On the Approach to the Critical Solution in Leading Order Thin-Film Coating and Rimming Flow

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Abstract

The approach to the critical solution in leading order coating and rimming flow of a thin fluid film on a uniformly rotating horizontal cylinder is investigated. In particular, it is shown that the weight of the leading order "full-film" solution approaches its critical maximum value with logarithmically infinite slope as the volume flux approaches its critical value.

1 Introduction

As well as being a fascinating problem in its own right, the flow of a thin fluid film on the exterior (usually called "coating flow") or the interior (usually called "rimming flow") of a uniformly rotating horizontal circular cylinder is a convenient "test-bed" for the analytical, numerical and experimental study of the often surprisingly subtle interplay between a variety of competing forces, notably those due to gravity, viscosity, surface tension and inertia, resulting in a wide variety of complex behaviour including, as the excellent experimental study by Thoroddsen and Mahadevan [1] describes, sloshing, pattern formation (including so-called "shark teeth" and "duck bill" patterns), fluid curtains, hydroplaning drops, air-entrainment and frontal avalanches. As a result, the problem has been the subject of ongoing study ever since the pioneering work by Moffatt [2]. Lack of space prohibits a complete review of the extensive and rapidly growing literature, but a representative selection of significant contributions published during the last decade are listed (chronologically) as references [3] to [16] below.

2 Problem Formulation

Consider steady two-dimensional flow of a thin film of Newtonian fluid of uniform density ρ and viscosity μ on either the exterior or the interior of a circular cylinder of radius a rotating about its horizontal axis with uniform angular speed ω (so that the circumferential speed is $U = a\omega$). Adopting polar coordinates (r, θ) with origin at the cylinder's axis and θ measured anti-clockwise from the right-hand side of the cylinder, we write $r = a \pm y$ for coating and rimming flow, respectively, (so that y is a local coordinate measured away from the cylinder), and denote the free surface of the film by $y = h(\theta)$. The velocity $\mathbf{u}(y, \theta) = v(y, \theta)\mathbf{e}_r + u(y, \theta)\mathbf{e}_\theta$ and pressure $p = p(y, \theta)$ satisfy the familiar mass-conservation and Navier–Stokes equations subject to the usual boundary conditions of continuity of stress and the kinematic condition at the free surface of the film and continuity of velocity at the cylinder. Non-dimensionalising y and h with δa, u with U, v with δU and p with $\delta\rho g a$, where $\delta = (\mu U/\rho g a^2)^{1/2} \ll 1$ is the small aspect ratio of the film, and taking the thin-film limit $\delta \to 0^+$, the leading-order versions of the governing equations and boundary conditions for coating

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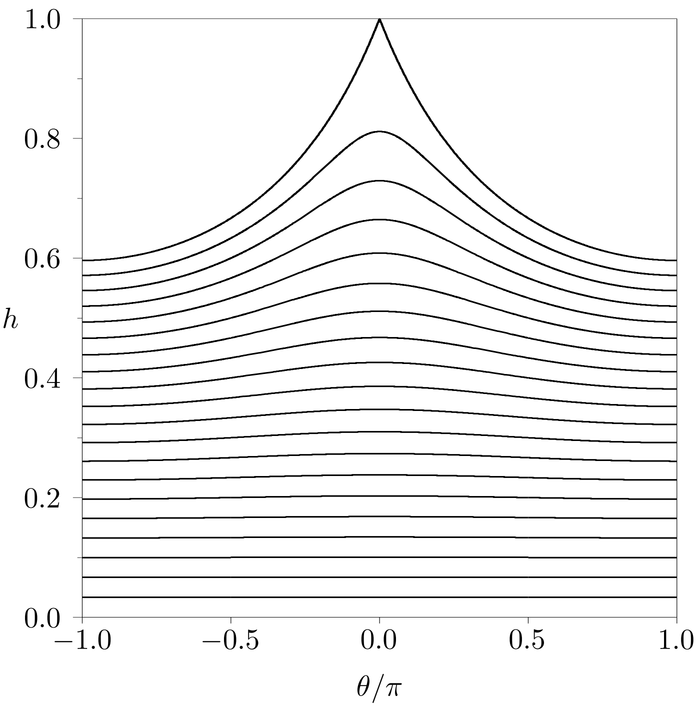

Figure 1: The free surfaces of the full-film solutions h plotted as functions of θ/π for $Q = 1/30$ (the lowest curve), $1/15$, $1/10$, ..., $2/3$ (the highest curve).

