An adaptive stabilized finite element method for the generalized Stokes problem

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Abstract

In this work we present an adaptive strategy (based on an a posteriori error estimator) for a stabilized finite element method for the Stokes problem, with and without a reaction term. The hierarchical type estimator is based on the solution of local problems posed on appropriate finite dimensional spaces of bubble-like functions. An equivalence result between the norm of the finite element error and the estimator is given, where the dependence of the constants on the physics of the problem is explicited. Several numerical results confirming both the theoretical results and the good performance of the estimator are given.

Key words: Stokes equation, a posteriori error estimator, bubble function, stabilized finite element method, adapted mesh

1 Introduction

A posteriori error analysis and adaptive finite element methods for problems in fluid dynamics has been a very active subject of research in the last decades. For instance, for the advective-diffusive model we can quote the works [22,17,5,6], among others. Now, for the Stokes problem, the works by Verfürth

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[1] Partially supported by CONICYT through FONDECYT Project 1040595 (Chile).
[2] Partially supported by CONICYT through FONDECYT Project 1061032 and by FONDAP Program in Applied Mathematics (Chile).

[20,21] and Bank and Welfert [7], laid the basic foundation for the mathematical analysis of practical methods (see also [11] for error estimators in the nonconforming case). More recently, in [3,2] and [12], a posteriori error estimators rigorously bounding the discretization errors have been addressed. All previous references deal with stable (in the sense of the discrete inf-sup condition [9]) discretizations for the Stokes problem. In [18] and [4], an a posteriori error analysis of stabilized formulations for the Stokes problem was performed, but the analysis was restricted to the pure Stokes case (i.e., without a reaction term).

In this paper we introduce and analyze from theoretical and experimental points of view an adaptive scheme to efficiently solve the generalized Stokes problem. The scheme is based on the unusual stabilized finite element method introduced in [8], combined with an error estimator which is based on an idea from [2], building an auxiliary problem, whose solution is equivalent with the norm of the finite element error. Since this auxiliary problem is posed on an infinite dimensional setting, we build a hierarchical estimation for the solution of this problem, which turns out to be equivalent with the norm of its solution, and hence the resulting finite element approximation is equivalent to the original finite element error.

An outline of the paper is as follows. The model problem is stated in Section 2, and the bases of the discrete approximation are settled in Section 3. Next, in Section 4 we propose the auxiliary problem and prove that we can define a norm based on the solution of this auxiliary problem, which is equivalent to the norm of the error. This auxiliary problem is applied to the solution of the residual equation and hence we state, at the end of Section 4.1, a first equivalence result between the norm of the error and the solution of the auxiliary problem (with the residual as right-hand side). As we told before, the auxiliary problem is posed on an infinite dimensional space, and hence in Section 5 we define a finite dimensional approximation (based on a hierarchical idea) of its solution. Finally, in Section 6 we present several numerical results confirming the theoretical results and showing the good performance of our estimator, and in Section 7 we give some conclusions.

2 The model problem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with polygonal boundary Γ. We denote by $H^m(\Omega)$ the usual Sobolev space of order $m \geq 0$, with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$, respectively (with the convention $H^0(\Omega) = L^2(\Omega)$ and $|\cdot|_{0,\Omega} = \|\cdot\|_{0,\Omega}$). Then, given $\boldsymbol{f} \in L^2(\Omega)^2$, $\sigma \geq 0$ and $\nu \in \mathbb{R}^+$, our generalized

Stokes problem reads: *Find a velocity* $\boldsymbol{u}$ *and the pressure field* p *such that*

$$(P)\quad \begin{cases} \sigma\,\boldsymbol{u} - \nu\,\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{div}\,\boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \Gamma. \end{cases}$$

Let then $\boldsymbol{H} := H_0^1(\Omega)^2$ and $Q := L_0^2(\Omega) := \{q \in L^2(\Omega) : (q,1)_\Omega = 0\}$, where $(\cdot\,,\cdot)_D$ stands for the inner product in $L^2(D)$ (or in $L^2(D)^2, L^2(D)^{2\times 2}$, if necessary) be the functional spaces to be used. The weak formulation of the problem (P) reads: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ *such that*

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + b(\boldsymbol{u},q) = (\boldsymbol{f},\boldsymbol{v})_\Omega \qquad \forall\,(\boldsymbol{v},q) \in \boldsymbol{H} \times Q, \tag{2.1}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) := \sigma\,(\boldsymbol{u},\boldsymbol{v})_\Omega + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})_\Omega\,, \tag{2.2}$$
$$b(\boldsymbol{v},q) := -(q,\operatorname{div}\boldsymbol{v})_\Omega\,. \tag{2.3}$$

Furthermore, let $c : Q \times Q \to \mathbb{R}$ be the symmetric bilinear form defined by:

$$c(p,q) := \frac{1}{\nu}\,(p,q)_\Omega.$$

Using bilinear forms a and c we define the following norms:

$$\|\boldsymbol{v}\|_a := a(\boldsymbol{v},\boldsymbol{v})^{1/2} \qquad \forall \boldsymbol{v} \in \boldsymbol{H},$$
$$\|q\|_c := c(q,q)^{1/2} \qquad \forall q \in Q\,,$$

and the following norm on the product space $\boldsymbol{H} \times Q$:

$$\|(\boldsymbol{v},q)\| := \left\{ \|\boldsymbol{v}\|_a^2 + \|q\|_c^2 \right\}^{1/2} \qquad \forall (\boldsymbol{v},q) \in \boldsymbol{H} \times Q. \tag{2.4}$$

The following result states the main properties of these bilinear forms.

Lemma 1 *Let* a *and* b *be the bilinear forms given by (2.2) and (2.3), respectively. Then*

$$|a(\boldsymbol{v},\boldsymbol{w})| \leq \|\boldsymbol{v}\|_a \|\boldsymbol{w}\|_a \qquad \forall\,\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}\,, \tag{2.5}$$
$$|b(\boldsymbol{v},q)| \leq \sqrt{2}\,\|\boldsymbol{v}\|_a \|q\|_c \qquad \forall (\boldsymbol{v},q) \in \boldsymbol{H} \times Q\,, \tag{2.6}$$
$$\sup_{\boldsymbol{v}\in\boldsymbol{H}} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_a} \geq \alpha_b \sqrt{\frac{\nu}{\sigma+\nu}}\,\|q\|_c \qquad \forall q \in Q\,, \tag{2.7}$$

where $\alpha_b > 0$ *is a constant depending only on* Ω.

PROOF. The proof follows from the norms definition and the well-known properties of these bilinear forms (see Theorem 4.1 in [15]). $\square$

Then, using the classical theory of Babuska-Brezzi (cf. [15]), we can state the following result.

Lemma 2 *The weak problem (2.1) has a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$.

3 Notations and preliminary results

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of Ω and let us denote by $\mathcal{E}_h$ the set of all sides of $\mathcal{T}_h$ with the usual splitting $\mathcal{E}_h = \mathcal{E}_\Omega \cup \mathcal{E}_\Gamma$, where $\mathcal{E}_\Omega$ stands for the sides lying on the interior of Ω. Also, for $T \in \mathcal{T}_h$, we denote by $\mathcal{N}(T)$ the set of nodes of T and by $\mathcal{E}(T)$ the set of sides of T. Also, for $T \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$ we define the following neighborhoods:

$$\omega_T := \bigcup_{\mathcal{E}(T)\cap\mathcal{E}(T')\neq\emptyset} T' \quad , \quad \widetilde{\omega}_T := \bigcup_{\mathcal{N}(T)\cap\mathcal{N}(T')\neq\emptyset} T',$$
$$\omega_F := \bigcup_{F\in\mathcal{E}(T')} T' \quad , \quad \widetilde{\omega}_F := \bigcup_{\mathcal{N}(F)\cap\mathcal{N}(T')\neq\emptyset} T'.$$

Next, for $T \in \mathcal{T}_h$ and $F \in \mathcal{E}_\Omega$, let h_T be the diameter of T, $h_F := |F|$, and let us define the following mesh-dependent constants:

$$\theta_T := \begin{cases} \nu^{-1/2}\, h_T & , \sigma = 0\,, \\ \sigma^{-1/2}\, \min\{h_T\, \sigma^{1/2}\, \nu^{-1/2}, 1\} & , \sigma > 0\,. \end{cases}$$
$$\theta_F := \begin{cases} \nu^{-1/2}\, h_F^{1/2} & , \sigma = 0\,, \\ \nu^{-1/4}\, \sigma^{-1/4}\, \min\{h_F\, \sigma^{1/2}\, \nu^{-1/2}, 1\}^{1/2} & , \sigma > 0\,. \end{cases}$$

In the rest of the paper we will use the notation

$$a \preceq b \Longleftrightarrow a \leq K\, b,$$
$$a \simeq b \Longleftrightarrow a \preceq b \text{ and } b \preceq a,$$

where the positive constant K is independent of h, σ and ν.

Finally, let $k, l \in \mathbb{N}$, and let us define the following finite element spaces

$$\boldsymbol{H}_h := \{\boldsymbol{\varphi} \in C(\overline{\Omega})^2 : \boldsymbol{\varphi}|_T \in \mathbb{P}_k(T)^2, \forall T \in \mathcal{T}_h\} \cap H_0^1(\Omega)^2,$$
$$Q_h := \{\varphi \in C(\overline{\Omega}) : \varphi|_T \in \mathbb{P}_l(T), \forall T \in \mathcal{T}_h\} \cap L_0^2(\Omega).$$

Lemma 3 *The following estimates hold for all* $\boldsymbol{v}_h \in \boldsymbol{H}_h$ *and* $\sigma \geq 0$*:*

$$\|\nabla \boldsymbol{v}_h\|_{0,T} \preceq h_T^{-1}\, \theta_T\, \|\boldsymbol{v}_h\|_{a,T}\,, \tag{3.1}$$
$$\|\Delta \boldsymbol{v}_h\|_{0,T} \preceq h_T^{-2}\, \theta_T\, \|\boldsymbol{v}_h\|_{a,T}\,. \tag{3.2}$$

PROOF. If $\sigma = 0$ the proof follows from the inverse inequality

$$\|\nabla \boldsymbol{v}_h\|_{0,T} \preceq h_T^{-1}\, \|\boldsymbol{v}_h\|_{0,T} \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{H}_h\,, \tag{3.3}$$

(see Lemma 1.138 in [14]) and the definition of θ_T. For $\sigma > 0$, from the definition of $\|\cdot\|_{a,T}$ we see that

$$\|\nabla \boldsymbol{v}_h\|_{0,T} \leq \nu^{-1/2}\, \|\boldsymbol{v}_h\|_{a,T}. \tag{3.4}$$

On the other hand, using the inverse inequality (3.3) we obtain

$$\|\nabla \boldsymbol{v}_h\|_{0,T} \preceq h_T^{-1}\, \|\boldsymbol{v}_h\|_{0,T} \preceq h_T^{-1}\, \sigma^{-1/2}\, \|\boldsymbol{v}_h\|_{a,T}. \tag{3.5}$$

