Population Dynamical Behavior of Lotka-Volterra System under Regime Switching *

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Abstract. In this paper, we investigate a Lotka-Volterra system under regime switching

$$dx(t) = \text{diag}(x_1(t), \ldots, x_n(t))[(b(r(t)) + A(r(t))x(t))dt + \sigma(r(t))dB(t)],$$

where $B(t)$ is a standard Brownian motion. The aim here is to find out what happens under regime switching. We first obtain the sufficient conditions for the existence of global positive solutions, stochastic permanence, extinction. We find out that both stochastic permanence and extinction have close relationships with the stationary probability distribution of the Morkov chain. The limit of the average in time of the sample path of the solution is then estimated by two constants related to the stationary distribution and the coefficients. Finally, the main results are illustrated by several examples.

Keywords. Brownian motion; Stochastic differential equation; Generalized Itô's formula; Markov chain; Stochastic permanence.

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1 Introduction

The classical Lotka-Volterra model for n interacting species is described by the n-dimensional ordinary differential equation (ODE)

$$\dot{x}(t) = \text{diag}(x_1(t), \ldots, x_n(t))[b + Ax(t)], \tag{1.1}$$

where

$$x = (x_1, \cdots, x_n)^T \in R^n, \quad b = (b_1, \cdots, b_n)^T \in R^n_+, \quad A = (a_{ij})_{n\times n} \in R^{n\times n}.$$

There are many extensive literatures concerned with the dynamics of this model and we here don't mention them in details.

Population systems are often subject to environmental noise. It is therefore useful to reveal how the noise affects the population systems. As we know, there are various types of environmental noise. First of all, let us consider one type of them, namely the white noise. Recall that the parameter b_i represents the intrinsic growth rate of species i. In practice we usually estimate it by an average value plus an error which follows a normal distribution. If we still use b_i to denote the average growth rate, then the intrinsic growth rate becomes

$$b_i \to b_i + \sigma_i \dot{B}(t),$$

where $\dot{B}(t)$ is a white noise, and σ_i is a positive constant representing the intensity of the noises respectively. Then this environmentally perturbed system can be described by the Itô equation

$$dx(t) = \text{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t))dt + \sigma dB(t)], \tag{1.2}$$

where $\sigma = (\sigma_1, \cdots, \sigma_n)^T$, $B(t)$ is an standard Brownian motion with $B(0) = 0$. As a matter of fact, population systems perturbed by the white noise have recently been studied by many authors, for example, [2], [3], [11]-[13], [14]-[17], [19], [21], [22], [24], [25], [28]. In particular, Mao, Marion and Renshaw [24], [25] revealed that the environmental noise can suppress a potential population explosion while Mao [21] showed that different structures of white noise may have different effects on the population systems.

Let us now take a further step by considering another type of environmental noise, namely, color noise, say telegraph noise (see e.g. [20], [31]). The telegraph noise can be illustrated as a switching between two or more regimes of environment, which differ by factors such as nutrition or as rain falls [7], [30]. The switching is memoryless and the waiting time for the next switch has an exponential distribution. We can hence model the regime switching by a finite-state Markov Chain. Assume that there are N regimes and the system obeys

$$dx(t) = \text{diag}(x_1(t), \dots, x_n(t))[(b(k) + A(k)x(t))dt + \sigma(k)dB(t)], \tag{1.3}$$

in regime k ($1 \leq k \leq N$), where $b(k) = (b_1(k), \cdots, b_n(k))^T$ etc. The switching between these N regimes is governed by a Markov chain $r(t)$ on the state space $S = \{1, 2, \dots, N\}$. The population system under regime switching can therefore be described by the following stochastic model (SDE)

$$dx(t) = \text{diag}(x_1(t), \dots, x_n(t))[(b(r(t)) + A(r(t))x(t))dt + \sigma(r(t))dB(t)]. \tag{1.4}$$

This system is operated as follows: If $r(0) = k_0$, the system obeys equation (1.3) with $k = k_0$ till time τ_1 when the Markov chain jumps to k_1 from k_0; the system will then obey equation (1.3) with $k = k_1$ from τ_1 till τ_2 when the Markov chain jumps to k_2 from k_1. The system will continue to switch as long as the Markov chain jumps. In other words, the SDE (1.4) can be regarded as equations (1.3) switching from one to another according to the law of the Markov Chain. Each of (1.3) ($1 \leq k \leq N$) is hence called a subsystem of the SDE (1.4).

Recently, Takeuchi et al. [31] have investigated a 2-dimensional autonomous predator-prey Lotka-Volterra system with regime switching and revealed a very interesting and surprising result: If two equilibrium states of the subsystems are different, all positive trajectories of this system always exit from any compact set of R_+^2 with probability one; on the other hand, if the two equilibrium states coincide, then the trajectory either leaves from any compact set of R_+^2 or converges to the equilibrium state. In practice, two equilibrium states are usually different whence Takeuchi et al. [31] showed that the stochastic population system is neither permanent nor dissipative (see e.g. [10]). This is an important result as it reveals the significant effect of the environmental noise to the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative. It is these factors that motivate us to consider the Lotka-Volterra system subject to both white noise and color noise described by the SDE (1.4) .

In this paper, in order to obtain better dynamic properties of the SDE (1.4), we will show that there exists a positive global solution with any initial positive value under some conditions in section 2. In the study of population dynamics, permanence and extinction are two of the important and interesting topics which mean that the population system will survive or die out in the future, respectively. One of our main aims is to investigate these properties. In sections 3 and 4, we give the sufficient conditions for stochastic permanence and extinction which have closed relations with the stationary probability distribution of the Markov chain. When the SDE (1.4) is stochastically permanent we estimate the limit of average in time of the sample path of its solution in section 5. Finally, we illustrate our main results through several examples in sections 6 and 7.

The key method used in this paper is the analysis of Lyapunov functions. This Lyapunov function analysis for stochastic differential equations was developed by Khasminskii (see e.g. [18]) and has been used by many authors (see e.g. [1, 9, 13, 22, 26, 27]).

2 Positive and Global Solutions

Throughout this paper, unless otherwise specified, let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ denote the standard Brownian motion defined on this probability space. We also denote by R^n_+ the positive cone in R^n, that is $R^n_+ = \{x \in R^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$, and denote by $\bar{R}^n_+$ the nonnegative cone in R^n, that is $\bar{R}^n_+ = \{x \in R^n : x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|A| :\ |x| = 1\}$.

In this paper we will use a lot of quadratic functions of the form $x^T Ax$ for the state $x \in R^n_+$ only. Therefore, for a symmetric $n \times n$ matrix A, we naturally introduce the following definition

$$\lambda^+_{\max}(A) = \sup_{x \in R^n_+, |x|=1} x^T Ax.$$

Let us emphasis that this is different from the largest eigenvalue $\lambda_{\max}(A)$ of the matrix A but $\lambda^+_{\max}(A)$ does have some similar properties as $\lambda_{\max}(A)$ has. It follows straightforward from the definition that $\lambda^+_{\max}(A) \leq \lambda_{\max}(A)$ and $x^T Ax \leq \lambda^+_{\max}(A)|x|^2$ for any $x \in R^n_+$. For more properties of $\lambda^+_{\max}(A)$ please see the Appendix in [3].

Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{uv})_{N\times N}$ given by

$$P\{r(t+\delta) = v | r(t) = u\} = \begin{cases} \gamma_{uv}\delta + o(\delta), & \text{if } u \neq v, \\ 1 + \gamma_{uv}\delta + o(\delta), & \text{if } u = v, \end{cases}$$

where $\delta > 0$. Here γ_{uv} is the transition rate from u to v and $\gamma_{uv} \geq 0$ if $u \neq v$ while

$$\gamma_{uu} = -\sum_{v\neq u} \gamma_{uv}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right continuous step function with a finite number of jumps in any finite subinterval of $\bar{R}_+$. As a standing hypothesis we assume in this paper that the Markov chain $r(t)$ is *irreducible*. This is very reasonable as it means that the system will switch from any regime to any other regime. This is equivalent to the condition that for any $u, v \in S$, one can find finite numbers $i_1, i_2, \ldots, i_k \in S$ such that $\gamma_{u,i_1}\gamma_{i_1,i_2}\ldots\gamma_{i_k,v} > 0$. Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\text{rank}(\Gamma) = N-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in R^{1\times N}$ which can be determined by soving the following linear equation

$$\pi\Gamma = 0 \tag{2.1}$$

subject to

$$\sum_{k=1}^{N} \pi_k = 1 \quad \text{and} \quad \pi_k > 0, \ \forall\, k \in S.$$

For convenience and simplicity in the following discussion, for any constant sequence $\{c_{ij}(k)\}$, $(1 \leq i, j \leq n, 1 \leq k \leq N)$ define

$$\check{c} = \max_{1\leq i,j\leq n, 1\leq k\leq N} c_{ij}(k), \qquad \check{c}(k) = \max_{1\leq i,j\leq n} c_{ij}(k),$$
$$\hat{c} = \min_{1\leq i,j\leq n, 1\leq k\leq N} c_{ij}(k), \qquad \hat{c}(k) = \min_{1\leq i,j\leq n} c_{ij}(k).$$

Moreover, let $C^{2,1}(R^n \times \bar{R}_+ \times S; \bar{R}_+)$ denote the family of all positive real-valued functions $V(x,t,k)$ on $R^n \times \bar{R}_+ \times S$ which are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(R^n \times \bar{R}_+ \times S; \bar{R}_+)$, define an operator LV from $R^n \times \bar{R}_+ \times S$ to R by

$$LV(x,t,k) = V_t(x,t,k) + V_x(x,t,k)\text{diag}(x_1, \cdots, x_n)(b(k) + A(k)x)$$

$$+\frac{1}{2}[\sigma^T(k)\text{diag}(x_1,\cdots,x_n)V_{xx}(x,t,k)\text{diag}(x_1,\cdots,x_n)\sigma(k)]$$
$$+\sum_{l=1}^{N}\gamma_{kl}V(x,t,l),$$

where

$$V_t(x,t,k)=\frac{\partial V(x,t,k)}{\partial t},\quad V_x(x,t,k)=\left(\frac{\partial V(x,t,k)}{\partial x_1},\cdots,\frac{\partial V(x,t,k)}{\partial x_n}\right),$$
$$V_{xx}(x,t,k)=\left(\frac{\partial^2 V(x,t,k)}{\partial x_i\partial x_j}\right)_{n\times n}.$$

As the ith state $x_i(t)$ of the SDE (1.4) is the size of ith species in the system at time t, it should be nonnegative. Moreover, in order for a stochastic differential equation with Markovian switching to have a unique global (i.e. no explosion in a finite time) solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. [26]). However, the coefficients of the SDE (1.4) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of the SDE (1.4) may explode at a finite time. It is therefore useful to establish some conditions under which the solution of the SDE (1.4) is not only positive but will also not explode to infinity at any finite time.