flow are $u_\theta + v_y = 0$, $p_y = -\sin\theta$ and $u_{yy} = \cos\theta$ subject to $u = 1$ and $v = 0$ on $y = 0$, and $p = 0$ and $u_y = 0$ on $y = h$. These equations and boundary conditions can readily be solved to yield

$$p = \sin\theta\,(h - y), \quad u = 1 - \frac{\cos\theta}{2}(2hy - y^2), \tag{1}$$

and hence the non-dimensional leading order volume flux per unit axial length crossing the station $\theta =$ constant (in the direction of increasing θ) is given by

$$Q = \int_0^h u\,\mathrm{d}y = h - \frac{\cos\theta}{3}h^3, \tag{2}$$

while the non-dimensional leading order weight of fluid is

$$W = \int_{-\pi}^{\pi} h\,\mathrm{d}\theta. \tag{3}$$

The corresponding expressions for rimming flow are obtained by simply replacing p with $-p$. In particular, the expression for Q in terms of h given by (2) is the same for both coating and rimming flow. Since the flow is steady Q must be constant, and so we may regard (2) as a cubic equation for h. Once h is known the entire solution (including W) is determined in terms of the parameter Q. In practice either Q or W (but not both) would be prescribed.

3 Full-Film Solutions

As Moffatt [2] showed, when $0 < Q \leq 2/3$ (but not otherwise) there is a one-parameter family of so-called "full film" solutions with finite, non-zero thickness satisfying $0 < h \leq 1$ for all values of θ

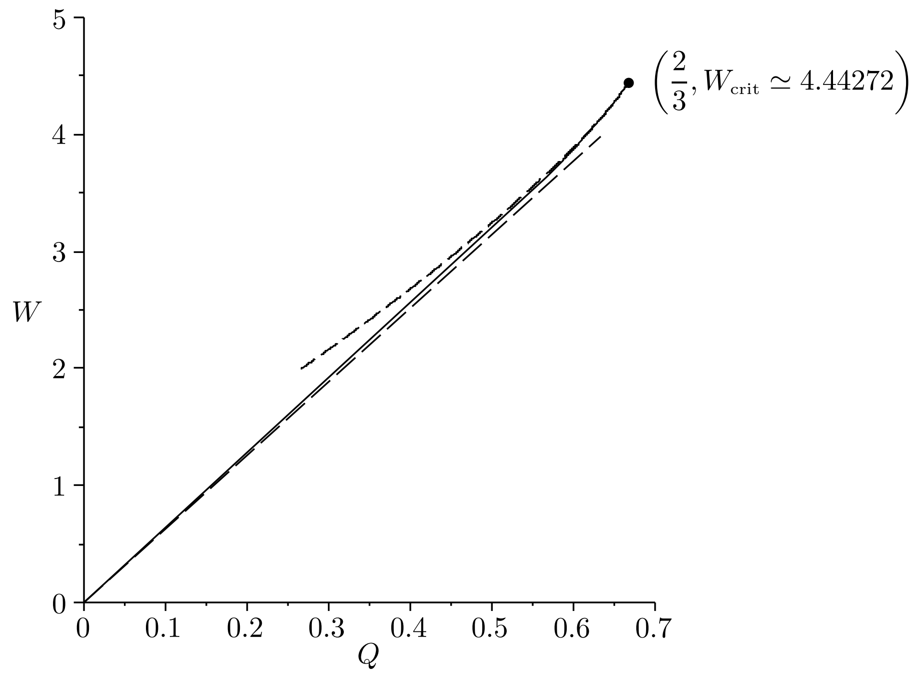

Figure 2: The weight of the full-film solutions W plotted as a function of Q for $0 < Q \leq 2/3$. The asymptotic expansion for W in the limit $Q \to 0^+$ given by $W = 2\pi Q + O(Q^5)$ and the asymptotic expansion for W in the limit $Q \to 2/3^-$ given by (21) are shown with dashed lines.

(i.e. that extend all the way around the cylinder). As Duffy and Wilson [3] describe, there are also so-called "curtain" solutions that become unbounded at the top and bottom of the cylinder, but these are not considered here. The full-film solutions have top-to-bottom symmetry (i.e. they are symmetric about the line $\theta = 0, \pi$). At a fixed value of θ their thickness is a monotonically increasing function of Q, while for a fixed value of Q their thickness decreases monotonically from its maximum value at $\theta = 0$ through the value Q at $\theta = \pm\pi/2$ to its minimum value at $\theta = \pi$. Figure 1 shows the free surfaces of the full-film solutions h plotted as functions of θ/π for various values of Q. In particular, as Figure 1 illustrates, when $0 < Q < 2/3$ the free surfaces are locally parabolic near $\theta = 0$ and $\theta = \pi$. However, as Figure 1 also shows, in the critical case $Q = 2/3$ in which $h = h_{\text{crit}}$, where $h_{\text{crit}} = h_{\text{crit}}(\theta)$ is the appropriate solution of