Then (3.1) arises using (3.4)-(3.5). For the second estimate, (3.3) and (3.1) lead to

$$\|\Delta \boldsymbol{v}_h\|_{0,T} \preceq h_T^{-1}\, \|\nabla \boldsymbol{v}_h\|_{0,T} \preceq h_T^{-2}\, \theta_T\, \|\boldsymbol{v}_h\|_{a,T}\,,$$

and the result follows. $\square$

Let now $I_h : \boldsymbol{H} \longrightarrow \boldsymbol{H}_h$ denote the Clément interpolation operator (cf. [10,15]). For all $T \in \mathcal{T}_h$ and all $F \in \mathcal{E}(T)$ this operator satisfies

$$|\boldsymbol{v} - I_h \boldsymbol{v}|_{m,T} \preceq h_T^{n-m}\, |\boldsymbol{v}|_{n,\widetilde{\omega}_T}\,, \tag{3.6}$$
$$\|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,F} \preceq h_F^{n-\frac{1}{2}}\, |\boldsymbol{v}|_{n,\widetilde{\omega}_F}\,, \tag{3.7}$$

for all $\boldsymbol{v} \in H^n(\Omega)^2$, and all $0 \leq m \leq 1$, $1 \leq n \leq k+1$. The following result holds for the Clément interpolation operator:

Lemma 4 *For all* $T \in \mathcal{T}_h$*,* $F \in \mathcal{E}(T)$*,* $\boldsymbol{v} \in H^1(\Omega)^2$*, there holds*

$$\|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,T} \preceq \theta_T\, \|\boldsymbol{v}\|_{a,\widetilde{\omega}_T}, \tag{3.8}$$
$$\|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,F} \preceq \theta_F\, \|\boldsymbol{v}\|_{a,\widetilde{\omega}_F}, \tag{3.9}$$
$$\|I_h \boldsymbol{v}\|_{a,T} \preceq \|\boldsymbol{v}\|_{a,\widetilde{\omega}_T}. \tag{3.10}$$

PROOF. First and third estimates arise using (3.6)-(3.7), the previous lemma and the mesh regularity. In order to prove the second one, from [22], Lemma 3.1, we obtain

$$\|\boldsymbol{v} - I_h\boldsymbol{v}\|_{0,F} \preceq h_T^{-1/2} \|\boldsymbol{v} - I_h\boldsymbol{v}\|_{0,T} + \|\boldsymbol{v} - I_h\boldsymbol{v}\|_{0,T}^{1/2} |\boldsymbol{v} - I_h\boldsymbol{v}|_{1,T}^{1/2}. \tag{3.11}$$

Then, using (3.11) and (3.8) we arrive at

$$\begin{aligned}\|\boldsymbol{v} - I_h\boldsymbol{v}\|_{0,F} &\preceq h_T^{-1/2}\theta_T \|\boldsymbol{v}\|_{a,\widetilde{\omega}_T} + \nu^{-1/4}\theta_T^{1/2} \|\boldsymbol{v}\|_{a,\widetilde{\omega}_T} \\ &\preceq \left[h_T^{-1/2}\theta_T + \nu^{-1/4}\theta_T^{1/2}\right] \|\boldsymbol{v}\|_{a,\widetilde{\omega}_T}\,.\end{aligned}$$

Finally, since the mesh is regular

$$\begin{aligned}h_T^{-1/2}\theta_T + \nu^{-1/4}\theta_T^{1/2} &= \nu^{-1/4}\,\sigma^{-1/4}\ \min\{h_T\nu^{-1/2}\sigma^{1/2}, 1\}^{1/2} \times \\ &\qquad \left[1 + \min\{1, h_T^{-1}\nu^{1/2}\sigma^{-1/2}\}^{1/2}\right] \\ &\preceq \theta_F\,,\end{aligned}$$

and the second estimate follows. □

Corollary 5 *For all $\boldsymbol{\phi} \in \boldsymbol{H}$, the following estimates hold*

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} \theta_T^{-2}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,T}^2 \preceq a(\boldsymbol{\phi},\boldsymbol{\phi}), \\ &\sum_{F\in\mathcal{E}_\Omega} \theta_F^{-2}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,F}^2 \preceq a(\boldsymbol{\phi},\boldsymbol{\phi})\,.\end{aligned}$$

PROOF. First, from (3.8) and the mesh regularity we obtain

$$\sum_{T\in\mathcal{T}_h} \theta_T^{-2}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,T}^2 \ \preceq\ \sum_{T\in\mathcal{T}_h} \theta_T^{-2}\,\theta_T^2\,\|\boldsymbol{\phi}\|_{a,\widetilde{\omega}_T}^2 \ \preceq\ a(\boldsymbol{\phi},\boldsymbol{\phi})\,.$$

Next, from (3.9) and the mesh regularity we obtain the second estimate. □

4 The auxiliary problem

Let $(\boldsymbol{e}, E) \in \boldsymbol{H} \times Q$, and let us define $(\boldsymbol{\phi}, \psi) \in \boldsymbol{H} \times Q$ as the solution of the weak problem:

$$a(\boldsymbol{\phi},\boldsymbol{v}) + c(\psi,q) = a(\boldsymbol{e},\boldsymbol{v}) + b(\boldsymbol{v},E) + b(\boldsymbol{e},q) \qquad \forall(\boldsymbol{v},q) \in \boldsymbol{H} \times Q. \tag{4.1}$$

The well-posedeness of this problem arises from the fact that a and c are elliptic bilinear forms on $\boldsymbol{H}$ and Q, respectively.

Let $\|\!|\cdot|\!\|: \boldsymbol{H}\times Q\to\mathbb{R}$ be the mapping defined by

$$(\boldsymbol{e},E)\longmapsto \|\!|(\boldsymbol{e},E)|\!\| := \left\{\|\boldsymbol{\phi}\|_a^2+\|\psi\|_c^2\right\}^{1/2}, \tag{4.2}$$

where $(\boldsymbol{\phi},\psi)$ is the solution of (4.1).

Lemma 6 *The mapping (4.2) defines a norm on $\boldsymbol{H}\times Q$.*

PROOF. Since $\|\cdot\|_a$ and $\|\cdot\|_c$ are norms on $\boldsymbol{H}$ and Q, respectively, we only have to prove that $\|\!|(\boldsymbol{e},E)|\!\|=0$ implies $(\boldsymbol{e},E)=\mathbf{0}$. If $\|\!|(\boldsymbol{e},E)|\!\|=0$, then

$$a(\boldsymbol{e},\boldsymbol{v})+b(\boldsymbol{v},E)+b(\boldsymbol{e},q)=0 \qquad \forall(\boldsymbol{v},q)\in\boldsymbol{H}\times Q\,. \tag{4.3}$$

If we consider $\boldsymbol{v}=\mathbf{0}$ in (4.3), then $\boldsymbol{e}\in\operatorname{Ker}(\operatorname{div})$. Next, if $q=0$ and $\boldsymbol{v}=\boldsymbol{e}$, then $b(\boldsymbol{e},E)=0$ and hence $a(\boldsymbol{e},\boldsymbol{e})=0$, which implies $\boldsymbol{e}=\mathbf{0}$. Finally, since $\boldsymbol{e}=\mathbf{0}$, we have

$$(E,\operatorname{div}\boldsymbol{v})_\Omega=0 \qquad \forall\boldsymbol{v}\in\boldsymbol{H}\,,$$

and, since $\operatorname{div}:\boldsymbol{H}\longrightarrow Q$ is a surjective operator, there exists $\boldsymbol{v}\in\boldsymbol{H}$, such that $\operatorname{div}\boldsymbol{v}=E$. Hence $E=0$. □

The next result shows the equivalence between $\|\!|\cdot|\!\|$ and (2.4).

Theorem 7 *There exists a positive constant K_2, independent of σ and ν, such that*

$$\frac{1}{4}\|\!|(\boldsymbol{e},E)|\!\|^2 \le \|\boldsymbol{e}\|_a^2+\|E\|_c^2\le K_2\left(\frac{\sigma+\nu}{\nu}\right)^2\|\!|(\boldsymbol{e},E)|\!\|^2,$$

for all $(\boldsymbol{e},E)\in\boldsymbol{H}\times Q$.

PROOF. *Upper bound:* Using (2.7), $q=0$ in (4.1), Cauchy-Schwarz's inequality and (2.5), we have

$$\begin{aligned}\alpha_b\sqrt{\frac{\nu}{\sigma+\nu}}\,\|E\|_c &\le \sup_{\boldsymbol{v}\in\boldsymbol{H}}\frac{|b(\boldsymbol{v},E)|}{\|\boldsymbol{v}\|_a}\\ &=\sup_{\boldsymbol{v}\in\boldsymbol{H}}\frac{|a(\boldsymbol{\phi},\boldsymbol{v})-a(\boldsymbol{e},\boldsymbol{v})|}{\|\boldsymbol{v}\|_a}\ \le\ \|\boldsymbol{\phi}\|_a+\|\boldsymbol{e}\|_a\,,\end{aligned}$$

and then

$$\|E\|_c \leq \alpha_b^{-1} \sqrt{\frac{\sigma+\nu}{\nu}} \Big\{ \|\boldsymbol{\phi}\|_a + \|\boldsymbol{e}\|_a \Big\}. \tag{4.4}$$

Now, considering $q = -E$, $\boldsymbol{v} = \boldsymbol{e}$ in (4.1), using (2.5), Cauchy-Schwarz's inequality and (4.4), we obtain

$$\begin{aligned}
\|\boldsymbol{e}\|_a^2 &= a(\boldsymbol{e}, \boldsymbol{e}) \\
&= a(\boldsymbol{\phi}, \boldsymbol{e}) - c(\psi, E) \\
&\leq \|\boldsymbol{\phi}\|_a \|\boldsymbol{e}\|_a + \|\psi\|_c \|E\|_c \\
&\leq \left\{ \|\boldsymbol{\phi}\|_a + \alpha_b^{-1} \sqrt{\frac{\sigma+\nu}{\nu}} \, \|\psi\|_c \right\} \|\boldsymbol{e}\|_a + \alpha_b^{-1} \sqrt{\frac{\sigma+\nu}{\nu}} \, \|\psi\|_c \|\boldsymbol{\phi}\|_a \\
&\leq \frac{1}{2} \left\{ \|\boldsymbol{\phi}\|_a + \alpha_b^{-1} \sqrt{\frac{\sigma+\nu}{\nu}} \, \|\psi\|_c \right\}^2 + \frac{1}{2} \, \|\boldsymbol{e}\|_a^2 + \\
&\quad \frac{1}{2} \, \frac{\sigma+\nu}{\alpha_b^2 \nu} \, \|\boldsymbol{\phi}\|_a^2 + \frac{1}{2} \, \|\psi\|_c^2 \\
&\leq \|\boldsymbol{\phi}\|_a^2 + \frac{\sigma+\nu}{\alpha_b^2 \nu} \, \|\psi\|_c^2 + \frac{1}{2} \, \|\boldsymbol{e}\|_a^2 + \frac{1}{2} \, \frac{\sigma+\nu}{\alpha_b^2 \nu} \, \|\boldsymbol{\phi}\|_a^2 + \frac{1}{2} \, \|\psi\|_c^2 \\
&\leq C \, \frac{\sigma+\nu}{\nu} \, \left\{ \|\boldsymbol{\phi}\|_a^2 + \|\psi\|_c^2 \right\} + \frac{1}{2} \, \|\boldsymbol{e}\|_a^2,
\end{aligned}$$

which leads to

$$\|\boldsymbol{e}\|_a^2 \leq C \, \frac{\sigma+\nu}{\nu} \, \left\{ \|\boldsymbol{\phi}\|_a^2 + \|\psi\|_c^2 \right\} . \tag{4.5}$$

Hence, from (4.4) and (4.5), we have

$$\|\boldsymbol{e}\|_a^2 + \|E\|_c^2 \, \leq \, K_2 \left(\frac{\sigma+\nu}{\nu} \right)^2 \, |||(\boldsymbol{e}, E)|||^2 .$$