Assumption 1 *Assume that there exist positive numbers $c_1(k),\cdots,c_n(k)$ for each $k\in S$ such that*

$$-\lambda:=\max_{k\in S}\left\{\lambda_{max}^{+}\left(\bar{C}(k)A(k)+A^T(k)\bar{C}(k)\right)\right\}\le 0,$$

where $\bar{C}(k)=\text{diag}(c_1(k),\cdots,c_n(k))$.

Theorem 2.1 *Under Assumption 1, for any given initial value $x(0)\in R_+^n$, there is an unique solution $x(t)$ to the SDE (1.4) on $t\ge 0$ and the solution will remain in R_+^n with probability 1, namely, $x(t)\in R_+^n$ for all $t\ge 0$ almost surely.*

The proof is a modification of that for the autonomous case (see e.g. [2, 3, 21]) but for the completeness of the paper we will give it in Appendix A.

3 Stochastic Permanence

Theorem 2.1 shows that the solution of the SDE (1.4) will remain in the positive cone R_+^n. This nice property provides us with a great opportunity to discuss how the solution varies in R^n_+ in more details. In this section we will give the definitions of stochastically ultimate boundedness and stochastic permanence of the SDE (1.4) and some sufficient conditions for them.

Definition 3.1 *The SDE (1.4) is said to be stochastically ultimately bounded, if for any* $\epsilon \in (0,\ 1)$, *there is a positive constant* $\chi(= \chi(\epsilon))$, *such that for any initial value* $x(0) \in R^n_+$, *the solution of the SDE (1.4) has the property that*

$$\limsup_{t\to+\infty} P\left\{|x(t)| > \chi\right\} < \epsilon.$$

Definition 3.2 *The SDE (1.4) is said to be stochastically permanent if for any* $\epsilon \in (0,\ 1)$, *there exist positive constants* $\delta = \delta(\epsilon)$, $\chi = \chi(\epsilon)$ *such that*

$$\liminf_{t\to+\infty} P\left\{|x(t)| \le \chi\right\} \ge 1-\epsilon, \qquad \liminf_{t\to+\infty} P\left\{|x(t)| \ge \delta\right\} \ge 1-\epsilon,$$

where $x(t)$ *is the solution of the equation with any initial value* $x(0) \in R^n_+$.

It is obvious that if the SDE is stochastically permanent, it must be stochastically ultimately bounded. Let us begin with the easier one.

Assumption 2 *Assume that there exist positive numbers* $c_1(k), \cdots, c_n(k)$ *for each* $k \in S$ *such that*

$$-\lambda := \max_{k\in S}\left\{\lambda^+_{max}\left(\bar{C}(k)A(k) + A^T(k)\bar{C}(k)\right)\right\} < 0,$$

where $\bar{C}(k) = \mathrm{diag}(c_1(k), \cdots, c_n(k))$.

Lemma 3.1 *Under Assumption 2, for any given positive constant p, there is a positive constant* $K(p)$ *such that for any initial value* $x(0) \in R^n_+$, *the solution* $x(t)$ *of the SDE (1.4) has the property that*

$$\limsup_{t\to\infty} E(|x(t)|^p) \le K(p). \tag{3.1}$$

Proof. By Theorem 2.1, the solution $x(t)$ will remain in R^n_+ for all $t \ge 0$ with probability 1. Define for any given positive constant p

$$V(x,t,k) = e^t(1 + C(k)x)^p = e^t\left(1 + \sum_{i=1}^n c_i(k)x_i\right)^p \quad \text{for } \ x \in R^n_+,$$

where $C(k) = (c_1(k), \cdots, c_n(k))$. Using the method of Lyapunov function analysis, we could obtain the required assertion. The left proof is rather standard and hence is omitted.

Theorem 3.1 *The solutions of the SDE (1.4) is stochastically ultimately bounded under Assumption 2.*

The proof of Theorem 3.1 is a simple application of Chebyshev's inequality and Lemma 3.1. Let us now discuss the more complicated stochastic permanence. For convenience, let

$$\hat{\beta}(k) = \hat{b}(k) - \frac{1}{2}\breve{\sigma}^2(k), \qquad \breve{\beta}(k) = \breve{b}(k) - \frac{1}{2}\hat{\sigma}^2(k), \tag{3.2}$$

and we impose the following assumptions:

Assumption 3 *For some* $u \in S, \quad \gamma_{iu} > 0, \qquad \forall i \neq u.$

Assumption 4 $\sum_{k=1}^{N} \pi_k \hat{\beta}(k) > 0.$

Assumption 5 $\hat{\beta}(k) > 0$ *(* $1 \le k \le N$*).*

To state our main result, we will need a few more notations. Let G be a vector or matrix. By $G \gg 0$ we mean all elements are positive. We also adopt here the traditional notation by letting

$$Z^{N\times N} = \{A = (a_{ij})_{N\times N} : \ a_{ij} \le 0, \ i \neq j\}.$$

We shall also need two classical results.

Lemma 3.2 (Mao and Yuan [26], Lemma 5.3) *If* $A = (a_{ij}) \in Z^{N\times N}$ *has all of its row sums positive, that is*

$$\sum_{j=1}^{N} a_{ij} > 0 \quad \text{for all } 1 \le i \le N,$$

then $detA > 0$.

Lemma 3.3 (Mao and Yuan [26], Theorem 2.10) *If* $A \in Z^{N\times N}$*, then the following statements are equivalent:*

(1) *A is a nonsingular M-matrix.*

(2) *All of the principal minors of A are positive; that is*

$$\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \dots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \dots, N.$$

(3) *A is semi-positive; that is, there exists* $x \gg 0$ *in* R^n *such that* $Ax \gg 0$.

The proof of the stochastic permanence is rather technical, so we prepare several useful lemmas.

Lemma 3.4 *Assumptions 3 and 4 imply that there exists a constant* $\theta > 0$ *such that the matrix*

$$A(\theta) = \mathrm{diag}\left(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_N(\theta)\right) - \Gamma \tag{3.3}$$

is a nonsingular M-matrix, where

$$\xi_k(\theta) = \theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k), \quad k \in S.$$

Proof. It is known that a determinant will not change its value by switching the ith row with the jth row and then switching the ith column with the jth column. It is also known that given a nonsingular M-matrix, if we switch the ith row with the jth row and then switch the ith column with the jth column, then the new matrix is still a nonsingular M-matrix. We may therefore assume $u = N$ without loss of generality, that is

$$\gamma_{kN} > 0 \qquad \forall\ 1 \le k \le N-1$$

instead of Assumption 3. It is easy to see that

$$\begin{aligned}
\det A(\theta) &= \begin{vmatrix} \xi_1(\theta), & -\gamma_{12}, & \ldots, & -\gamma_{1N} \\ \xi_2(\theta), & \xi_2(\theta)-\gamma_{22}, & \ldots, & -\gamma_{2N} \\ \vdots & \vdots & \ldots, & \vdots \\ \xi_{N-1}(\theta), & -\gamma_{N-1,2}, & \ldots, & -\gamma_{N-1,N} \\ \xi_N(\theta), & -\gamma_{N2}, & \ldots, & \xi_N(\theta)-\gamma_{NN} \end{vmatrix} \\
&= \sum_{k=1}^{N} \xi_k(\theta) M_k(\theta),
\end{aligned} \tag{3.4}$$

where $M_k(\theta)$ is the corresponding minor of $\xi_k(\theta)$ in the first column. More precisely,

$$M_1(\theta) = (-1)^{1+1} \begin{vmatrix} \xi_2(\theta)-\gamma_{22}, & \ldots, & -\gamma_{2N} \\ \vdots & \ldots, & \vdots \\ -\gamma_{N-1,2}, & \ldots, & -\gamma_{N-1,N} \\ -\gamma_{N2}, & \ldots, & \xi_N(\theta)-\gamma_{NN} \end{vmatrix},$$

$$\vdots$$

$$M_N(\theta) = (-1)^{N+1} \begin{vmatrix} -\gamma_{12}, & \dots, & -\gamma_{1N} \\ \xi_2(\theta) - \gamma_{22}, & \dots, & -\gamma_{2N} \\ \vdots & \dots, & \vdots \\ -\gamma_{N-1,2}, & \dots, & -\gamma_{N-1,N} \end{vmatrix}.$$

Noting that

$$\xi_k(0) = 0 \quad \text{and} \quad \frac{d}{d\theta}\xi_k(0) = \hat{\beta}(k),$$

we have

$$\frac{d}{d\theta}\det A(0) = \sum_{k=1}^{N} \hat{\beta}(k) M_k(0),$$

which means that

$$\frac{d}{d\theta}\det A(0) = \begin{vmatrix} \hat{\beta}(1), & -\gamma_{12}, & \dots, & -\gamma_{1N} \\ \hat{\beta}(2), & -\gamma_{22}, & \dots, & -\gamma_{2N} \\ \vdots & \vdots & \dots, & \vdots \\ \hat{\beta}(N), & -\gamma_{N2}, & \dots, & -\gamma_{NN} \end{vmatrix}. \tag{3.5}$$

