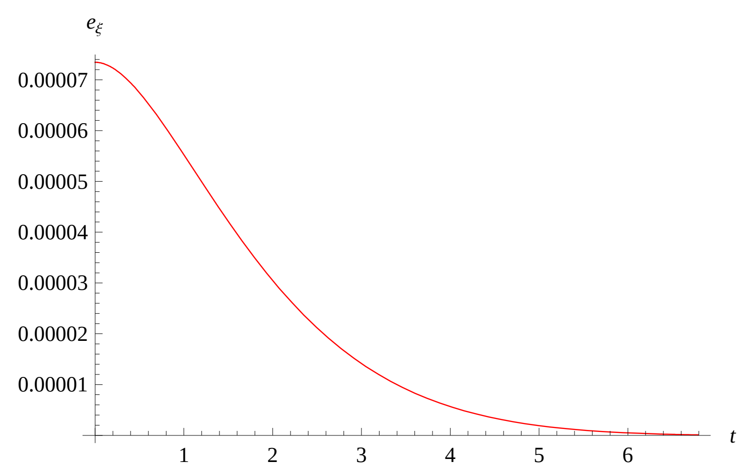
Figure 4: (a) Acceleration derived from the solar sail about the L_2 point; (b) Acceleration derived from the thruster about the L_2 point.

Evaluation of Hybrid Sail Performance

- Figure 5 (a) (resp. Figure 5 (b)) illustrates the position error components, denoted by e_ξ, e_η, e_ζ under the nonlinear control and the SEP thruster around L_1 (resp. L_2).

- These Figures show that the motion is bounded and periodic. This observation implies that the augmented thrust acceleration ensures a constant displacement orbit.