Lower bound: Taking $\boldsymbol{v} = \boldsymbol{\phi}$, $q = 0$ in (4.1) and using (2.6), we obtain

$$\|\boldsymbol{\phi}\|_a^2 \, = \, a(\boldsymbol{\phi}, \boldsymbol{\phi}) \, = \, a(\boldsymbol{e}, \boldsymbol{\phi}) + b(\boldsymbol{\phi}, E) \, \leq \, \|\boldsymbol{e}\|_a \|\boldsymbol{\phi}\|_a + \sqrt{2} \, \|\boldsymbol{\phi}\|_a \|E\|_c \, ,$$

and then, dividing by $\|\boldsymbol{\phi}\|_a$ we obtain

$$\|\boldsymbol{\phi}\|_a \leq \|\boldsymbol{e}\|_a + \sqrt{2} \, \|E\|_c. \tag{4.6}$$

Next, taking $\boldsymbol{v} = \boldsymbol{0}$, $q = \psi$ in (4.1) and using (2.6), we obtain

$$\|\psi\|_c^2 \, = \, c(\psi, \psi) \, = \, b(\boldsymbol{e}, \psi) \, \leq \, \sqrt{2} \, \|\boldsymbol{e}\|_a \|\psi\|_c \, \leq \, \|\boldsymbol{e}\|_a^2 + \frac{1}{2} \, \|\psi\|_c^2 \, ,$$

which leads to

$$\|\psi\|_c^2 \leq 2 \, \|\boldsymbol{e}\|_a^2 . \tag{4.7}$$

Hence, from (4.6) and (4.7), we finally obtain

$$|||(\boldsymbol{e}, E)|||^2 \, \leq \, 4 \, \{ \|\boldsymbol{e}\|_a^2 + \|E\|_c^2 \} \, ,$$

and the result follows. $\square$

Remark 8 *It is worth remarking that if we are dealing with a "pure" Stokes problem, i.e., if* $\sigma = 0$*, then the previous result gives an equivalence result with constants independent of* ν *(of course,* ν *is present in the definition of* $\|\cdot\|_a$ *and* $\|\cdot\|_c$*).*

4.1 Application to the residual equation

The finite element method to be considered in this paper is the following stabilized finite element method for (2.1) (cf. [8]): *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ *such that:*

$$A_\delta((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) \;=\; F_\delta(\boldsymbol{v}_h, q_h) \qquad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h\,, \tag{4.8}$$

where

$$\begin{aligned} A_\delta((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) &:= a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + b(\boldsymbol{u}_h, q_h) \\ - \sum_{T \in \mathcal{T}_h} \delta_T\, (\sigma\, \boldsymbol{u}_h &- \nu\, \Delta \boldsymbol{u}_h + \nabla p_h, \sigma\, \boldsymbol{v}_h - \nu\, \Delta \boldsymbol{v}_h + \nabla q_h)_T, \end{aligned}$$

and

$$F_\delta(\boldsymbol{v}_h, q_h) := (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega - \sum_{T \in \mathcal{T}_h} \delta_T\, (\boldsymbol{f}, \sigma\, \boldsymbol{v}_h - \nu\, \Delta \boldsymbol{v}_h + \nabla q_h)_T.$$

If $\sigma > 0$, the stabilization parameter δ_T is given by:

$$\delta_T := \frac{h_T^2}{\sigma\, h_T^2\, \max\{\lambda_T, 1\} + 4\,\nu/m_k}, \tag{4.9}$$

where

$$\begin{aligned} \lambda_T &:= \frac{4\,\nu}{m_k\, \sigma\, h_T^2}\,, \\ m_k &:= \min\left\{\frac{1}{3}, K_k\right\}\,, \end{aligned} \tag{4.10}$$

and K_k is the positive constant appearing in the inverse inequality

$$K_k\, h_T^2\, \|\Delta \boldsymbol{v}_h\|_{0,T}^2 \le \|\nabla \boldsymbol{v}_h\|_{0,T}^2 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{H}_h,$$

which depends only on k and the mesh regularity. If $\sigma = 0$, we recover the GLS method [16] with $\delta_T = h_T^2 m_k / 8\nu$.

Remark 9 *The choice of a continuous finite element space for the pressure is made only for simplicity of the presentation. Discontinuous spaces for the* *pressure may be also be considered, but in that case appropriate jump terms on the interelement boundaries should be added (see [13] for a discussion on the subject and [4] for a residual a posteriori error analysis for a stabilized method using discontinuous pressures).*

Next, let $\boldsymbol{e}$ and E be the errors in approximating the velocity and pressure, respectively, i.e.

$$\begin{aligned} \boldsymbol{e} &:= \boldsymbol{u} - \boldsymbol{u}_h\,, \\ E &:= p - p_h\,. \end{aligned}$$

Then, with this choice for $(\boldsymbol{e}, E)$, the variational problem (4.1) reads:

$$a(\boldsymbol{\phi}, \boldsymbol{v}) + c(\psi, q) = (\boldsymbol{f}, \boldsymbol{v})_\Omega - a(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) - b(\boldsymbol{u}_h, q)\,, \tag{4.11}$$

for all $(\boldsymbol{v}, q)$ in $\boldsymbol{H} \times Q$, or, written in another way

$$a(\boldsymbol{\phi}, \boldsymbol{v}) + c(\psi, q) = \mathcal{R}_h(\boldsymbol{v}, q) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H} \times Q\,, \tag{4.12}$$

where $\mathcal{R}_h : \boldsymbol{H} \times Q \longrightarrow \mathbb{R}$ stands for the residual functional given by

$$\mathcal{R}_h(\boldsymbol{v}, q) := (\boldsymbol{f}, \boldsymbol{v})_\Omega - a(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) - b(\boldsymbol{u}_h, q)\,.$$

This auxiliary problem is clearly uncoupled. Indeed, defining the linear bounded operators $A : \boldsymbol{H} \to \boldsymbol{H}'$, $A\boldsymbol{u}(\boldsymbol{v}) := a(\boldsymbol{u}, \boldsymbol{v})$, and $C : Q \to Q'$, $Cp(q) := c(p, q)$, then (4.12) may be rewritten as:

$$\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi} \\ \psi \end{bmatrix} = \begin{bmatrix} \mathcal{R}_h^1 \\ \mathcal{R}_h^2 \end{bmatrix}\,, \tag{4.13}$$

where $\mathcal{R}_h^1 \in \boldsymbol{H}'$ and $\mathcal{R}_h^2 \in Q'$ are given by

$$\begin{aligned} \mathcal{R}_h^1(\boldsymbol{v}) &:= (\boldsymbol{f}, \boldsymbol{v})_\Omega - a(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h)\,, \\ \mathcal{R}_h^2(q) &:= -b(\boldsymbol{u}_h, q)\,. \end{aligned}$$

Remark 10 *Considering $\boldsymbol{v} = \boldsymbol{0}$ in (4.11), we have*

$$\int_\Omega (\nu^{-1}\psi - \operatorname{div} \boldsymbol{u}_h)\, q\, dx = 0 \qquad \forall q \in Q\,,$$

and hence, since $\nu^{-1}\psi - \operatorname{div} \boldsymbol{u}_h \in Q$, we can see that

$$\psi = \nu \operatorname{div} \boldsymbol{u}_h\,,$$

and then we know the explicit solution for ψ.

Now, from the previous remark, in (4.13) we only need to solve

$$A\,\boldsymbol{\phi} = \mathcal{R}_h^1,$$

which is equivalent to the following variational equation:

$$a(\boldsymbol{\phi}, \boldsymbol{v}) \;=\; \mathcal{R}_h^1(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}\,. \tag{4.14}$$

In order to give a more precise (and useful in what follows) expression for $\mathcal{R}_h^1$, denoting $\boldsymbol{\varepsilon}_h := \nu\,\nabla \boldsymbol{u}_h - p_h\,\boldsymbol{I}$ (where $\boldsymbol{I}$ stands for the $\mathbb{R}^{2\times 2}$ identity matrix), integration by parts leads to

$$\mathcal{R}_h^1(\boldsymbol{v}) \;=\; \sum_{T\in\mathcal{T}_h} (\boldsymbol{R}_T, \boldsymbol{v})_T + \sum_{F\in\mathcal{E}_\Omega} (\boldsymbol{R}_F, \boldsymbol{v})_F\,, \tag{4.15}$$

where $\boldsymbol{R}_T \in L^2(T)^2$ and $\boldsymbol{R}_F \in L^2(F)^2$ are given by

$$\boldsymbol{R}_T := (\boldsymbol{f} - \sigma\,\boldsymbol{u}_h + \nu\,\Delta\boldsymbol{u}_h - \nabla p_h)|_T\,,$$

and

$$\boldsymbol{R}_F \;:=\; -\left[\!\!\left[\boldsymbol{\varepsilon}_h \cdot \boldsymbol{n}\right]\!\!\right]_F,$$

$[\![v]\!]_F$ being the jump of v across F. Note that in our case p_h is a continuous function, and then $\boldsymbol{R}_F$ reduces to $-[\![\nu\nabla\boldsymbol{u}_h \cdot \boldsymbol{n}]\!]_F$.

Finally, we remark that if $(\boldsymbol{\phi}, \psi)$ is the solution of (4.12), then $\psi = \nu\,\mathrm{div}\,\boldsymbol{u}_h$, and hence, applying Theorem 7 we see that

$$\|\boldsymbol{\phi}\|_a^2 \,+\, \nu\,\|\mathrm{div}\,\boldsymbol{u}_h\|_{0,\Omega}^2 \;\preceq\; \|\boldsymbol{e}\|_a^2 \,+\, \|E\|_c^2 \;\preceq\; \left(\frac{\sigma+\nu}{\nu}\right)^2 \left[\|\boldsymbol{\phi}\|_a^2 \,+\, \nu\,\|\mathrm{div}\,\boldsymbol{u}_h\|_{0,\Omega}^2\right].$$

Based on this remark in the next section we will build an a posteriori error estimator for $\boldsymbol{\phi}$.

5 The hierarchical error estimator

Let $\boldsymbol{W}_h$ be a finite element space such that $\boldsymbol{H}_h \subseteq \boldsymbol{W}_h \subseteq \boldsymbol{H}$. Let us suppose that there exist M subspaces $\boldsymbol{H}_i$ of $\boldsymbol{W}_h$ such that

$$\boldsymbol{W}_h = \boldsymbol{H}_0 + \sum_{i=1}^{M} \boldsymbol{H}_i\,,$$

where $\boldsymbol{H}_0 := \boldsymbol{H}_h$. Associated with each subspace $\boldsymbol{H}_i$ there exists a projection operator $P_i : \boldsymbol{H} \longrightarrow \boldsymbol{H}_i$ given by the solution of the local problem

$$a(P_i\boldsymbol{v}, \boldsymbol{w}_i) = a(\boldsymbol{v}, \boldsymbol{w}_i) \qquad \forall \boldsymbol{w}_i \in \boldsymbol{H}_i\,, P_i\boldsymbol{v} \in \boldsymbol{H}_i\,.$$

Using these notations we define our *hierarchical a posteriori error estimator* η_H by

$$\eta_H := \left\{ \sum_{i=1}^{M} a(P_i\boldsymbol{\phi}, P_i\boldsymbol{\phi}) \right\}^{1/2} ,$$

where $\boldsymbol{\phi}$ is the solution of (4.14). Let us recall that $P_i\boldsymbol{\phi}$ is the solution of the local problem: *Find $P_i\boldsymbol{\phi} \in \boldsymbol{H}_i$ such that*

$$a(P_i\boldsymbol{\phi}, \boldsymbol{v}_i) \,=\, \mathcal{R}_h^1(\boldsymbol{v}_i) \qquad \forall \boldsymbol{v}_i \in \boldsymbol{H}_i \,.$$

We remark that, if $\boldsymbol{H}_i$ is local enough and of small dimension, then the computation of $P_i\boldsymbol{\phi}$ is easy and cheap. In which follows, we will define a space $\boldsymbol{H}_i$ associated to each element $T \in \mathcal{T}_h$ and each side $F \in \mathcal{E}_\Omega$. In this way, our a posteriori error estimator η_H reduces to:

$$\eta_H = \left\{ \sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi}) \right\}^{1/2} . \tag{5.1}$$

These finite element spaces $\boldsymbol{H}_i$ may be spanned by appropriate bubble functions. Let us define the finite dimensional spaces $\boldsymbol{H}^b$, called *bubble function spaces*, by

$$\boldsymbol{H}^b = \begin{cases} \boldsymbol{H}_T^b & \text{for each } T \in \mathcal{T}_h, \\ \boldsymbol{H}_F^b & \text{for each } F \in \mathcal{E}_\Omega \,, \end{cases}$$

with the restriction $\boldsymbol{H}_T^b \subset H_0^1(T)^2$ and $\boldsymbol{H}_F^b \subset H_0^1(\omega_F)^2$. Moreover, we will suppose that these bubble spaces are affine-equivalent to fixed finite dimensional spaces on a reference configuration, so that the following estimate holds

$$\|\mathbf{b}\|_{0,T}^2 \,\preceq\, h_T^2\, |\mathbf{b}|_{1,T}^2 \,, \tag{5.2}$$

for all $\mathbf{b} \in \boldsymbol{H}^b$, and all $T \in \mathcal{T}_h$.