By Appendix A in literature [23], under Assumption 3, Assumption 4 is equivalent to

$$\begin{vmatrix} \hat{\beta}(1), & -\gamma_{12}, & \dots, & -\gamma_{1N} \\ \hat{\beta}(2), & -\gamma_{22}, & \dots, & -\gamma_{2N} \\ \vdots & \vdots & \dots, & \vdots \\ \hat{\beta}(N), & -\gamma_{N2}, & \dots, & -\gamma_{NN} \end{vmatrix} > 0.$$

Together with (3.5), we obtain that

$$\frac{d}{d\theta}\det A(0) > 0. \tag{3.6}$$

It is easy to see that $\det A(0) = 0$. Hence, we can find a $\theta > 0$ sufficiently small for $\det A(\theta) > 0$ and

$$\xi_k(\theta) = \theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k) > -\gamma_{kN}, \quad 1 \le k \le N-1. \tag{3.7}$$

For each $k = 1, 2, \dots, N-1$, consider the leading principle sub-matrix

$$A_k(\theta) := \begin{vmatrix} \xi_1(\theta) - \gamma_{11}, & -\gamma_{12}, & \dots, & -\gamma_{1k} \\ -\gamma_{21}, & \xi_2(\theta) - \gamma_{22}, & \dots, & -\gamma_{2k} \\ \vdots & \vdots & \dots, & \vdots \\ -\gamma_{k1}, & -\gamma_{k2}, & \dots, & \xi_k(\theta) - \gamma_{kk} \end{vmatrix}$$

of $A(\theta)$. Clearly $A_k(\theta) \in Z^{k\times k}$. Moreover, by (3.7) , each row of this sub-matrix has the sum

$$\xi_u(\theta) - \sum_{v=1}^{k} \gamma_{uv} \geq \xi_u(\theta) + \gamma_{uN} > 0.$$

By Lemma 3.2, $\det A_k(\theta) > 0$. In other words, we have shown that all the leading principal minors of $A(\theta)$ are positive. By Lemma 3.3, we obtain the required assertion.

Lemma 3.5 *Assumption 5 imply that there exists a constant $\theta > 0$ such that the matrix $A(\theta)$ is a nonsingular M-matrix.*

Proof. Note that for every $k \in S$,

$$\xi_k(0) = 0 \quad \text{and} \quad \frac{d}{d\theta}\xi_k(0) = \hat{\beta}(k) > 0.$$

we can then choose $\theta > 0$ so small that $\xi_k(\theta) > 0$ for all $1 \leq k \leq N$. Consequently, every row of $A(\theta)$ has a positive sum. By Lemma 3.2, we see easily that all the leading principal minors of $A(\theta)$ are positive. So $A(\theta)$ is a nonsingular M-matrix.

Lemma 3.6 *Let Assumption 1 hold. If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix, then the solution $x(t)$ of the SDE (1.4) with any initial value $x(0) \in R^n_+$ has the property that*

$$\limsup_{t\to\infty} E(\frac{1}{|x(t)|^{\theta}}) \leq H, \tag{3.8}$$

where H is a positive constant.

Proof. By Theorem 2.1, the solution $x(t)$ with initial value $x(0) \in R^n_+$ will remain in R^n_+ with probability one. Define

$$V(x) = \sum_{i=1}^{n} x_i \qquad \text{on} \quad t \geq 0. \tag{3.9}$$

Then

$$dV(x) = x^T\{[b(r(t)) + A(r(t))x]dt + \sigma(r(t))dB(t)\}. \tag{3.10}$$

Define also

$$U(x) = \frac{1}{V(x)} \qquad \text{on} \quad t \geq 0. \tag{3.11}$$

By the generalized Itô formula, we derive from (3.10) that

$$\begin{aligned} dU &= -U^2dV + U^3(dV)^2 \\ &= -U^2x^T\{[b(r(t)) + A(r(t))x]dt + \sigma(r(t))dB(t)\} + U^3|x^T\sigma(r(t))|^2dt \\ &= \{-U^2x^T[b(r(t)) + A(r(t))x] + U^3|x^T\sigma(r(t))|^2\}dt \\ &\quad -U^2x^T\sigma(r(t))dB(t), \end{aligned} \tag{3.12}$$

dropping $x(t)$ from $U(x(t))$, $V(x(t))$ and t from $x(t)$ respectively. By Lemma 3.3, for given θ, there is a vector $\vec{q} = (q_1, \dots, q_N)^T \gg 0$ such that

$$\vec{\lambda} = (\lambda_1, \cdots, \lambda_N)^T := A(\theta)\vec{q} \gg 0,$$

namely,

$$q_k\left(\theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k)\right) - \sum_{l=1}^{N}\gamma_{kl}q_l > 0 \quad \text{for all } 1 \le k \le N. \tag{3.13}$$

Define the function $\bar{V}: R_+^n \times S \to R_+$ by

$$\bar{V}(x,k) = q_k(1+U)^\theta. \tag{3.14}$$

Applying the generalized Itô formula, we have

$$\begin{aligned} L\bar{V}(x,k) &= q_k\theta(1+U)^{\theta-1}\left\{-U^2x^T[b(k) + A(k)x] + U^3|x^T\sigma(k)|^2\right\} \\ &\quad +\frac{1}{2}q_k\theta(\theta-1)(1+U)^{\theta-2}U^4|x^T\sigma(k)|^2 + \sum_{l=1}^{N}\gamma_{kl}q_l(1+U)^\theta \\ &= q_k\theta(1+U)^{\theta-2}\left\{-(1+U)U^2x^T[b(k)+A(k)x] + (1+U)U^3|x^T\sigma(k)|^2 \right. \\ &\quad \left. +\frac{1}{2}(\theta-1)U^4|x^T\sigma(k)|^2\right\} + \sum_{l=1}^{N}\gamma_{kl}q_l(1+U)^\theta. \end{aligned} \tag{3.15}$$

We compute that

$$\begin{aligned} &-(1+U)U^2x^T[b(k)+A(k)x] + (1+U)U^3|x^T\sigma(k)|^2 + \frac{1}{2}(\theta-1)U^4|x^T\sigma(k)|^2 \\ = &-U^2x^Tb(k) - U^3x^Tb(k) - U^2x^TA(k)x - U^3x^TA(k)x \\ &+U^3|x^T\sigma(k)|^2 + \frac{1}{2}(\theta+1)U^4|x^T\sigma(k)|^2 \\ = &-\frac{x^TA(k)x}{V^2} + \left\{-\frac{x^Tb(k)}{V} + \frac{|x^T\sigma(k)|^2 - x^TA(k)x}{V^2}\right\}U \\ &-\left\{\frac{x^Tb(k)}{V} - \frac{1}{2}(\theta+1)\frac{|x^T\sigma(k)|^2}{V^2}\right\}U^2. \end{aligned}$$

It is easy to see that for all $x \in R_+^n$,

$$-\frac{x^TA(k)x}{V^2} \le K_1 \quad \text{and} \quad -\frac{x^Tb(k)}{V} + \frac{|x^T\sigma(k)|^2 - x^TA(k)x}{V^2} \le K_1,$$

where K_1 is a positive constant, while

$$\frac{x^T b(k)}{V} - \frac{1}{2}(\theta+1)\frac{|x^T\sigma(k)|^2}{V^2} \geq \hat{b}(k) - \frac{1}{2}(\theta+1)\breve{\sigma}^2(k) = \hat{\beta}(k) - \frac{1}{2}\theta\breve{\sigma}^2(k), \qquad (3.16)$$

where $\vec{1}^T = (1, \cdots, 1)^T \in R^n_+$. Hence

$$\begin{aligned} & -(1+U)U^2 x^T[b(k) + A(k)x] + (1+U)U^3|x^T\sigma(k)|^2 + \frac{1}{2}(\theta-1)U^4|x^T\sigma(k)|^2 \\ \leq & -\left(\hat{\beta}(k) - \frac{1}{2}\theta\breve{\sigma}^2(k)\right)U^2 + K_1(1+U). \end{aligned}$$

Substituting this into (3.15) yields

$$\begin{aligned} L\bar{V}(x,k) &= q_k\theta(1+U)^{\theta-2}\left\{-\left(\hat{\beta}(k) - \frac{1}{2}\theta\breve{\sigma}^2(k)\right)U^2 + K_1(1+U)\right\} + \sum_{l=1}^N \gamma_{kl}q_l(1+U)^\theta \\ &\leq (1+U)^{\theta-2}\left\{-\left[q_k\left(\theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k)\right) - \sum_{l=1}^N \gamma_{kl}q_l\right]U^2\right. \\ &\quad \left. + \left(q_k\theta K_1 + 2\sum_{l=1}^N \gamma_{kl}q_l\right)U + \left(q_k\theta K_1 + \sum_{l=1}^N \gamma_{kl}q_l\right)\right\}. \qquad (3.17) \end{aligned}$$

Now, choose a constant $\kappa > 0$ sufficiently small such that it satisfies

$$\vec{\lambda} - \kappa\vec{q} \gg 0,$$

i.e.

$$q_k\left(\theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k)\right) - \sum_{l=1}^N \gamma_{kl}q_l - \kappa q_k > 0 \quad \text{for all } 1 \leq k \leq N. \qquad (3.18)$$