$$\frac{2}{3} = h_{\text{crit}} - \frac{\cos\theta}{3} h_{\text{crit}}^3, \tag{4}$$

the free surface has a corner at $\theta = 0$ given by

$$h_{\text{crit}} = 1 - \frac{|\theta|}{\sqrt{6}} + \frac{2\theta^2}{9} + O(\theta^3) \quad \text{as} \quad \theta \to 0, \tag{5}$$

but remains locally parabolic near $\theta = \pi$. The weight of the full-film solutions W is plotted as a function of Q for $0 < Q \leq 2/3$ in Figure 2, which shows that W is a monotonically increasing function of Q. In the limit of small flux $Q \to 0^+$ the film becomes uniformly thin with thickness Q according to

$$h = Q + \frac{\cos\theta}{3} Q^3 + O(Q^5) \tag{6}$$

and so the weight approaches zero linearly according to $W = 2\pi Q + O(Q^5)$. This asymptotic expansion for W is also plotted in Figure 2 showing that it is a good approximation to the exact value provided that Q is not too close to 2/3. As Moffatt [2] showed, the critical maximum value of the weight is $W = W_{\mathrm{crit}}$ at $Q = 2/3$, where the value of W_{crit}, namely

$$W_{\mathrm{crit}} = \int_{-\pi}^{\pi} h_{\mathrm{crit}}\,\mathrm{d}\theta \simeq 4.44272, \tag{7}$$

has to be determined numerically. (Note that the numerical value of the critical weight actually given by Moffatt [2] was 4.428; the disparity with the correct value is presumably the result of a typographical error.)

4 Approach to the Critical Solution

The purpose of the present work is to investigate the approach of the solution h to the critical solution h_{crit} as the flux Q approaches its critical value of 2/3 and hence, in particular, to determine the approach of the weight W to its critical maximum value W_{crit}. In order to do this, we introduce a new small parameter $\epsilon \ll 1$ measuring how close the flux is to its critical value by writing $Q = 2/3 - \epsilon$ and take the limit $\epsilon \to 0^+$. Note that the small parameter ϵ should not to be confused with the small aspect ratio δ introduced previously. Thus the present analysis, which determines the approach of the leading order solution to the critical leading order solution as $\epsilon \to 0^+$, complements (but differs from) that by Wilson, Hunt and Duffy [8] who went to higher order in the aspect ratio δ to obtain higher order corrections to the critical leading order solution as $\delta \to 0^+$.

4.1 Outer Solution Valid Away From $\theta = 0$

We seek a naive regular outer asymptotic solution for h in the form

$$h = h_{\mathrm{crit}}(\theta) + \epsilon h_1(\theta) + o(\epsilon), \tag{8}$$

and substituting (8) into (2) and expanding for small ϵ recovers (4) at leading order and yields

$$h_1 = -\frac{1}{1 - \cos\theta\, h_{\mathrm{crit}}^2} \tag{9}$$

at first order. Away from $\theta = 0$ the outer expansion (8) is uniform, but as $\theta \to 0$ we have $h_{\mathrm{crit}} \sim 1 - |\theta|/\sqrt{6}$ from (5) and hence

$$h_1 \sim -\frac{\sqrt{6}}{2|\theta|} \to -\infty \quad \text{as} \quad \theta \to 0, \tag{10}$$

and so the outer expansion (8) is non-uniform near $\theta = 0$, and the corresponding naive expansion for the weight $W = W_{\mathrm{crit}} + O(\epsilon)$ fails because, as (10) shows, h_1 is non-integrable at $\theta = 0$. In order to resolve the non-uniformity in the outer expansion of h, and hence obtain the correct asymptotic expansion for W, we need to resolve the solution in an appropriate inner region near $\theta = 0$.

4.2 Inner Solution Valid Near $\theta = 0$

Perhaps the simplest way to determine the unknown width of the inner region is to examine the behaviour of h at $\theta = 0$ in more detail. Setting $\theta = 0$ in (2) shows that $h(0)$ satisfies the cubic equation $Q = h(0) - h(0)^3/3$ and hence (taking the appropriate root) that $h(0) \sim 1 - \sqrt{\epsilon}$ as $\epsilon \to 0^+$. As a consequence we expect the outer solution (8) to fail when ϵh_1 is the same size as $\sqrt{\epsilon}$, i.e. when