Finally, we will suppose that these bubble function spaces satisfy the following inf-sup condition (LBB): *There exists $\beta > 0$, independent of h, σ and ν, such that*

$$\sup_{\mathcal{B}_T \in \boldsymbol{H}_T^b} \frac{(\mathcal{B}_T, \boldsymbol{R}_T)_T}{a_T(\mathcal{B}_T, \mathcal{B}_T)^{1/2}} \geq \beta\, \theta_T \|\boldsymbol{R}_T\|_{0,T} \qquad \forall T \in \mathcal{T}_h \,,$$

$$\sup_{\mathcal{B}_F \in \boldsymbol{H}_F^b} \frac{(\mathcal{B}_F, \boldsymbol{R}_F)_F}{a_{\omega_F}(\mathcal{B}_F, \mathcal{B}_F)^{1/2}} \geq \beta\, \theta_F \|\boldsymbol{R}_F\|_{0,F} \qquad \forall F \in \mathcal{E}_\Omega \,,$$

where $a_D(\cdot\,,\cdot\,)$ stands for integration over $D \subseteq \mathbb{R}^2$.

Remark 11 *Later, in Appendix B, we will give a concrete example of bubble function spaces satisfying (LBB).*

Lemma 12 *If (LBB) holds, then*

$$\mathcal{R}_h^1(\boldsymbol{v}) \preceq \sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi})^{1/2}\theta_T^{-1}\|\boldsymbol{v}\|_{0,T} + \sum_{F\in\mathcal{E}_\Omega}\left[a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})^{1/2} + \sum_{T'\subset\omega_F} a(P_{T'}\boldsymbol{\phi}, P_{T'}\boldsymbol{\phi})^{1/2}\right]\theta_F^{-1}\|\boldsymbol{v}\|_{0,F},$$

for all $\boldsymbol{v}$ in $\boldsymbol{H}$.

PROOF. We first note that from (4.15) and Cauchy-Schwarz's inequality we arrive at

$$\mathcal{R}_h^1(\boldsymbol{v}) \preceq \sum_{T\in\mathcal{T}_h}\|\boldsymbol{R}_T\|_{0,T}\|\boldsymbol{v}\|_{0,T} + \sum_{F\in\mathcal{E}_\Omega}\|\boldsymbol{R}_F\|_{0,F}\|\boldsymbol{v}\|_{0,F}\,.$$

Next, using Cauchy-Schwarz's inequality, (LBB) condition and the definition of $P_T\boldsymbol{\phi}$ we obtain

$$\begin{aligned}
\theta_T\|\boldsymbol{R}_T\|_{0,T} &\le \frac{1}{\beta}\sup_{\mathcal{B}_T\in\boldsymbol{H}_T^b}\frac{(\mathcal{B}_T,\boldsymbol{R}_T)_T}{a_T(\mathcal{B}_T,\mathcal{B}_T)^{1/2}}\\
&= \frac{1}{\beta}\sup_{\mathcal{B}_T\in\boldsymbol{H}_T^b}\frac{\mathcal{R}_h^1(\mathcal{B}_T)}{a_T(\mathcal{B}_T,\mathcal{B}_T)^{1/2}}\\
&= \frac{1}{\beta}\sup_{\mathcal{B}_T\in\boldsymbol{H}_T^b}\frac{a(P_T\boldsymbol{\phi},\mathcal{B}_T)}{a(\mathcal{B}_T,\mathcal{B}_T)^{1/2}}\\
&\le \frac{1}{\beta}\,a(P_T\boldsymbol{\phi},P_T\boldsymbol{\phi})^{1/2}\,.
\end{aligned}\tag{5.3}$$

Moreover, for each $F\in\mathcal{E}_\Omega$ we have

$$\begin{aligned}
\theta_F\|\boldsymbol{R}_F\|_{0,F} &\le \frac{1}{\beta}\sup_{\mathcal{B}_F\in\boldsymbol{H}_F^b}\frac{(\mathcal{B}_F,\boldsymbol{R}_F)_F}{a(\mathcal{B}_F,\mathcal{B}_F)^{1/2}}\\
&= \frac{1}{\beta}\sup_{\mathcal{B}_F\in\boldsymbol{H}_F^b}\frac{\mathcal{R}_h^1(\mathcal{B}_F) - \sum_{T'\subset\omega_F}(\boldsymbol{R}_{T'},\mathcal{B}_F)_{T'}}{a(\mathcal{B}_F,\mathcal{B}_F)^{1/2}}\\
&\le \frac{1}{\beta}\sup_{\mathcal{B}_F\in\boldsymbol{H}_F^b}\frac{a(P_F\boldsymbol{\phi},\mathcal{B}_F)}{a(\mathcal{B}_F,\mathcal{B}_F)^{1/2}} + \frac{1}{\beta}\sup_{\mathcal{B}_F\in\boldsymbol{H}_F^b}\sum_{T'\subset\omega_F}\frac{\|\boldsymbol{R}_{T'}\|_{0,T'}\|\mathcal{B}_F\|_{0,T'}}{a(\mathcal{B}_F,\mathcal{B}_F)^{1/2}}\\
&\preceq a(P_F\boldsymbol{\phi},P_F\boldsymbol{\phi})^{1/2} + \sum_{T'\subset\omega_F}\theta_{T'}\|\boldsymbol{R}_{T'}\|_{0,T'}\,,
\end{aligned}$$

since, on each $T'\subset\omega_F$ there holds

$$\frac{\|\mathcal{B}_F\|_{0,T'}^2}{a_{T'}(\mathcal{B}_F,\mathcal{B}_F)} \preceq \theta_{T'}^2.$$

In fact, if $\sigma > 0$, applying (5.2) and the definition of θ_T yields to

$$
\begin{aligned}
\frac{\|\mathcal{B}_F\|_{0,T'}^2}{a_{T'}(\mathcal{B}_F,\mathcal{B}_F)} &= \frac{\int_{T'} \mathcal{B}_F \cdot \mathcal{B}_F}{\sigma \int_{T'} \mathcal{B}_F \cdot \mathcal{B}_F + \nu \int_{T'} \nabla \mathcal{B}_F : \nabla \mathcal{B}_F} \\
&\preceq \frac{\int_{T'} \mathcal{B}_F \cdot \mathcal{B}_F}{\sigma \int_{T'} \mathcal{B}_F \cdot \mathcal{B}_F + \nu\, h_{T'}^{-2} \int_{T'} \mathcal{B}_F \cdot \mathcal{B}_F} \\
&\preceq \frac{1}{\sigma + \nu\, h_{T'}^{-2}} \\
&\preceq \frac{\sigma^{-1}}{\max\{1, \nu\, \sigma^{-1}\, h_{T'}^{-2}\}} \\
&\preceq \sigma^{-1} \min\{1, \nu^{-1}\, \sigma\, h_{T'}^{2}\} \\
&\preceq \theta_{T'}^2 \,.
\end{aligned}
$$

The result for $\sigma = 0$ follows in an analogous way. $\square$

Up to now we have not used any particular feature of the stabilized finite element method (4.8). The following technical result, whose proof may be found in Appendix A, will be useful in the proof of the reliability of our error estimator (5.1) (see Lemma 14 below).

Lemma 13 *For all $\boldsymbol{v}_h \in \boldsymbol{H}_h$ there holds*

$$
\mathcal{R}_h^1(\boldsymbol{v}_h) \preceq \sum_{T \in \mathcal{T}_h} \theta_T \, \|\boldsymbol{R}_T\|_{0,T} \|\boldsymbol{v}_h\|_{a,T} \,.
$$

Lemma 14 *Let $\boldsymbol{\phi}$ be the solution of (4.14). Then, if (LBB) holds, then*

$$
a(\boldsymbol{\phi}, \boldsymbol{\phi}) \;\preceq\; \eta_H^2 \,.
$$

PROOF. From Lemma 12 applied to $\boldsymbol{v} = \boldsymbol{\phi} - I_h \boldsymbol{\phi}$, Cauchy-Schwarz's inequality and Corollary 5 we obtain

$$
\begin{aligned}
&\mathcal{R}_h^1(\boldsymbol{\phi} - I_h\boldsymbol{\phi}) \preceq \sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi})^{1/2}\theta_T^{-1}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,T} \\
&+ \sum_{F\in\mathcal{E}_\Omega} \left[a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})^{1/2} + \sum_{T'\subset\omega_F} a(P_{T'}\boldsymbol{\phi}, P_{T'}\boldsymbol{\phi})^{1/2}\right]\theta_F^{-1}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,F} \\
\preceq& \left\{\sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})\right\}^{1/2} \times \\
&\left\{\sum_{T\in\mathcal{T}_h} \theta_T^{-2}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,T}^2 + \sum_{F\,\in\,\mathcal{E}_\Omega} \theta_F^{-2}\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{0,F}^2\right\}^{1/2} \\
\preceq& \left\{\sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})\right\}^{1/2} \|\boldsymbol{\phi}\|_a.
\end{aligned}
$$

Hence, from Lemmas 12, 13, (5.3), (3.10) and Cauchy-Schwarz's inequality we obtain:

$$
\begin{aligned}
&a(\boldsymbol{\phi}, \boldsymbol{\phi}) = \mathcal{R}_h^1(\boldsymbol{\phi}) \\
=& \mathcal{R}_h^1(\boldsymbol{\phi} - I_h\,\boldsymbol{\phi}) + \mathcal{R}_h^1(I_h\boldsymbol{\phi}) \\
\preceq& \left\{\sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})\right\}^{1/2} \|\boldsymbol{\phi}\|_a + \\
&\sum_{T\,\in\,\mathcal{T}_h} \theta_T\,\|\boldsymbol{R}_T\|_{0,T}\|I_h\boldsymbol{\phi}\|_{a,T} \\
\preceq& \left\{\sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})\right\}^{1/2} \|\boldsymbol{\phi}\|_a + \\
&\sum_{T\,\in\,\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi})^{1/2}\|\boldsymbol{\phi}\|_{a,\widetilde{\omega}_T} \\
\preceq& \left\{\sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi})\right\}^{1/2} \|\boldsymbol{\phi}\|_a\,,
\end{aligned}
$$

and the result follows from the definition of $\|\cdot\|_a$. $\square$

Using the previous results we can state the following equivalence theorem:

Theorem 15 *Let $\boldsymbol{\phi}$ be the solution of (4.14). If (LBB) holds, then*

$$a(\boldsymbol{\phi}, \boldsymbol{\phi}) \simeq \eta_H^2,$$

where η_H is given by (5.1) and the equivalence constants are independent of h, σ and ν.