Then, by the generalized Itô formula again and (3.18)

$$\begin{aligned} & L\left[e^{\kappa t}\bar{V}(x,k)\right] \\ = & \kappa e^{\kappa t}q_k(1+U)^\theta + e^{\kappa t}L\bar{V}(x,k) \\ \leq & e^{\kappa t}(1+U)^{\theta-2}\left\{\kappa q_k(1+U)^2 - \left[q_k\left(\theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k)\right) - \sum_{l=1}^N \gamma_{kl}q_l\right]U^2\right. \\ & \left. + \left(q_k\theta K_1 + 2\sum_{l=1}^N \gamma_{kl}q_l\right)U + \left(q_k\theta K_1 + \sum_{l=1}^N \gamma_{kl}q_l\right)\right\} \\ = & e^{\kappa t}(1+U)^{\theta-2}\left\{-U^2\left[q_k\left(\theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k)\right) - \sum_{l=1}^N \gamma_{kl}q_l - \kappa q_k\right]\right. \\ & \left. + \left(q_k\theta K_1 + 2\sum_{l=1}^N \gamma_{kl}q_l + 2\kappa q_k\right)U + \left(q_k\theta K_1 + \sum_{l=1}^N \gamma_{kl}q_l + \kappa q_k\right)\right\} \\ \leq & n^{-\theta}\hat{q}\kappa H e^{\kappa t}, \qquad (3.19) \end{aligned}$$

where

$$H = \frac{1}{\hat{q}\kappa} n^\theta \max_{1\le k\le N} \left\{ \sup_{x\in R^+} \left\{ (1+x)^{\theta-2} \left\{ -x^2 \left[q_k \left(\theta\hat{\beta}(k) - \frac{1}{2}\theta^2\breve{\sigma}^2(k) \right) - \sum_{l=1}^N \gamma_{kl} q_l - \kappa q_k \right] + \left(q_k\theta K_1 + 2\sum_{l=1}^N \gamma_{kl} q_l + 2\kappa q_k \right) x + \left(q_k \theta K_1 + \sum_{l=1}^N \gamma_{kl} q_l + \kappa q_k \right) \right\} \right\}, 1 \right\} \quad (3.20)$$

in which we put 1 in order to make H positive. (3.19) implies

$$\limsup_{t\to\infty} E\left[U^\theta(x(t))\right] \le \limsup_{t\to\infty} E\left[(1+U(x(t)))^\theta\right] \le n^{-\theta} H. \quad (3.21)$$

For $x(t) \in R^n_+$, note that

$$\left(\sum_{i=1}^n x_i(t) \right)^\theta \le \left(n \max_{1\le i\le n} x_i(t) \right)^\theta = n^\theta \left(\max_{1\le i\le n} x_i^2(t) \right)^{\frac{\theta}{2}} \le n^\theta |x(t)|^\theta. \quad (3.22)$$

Consequently,

$$\limsup_{t\to\infty} E(\frac{1}{|x(t)|^\theta}) \le H.$$

We obtain the required assertion (3.8) .

Theorem 3.2 *Under Assumptions 2, 3 and 4, the SDE (1.4) is stochastically permanent.*

The proof is a simple application of the Chebyshev inequality, Lemmas 3.4 and 3.6. Similarly, we have the following result.

Theorem 3.3 *Under Assumptions 2 and 5, the SDE (1.4) is stochastically permanent.*

4 Extinction

In the previous sections we have showed that under certain conditions, the original autonomous equations (1.1) and the associated SDE (1.4) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded and permanent. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will show in this section that if the noise is sufficiently large, the solution to the associated SDE (1.4) will become extinct with probability one, although the solution to the original equation (1.1) may be persistent. For example, recall a simple case, namely the scalar logistic equation

$$dN(t) = N(t)(b - aN(t))dt, \;\; t \ge 0. \quad (4.1)$$

It is well known that if $b > 0,\ a > 0$, then its solution $N(t)$ is persistent because

$$\lim_{t\to\infty} N(t) = \frac{a}{b}.$$

However, consider its associated stochastic equation

$$dN(t) = N(t)[(b - aN(t))dt + \sigma dB(t)], \quad t \geq 0, \tag{4.2}$$

where $\sigma > 0$. We will see from the following theorem that if $\sigma^2 > 2b$, then the solution to this stochastic equation will become extinct with probability one, namely

$$\lim_{t\to\infty} N(t) = 0 \qquad \text{a.s.}$$

In other words, the following theorem reveals the important fact that the environmental noise may make the population extinct.

Assumption 6 *Assume that there exist positive numbers $c_1, \cdots, c_n$ such that*

$$-\lambda := \max_{k\in S} \left\{\lambda^+_{max}\left(\bar{C}A(k) + A^T(k)\bar{C}\right)\right\} \leq 0,$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$.

Theorem 4.1 *Let Assumption 6 hold. For any given initial value $x(0) \in R^n_+$, the solution $x(t)$ of the SDE (1.4) has the property that*

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{t} \leq \sum_{k=1}^{N} \pi_k \breve{\beta}(k) \qquad a.s. \tag{4.3}$$

Particularly, if $\sum_{k=1}^{N} \pi_k \breve{\beta}(k) < 0$, then

$$\lim_{t\to\infty} |x(t)| = 0 \qquad a.s.$$

Proof. Define $V(x) = Cx = \sum_{i=1}^{n} c_i x_i,\ x \in R^n_+$, where $C = (c_1, \cdots, c_n)$. By the generalized Itô formula, we have

$$dV(x(t)) = x^T(t)\bar{C}\{[b(r(t)) + A(r(t))x(t)]dt + \sigma(r(t))dB(t)\}.$$

Thus

$$\begin{aligned} d\log V(x(t)) &= \frac{1}{V}dV - \frac{1}{2V^2}(dV)^2 \\ &= \frac{1}{V}x^T\bar{C}\{[b(r(t)) + A(r(t))x]dt + \sigma(r(t))dB(t)\} \\ &\quad -\frac{1}{2V^2}|x^T\bar{C}\sigma(r(t))|^2 dt, \end{aligned} \tag{4.4}$$

dropping $x(t)$ from $V(x(t))$ and t from $x(t)$ respectively. We compute

$$\frac{x^T\bar{C}A(r(t))x}{V} = \frac{x^T[\bar{C}A(r(t)) + A^T(r(t))\bar{C}]x}{2V} \le -\frac{\lambda|x|^2}{2V} \le -\frac{\lambda}{2|C|}|x| \le 0.$$

We compute also

$$\frac{x^T\bar{C}b(r(t))}{V} - \frac{|x^T\bar{C}\sigma(r(t))|^2}{2V^2} \le \breve{b}(r(t)) - \frac{1}{2}\hat{\sigma}^2(r(t)) = \breve{\beta}(r(t)). \tag{4.5}$$

Substituting these two inequalities into (4.4) yields

$$d\log V(x(t)) \le \breve{\beta}(r(t))dt + \frac{x^T(t)\bar{C}\sigma(r(t))}{V(x(t))}dB(t).$$

This implies

$$\log V(x(t)) \;\; \le \;\; \log V(x(0)) + \int_0^t \breve{\beta}(r(s))ds + M(t), \tag{4.6}$$

where $M(t)$ is a martingale defined by

$$M(t) = \int_0^t \frac{x^T(s)\bar{C}\sigma(r(s))}{V(x(s))}dB(s).$$

The quadratic variation of this martingale is

$$\langle M, M\rangle_t = \int_0^t \frac{|x^T(s)\bar{C}\sigma(r(s))|^2}{V^2(x(s))}ds \le \frac{\breve{\sigma}^2|\bar{C}|^2}{\hat{c}^2}t.$$

By the strong law of large numbers for martingales (see [22], [26]), we therefore have

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0 \quad \text{a.s.}$$

It finally follows from (4.6) by dividing t on the both sides and then letting $t \to \infty$ that

$$\limsup_{t\to\infty}\frac{\log V(x(t))}{t} \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t \breve{\beta}(r(s))ds = \sum_{k=1}^N \pi_k\breve{\beta}(k) \quad \text{a.s.}$$

which implies the required assertion (4.3) .

5 Asymptotic Boundedness of Integral Average

Lemma 5.1 *Under Assumption 2, for any given initial value $x(0) \in R_+^n$, the solution $x(t)$ of the SDE (1.4) has the property that*

$$\limsup_{t\to\infty}\frac{\log(|x(t)|)}{\log t} \le 1 \quad a.s. \tag{5.1}$$

The proof is somehow standard so we only give a brief one in Appendix B.

Lemma 5.2 *Let Assumption 1 hold. If there exists a constant* $\theta > 0$ *such that* $A(\theta)$ *is a nonsingular M-matrix, then the solution* $x(t)$ *of the SDE (1.4) with any initial value* $x(0) \in R^n_+$ *has the property that*

$$\liminf_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \geq -\frac{1}{\theta} \qquad a.s \tag{5.2}$$

Proof. Let $U : R^n_+ \to R_+$ be the same as defined by (3.11) , for convenience, we write $U(x(t)) = U(t)$. Applying the generalized Itô formula, for the fixed constant $\theta > 0$, we derive from (3.17) that

$$\begin{aligned} & d\left[(1+U(t))^{\theta}\right] \\ \leq \ & \theta(1+U(t))^{\theta-2}\left\{-\left(\hat{\beta}(r(t)) - \frac{1}{2}\theta\breve{\sigma}^2(r(t))\right)U^2(t) + K_1U(t) + K_1\right\}. \\ & -\theta(1+U(t))^{\theta-1}U^2(t)x^T\sigma(r(t))dB(t). \end{aligned} \tag{5.3}$$

Under given condition, by (3.21) of Lemma 3.6, there exists a positive constant M such that

$$E\left[(1+U(t))^{\theta}\right] \leq M \quad \text{ on } \ t \geq 0. \tag{5.4}$$

Let $\delta > 0$ be sufficiently small for

$$\theta\left[(\hat{\beta} + \frac{1}{2}\theta\breve{\sigma}^2 + K_1)\delta + \frac{3}{\hat{c}}\max_{k\in S}\{|\sigma(k)|\}\delta^{\frac{1}{2}}\right] < \frac{1}{2}. \tag{5.5}$$

Let $k = 1, 2, \cdots$. (5.3) implies that

$$\begin{aligned} & E\left[\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}\right] \\ \leq \ & E\left[(1+U((k-1)\delta))^{\theta}\right] \\ + \ & E\left(\sup_{(k-1)\delta\leq t\leq k\delta}|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-2}\right. \\ & \quad \times\left.\left\{-\left(\hat{\beta}(r(s)) - \frac{1}{2}\theta\breve{\sigma}^2(r(s))\right)U^2(s) + K_1(U(s)+1)\right\}ds|\right) \\ + \ & E\left(\sup_{(k-1)\delta\leq t\leq k\delta}|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-1}U^2(s)x^T(s)\sigma(r(s))dB(s)|\right). \end{aligned} \tag{5.6}$$