Evaluation of Hybrid Sail Performance

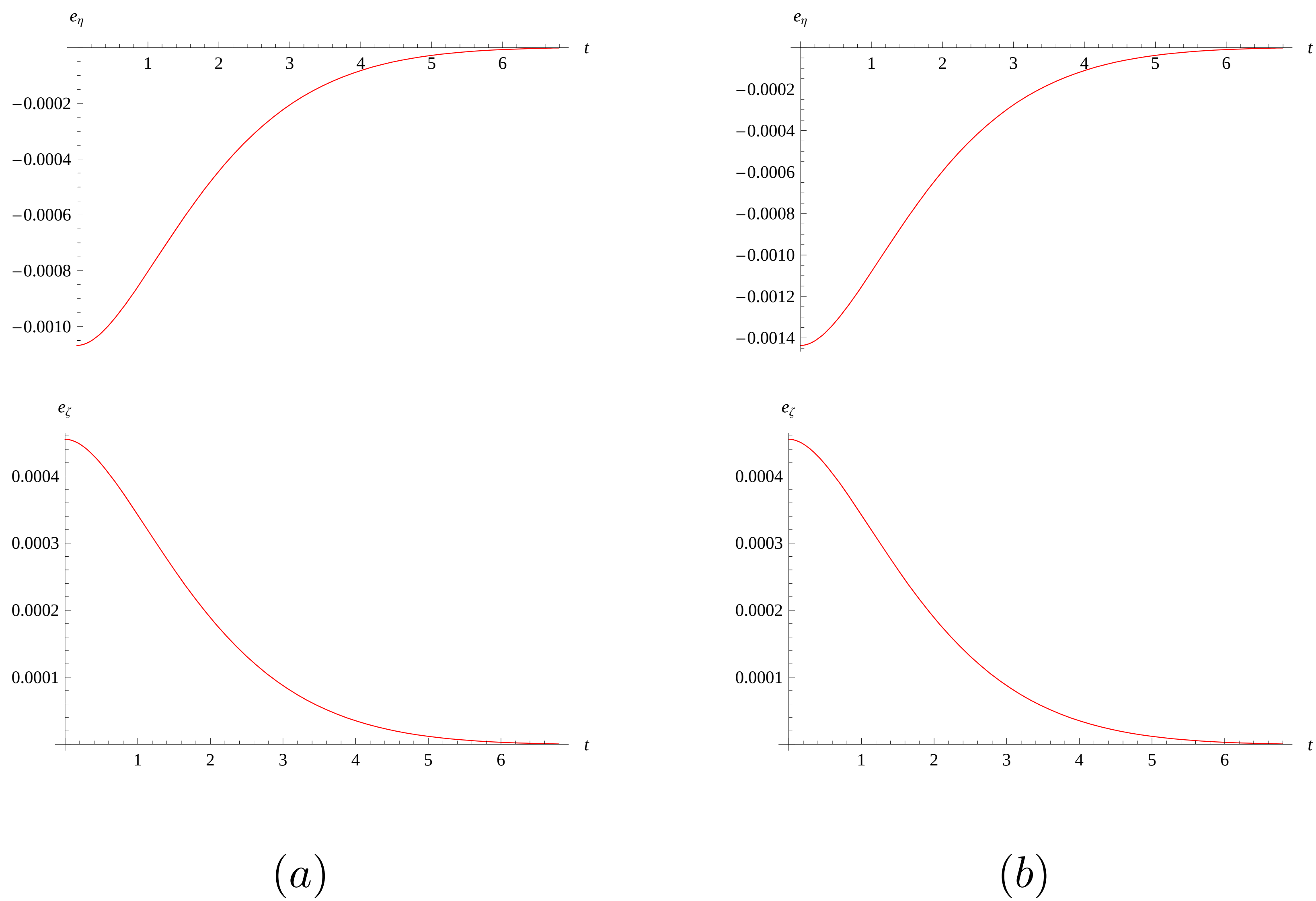
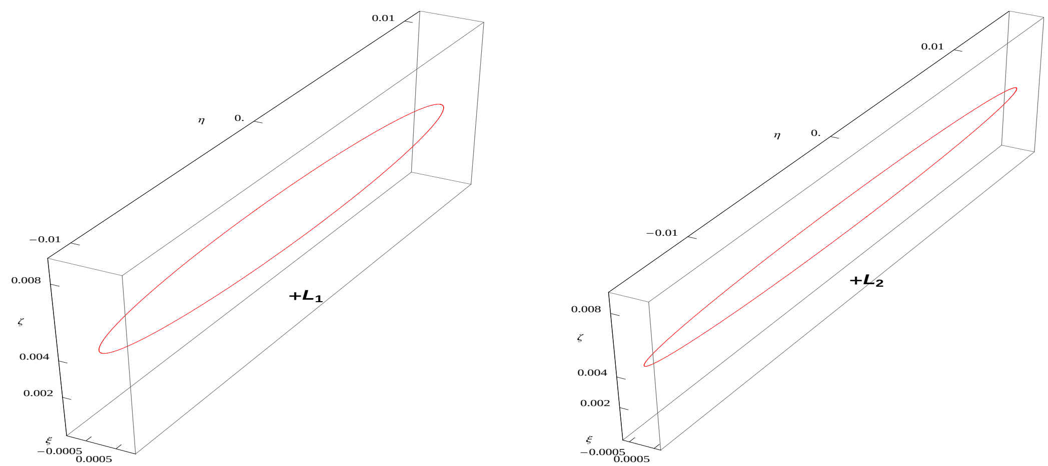


Figure 5: (a) Position error components about the L_1 point with $e(0) = (-0.00011, -0.0010, 0.00045)^T$ (critically damped motion); (b) Position errors components about the L_2 point with $e(0) = (0.000073, -0.0014, 0.00045)^T$ (critically damped motion).

Evaluation of Hybrid Sail Performance



(a) (b)

Figure 6: Orbit resulting from tracking the reference orbit using the nonlinear control and SEP thruster: (a) Above L_1; (b) Above L_2.

Propellant Usage

- Assume a specific impulse of $I_{sp} = 3000\ sec$ and an initial mass of $m_i = 500\ kg$, we have the average ΔV per orbit of approximately 23 m/s.

- Then, the total ΔV per orbit over 5 years is $1536\ m/s$. The consumed propellant mass is then $m_{prop} = 25\ kg$.

Applications

- A hybrid concept for displaced periodic orbits in the Earth-Moon system has been developed.

- A feedback linearization was used to perform stabilization and trajectory tracking for nonlinear systems.

- The augmented thrust acceleration is than applied to ensure a constant displacement periodic orbit, which provides key advantages for lunar polar telecommunications.

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