A highly sensitive mean-reverting process in finance and the Euler–Maruyama approximations

Fuke Wu[1*] Xuerong Mao[2†] Kan Chen[2‡]

[1] *Department of Mathematics*

Huazhong University of Science and Technology, Wuhan, Hubei 430074, P.R.China

[2] *Department of Statistics and Modelling Science, University of Strathclyde, Glasgow G1 1XH, UK*

Abstract

Empirical studies show that the most successful continuous-time models of the short-term rate in capturing the dynamics are those that allow the volatility of interest changes to be highly sensitive to the level of the rate. However, from the mathematics, the high sensitivity to the level implies that the coefficients do not satisfy the linear growth condition, so we can not examine its properties by traditional techniques. This paper overcomes the mathematical difficulties due to the nonlinear growth and examines its analytical properties and the convergence of numerical solutions in probability. The convergence result can be used to justify the method within Monte–Carlo simulations that compute the expected payoff of financial products. For illustration, we apply our results compute the value of a bond with interest rate given by the highly sensitive mean-reverting process as well as the value of a single barrier call option with the asset price governed by this process.

Keywords: Structure of interest rate; Stochastic differential equation; Convergence in probability; Euler–Maruyama method; Monte–Carlo simulation.

1 Introduction

The short-term riskless interest rate is one of the most fundamental and important quantities in financial markets. Many models have been put forward to explain its behavior. Letting

*Corresponding author, *E-mail*: wufuke@mail.hust.edu.cn

†*E-mail*: xuerong@stams.strath.ac.uk

‡*E-mail*: kan@stams.strath.ac.uk

$R(t)$ represent the short-term interest rate, we list some well-known models as follows:

1. Merton (1973 [12])	$dR(t) = \mu dt + \sigma dw(t)$
2. Vasicek (1977 [14])	$dR(t) = (\mu + \lambda R(t))dt + \sigma dw(t)$
3. Cox, Ingersoll and Ross (1985 [5])	$dR(t) = (\mu + \lambda R(t))dt + \sigma\sqrt{R(t)}dw(t)$
4. Dothan (1978 [6])	$dR(t) = \sigma R(t)dw(t)$
5. Geometric Brownian motion	$dR(t) = \lambda R(t)dt + \sigma R(t)dw(t)$
6. Brennan and Schwartz (1980 [1])	$dR(t) = (\mu + \lambda R(t))dt + \sigma R(t)dw(t)$
7. Cox, Ingersoll and Ross (1980 [4])	$dR(t) = \sigma R(t)^{\frac{3}{2}}dw(t)$
8. Constant Elasticity of Variance (1975 [3])	$dR(t) = \lambda R(t)dt + \sigma R(t)^{\gamma}dw(t)$

where λ, μ and σ are constants.

As in [2] and [13], these eight models may be nested within the following stochastic differential equation:

$$dR(t) = \lambda(\mu - R(t))dt + \sigma R^{\gamma}(t)dw(t) \tag{1.1}$$

by simply placing the appropriate restrictions on the three parameters λ, μ and γ.

When $\gamma = 1/2$, the solution of Eq. (1.1) is the well-known mean-reverting square root process [5]. It is widely used to model volatility, interest rates and other financial quantities. Many papers (e.g. [8, 9],) discuss its analytical properties. Higham and Mao [7] examine strong convergence of its Monte–Carlo simulation. When $\gamma \in [1/2, 1]$, Mao *et al* [10] discuss its analytical properties and strong convergence of numerical solutions.

Some empirical studies show that the most successful continuous-time models of the short-term rate in capturing the dynamics are those that allow the volatility of interest changes to be highly sensitive to the level of the rate. By χ^2 tests to U.S. T-bill data, the above models which assume $\gamma < 1$ are rejected and those which assume $\gamma \geq 1$ are not rejected. Applying the Generalized Method Moment, Chan *et al* [2] give $\gamma = 1.449$. Using the same data, by the Gaussian Estimation methods, Nowman [13] estimates $\gamma = 1.361$ [1]. Therefore, it is more evident to consider $\gamma \geq 1$.

However, $\gamma > 1$ implies that the diffusion coefficient does not satisfy the linear growth condition so we cannot apply the classical results (e.g. [9]) on the existence and uniqueness of the solutions, boundedness of the moment, the convergence of its Euler–Maruyama approximate solutions, and so on. This paper develops new techniques to overcome these difficulties.

Many empirical studies, including [2] and [13], estimate the parameters of the continuous-time model by the Euler–Maruyama discrete approximation. However, they can not ensure

[1] In this paper, using one month British sterling rate data, Norman obtain $\gamma = 0.2898$. The difference between U.S and British data show the volatility of short-term rates is highly sensitive to the level of rates in the U.S., while it is not in U.K.

that these parameters are precise enough because it has not been proved yet that the approximate solution will converge to the exact solution when the discrete step size tends to zero. This paper will fill the gap.