PROOF. The upper bound has already been stated in Lemma 14. For the lower bound, for simplicity let us write

$$\sum_{T\in\mathcal{T}_h} a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \sum_{F\in\mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi}) = \sum_{i=1}^{M} a(P_i\boldsymbol{\phi}, P_i\boldsymbol{\phi})\,,$$

for some positive integer M. From the definition of $P_i\boldsymbol{\phi}$ and Cauchy-Schwarz's inequality we have

$$\begin{aligned}\left[\sum_{i=1}^{M} a(P_i\boldsymbol{\phi}, P_i\boldsymbol{\phi})\right]^2 &= \left[\sum_{i=1}^{M} a(\boldsymbol{\phi}, P_i\boldsymbol{\phi})\right]^2\\ &= \left[a(\boldsymbol{\phi}, \sum_{i=1}^{M} P_i\boldsymbol{\phi})\right]^2\\ &\le a(\boldsymbol{\phi},\boldsymbol{\phi})\, a(\sum_{i=1}^{M} P_i\boldsymbol{\phi}, \sum_{i=1}^{M} P_i\boldsymbol{\phi})\,. \end{aligned} \tag{5.4}$$

Using Cauchy-Schwarz's inequality once more we arrive at

$$\begin{aligned} a(\sum_{i=1}^{M} P_i\boldsymbol{\phi}, \sum_{i=1}^{M} P_i\boldsymbol{\phi}) &= \sum_{i=1}^{M}\sum_{j\in I_i} a(P_i\boldsymbol{\phi}, P_j\boldsymbol{\phi})\\ &\le \sum_{i=1}^{M}\sum_{j\in I_i}\left\{\frac{1}{2}a(P_i\boldsymbol{\phi}, P_i\boldsymbol{\phi}) + \frac{1}{2}a(P_j\boldsymbol{\phi}, P_j\boldsymbol{\phi})\right\}\\ &\le K_{max}\sum_{i=1}^{M} a(P_i\boldsymbol{\phi}, P_i\boldsymbol{\phi})\,, \end{aligned} \tag{5.5}$$

where I_i denotes the set of spaces $\boldsymbol{H}_j$ which are neighbors of $\boldsymbol{H}_i$, i.e.

$$I_i := \{\, j \,:\, \exists\, \boldsymbol{v}_j \in \boldsymbol{H}_j \text{ and } \boldsymbol{v}_i \in \boldsymbol{H}_i \text{ such that } a(\boldsymbol{v}_i, \boldsymbol{v}_j) \neq 0\}\,,$$

and where K_{max} is the maximum number of neighbors, i.e.

$$K_{max} := \max\{card(I_l) \,:\, 1 \le l \le M\}\,,$$

which is uniformly bounded from the mesh regularity. Hence, from (5.4) and (5.5) we obtain

$$\sum_{i=1}^{M} a(P_i\boldsymbol{\phi}, P_i\boldsymbol{\phi}) \le K_{max}\, a(\boldsymbol{\phi},\boldsymbol{\phi})\,,$$

and the result follows. □

Finally, from the discussion at the end of the last section and Theorem 15, we can prove the following main result.

Theorem 16 *Let* $(\boldsymbol{u}, p)$, $(\boldsymbol{u}_h, p_h)$ *and* $\boldsymbol{\phi}$ *be the solutions of (2.1), (4.8) and (4.14), respectively. If (LBB) holds, then the following equivalence holds*

$$\sum_{T \in \mathcal{T}_h} \tilde{\eta}_{H,T}^2 \preceq \|\boldsymbol{u} - \boldsymbol{u}_h\|_a^2 + \|p - p_h\|_c^2 \preceq \left(\frac{\sigma + \nu}{\nu}\right)^2 \sum_{T \in \mathcal{T}_h} \tilde{\eta}_{H,T}^2 ,$$

where

$$\tilde{\eta}_{H,T} := \left\{ a(P_T\boldsymbol{\phi}, P_T\boldsymbol{\phi}) + \frac{1}{2} \sum_{F \in \mathcal{E}(T) \cap \mathcal{E}_\Omega} a(P_F\boldsymbol{\phi}, P_F\boldsymbol{\phi}) + \nu \, \|\mathrm{div}\, \boldsymbol{u}_h\|_{0,T}^2 \right\}^{1/2} .$$

Remark 17 *It is worth remarking that the above results hold supposing only the (LBB) condition, which is simpler to verify than the saturation assumption. In fact, Appendix B is devoted to show a concrete example of bubble function spaces satisfying the (LBB) condition.*

6 Numerical results

In this section we report some results obtained for the standard Stokes problem (i.e. $\sigma = 0$), and the generalized one ($\sigma \neq 0$). In both cases we show the ability of the adaptive scheme based on our a posteriori error estimator to generate adapted meshes and to improve the discrete solution without using a highly refined uniform mesh. We first test the theoretical results concerning the reliability and efficiency of the a posteriori error estimator given by (5.1) using an analytical solution as reference and comparing the exact finite element error and the estimated error. Afterward, we test the adaptive finite element scheme in test cases for which we do not know the exact solution, but we have some a priori information about the location of singularities and/or boundary layers. All the numerical results of this section have been obtained using equal-order $[\mathbb{P}_1]^2 \times \mathbb{P}_1$ elements, and from now on d.o.f. will denote the degrees of freedom associated with a particular mesh.

The adaptive procedure consists of solving problem (4.8) on a sequence of meshes up to finally attain a solution with an estimated error within a prescribed tolerance. To attain this purpose, we initiate the process with a quasi-uniform mesh and, at each step, a new mesh better adapted to the solution of problem (2.1) is created. This is done by computing the local error estimators $\tilde{\eta}_{H,T}$ for all T in the "old" mesh $\mathcal{T}_h$, and refining those elements T with $\tilde{\eta}_{H,T} \geq \theta \max\{\tilde{\eta}_{H,T} : \; T \in \mathcal{T}_h\}$, where $\theta \in (0, 1)$ is a prescribed parameter. In all our experiments we have chosen $\theta = \frac{1}{2}$.

We have used the mesh generator `Triangle`. This generator allows us to create successively refined meshes based on a hybrid Delaunay refinement algorithm.

This process provides a sequence of refined meshes that form a hierarchy of nodes, but not a hierarchy of elements (for details, see [19]).

6.1 The Stokes problem ($\sigma = 0$)

6.1.1 An analytical solution

For this test case, the domain is taken as the square $\Omega = (0,1) \times (0,1)$, $\nu = 1$, and $\boldsymbol{f}$ is set such as the exact solution of our Stokes problem is given by

$$\begin{aligned} u_1(x,y) &= -256x^2(x-1)^2y(y-1)(2y-1)\,, \\ u_2(x,y) &= -u_1(y,x)\,, \\ p(x,y) &= 150(x-0.5)(y-0.5)\,. \end{aligned}$$

In order to test our a posteriori error estimator in Figure 1 we depict the error, in the norm defined in (2.4), and the estimator $\tilde{\eta}_H$ as $h \to 0$. We can observe that both values are in good accordance, which is confirmed in Table 1 where we show the effectivity index

$$E_i := \frac{\tilde{\eta}_H}{\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|}\,,$$

which remains bounded as $h \to 0$. Finally, in order to study the sensitivity of the effectivity index as $\nu \to 0$, we present in Table 2 the behavior of $\tilde{\eta}_H$ and $\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$ for a fixed mesh and for $\nu = 1, 10^{-1}, \ldots, 10^{-6}$. We observe that, as was predicted by Theorem 16, the estimator $\tilde{\eta}_H$ follows the same pattern of $\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$, and hence, the effectivity index remains bounded independently of the value of ν (as a matter of fact, even if we observe that the actual effectivity index varies, its variation can not be compared to the variation of ν).

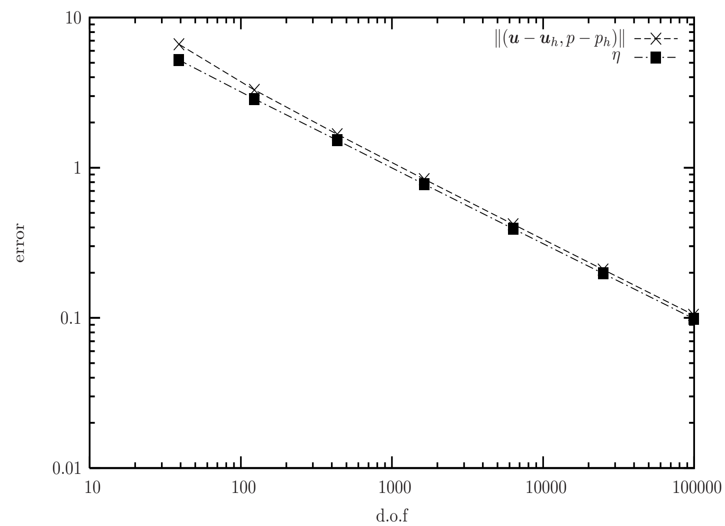

Fig. 1. Exact error and the a posteriori error estimate.

d.o.f	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	$\tilde{\eta}_H$	E_i
39	6.641955	5.216376	0.785367
123	3.292848	2.873238	0.872569
435	1.671618	1.523188	0.911205
1635	0.838908	0.775193	0.924050
6339	0.419710	0.392412	0.934960
24963	0.209854	0.197351	0.940422
99075	0.104919	9.900770e-02	0.943655

Table 1
Exact error, a posteriori error estimator and effectivity index.

ν	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	$\tilde{\eta}_H$	E_i
1	0.209854	0.197351	0.940422
1e-01	6.643132e-02	6.244997e-02	0.940068
1e-02	2.309899e-02	2.105384e-02	0.911461
1e-03	3.123896e-02	2.392909e-02	0.766001
1e-04	9.655438e-02	7.305909e-02	0.756662
1e-05	0.305260	0.227342	0.744750
1e-06	0.965315	0.645566	0.668762

Table 2
Sensitivity of the estimator to ν.

6.1.2 The lid-driven cavity problem

For this case we use the same domain as in previous section, we set $\boldsymbol{f} = \mathbf{0}$, and the boundary conditions $\boldsymbol{u} = \mathbf{0}$ on $[\{0\} \times (0,1)] \cup [(0,1) \times \{0\}] \cup [\{1\} \times (0,1)]$ and $\boldsymbol{u} = (1,0)^t$ on $(0,1) \times \{1\}$. We show in Figure 2 the initial mesh and the adapted one obtained using our error estimate. In Figure 3 we depict the discrete pressure field obtained using the initial and adapted meshes where we note the improvement in the quality of the computed solution since the singular nature of the pressure is better captured in the adapted mesh.