We compute

$$E\left(\sup_{(k-1)\delta\leq t\leq k\delta}|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-2}\right.$$

$$\times \left\{-\left(\hat{\beta}(r(s))-\frac{1}{2}\theta\breve{\sigma}^2(r(s))\right)U^2(s)+K_1(U(s)+1)\right\}ds|\Bigg)$$

$$
\begin{aligned}
&\leq E\left(\int_{(k-1)\delta}^{k\delta}|\theta(1+U(s))^{\theta-2}\left\{-\left(\hat{\beta}-\frac{1}{2}\theta\breve{\sigma}^2\right)U^2(s)+K_1U(s)+K_1\right\}|ds\right)\\
&\leq \theta E\left(\int_{(k-1)\delta}^{k\delta}(\hat{\beta}+\frac{1}{2}\theta\breve{\sigma}^2+K_1)(1+U(s))^{\theta}ds\right)\\
&\leq \theta(\hat{\beta}+\frac{1}{2}\theta\breve{\sigma}^2+K_1)E\left(\int_{(k-1)\delta}^{k\delta}\sup_{(k-1)\delta\leq s\leq k\delta}(1+U(s))^{\theta}ds\right)\\
&\leq \theta(\hat{\beta}+\frac{1}{2}\theta\breve{\sigma}^2+K_1)\delta E\left(\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}\right) \qquad (5.7)
\end{aligned}
$$

By the well-known Burkholder-Davis-Gundy inequality, we derive that

$$
\begin{aligned}
&E\left(\sup_{(k-1)\delta\leq t\leq k\delta}|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-1}U^2(s)x^T(s)\sigma(r(s))dB(s)|\right)\\
&\leq 3E\left(\int_{(k-1)\delta}^{k\delta}\left(\theta(1+U(s))^{\theta-1}U^2(s)\right)^2\left|x^T(s)\sigma(r(s))\right|^2ds\right)^{\frac{1}{2}}\\
&\leq 3\theta E\left(\int_{(k-1)\delta}^{k\delta}(1+U(s))^{2(\theta-1)}U^2(s)\frac{|x(s)|^2|\sigma(r(s))|^2}{\hat{c}^2|x(s)|^2}ds\right)^{\frac{1}{2}}\\
&\leq \frac{3}{\hat{c}}\theta\max_{k\in S}\{|\sigma(k)|\}E\left(\int_{(k-1)\delta}^{k\delta}(1+U(s))^{2\theta}ds\right)^{\frac{1}{2}}\\
&\leq \frac{3}{\hat{c}}\theta\max_{k\in S}\{|\sigma(k)|\}\delta^{\frac{1}{2}}E\left(\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{2\theta}\right)^{\frac{1}{2}}\\
&\leq \frac{3}{\hat{c}}\theta\max_{k\in S}\{|\sigma(k)|\}\delta^{\frac{1}{2}}E\left(\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}\right).
\end{aligned}
$$

Substituting this and (5.7) into (5.6) gives

$$
\begin{aligned}
&E\left[\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}\right]\leq E\left[(1+U((k-1)\delta))^{\theta}\right]\\
&+\theta\left[(\hat{\beta}+\frac{1}{2}\theta\breve{\sigma}^2+K_1)\delta+\frac{3}{\hat{c}}\max_{k\in S}\{|\sigma(k)|\}\delta^{\frac{1}{2}}\right]E\left(\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}\right). \qquad (5.8)
\end{aligned}
$$

Make use of (5.4) and (5.5) we obtain that

$$E\left[\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}\right]\leq 2M. \qquad (5.9)$$

Let $\epsilon>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$P\left\{\omega:\sup_{(k-1)\delta\leq t\leq k\delta}(1+U(t))^{\theta}>(k\delta)^{1+\epsilon}\right\}\leq\frac{2M}{(k\delta)^{1+\epsilon}},\qquad k=1,2,\ldots$$

Applying the well-known Borel-Cantelli lemma (see e.g. [22], [26]), we obtain that for almost all $\omega \in \Omega$

$$\sup_{(k-1)\delta \le t \le k\delta} (1+U(t))^\theta \le (k\delta)^{1+\epsilon} \tag{5.10}$$

holds for all but finitely many k. Hence, there exists an integer $k_0(\omega) > 1/\delta + 2$, for almost all $\omega \in \Omega$, for which (5.10) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \ge k_0$ and $(k-1)\delta \le t \le k\delta$,

$$\frac{\log(1+U(t))^\theta}{\log t} \le \frac{(1+\epsilon)\log(k\delta)}{\log((k-1)\delta)} = 1+\epsilon.$$

Therefore

$$\limsup_{t\to\infty} \frac{\log(1+U(t))^\theta}{\log t} \le 1+\epsilon. \quad \text{a.s.}$$

Letting $\epsilon \to 0$, we obtain the desired assertion

$$\limsup_{t\to\infty} \frac{\log(1+U(t))^\theta}{\log t} \le 1. \quad \text{a.s.}$$

Recalling the definition of $U(t)$, we yield

$$\limsup_{t\to\infty} \frac{\log\left(\frac{1}{|x(t)|^\theta}\right)}{\log t} \le 1 \quad \text{a.s.}$$

which further implies

$$\liminf_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \ge -\frac{1}{\theta} \quad \text{a.s.}$$

This is our required assertion (5.2).

Assumption 7 *Assume that there exist positive numbers $c_1, \cdots, c_n$ such that*

$$-\lambda := \max_{k\in S} \left\{\lambda_{max}^{+}\left(\bar{C}A(k) + A^T(k)\bar{C}\right)\right\} < 0,$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$.

Theorem 5.1 *Under Assumptions 3, 4 and 7, for any initial value $x(0) \in R_+^n$, the solution $x(t)$ of the SDE (1.4) obeys*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \frac{2|C|}{\lambda}\sum_{k=1}^N \pi_k \breve{\beta}(k) \quad a.s. \tag{5.11}$$

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t |x(s)|ds \ge \frac{2\hat{c}}{\hat{\lambda}}\sum_{k=1}^N \pi_k \hat{\beta}(k) \quad a.s. \tag{5.12}$$

where

$$-\hat{\lambda} := \min_{k\in S} \left\{\lambda_{max}^{+}\left(\bar{C}A(k) + A^T(k)\bar{C}\right)\right\} < 0. \tag{5.13}$$

Proof. Define

$$V(x) = Cx = \sum_{i=1}^{n} c_i x_i \quad \text{for} \quad x \in R_+^n.$$

By the generalized Itô formula, we have

$$dV(x(t)) = x^T(t)\bar{C}\{[b(r(t)) + A(r(t))x(t)]dt + \sigma(r(t))dB(t)\}. \tag{5.14}$$

It is easy to observe from the inequality (5.1) of Lemma 5.1 and (5.2) of Lemma 5.2 that

$$\lim_{t\to+\infty} \frac{\log V(x(t))}{t} = 0 \quad \text{a.s.} \tag{5.15}$$

We derive from (5.14) that

$$\begin{aligned} d\log V(x(t)) &= \frac{1}{V(x(t))} x^T(t)\bar{C}\{[b(r(t)) + A(r(t))x(t)]dt + \sigma(r(t))dB(t)\} \\ &\quad - \frac{1}{2V^2(x(t))}|x^T(t)\bar{C}\sigma(r(t))|^2 dt. \end{aligned} \tag{5.16}$$

We compute

$$-\frac{\hat{\lambda}}{2\hat{c}}|x| \le \frac{x^T\bar{C}A(r(t))x}{V(x)} = \frac{x^T[\bar{C}A(r(t)) + A^T(r(t))\bar{C}]x}{2V(x)} \le -\frac{\lambda}{2|C|}|x| < 0. \tag{5.17}$$

By (3.16) and (4.5) , we know

$$\hat{\beta}(r(t)) \le \frac{x^T\bar{C}b(r(t))}{V(x)} - \frac{|x^T\bar{C}\sigma(r(t))|^2}{2V^2(x)} \le \breve{\beta}(r(t)). \tag{5.18}$$

Substituting these into (5.16) yields

$$d\log V(x(t)) \le \breve{\beta}(r(t))dt - \frac{\lambda}{2|C|}|x(t)|dt + \frac{x^T(t)\bar{C}\sigma(r(t))}{V(x(t))}dB(t).$$

Hence

$$\log V(x(t)) + \frac{\lambda}{2|C|}\int_0^t |x(s)|ds \le \log V(x(0)) + \int_0^t \breve{\beta}(r(s))ds + \int_0^t \frac{x^T(s)\bar{C}\sigma(r(s))}{V(x(s))}dB(s). \tag{5.19}$$

By the strong law of large numbers for martingales (see [22], [26]), we therefore have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{x^T(s)\bar{C}\sigma(r(s))}{V(x(s))}dB(s) = 0 \quad \text{a.s.}$$

We can therefore divide both sides of (5.19) by t and then let $t \to \infty$ to obtain

$$\frac{\lambda}{2|C|}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t |x(s)|ds \le \sum_{k=1}^{N} \pi_k \breve{\beta}(k) \quad \text{a.s.}$$

which implies the required assertion (5.11).

On the other hand, we observe from (5.16), (5.17) and (5.18) that

$$d\log V(x(t)) \geq \hat{\beta}(r(t))dt - \frac{\hat{\lambda}}{2\hat{c}}|x(t)|dt + \frac{x^T(t)\bar{C}\sigma(r(t))}{V(x)}dB(t). \tag{5.20}$$

Hence

$$\log V(x(t)) + \frac{\hat{\lambda}}{2\hat{c}}\int_0^t |x(s)|ds \geq \log V(x(0)) + \int_0^t \hat{\beta}(r(s))ds + \int_0^t \frac{x^T(s)\bar{C}\sigma(r(s))}{V(x(s))}dB(s).$$

So we have

$$\frac{\hat{\lambda}}{2\hat{c}}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t |x(s)|ds \geq \sum_{k=1}^N \pi_k\hat{\beta}(k) \quad \text{a.s.}$$

which implies the other required assertion (5.12).