$\epsilon\sqrt{6}/2|\theta| \sim \sqrt{\epsilon}$, i.e. when $|\theta| = O(\sqrt{\epsilon})$, suggesting that the width of the inner region is $O(\sqrt{\epsilon})$. Thus we introduce an appropriately scaled inner variable $\phi = \theta/\sqrt{\epsilon}$ and seek an inner solution valid near $\theta = 0$ in the form

$$h = 1 + \sqrt{\epsilon}H_1(\phi) + o(\sqrt{\epsilon}). \tag{11}$$

Substituting (11) into (2) and expanding for small ϵ yields $H_1^2 = 1 + \phi^2/6$ with the appropriate solution

$$H_1 = -\sqrt{1 + \frac{\phi^2}{6}}, \tag{12}$$

where the minus sign has been chosen to give the correct value at $\phi = 0$ (namely $H_1(0) = -1$) in order to agree with the expansion for $h(0)$ given above and to match with (5) as $|\phi| \to \infty$. In particular, (12) reveals that the free surface is hyperbolic in the inner region.

4.3 Uniformly Valid Composite Solution

The complete solution consists of the outer solution away from $\theta = 0$ and the inner solution near $\theta = 0$, and a uniformly valid leading order composite solution h_{comp} can be constructed in the usual way by writing $h_{\text{comp}} = h_{\text{outer}} + h_{\text{inner}} - h_{\text{common}}$, where the "common part"

$$h_{\text{common}} = 1 - \sqrt{\epsilon}\left(\frac{|\phi|}{\sqrt{6}} + \frac{\sqrt{6}}{2|\phi|}\right) = 1 - \frac{|\theta|}{\sqrt{6}} - \frac{\epsilon\sqrt{6}}{2|\theta|} \tag{13}$$

is subtracted to avoid double counting in the overlap region, to yield

$$h_{\text{comp}} = h_{\text{crit}} + \frac{\epsilon\sqrt{6}}{2|\theta|} - \frac{\epsilon}{1 - \cos\theta\, h_{\text{crit}}^2} + \frac{|\theta|}{\sqrt{6}} - \sqrt{\epsilon + \frac{\theta^2}{6}} \quad \text{as} \quad \epsilon \to 0^+. \tag{14}$$

4.4 Asymptotic Expansion for the Weight

Armed with a uniformly valid leading order composite solution h_{comp} given by (14) we can obtain the corresponding asymptotic expansion for the weight by writing

$$W = W_{\text{crit}} + \epsilon W_1 + \hat{W} + o(\epsilon), \tag{15}$$

where

$$W_1 = \int_{-\pi}^{\pi} \left[\frac{\sqrt{6}}{2|\theta|} - \frac{1}{1 - \cos\theta\, h_{\text{crit}}^2}\right] \mathrm{d}\theta \simeq -1.45785 \tag{16}$$

and

$$\hat{W} = \int_{-\pi}^{\pi} \left[\frac{|\theta|}{\sqrt{6}} - \sqrt{\epsilon + \frac{\theta^2}{6}}\right] \mathrm{d}\theta. \tag{17}$$

Evaluating $\hat{W}$ yields

$$\hat{W} = \epsilon\sqrt{6}\log\left(\frac{\sqrt{\epsilon + \frac{\pi^2}{6}} - \frac{\pi}{\sqrt{6}}}{\sqrt{\epsilon}}\right) - \pi\left(\sqrt{\epsilon + \frac{\pi^2}{6}} - \frac{\pi}{\sqrt{6}}\right), \tag{18}$$

and expanding (18) for small ϵ gives

$$\hat{W} = \frac{\sqrt{6}}{2}\epsilon\log\left(\frac{3\epsilon}{2\pi^2\mathrm{e}}\right) - \frac{3\sqrt{6}}{4\pi^2}\epsilon^2 + O(\epsilon^3). \tag{19}$$

Thus the asymptotic expansion for the weight is

$$W = W_{\text{crit}} + \epsilon \left[W_1 + \frac{\sqrt{6}}{2} \log \left(\frac{3\epsilon}{2\pi^2 \mathrm{e}} \right) \right] + o(\epsilon) \quad \text{as} \quad \epsilon \to 0^+, \tag{20}$$

which can be expressed in terms of numerical values as

$$W \simeq 4.44272 + \epsilon\,(-4.99001 + 1.22474 \log \epsilon) + o(\epsilon) \quad \text{as} \quad \epsilon \to 0^+. \tag{21}$$

The correctness of the expansion (21) was confirmed by comparison with exact values of W calculated by solving (2) for h and then evaluating W numerically, and this is illustrated in Figure 2 which shows the excellent agreement between the asymptotic and the exact values of W provided that Q is sufficiently close to 2/3. In particular, (21) shows, rather unexpectedly (and despite what Figure 2 might appear to suggest), that W approaches W_{crit} with logarithmically infinite slope as $Q \to 2/3^-$.

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