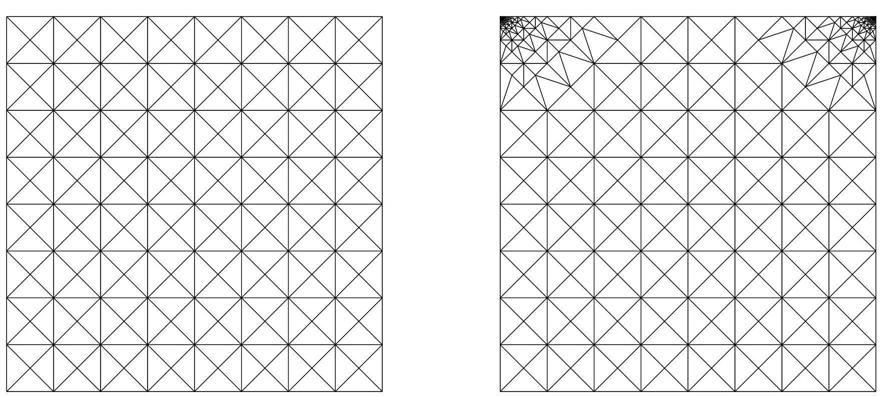

Fig. 2. Initial and adapted meshes.

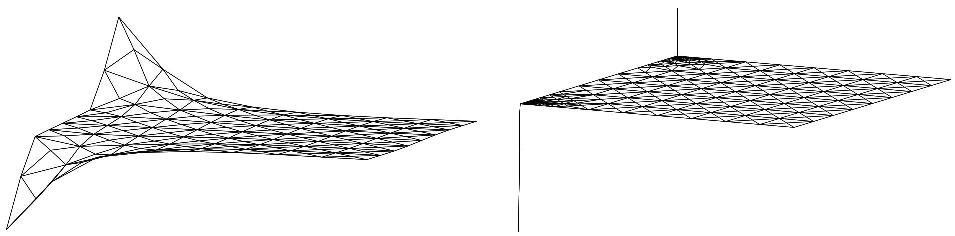

Fig. 3. The pressure in the initial and adapted meshes.

6.1.3 The backward facing step problem

This test case is posed on the backward facing step configuration. The step is located at $(x, y) = (2.5, 0)$, the entry of the channel is at $x = 0$ and the exit of the channel at $x = 22$. The channel width is 1 at entry and 2 at exit. The boundary conditions are inflow parabolic profiles and free outflow. We assume $\boldsymbol{f} = \mathbf{0}$. In this case a singularity arises at the step from the re-entrant corner. Hence we can expect the meshes to be locally refined around the corner. In Figure 4 we depict the initial mesh, and in Figure 5 we show a zoom of the adapted mesh where we can observe the local behavior of the adapted mesh. Isovalues of the vertical component of the velocity are depicted in Figure 6 for both meshes. We remark the improvement in the quality of the discrete solution if we use the adapted mesh.

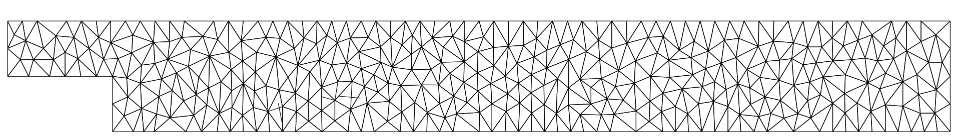

Fig. 4. Initial mesh.

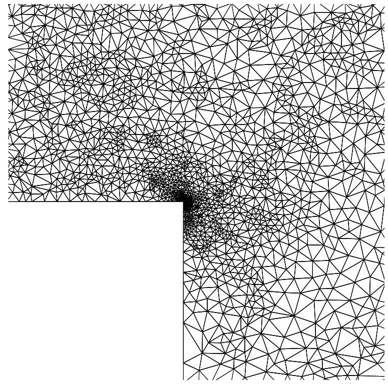

Fig. 5. A zoom, near the singularity, of the adapted mesh.

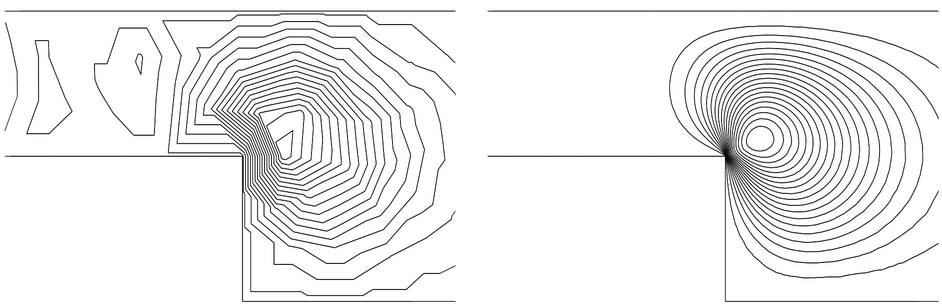

Fig. 6. A zoom, near the singularity, of the vertical velocity in the initial and the adapted meshes.

6.2 The generalized problem ($\sigma \neq 0$)

6.2.1 An analytical solution

For this test case we consider $\Omega = (0,1) \times (0,1)$, and with the aim of testing our approach using non-polynomial solutions, we set $\boldsymbol{f}$ such that the exact solution of our generalized Stokes problem is given by

$$\begin{aligned} u_1(x,y) &= \sin(\pi x)\sin(\pi y)\,, \\ u_2(x,y) &= \cos(\pi x)\cos(\pi y)\,, \\ p(x,y) &= 150(x-0.5)(y-0.5)\,. \end{aligned}$$

In Figures 7 and 8 we present the behavior, when $\nu = 1$ and $\sigma = 1, 10^6$, of the true error and the error estimate when h goes to 0. In Tables 3 and 4 we show the same kind of information plus the effectivity index. Note that this case is not covered by our theoretical results since condition (F) is not satisfied. Nevertheless, the exact error follows the same pattern of our a posteriori error estimator.

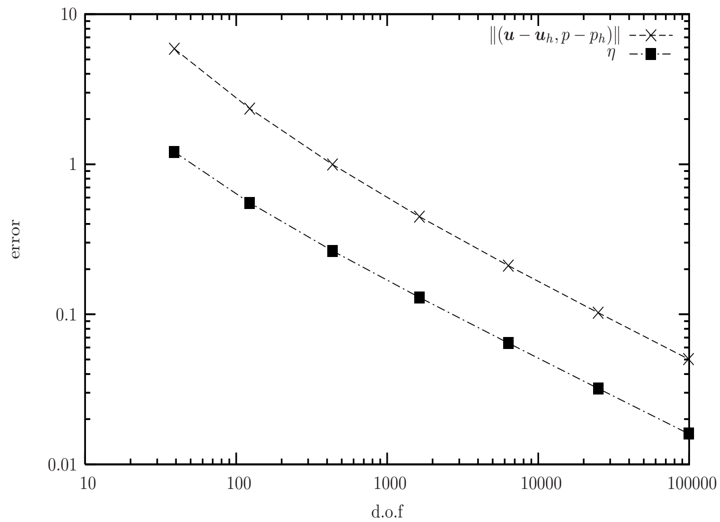

Fig. 7. Exact error and the a posteriori error estimate ($\nu = 1$ and $\sigma = 1$).

d.o.f	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	η	E_i
39	5.883058	1.207587	0.205265
123	2.351148	0.552966	0.235189
435	0.995980	0.264470	0.265538
1635	0.447907	0.129655	0.289470
6339	0.211081	6.438188E-02	0.305009
24963	0.102311	3.209443E-02	0.313694
99075	5.035011E-02	1.602415E-02	0.318254

Table 3
Error, a posteriori error estimator and effectivity index ($\nu = 1$ and $\sigma = 1$).

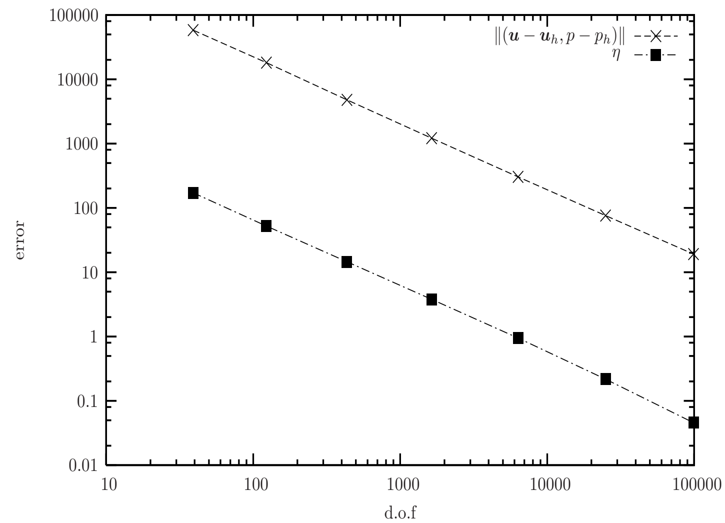

Fig. 8. Exact error and the a posteriori error estimate ($\nu = 1$ and $\sigma = 10^6$).

d.o.f	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	η	E_i
39	58179.411175	171.910627	2.954836E-03
123	18164.167330	52.292356	2.878874E-03
435	4778.880002	14.550425	3.044735E-03
1635	1210.276370	3.799917	3.139710E-03
6339	303.836382	0.944739	3.109368E-03
24963	76.196100	0.217320	2.852117E-03
99075	19.142527	4.566340E-02	2.385442E-03

Table 4
Error, a posteriori error estimator and effectivity index ($\nu = 1$ and $\sigma = 10^6$).

6.2.2 The lid-driven cavity problem

Again, we consider the problem described in Section 6.1.2, but in this case we assume $\nu = 1$ and $\sigma = 10^6$. In Figure 9 we depict the initial and final adapted meshes. We note that our a posteriori error estimate is able to detect correctly the boundary layer of the solution. In Figure 10 we show a vertical cross section of the first component of the velocity field. This cross section shows us the quality of the discrete solution computed using the adapted mesh. Note that the exponential boundary layer is clearly captured.

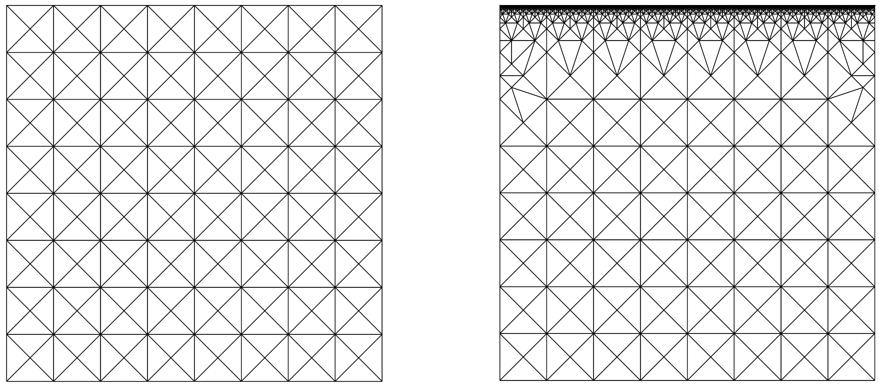

Fig. 9. Initial and final adapted meshes.

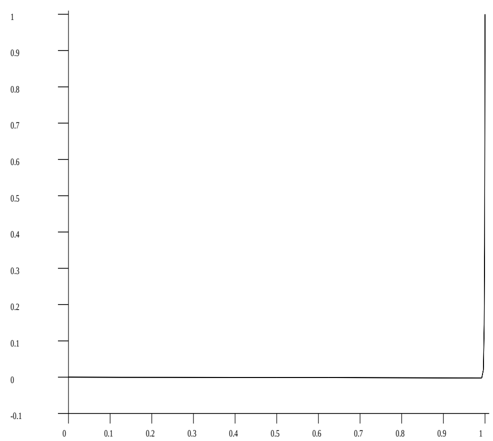

Fig. 10. A cross section of the tangential velocity at $x = \frac{1}{2}$.