Similarly, using Lemmas 3.5, 5.1 and 5.2, we can show:

Theorem 5.2 *Under Assumptions 5 and 7, for any initial value $x(0) \in R^n_+$, the solution $x(t)$ of the SDE (1.4) obeys*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t |x(s)|ds \leq \frac{2|C|}{\lambda}\sum_{k=1}^N \pi_k\breve{\beta}(k) \quad a.s. \tag{5.21}$$

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t |x(s)|ds \geq \frac{2\hat{c}}{\hat{\lambda}}\sum_{k=1}^N \pi_k\hat{\beta}(k) \quad a.s. \tag{5.22}$$

6 Conclusions and Examples

Let Assumptions 3 and 7 hold. It is interesting to point out that if $\hat{\beta}(k) > 0$ for some $k \in S$, then the equation

$$dx(t) = \text{diag}(x_1(t), \ldots, x_n(t))[(b(k) + A(k)x(t))dt + \sigma(k)dB(t)] \tag{6.1}$$

is stochastically permanent. Hence Theorems 3.3 tells us if every individual equation

$$dx(t) = \text{diag}(x_1(t), \ldots, x_n(t))[(b(k) + A(k)x(t))dt + \sigma(k)dB(t)] \tag{6.2}$$

is stochastically permanent, then as the result of Markovian switching, the overall behavior, i.e. the SDE (1.4) remains stochastically permanent. On the other hand, if $\breve{\beta}(k) < 0$ for some $k \in S$, then equation (6.1) is extinctive. Hence Theorem 4.1 tell us if every individual equation (6.2) is extinctive, then as the result of Markovian switching, the overall behavior, i.e. the SDE (1.4) remains extinctive. However, Theorems 3.2 and 4.1 tell us a more interesting result that some individuals in (6.2) are stochastically permanent while some are extinctive, but as the results of Markovian switching, the overall behavior, i.e. the SDE (1.4) may be stochastically permanent or extinctive which depends on the sign of $\sum_{k=1}^{n} \pi_k \hat{\beta}(k)$ and $\sum_{k=1}^{n} \pi_k \breve{\beta}(k)$ respectively. Moreover, if the SDE (1.4) is stochastically permanent, the limit of the average in time of the sample path of the solution could be estimated making use of Theorems 5.1 and 5.2. We shall illustrate these conclusions through the following examples.

Example 6.1 First of all, let us consider the following one-dimensional logistic system with regime switching

$$dN(t) = N(t)[(b(r(t)) - a(r(t))N(t))dt + \sigma(r(t))dB(t)], \quad t \geq 0, \tag{6.3}$$

where $r(t)$ is a right-continuous Markov chain taking value in $S = \{1,\ 2,\ 3\}$. As pointed out in Section 1, we may regard the SDE (6.3) as the result of the following three equations switching from one to another according to the movement of the Markovian chain:

$$dN(t) = N(t)[(b(1) - a(1)N(t))dt + \sigma(1)dB(t)], \tag{6.4}$$

where $b(1) = 11,\ a(1) = 1,\ \sigma(1) = 2$;

$$dN(t) = N(t)[(b(2) - a(2)N(t))dt + \sigma(2)dB(t)], \tag{6.5}$$

where $b(2) = 1,\ a(2) = \frac{1}{2},\ \sigma(2) = 2\sqrt{2}$;

$$dN(t) = N(t)[(b(3) - a(3)N(t))dt + \sigma(3)dB(t)], \tag{6.6}$$

where $b(3) = 3,\ a(3) = \frac{1}{3},\ \sigma(3) = \sqrt{14}$. Compute

$$\hat{\beta}(1) = \breve{\beta}(1) = 9 > 0, \quad \hat{\beta}(2) = \breve{\beta}(2) = -3 < 0, \quad \hat{\beta}(3) = \breve{\beta}(3) = -4 < 0;$$

$$\lambda = \frac{2}{3} > 0, \quad \hat{\lambda} = 2 > 0.$$

We observe that the SDE (6.4)(blue) is stochastically permanent while the SDEs (6.5) (red) and (6.6) (green) are extinctive, see Figure 1. To see how the Markovian switing affect the system, let us discuss two cases.

Case 1. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Figure 1. SDE (6.4) (blue): $N(0) = 5$; SDE (6.5)(red): $N(0) = 13$; SDE(6.6) (green): $N(0) = 10$.

Figure 2. $N(0) = 2,\ r(0) = 3$.

By solving the linear equation (2.1) we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = (\frac{7}{15}, \frac{1}{5}, \frac{1}{3}).$$

Then

$$\sum_{k=1}^{3} \pi_k \hat{\beta}(k) = \frac{34}{15} > 0.$$

Therefore, by Theorems 3.2 and 5.1, as the result of Markovian switching, the overall behavior, i.e. the SDE (6.3) is stochastically permanent, see Figure 2, and its solution $N(t)$ with any positive initial value has the following property:

$$\frac{34}{15} \leq \liminf_{t\to+\infty} \frac{1}{t}\int_0^t N(s)ds \leq \limsup_{t\to+\infty} \frac{1}{t}\int_0^t N(s)ds \leq \frac{34}{5} \qquad \text{a.s.}$$

Case 2. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -5 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{pmatrix}.$$

Figure 3. $N(0) = 5,\ r(0) = 3.$

By solving the linear equation (2.1) we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = (\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}).$$

Then

$$\sum_{k=1}^{3} \pi_k \breve{\beta}(k) = -\frac{1}{4} < 0.$$

Therefore, by Theorems 4.1, as the result of Markovian switching, the overall behavior, i.e. the SDE (6.3) is extinctive, see Figure 3.

Example 6.2 Consider the two-species Lotka-Volterra system of facultative mutualism with regime switching described by

$$\begin{cases} dx_1(t) &= x_1(t)\,[(b_1(r(t)) - a_{11}(r(t))x_1(t) + a_{12}(r(t))x_2(t))\,dt + \sigma_1(r(t))dB(t)] \\ dx_2(t) &= x_2(t)\,[(b_2(r(t)) + a_{21}(r(t))x_1(t) - a_{22}(r(t))x_2(t))\,dt + \sigma_2(r(t))dB(t)] \end{cases} \tag{6.7}$$

for $t \geq 0$, where $r(t)$ is a right-continuous Markov chain taking values in $S = \{1,\ 2\}$. As pointed out in Section 1, we may regard the SDE (6.7) as the result of the following two equations:

$$\begin{cases} dx_1(t) &= x_1(t)\,[(b_1(1) - a_{11}(1)x_1(t) + a_{12}(1)x_2(t))\,dt + \sigma_1(1)dB(t)] \\ dx_2(t) &= x_2(t)\,[(b_2(1) + a_{21}(1)x_1(t) - a_{22}(1)x_2(t))\,dt + \sigma_2(1)dB(t)] \end{cases} \tag{6.8}$$

and

$$\begin{cases} dx_1(t) &= x_1(t)\,[(b_1(2) - a_{11}(2)x_1(t) + a_{12}(2)x_2(t))\,dt + \sigma_1(2)dB(t)] \\ dx_2(t) &= x_2(t)\,[(b_2(2) + a_{21}(2)x_1(t) - a_{22}(2)x_2(t))\,dt + \sigma_2(2)dB(t)] \end{cases} \tag{6.9}$$

switching from one to the other according to the movement of the Markovian chain $r(t)$.

Assume that

$$
\begin{aligned}
&b_1(1) = 5, \quad a_{11}(1) = 2, \quad a_{12}(1) = 1, \quad \sigma_1(1) = \sqrt{2};\\
&b_2(1) = 8, \quad a_{21}(1) = 6, \quad a_{22}(1) = 2, \quad \sigma_2(1) = 2;\\
&b_1(2) = 4, \quad a_{11}(2) = 1, \quad a_{12}(2) = 0, \quad \sigma_1(2) = \sqrt{14};\\
&b_2(2) = 5, \quad a_{21}(2) = 1, \quad a_{22}(2) = 2, \quad \sigma_2(2) = 4.
\end{aligned}
$$

Let $\bar{C} = I \in R^{2\times 2}$ and compute

$$
\lambda_{1,2}\left(IA(1) + A^T(1)I\right) = -5 \pm \sqrt{10}, \qquad \lambda_{1,2}\left(IA(2) + A^T(2)I\right) = -3 \pm \sqrt{2}.
$$

Then

$$
-\lambda = \lambda^+_{max}\left(IA(1) + A^T(1)I\right) \le -3+\sqrt{2} < 0, \quad -\hat{\lambda} = \lambda^+_{min}\left(IA(2) + A^T(2)I\right) \ge -5-\sqrt{10},
$$

whence Assumption 2 holds. Moreover,

$$
\lambda \ge 3 - \sqrt{2}, \qquad \hat{\lambda} \le 5 + \sqrt{10},
$$

and

$$
\hat{\beta}(1) = 3, \quad \hat{\beta}(2) = -4, \ \breve{\beta}(1) = 7, \quad \breve{\beta}(2) = -2.
$$

To see if the SDE (6.7) is stochastically permanent or extinctive, we consider two cases:

Case 1. Let the generator of the Markov chain $r(t)$ be

$$
\Gamma = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}.
$$

It is easy to see that the Markov chain has its stationary probability distribution $\pi = (\pi_1, \pi_2) = (\frac{3}{5}, \frac{2}{5})$. We observe that the SDE (6.8) is stochastically permanent while the SDE (6.9) is extinctive. However, as the result of Markovian switching, the overall behavior, i.e. the SDE (6.7) will be stochastically permanent noting that

$$
\sum_{k=1}^{2} \pi_k \hat{\beta}(k) = \frac{3}{5} \times 3 + \frac{2}{5} \times (-4) = \frac{1}{5} > 0.
$$

$$
\sum_{k=1}^{2} \pi_k \breve{\beta}(k) = \frac{3}{5} \times 7 + \frac{2}{5} \times (-2) = \frac{17}{5}.
$$