7 Concluding remarks

An adaptive finite element scheme for the generalized Stokes equation has been introduced and analyzed. This scheme is based on a stabilized finite element method combined with an a posteriori error estimator. This error estimator is cheap and easy to calculate once we have chosen the bubble function spaces to be used. The equivalence between the estimator and the finite element error has been proved using a general hypothesis on the auxiliary bubble function spaces, thus avoiding the use of a saturation assumption, and we have provided a concrete pair of bubble spaces satisfying this requirement.

Even if the theoretical results concerning the estimator include constants depending on the physics of the problem, we remark that, for the pure Stokes problem, they provide equivalence constants which are independent of the viscosity. We also note that this dependence arises from the auxiliary problem posed on the continuous setting, and not from the hierarchical approach.

Finally, it is worth remarking that, even if the basic idea is closely related to the idea from [3] (see also [18]), our presentation is more general and the actual error estimator is quite different, and easier to compute. The extension of this idea to the Oseen and to the fully nonlinear Navier-Stokes equations will be the subject of future research.

A The proof of Lemma 13

First, we give the following result concerning the stabilization parameter δ_T.

Lemma 18 *Let $T \in \mathcal{T}_h$ and let δ_T be given by (4.9). Then, the following estimates hold*

$$\nu\,\delta_T \le \frac{1}{12}\,h_T^2\,, \tag{A.1}$$

$$\sigma\,\delta_T \le \min\{h_T\,\nu^{-1/2}\,\sigma^{1/2}, 1\}. \tag{A.2}$$

PROOF. In order to prove the first estimate we use (4.9) and (4.10) to obtain

$$\delta_T \le \frac{h_T^2}{4\,\nu/m_k} \le \frac{1}{12}\,\nu^{-1}\,h_T^2\,,$$

estimate which is valid independently of the value of σ. Second estimate is obvious if $\sigma = 0$, hence we will suppose from now on that $\sigma > 0$. First, we use (4.9) to get

$$\sigma\,\delta_T \le \frac{1}{\max\{\lambda_T, 1\}} \le 1\,. \tag{A.3}$$

On the other hand, we know from (A.1) that $\delta_T \le \frac{1}{12}\nu^{-1}\,h_T^2$, and then

$$\sigma\,\delta_T \le \frac{1}{12}\,h_T^2\,\nu^{-1}\,\sigma\,. \tag{A.4}$$

Taking then the geometric mean of (A.3) and (A.4), we have

$$\sigma\,\delta_T \le \frac{1}{\sqrt{12}}\,h_T\,\nu^{-1/2}\,\sigma^{1/2}. \tag{A.5}$$

Finally, from (A.3) and (A.5), we obtain (A.2). □

Now, we are ready to prove Lemma 13. We will prove the result only for the case $\sigma > 0$, the other one being completely analogous. From the definition of $\mathcal{R}_h^1$, using (4.8) with $q_h = 0$, we have

$$\mathcal{R}_h^1(\boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot \boldsymbol{v}_h - a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = \sum_{T\in\mathcal{T}_h} \int_T \delta_T\, \boldsymbol{R}_T\,(\sigma\, \boldsymbol{v}_h - \nu\,\Delta \boldsymbol{v}_h)\,.$$

Next, from Cauchy-Schwarz's inequality, (3.2) and (A.1), we obtain

$$\begin{aligned}\nu\delta_T \int_T \boldsymbol{R}_T\,\Delta\boldsymbol{v}_h &\le \nu\delta_T \int_T |\boldsymbol{R}_T||\Delta\boldsymbol{v}_h|\\ &\le \frac{h_T^2}{12}\,\|\boldsymbol{R}_T\|_{0,T}\|\Delta\boldsymbol{v}_h\|_{0,T}\\ &\preceq \theta_T\,\|\boldsymbol{R}_T\|_{0,T}\|\boldsymbol{v}_h\|_{a,T}\,.\end{aligned}$$

On the other hand, using (A.2), Cauchy-Schwarz's inequality and the definition of θ_T we obtain

$$\begin{aligned}\sigma\,\delta_T \int_T \boldsymbol{R}_T\boldsymbol{v}_h &\le \min\{\nu^{-1/2}\sigma^{1/2}\,h_T, 1\}\,\|\boldsymbol{R}_T\|_{0,T}\,\|\boldsymbol{v}_h\|_{0,T}\\ &\le \theta_T\,\|\boldsymbol{R}_T\|_{0,T}\,\|\boldsymbol{v}_h\|_{a,T}\,,\end{aligned}$$

and the result follows.

B Bubble function spaces satisfying (LBB) condition

For each element $T \in \mathcal{T}_h$ we define the *element bubble function* b_T by

$$b_T := 27 \prod_{x\in\mathcal{N}(T)} \lambda_x, \tag{B.1}$$

where λ_x denotes the barycentric coordinate associated to node x. Following Verfürth [22], let $\widehat{T}$ be the standard reference element, of vertices $(1,0),(0,1)$ and $(0,0)$. Given any number $\alpha \in (0,1]$ let us denote by $\Phi_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ the transformation which maps (x,y) onto $(x,\alpha y)$. Let

$$\widehat{T}_\alpha := \Phi_\alpha(\widehat{T}),$$

and let us denote by $\hat\lambda_{1,\alpha}, \hat\lambda_{2,\alpha}$ and $\hat\lambda_{3,\alpha}$ its barycentric coordinates (see Figure B.1).

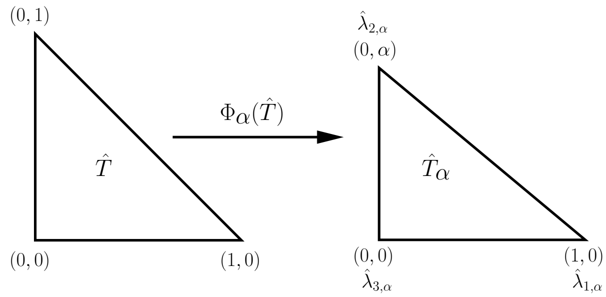

Fig. B.1. Triangles $\widehat{T}$ and $\widehat{T}_\alpha$.

Set

$$b_{\widehat{F},\alpha} := \begin{cases} 4\,\hat{\lambda}_{3,\alpha}\hat{\lambda}_{1,\alpha} & \text{on } \widehat{T}_\alpha, \\ 0 & \text{on } \widehat{T} \setminus \widehat{T}_\alpha, \end{cases}$$

where $\widehat{F} := \{(t,0) \in \mathbb{R}^2 : 0 \le t \le 1\}$. Let $F \in \mathcal{E}_\Omega$ and let us denote by T_1, T_2 two triangles which have F in common. Let $G_{F,i}$, $i = 1,2$, be the orientation preserving affine transformation which maps $\widehat{T}$ onto T_i and $\widehat{F}$ onto F (see Figure B.2).

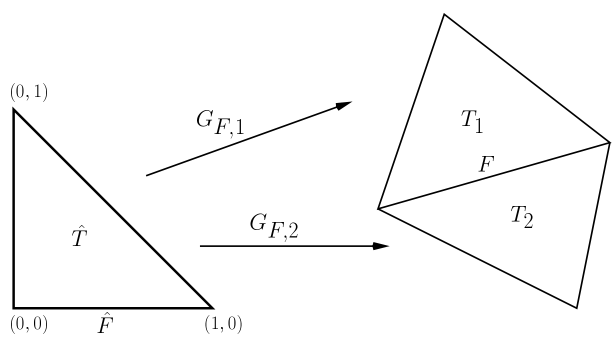

Fig. B.2. Affine transformation $G_{F,i}$, $i = 1,2$.

Set

$$b_{F,\alpha} := \begin{cases} b_{\widehat{F},\alpha} \circ G_{F,i}^{-1} & \text{on } T_i,\ i = 1,2, \\ 0 & \text{on } \Omega \setminus \omega_F. \end{cases} \tag{B.2}$$

Let $\hat{\Pi} := \{(x,0) : x \in \mathbb{R}\}$ and let $\hat{Q} : \mathbb{R}^2 \to \hat{\Pi}$ be the orthogonal projection from $\mathbb{R}^2$ to $\hat{\Pi}$. We introduce the lifting operator $\hat{P}_{\hat{F}} : \mathbb{P}_k(\hat{F}) \to \mathbb{P}_k(\hat{T})$ by

$$\hat{P}_{\hat{F}}(\hat{s}) = \hat{s} \circ \hat{Q}.$$

Let $T_i \subseteq \omega_F$ and let $G_{F,i}$ be the affine transformation defined in Figure B.2.

We define the lifting operator $P_{F,T_i} : \mathbb{P}_k(F) \to \mathbb{P}_k(T_i)$ by

$$P_{F,T_i}(s) = \hat{P}_{\hat{F}}(s \circ G_{F,i}) \circ G_{F,i}^{-1}.$$

Using these notations, we can define a lifting operator

$$s \in \mathbb{P}_k(F) \longrightarrow P_F(s) := \begin{cases} P_{F,T_1}(s) & \text{in } T_1\,, \\ P_{F,T_2}(s) & \text{in } T_2\,, \end{cases}$$

and, for $\boldsymbol{s} = (s_1, s_2) \in \mathbb{P}_k(F)^2$, we denote

$$\boldsymbol{P}_F(\boldsymbol{s}) = (P_F(s_1), P_F(s_2))\,.$$

Finally, for all $F \in \mathcal{E}_\Omega$ let α_F be the positive parameter given by

$$\alpha_F := \begin{cases} \min\{\nu^{1/2}\,\sigma^{-1/2}\,h_F^{-1}, 1\} & , \sigma > 0\,, \\ 1 & , \sigma = 0\,. \end{cases}$$

Theorem 19 *Let $k \in \mathbb{N}$. For all $\sigma \geq 0$, the following estimates hold*

$$\begin{aligned} \|\boldsymbol{v}\|_{0,T}^2 &\preceq (\boldsymbol{v}, b_T \boldsymbol{v})_T\,, \\ \|\boldsymbol{s}\|_{0,F}^2 &\preceq (\boldsymbol{s}, b_{F,\alpha_F} \boldsymbol{s})_F\,, \\ \|b_T\,\boldsymbol{v}\|_{a,T} &\preceq \theta_T^{-1}\,\|\boldsymbol{v}\|_{0,T}, \qquad \text{(B.3)} \\ \|b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{s})\|_{a,\omega_F} &\preceq \theta_F^{-1}\,\|\boldsymbol{s}\|_{0,F}, \qquad \text{(B.4)} \end{aligned}$$

for all $T \in \mathcal{T}_h$, $F \in \mathcal{E}_\Omega$, and every polynomial $\boldsymbol{v}$, $\boldsymbol{s}$ of degree k defined in T and F, respectively.