Moreover, by Theorem 5.1, the solution $x(t)$ with any initial value $x(0) \in R^n_+$ has the following property:

$$
\frac{2}{5(5+\sqrt{10})} \le \liminf_{t\to+\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \limsup_{t\to+\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \frac{34\sqrt{2}}{5(3-\sqrt{2})} \qquad \text{a.s.}
$$

Case 2. Assume that the generator of the Markov chain $r(t)$ is

$$\Gamma = \begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix}.$$

It is easy to see that the Markov chain has its stationary probability distribution $\pi = (\pi_1, \pi_2) = (\frac{1}{5}, \frac{4}{5})$. As the result of Markovian switching, the overall behavior, i.e. the SDE (6.7) will extinct almost surely because

$$\sum_{k=1}^{2} \pi_k \breve{\beta}(k) = \frac{1}{5} \times 7 + \frac{4}{5} \times (-2) = -\frac{1}{5} < 0.$$

Example 6.3 Consider the two-species Lotka-Volterra competitive system with regime switching described by

$$\begin{cases} dx_1(t) &= x_1(t)\left[(b_1(r(t)) - a_{11}(r(t))x_1(t) - a_{12}(r(t))x_2(t))\,dt + \sigma_1(r(t))dB(t)\right] \\ dx_2(t) &= x_2(t)\left[(b_2(r(t)) - a_{21}(r(t))x_1(t) - a_{22}(r(t))x_2(t))\,dt + \sigma_2(r(t))dB(t)\right] \end{cases} \tag{6.10}$$

for $t \geq 0$. Assume that the Markov chain $r(t)$ is on the state space $S = \{1,\ 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.$$

It is easy to see that the Markov chain has its stationary probability distribution $\pi = (\pi_1, \pi_2) = (\frac{1}{4}, \frac{3}{4})$. As pointed out in Section 1, we may regard the SDE (6.10) as the result of the following two equations:

$$\begin{cases} dx_1(t) &= x_1(t)\left[(b_1(1) - a_{11}(1)x_1(t) - a_{12}(1)x_2(t))\,dt + \sigma_1(1)dB(t)\right] \\ dx_2(t) &= x_2(t)\left[(b_2(1) - a_{21}(1)x_1(t) - a_{22}(1)x_2(t))\,dt + \sigma_2(1)dB(t)\right] \end{cases} \tag{6.11}$$

and

$$\begin{cases} dx_1(t) &= x_1(t)\left[(b_1(2) - a_{11}(2)x_1(t) - a_{12}(2)x_2(t))\,dt + \sigma_1(2)dB(t)\right] \\ dx_2(t) &= x_2(t)\left[(b_2(2) - a_{21}(2)x_1(t) - a_{22}(2)x_2(t))\,dt + \sigma_2(2)dB(t)\right] \end{cases} \tag{6.12}$$

switching from one to the other according to the movement of the Markovian chain $r(t)$.

Case 1. Assume that

$$b_1(1) = 9, \quad a_{11}(1) = 4, \quad a_{12}(1) = 2, \quad \sigma_1(1) = 2;$$

$$b_2(1) = 10, \quad a_{21}(1) = 6, \quad a_{22}(1) = 3, \quad \sigma_2(1) = 1;$$

$$b_1(2) = 4, \quad a_{11}(2) = 2, \quad a_{12}(2) = \frac{3}{2}, \quad \sigma_1(2) = 3;$$

$$b_2(2) = 6, \quad a_{21}(2) = 3, \quad a_{22}(2) = 1, \quad \sigma_2(2) = 2\sqrt{3}.$$

Then we know

$$\hat{\beta}(1) = 7,\ \hat{\beta}(2) = -2,\ \breve{\beta}(1) = \frac{19}{2},\ \breve{\beta}(2) = \frac{3}{2};$$
$$\hat{b}(1) = 9,\ \hat{b}(2) = 4,\ \breve{b}(1) = 10,\ \breve{b}(2) = 6,\ \hat{a} = 1,\ \breve{a} = 6.$$

The Appendix in [3] tell us that for a matrix $D = (d_{ij})_{n\times n}$

$$\lambda^+_{\max}(D) \le \max_{1\le i\le n}\left(d_{ii} + \sum_{j\ne i}(0 \vee d_{ij})\right); \qquad \lambda^+_{\max}(D) \ge \min_{1\le i\le n}\left(d_{ii} + \sum_{j\ne i}(0 \wedge d_{ij})\right).$$

Let $\bar{C} = I \in R^{2\times 2}$, then we know

$$\lambda \ge 4, \qquad \hat{\lambda} \le 36.$$

We observe that the SDE (6.11) is stochastically permanent while we are not sure that the SDE (6.12) is stochastically permanent or extinctive. However, as the result of Markovian switching, the overall behavior, i.e. the SDE (6.10) will be stochastically permanent noting that

$$\sum_{k=1}^{2} \pi_k \hat{\beta}(k) = \frac{1}{4} \times 7 + \frac{3}{4} \times (-2) = \frac{1}{4} > 0$$

and

$$\sum_{k=1}^{2} \pi_k \breve{\beta}(k) = \frac{1}{4} \times \frac{19}{2} + \frac{3}{4} \times \frac{3}{2} = \frac{7}{2}.$$

Moreover, by Theorem 5.1, the solution $x(t)$ with any initial value $(x(0), r(0)) \in R^2_+$ has the following property:

$$\frac{1}{72} \le \liminf_{t\to+\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \limsup_{t\to+\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \frac{7\sqrt{2}}{4} \qquad \text{a.s.}$$

Case 2. Assume that

$$b_1(1) = 6,\ \ a_{11}(1) = 1,\ \ a_{12}(1) = 3,\ \ \sigma_1(1) = \sqrt{2};$$
$$b_2(1) = 5,\ \ a_{21}(1) = 0,\ \ a_{22}(1) = 2,\ \ \sigma_2(1) = 2;$$
$$b_1(2) = 1,\ \ a_{11}(2) = 0,\ \ a_{12}(2) = 4,\ \ \sigma_1(2) = 3;$$
$$b_2(2) = 1,\ \ a_{21}(2) = 2,\ \ a_{22}(2) = 3,\ \ \sigma_2(2) = \sqrt{6}.$$

Then we know

$$\hat{\beta}(1) = 3,\ \hat{\beta}(2) = -\frac{7}{2},\ \breve{\beta}(1) = 5,\ \breve{\beta}(2) = -2.$$

We observe that the SDE (6.11) is stochastic permanent while the SDE (6.12) is extinctive. However, noting that

$$\sum_{k=1}^{2} \pi_k \breve{\beta}(k) = \frac{1}{4} \times 5 + \frac{3}{4} \times (-2) = -\frac{1}{4} < 0,$$

as the result of Markovian switching, the overall behavior, i.e. the SDE (6.10) will be extinctive by Theorem 4.1.

7 Application to Stochastic Harvest

It is clearly necessary to develop an ecologically optimal strategy for harvesting any renewable resource be it animals, fish, plants or whatever. Clark in [5, 6] introduced some important economic constraints and examples in population models of renewable resource. The collection of papers edited by Vincent and Skowronski [32] specially deals with renewable resource management. The results about the harvesting policy of resources, which has a direct relationship to sustainable development, are increasing, for example, see reference [4, 5, 6, 8, 29]. In the real world the natural growth of every renewable population has itself rule and is always affected inevitably by some random disturbance. Therefore we discuss an harvesting policy of single population modeled by randomized logistic equation

$$dN(t) = N(t)[(b - aN(t))dt + \sigma dB(t)], \quad t \geq 0, \tag{7.1}$$

which has a direct relationship to sustainable development, where b, a, σ are positive constants. Following Clark [5], we also assume that the harvest rate for the population would be proportional to its stock level $N(t)$. Thus, as a result of harvesting, the population growth obeys

$$dN(t) = N(t)[(b - h - aN(t))dt + \sigma dB(t)], \quad t \geq 0, \tag{7.2}$$

where $h > 0$ is the harvesting effort. However, whether to harvest is depend on weather factors, such as wind power, wave height. Therefore, the population switches between harvest regime and unharvest regime. The switching is memoryless and the waiting time for the next switch has an exponential distribution. The population system under regime switching can therefore be described by the following stochastic model

$$dN(t) = N(t)[(b(r(t)) - aN(t))dt + \sigma dB(t)], \quad t \geq 0, \tag{7.3}$$

where $b(1) = b,\ b(2) = b - h,\ r(t)$ is a right-continuous Markov chain taking value in $S = \{1,\ 2\}$. Assume that the Markov chain has the stationary distribution (π_1, π_2).

By Theorem 3.2, we see that given $b - \frac{\sigma^2}{2} > 0$, as the results of harvesting, the SDE (7.3) may be stochastically permanent or extinctive dependent on the power of harvesting effort h. More precisely, if $h < \frac{1}{\pi_2}(b - \frac{\sigma^2}{2})$ the population will develop stochastically permanently, while if $h > \frac{1}{\pi_2}(b - \frac{\sigma^2}{2})$ the population will be extinctive. Moreover, if the SDE (7.3) is stochastically permanent, the limit of the average in time of the sample path of the solution could be estimated by using Theorems 5.1 and 5.2.

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Appendix

A Proof of Theorem 2.1

Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $x(0) \in R_+^n$ there is an unique maximal local solution $x(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (cf. [26]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficiently large for every component of $x(0)$ lying within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\{t \in [0,\ \tau_e) : x_i(t) \notin (\frac{1}{m},\ m) \ \text{ for some } \ i = 1,\ \cdots,\ n\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m \to \infty} \tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+^n$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, there is a pair of constants $T > 0$ and $\epsilon \in (0,\ 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \epsilon \qquad \text{for all} \quad m \geq m_1. \tag{A.1}$$

Define a C^2-function $V : R_+^n \times S \to R_+$ by

$$V(x, k) = \sum_{i=1}^{n} c_i(k)[x_i - 1 - \log(x_i)].$$

The nonnegativity of this function can be seen from

$$u - 1 - \log(u) \geq 0 \qquad \text{on} \ \ u > 0.$$

If $x(t) \in R^n_+$, we compute that

$$
\begin{aligned}
LV(x,k) &= x^T\bar{C}(k)b(k) + x^T\bar{C}(k)A(k)x - C(k)[b(k)+A(k)x] \\
&\quad + \frac{1}{2}[\sigma^T(k)\bar{C}(k)\sigma(k)] + \sum_{l=1}^{N}\gamma_{kl}V(x,l) \\
&\le -\frac{1}{2}\lambda|x|^2 + x^T\bar{C}(k)b(k) - C(k)A(k)x - C(k)b(k) \\
&\quad + \frac{1}{2}[\sigma^T(k)\bar{C}(k)\sigma(k)] + \sum_{l=1}^{N}\gamma_{kl}V(x,l), \qquad \text{(A.2)}
\end{aligned}
$$

where we write $x(t) = x$ and $C(k) = (c_1(k), \cdots, c_n(k))$. Moreover, there is clearly a constant $K_1^* > 0$ such that

$$
\max_{k\in S}\left\{x^T\bar{C}(k)b(k) - C(k)A(k)x - C(k)b(k) + \frac{1}{2}[\sigma^T(k)\bar{C}(k)\sigma(k)]\right\} \le K_1^*(1+|x|).
$$

Substituting this into (A.2) yields

$$
LV(x,k) \le K_1^*(1+|x|) + \sum_{l=1}^{N}\gamma_{kl}V(x,l). \qquad \text{(A.3)}
$$

Noticing that $u \le 2(u-1-\log u)+2$ on $u>0$, we compute

$$
\begin{aligned}
|x| &\le \sum_{i=1}^{n} x_i \le \sum_{i=1}^{n}[2(x_i - 1 - \log x_i) + 2] \\
&\le 2n + \frac{2}{\hat{c}}\sum_{i=1}^{n} c_i(k)(x_i - 1 - \log x_i) \\
&= 2n + \frac{2}{\hat{c}}V(x,k). \qquad \text{(A.4)}
\end{aligned}
$$

Let

$$
\breve{q} = \max\left\{\frac{c_i(k)}{c_i(l)} : \ 1 \le i \le n, 1 \le k, \ l \le N\right\}. \qquad \text{(A.5)}
$$

By the definition of V, for any $k, \ l \in S$, we have

$$
V(x,l) = \sum_{i=1}^{n} c_i(l)[x_i - 1 - \log(x_i)] \le \sum_{i=1}^{n}\breve{q}c_i(k)[x_i - 1 - \log(x_i)] \le \breve{q}V(x,k).
$$

Thus

$$
\sum_{l=1}^{N}\gamma_{kl}V(x,l) \le \breve{q}\left(\sum_{l=1}^{N}|\gamma_{kl}|\right)V(x,k). \qquad \text{(A.6)}
$$

We therefore obtain from (A.3) , (A.4) and (A.6) that

$$
LV(x,k) \le K_2^*[1+V(x,k)], \qquad \text{(A.7)}
$$

where K_2^* is a positive constant. Making use of the generalized Itô formula, yields

$$\begin{aligned} EV(x(\tau_m \wedge T), r(\tau_m \wedge T)) &\le V(x(0), r(0)) + K_2^* E(\tau_m \wedge T) + K_2^* E \int_0^{\tau_m \wedge T} V(x(t), r(t))dt \\ &\le V(x(0), r(0)) + K_2^* T + K_2^* \int_0^T EV(x(\tau_m \wedge t), r(\tau_m \wedge t))dt. \end{aligned}$$

The Gronwall inequality implies that

$$EV(x(\tau_m \wedge T), r(\tau_m \wedge T)) \le [V(x(0), r(0)) + K_2^* T]e^{K_2^* T}. \tag{A.8}$$

Set $\Omega_m = \{\tau_m \le T\}$ for $m \ge m_1$ and by (A.1) , $P(\Omega_m) \ge \epsilon$. Note that for every $\omega \in \Omega_m$, there is some i such that $x_i(\tau_m,\ \omega)$ equals either m or $\frac{1}{m}$, and hence $V(x(\tau_m,\ \omega))$ is no less than either

$$\hat{c}\left(\sqrt{m} - 1 - 0.5\log(m)\right)$$

or

$$\hat{c}\left(\sqrt{\frac{1}{m}} - 1 - 0.5\log(\frac{1}{m})\right) = \hat{c}\left(\sqrt{\frac{1}{m}} - 1 + 0.5\log(m)\right).$$

Consequently,

$$V(x(\tau_m, \omega), r(\tau_m, \omega)) \ge \hat{c}\left([\sqrt{m} - 1 - 0.5\log(m)\ ] \wedge [0.5\log(m) - 1 + \sqrt{\frac{1}{m}}\ ]\right).$$

It then follows from (A.8) that

$$\begin{aligned} [V(x(0), r(0)) + K_2^* T]e^{K_2^* T} &\ge E[1_{\Omega_m}(\omega)V(x(\tau_m, \omega), r(\tau_m, \omega))] \\ &\ge \epsilon\hat{c}\left([\sqrt{m} - 1 - 0.5\log(m)] \wedge [0.5\log(m) - 1 + \sqrt{\tfrac{1}{m}}\ ]\right), \end{aligned}$$

where 1_{Ω_m} is the indicator function of Ω_m. Letting $k \to \infty$ leads to the contradiction

$$\infty > [V(x(0), r(0)) + K_2^* T]e^{K_2^* T} = \infty.$$

So we must have $\tau_\infty = \infty$ a.s. This completes the proof of Theorem 2.1.

B Proof of Lemma 5.1

Let $V : R_+^n \to R_+$ be defined as (3.9) , by the generalized Itô formula, we can show that

$$\begin{aligned} & E\left(\sup_{t \le r \le t+1} V(x(r))\right) \\ \le\ & EV(x(t)) + \max_{k \in S}\{|b(k)|\} \int_t^{t+1} E(|x(s)|)ds + \max_{k \in S}\{|A(k)|\} \int_t^{t+1} E(|x(s)|^2)ds \\ & + E\left(\sup_{t \le r \le t+1} \int_t^r x^T(s)\sigma(r(s))dB(s)\right). \end{aligned}$$

From (3.1) of Lemma 3.1, we know that

$$\limsup_{t\to\infty} EV(x(t)) \le n^{\frac{1}{2}} \limsup_{t\to\infty} E(|x(t)|) \le n^{\frac{1}{2}} K(1). \tag{B.1}$$

and

$$\limsup_{t\to\infty} E\int_t^{t+1} |x(s)|^2 ds \le K(2). \tag{B.2}$$

But, by the well-known Burkholder-Davis-Gundy inequality (see [22], [26]) and the Hölder inequality, we derive that

$$E\left(\sup_{t\le r\le t+1}\int_t^r x^T(s)\sigma(r(s))dB(s)\right) \;\le\; 3\max_{k\in S}\{|\sigma(k)|\}E\left(\int_t^{t+1}|x(s)|^2 ds\right)^{\frac{1}{2}}$$

Therefore

$$\begin{aligned} & E\left(\sup_{t\le r\le t+1} V(x(r))\right) \\ \le\;& EV(x(t)) + \max_{k\in S}\{|b(k)|\}\int_t^{t+1} E(|x(s)|)ds + \max_{k\in S}\{|A(k)|\}\int_t^{t+1} E(|x(s)|^2)ds \\ & +3\max_{k\in S}\{|\sigma(k)|\}\left[E\int_t^{t+1}|x(s)|^2 ds\right]^{\frac{1}{2}}. \end{aligned}$$

This, together with (B.1) and (B.2), yields

$$\begin{aligned} & \limsup_{t\to\infty} E\left(\sup_{t\le r\le t+1} V(x(r))\right) \\ \le\;& \left[n^{\frac{1}{2}} + \max_{k\in S}\{|b(k)|\}\right]K(1) + \max_{k\in S}\{|A(k)|\}K(2) + 3\max_{k\in S}\{|\sigma(k)|\}[K(2)]^{\frac{1}{2}}. \end{aligned}$$

Recalling the following inequality

$$|x(t)| \le \sum_{i=1}^n x_i(t) \le V(x(t)) \quad \text{ for any } x(t)\in R_+^n,$$

we obtain

$$\begin{aligned} & \limsup_{t\to\infty} E\left(\sup_{t\le r\le t+1} |x(r)|\right) \\ \le\;& \left[n^{\frac{1}{2}} + \max_{k\in S}\{|b(k)|\}\right]K(1) + \max_{k\in S}\{|A(k)|\}K(2) + 3\max_{k\in S}\{|\sigma(k)|\}[K(2)]^{\frac{1}{2}}. \end{aligned} \tag{B.3}$$

To prove assertion (5.1) we observe from (B.3) there is a positive constant $\bar{K}$ such that

$$E\left(\sup_{k\le t\le k+1}|x(t)|\right) \le \bar{K}, \quad k = 1,2,\dots$$

Let $\epsilon > 0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$P\left\{\sup_{k\le t\le k+1}|x(t)| > k^{1+\epsilon}\right\} \le \frac{\bar{K}}{k^{1+\epsilon}}, \quad k = 1,2,\dots$$

Applying the well-known Borel-Cantelli lemma (see e.g. [22]), we obtain that for almost all $\omega \in \Omega$

$$\sup_{k \le t \le k+1} |x(t)| \le k^{1+\epsilon} \tag{B.4}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega \in \Omega$, for which (B.4) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \ge k_0$ and $k \le t \le k+1$,

$$\frac{\log(|x(t)|)}{\log t} \le \frac{(1+\epsilon)\log k}{\log k} = 1 + \epsilon.$$

Therefore

$$\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \le 1 + \epsilon. \quad \text{a.s.}$$

Letting $\epsilon \to 0$ we obtain the desired assertion (5.1). The proof is therefore complete.