PROOF. The first two inequalities are proved (for the scalar case) in [22], Lemma 3.3. To prove the latter ones, let us first suppose that $\sigma > 0$. Using the inverse inequality (3.3) and the fact that $b_T \leq 1$,

$$\begin{aligned} \|b_T\,\boldsymbol{v}\|_{a,T}^2 &= \nu\,\|\nabla(b_T\,\boldsymbol{v})\|_{0,T}^2 + \sigma\,\|b_T\,\boldsymbol{v}\|_{0,T}^2 \\ &\preceq (\nu\,h_T^{-2} + \sigma)\,\|\boldsymbol{v}\|_{0,T}^2 \\ &\preceq \sigma\,(\nu\,\sigma^{-1}\,h_T^{-2} + 1)\,\|\boldsymbol{v}\|_{0,T}^2 \\ &\preceq \sigma\,\max\{\nu^{1/2}\,\sigma^{-1/2}\,h_T^{-1}, 1\}^2\,\|\boldsymbol{v}\|_{0,T}^2 \\ &\preceq \sigma\,\min\{\nu^{-1/2}\,\sigma^{1/2}\,h_T, 1\}^{-2}\,\|\boldsymbol{v}\|_{0,T}^2\,, \end{aligned}$$

and (B.3) follows. To prove (B.4), we first see that

$$
\begin{aligned}
\|b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{s})\|^2_{0,\omega_F} &= \sum_{T_i\subset\omega_F} \|b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{s})\|^2_{0,T_i} \\
&\preceq \sum_{T_i\subset\omega_F} h^2_{T_i}\|b_{\widehat{F},\alpha_F}\,\hat{\boldsymbol{P}}_{\hat{F}}(\hat{\boldsymbol{s}})\|^2_{0,\widehat{T}}
\end{aligned} \tag{B.5}
$$

Now, Lemma 3.3 in [22] applied to the vectorial case leads to

$$
\|b_{\widehat{F},\alpha_F}\hat{\boldsymbol{P}}_{\hat{F}}(\hat{\boldsymbol{s}})\|_{0,\widehat{T}} \preceq \sqrt{\alpha_F}\,\|\hat{\boldsymbol{s}}\|_{0,\widehat{F}}\,, \tag{B.6}
$$

$$
\|\widehat{\nabla}(b_{\widehat{F},\alpha_F}\hat{\boldsymbol{P}}_{\hat{F}}(\hat{\boldsymbol{s}}))\|_{0,\widehat{T}} \preceq \sqrt{\alpha_F + \frac{1}{\alpha_F}}\,\|\hat{\boldsymbol{s}}\|_{0,\widehat{F}}\,, \tag{B.7}
$$

and hence, using (B.5),(B.6) and the mesh regularity, we obtain

$$
\|b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{s})\|^2_{0,\omega_F} \;\preceq\; \alpha_F\,h_F^2\,\|\hat{\boldsymbol{s}}\|^2_{0,\widehat{F}} \;\preceq\; \alpha_F\,h_F\,\|\boldsymbol{s}\|^2_{0,F}\,.
$$

Moreover

$$
\begin{aligned}
\sigma\,h_F\,\alpha_F &= \nu^{1/2}\,\sigma^{1/2}\,\min\{1,\nu^{-1/2}\,\sigma^{1/2}\,h_F\} \\
&\le \nu^{1/2}\,\sigma^{1/2}\,\min\{1,\nu^{-1/2}\,\sigma^{1/2}\,h_F\}^{-1} \;=\; \theta_F^{-2}\,,
\end{aligned}
$$

and then

$$
\sigma\,\|b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{s})\|^2_{0,\omega_F} \;\preceq\; \theta_F^{-2}\,\|\boldsymbol{s}\|^2_{0,F}\,. \tag{B.8}
$$

On the other hand, from (B.7) and $\alpha_F \le 1$, it holds

$$
\begin{aligned}
\|\nabla(b_{F,\alpha_F}\boldsymbol{P}_F(\boldsymbol{s}))\|^2_{0,\omega_F} &= \sum_{T_i\subset\omega_F} \|\nabla(b_{F,\alpha_F}\boldsymbol{P}_F(\boldsymbol{s}))|^2_{0,T_i} \\
&\preceq \|\hat{\nabla}(\hat{b}_{F,\alpha_F}\hat{\boldsymbol{P}}_{\hat{F}}(\hat{\boldsymbol{s}}))\|^2_{0,\widehat{T}} \\
&\preceq \alpha_F^{-1}\,\|\hat{\boldsymbol{s}}\|^2_{0,\widehat{F}} \\
&\preceq h_F^{-1}\,\alpha_F^{-1}\,\|\boldsymbol{s}\|^2_{0,F}\,,
\end{aligned}
$$

and using that $\nu\,h_F^{-1}\,\alpha_F^{-1} \;=\; \theta_F^{-2}$, we obtain

$$
\nu\,\|\nabla(b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{s}))\|^2_{0,\omega_F} \;\preceq\; \theta_F^{-2}\,\|\boldsymbol{s}\|^2_{0,F}\,. \tag{B.9}
$$

Hence, the result for $\sigma > 0$ follows from (B.8) and (B.9). The proof for $\sigma = 0$ follows in an analogous way. $\square$

In order to satisfy the (LBB) condition we need to impose the following condition on $\boldsymbol{f}$:

(F) $\boldsymbol{f}$ *is a piecewise polynomial function,* i.e., there exists a positive integer t such that

$$
\boldsymbol{f} \in \{\boldsymbol{g} \in L^2(\Omega)^2 \;:\; \boldsymbol{g}|_T \in \mathbb{P}_t(T)^2\;,\,\forall\, T \in \mathcal{T}_h\}\,.
$$

Remark 20 *One possibility to overcome condition (F) above is to split the error between the error due to data approximation and the error due to the numerical method, as it has been done, for instance, in [1]. In any case, we remark that, since the degree t of the polynomial from condition (F) is not upper bounded, the error between $\boldsymbol{f}$ and its local projection onto the piecewise polynomial space may be seen as a higher order term.*

Next, we define the following bubble function spaces:

$$\begin{aligned}
&\boldsymbol{H}_T^b := \langle\{b_T\,\boldsymbol{R}_T\}\rangle \qquad \forall\, T \in \mathcal{T}_h\,,\\
&\boldsymbol{H}_F^b := \langle\{b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{R}_F)\}\rangle \qquad \forall F \in \mathcal{E}_\Omega\,,
\end{aligned}$$

where b_T and b_{F,α_F} are the bubble functions given by (B.1) and (B.2), respectively.

Remark 21 *We remark that this definition of bubble functions allows us use any polynomial order to approximate the velocity and the pressure. In fact, for every $k, l \geq 1$ the bubble function $b_T\boldsymbol{R}_T$ belongs to $\mathbb{P}_{\max\{t,k,l-1\}+3}(T)$ and $b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{R}_F)$ belongs to $\mathbb{P}_{k+1}(T)$. Hence, $\boldsymbol{H}_T^b$ and $\boldsymbol{H}_F^b$ are not subspaces of $\boldsymbol{H}_h$.*

Since $b_T\,\boldsymbol{R}_T \in \boldsymbol{H}_T^b$, using Theorem 19 we arrive at

$$\begin{aligned}
\sup_{\mathcal{B}_T \in \boldsymbol{H}_T^b} \frac{(\boldsymbol{R}_T, \mathcal{B}_T)_T}{\theta_T\,\|\boldsymbol{R}_T\|_{0,T}\,a_T(\mathcal{B}_T,\mathcal{B}_T)^{1/2}} &\geq \frac{(\boldsymbol{R}_T, b_T\,\boldsymbol{R}_T)_T}{\theta_T\,\|\boldsymbol{R}_T\|_{0,T}\,a_T(b_T\,\boldsymbol{R}_T, b_T\,\boldsymbol{R}_T)^{1/2}}\\
&\succeq \frac{\|\boldsymbol{R}_T\|_{0,T}^2}{\theta_T\,\|\boldsymbol{R}_T\|_{0,T}\;\theta_T^{-1}\|\boldsymbol{R}_T\|_{0,T}} \;\succeq\; \beta\,.
\end{aligned}$$

The same analysis may be carried out for every $F \in \mathcal{E}_\Omega$. In fact, we have

$$\begin{aligned}
&\sup_{\mathcal{B}_F \in \boldsymbol{H}_F^b} \frac{(\boldsymbol{R}_F, \mathcal{B}_F)_F}{\theta_F\,\|\boldsymbol{R}_F\|_{0,F}\,a_{\omega_F}(\mathcal{B}_F,\mathcal{B}_F)^{1/2}}\\
&\geq \frac{(\boldsymbol{R}_F, b_{F,\alpha_F}\,\boldsymbol{R}_F)_F}{\theta_F\,\|\boldsymbol{R}_F\|_{0,F}\,a_{\omega_F}(b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{R}_F), b_{F,\alpha_F}\,\boldsymbol{P}_F(\boldsymbol{R}_F))^{1/2}}\\
&\succeq \frac{\|\boldsymbol{R}_F\|_{0,F}^2}{\theta_F\,\|\boldsymbol{R}_F\|_{0,F}\;\theta_F^{-1}\,\|\boldsymbol{R}_F\|_{0,F}}\\
&\succeq \beta\,.
\end{aligned}$$

References

[1] M. Ainsworth. A posteriori error estimation for lowest order Raviart-Thomas mixed finite elements. Technical Report 23, University of Strathclyde, Department of Mathematics, 2006.

[2] M. Ainsworth and J. T. Oden. *A posteriori error estimation in finite element analysis*. John Wiley & Sons, New York, 2000.

[3] M. Ainsworth and J.T. Oden. A posteriori error estimators for the Stokes and Oseen equations. *SIAM J. Numer. Anal.*, 34:228–245, 1997.

[4] R. Araya, G.R. Barrenechea, and F. Valentin. A stabilized finite-element method for the Stokes problem including element and edge residuals. *IMA J. Numer. Anal.*, 27(1):172–197, 2007.

[5] R. Araya, E. Behrens, and R. Rodríguez. An adaptive stabilized finite element scheme for the advection-reaction-diffusion equation. *Appl. Num. Math.*, 54:491–503, 2005.

[6] R. Araya, A. Poza, and E.P. Stephan. A hierarchical a posteriori error estimate for an advection-diffusion-reaction problem. *Math. Models Methods Appl. Sci.*, 15(7):1119–1139, 2005.

[7] R.E. Bank and B.D. Welfert. A posteriori error estimators for the Stokes problem. *SIAM J. Numer. Anal.*, 28:591–623, 1991.

[8] G. Barrenechea and F. Valentin. An unusual stabilized finite element method for a generalized Stokes problem. *Numer. Math.*, 92(4):653–677, 2002.

[9] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer-Verlag, New York, 1991.

[10] P. Clément. Approximation by finite element functions using local regularization. *R.A.I.R.O. Anal. Numer.*, 9:77–84, 1975.

[11] E. Dari, R. Durán, and C. Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Math. Comp.*, 64:1017–1033, 1995.

[12] W. Dörfler and M. Ainsworth. Reliable a posteriori error control for nonconformal finite element approximation of Stokes flow. *Math. Comp.*, 54(252):1599–1619, 2005.

[13] H. Elman, D. Silvester, and A. Wathen. *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*. Oxford University Press, New York, 2005.

[14] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*. Springer-Verlag, New York, 2004.

[15] V. Girault and P.A. Raviart. *Finite element methods for the Navier-Stokes equations*. Springer-Verlag, Berlin, 1986.

[16] T.J.R. Hughes and L.P. Franca. A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces. *Comput. Methods Appl. Mech. Engrg.*, 65(1):85–96, 1987.

[17] V. John. A numerical study of a posteriori error estimators for convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 190:757–781, 2000.

[18] D. Kay and D. Silvester. A posteriori error estimation for stabilized mixed approximations of the Stokes equations. *SIAM J. Scientific Computing*, 21:1321–1336, 1999.

[19] J.R. Shewchuk. Delaunay refinement algorithms for triangular mesh generation. *Computational Geometry: Theory and Applications*, 22(1–3):21–74, 2002.

[20] R. Verfürth. A posteriori error estimators for the Stokes problem. *Numer. Math.*, 55:309 – 325, 1989.

[21] R. Verfürth. A posteriori error estimators for the Stokes problem II. Non-conforming discretizations. *Numer. Math.*, 60:235 – 249, 1991.

[22] R. Verfürth. A posteriori error estimators for convection-diffusion equations. *Numer. Math.*, 80:641–663, 1998.