In the next section, we first consider the existence and nonnegativity of the solution of Eq. (1.1). This is a natural requirement since Eq. (1.1) is frequently used to model the interest rate. In section three, we consider various boundedness of the solution of Eq. (1.1), including the moment boundedness, the stochastic boundedness and the pathwise estimations. In section four, we introduce the Euler–Maruyama approximations to the solution of Eq. (1.1) and examine its convergence in probability. Finally, we choose bonds and a single barrier call option to show that the numerical solution can be used to compute the expected payoffs.

2 Positive and global solutions

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (namely, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t)$ be a scalar Brownian motion defined on the probability space. We consider the mean-reverting γ-process

$$dR(t) = \lambda(\mu - R(t))dt + \sigma R^{\gamma}(t)dw(t) \tag{2.1}$$

with the initial value $R(0) > 0$, where λ, μ and σ are positive and $\gamma > 1$.

In order for a stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see [9]). However, the diffusion coefficient of Eq. (2.1) does not satisfy the linear growth condition, though it is locally Lipschitz continuous. We wonder if the solution of Eq. (2.1) may explode at a finite time. Furthermore, since Eq. (2.1) is used to model interest rate and other quantities, it is critical that the solution $R(t)$ will never become negative. The following theorem reveals the existence of the positive solution.

Theorem 2.1 *For any given initial value $R(0) > 0$, λ, μ and $\sigma > 0$, there exists a unique positive global solution $R(t)$ to Eq. (2.1) on $t \geq 0$.*

Proof Since the coefficients of (2.1) satisfy the local Lipschitz condition, for any given initial value $R(0) > 0$, there must exist a unique local solution $R(t) \in [0, \tau_e]$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = \infty$ a.s. For a sufficient large integer $k > 0$, namely $1/k < R(0) < k$, define the following stopping time,

$$\tau_k = \inf\{t \in [0, \tau_e] : \ R(t) \notin [1/k, k]\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \le \tau_e$ a.s. If we can prove $\tau_k \to \infty$ a.s. as $k \to \infty$, then $\tau_e = \infty$ a.s. and $R(t) > 0$ a.s. for all $t \ge 0$. In other words, to complete the proof what we need to show is that $\tau_\infty = \infty$ a.s. To prove this, for any constant T, if $\mathbb{P}\{\tau_k \le T\} \to 0$ as $k \to \infty$, then we have $\mathbb{P}\{\tau_\infty = \infty\} = 1$, which is the required assertion.

For $\theta \in (0, 1)$, define a C^2-function $V : (0,\infty) \to (0,\infty)$ by

$$V(R) = R^\theta - 1 - \theta \log R. \tag{2.2}$$

It is easy to see that $V(\cdot) \ge 0$ and $V(R) \to \infty$ as $R \to \infty$ or $R \to 0$. Applying the Itô formula yields

$$\begin{aligned} dV(R(t)) \;=\; & \Big\{\lambda\theta(\mu R^{\theta-1}(t) - R^\theta(t) - \mu R^{-1}(t) + 1) + \frac{\theta\sigma^2}{2}[(\theta-1)R^{\theta+2(\gamma-1)}(t) \\ & + R^{2(\gamma-1)}(t)]\Big\}dt + \sigma\theta(R^{\theta+\gamma-1}(t) - R^{\gamma-1}(t))dw(t). \end{aligned} \tag{2.3}$$

For $\theta \in (0, 1)$, there exists a constant K_1 such that

$$\lambda\theta(\mu R^{\theta-1} - R^\theta - \mu R^{-1} + 1) + \frac{\theta\sigma^2}{2}[(\theta-1)R^{\theta+2(\gamma-1)} + R^{2(\gamma-1)}] \le K_1.$$

Therefore, for any $t \in [0, T]$,

$$\mathbb{E}V(R(t \wedge \tau_k)) \le V(R(0)) + K_1 T,$$

so

$$\mathbb{P}(\tau_k \le T)[V(1/k) \wedge V(k)] \le \mathbb{E}V(R(T \wedge \tau_k)) \le V(R(0)) + K_1 T.$$

Therefore $\mathbb{P}(\tau_k \le T) \to 0$ since $V(1/k) \wedge V(k) \to \infty$ as $k \to \infty$. This implies $\mathbb{P}(\tau_\infty = \infty) = 1$, as required.

3 Boundedness

For the interest rates and other assets' prices, boundedness is a natural requirement. In this section, we will establish various boundedness for the solution to Eq. (2.1).

3.1 Moment boundedness

We mainly focus on the boundedness of the first moment and the second moment.

Theorem 3.1 *The solution of Eq. (2.1) obeys*

$$\mathbb{E}R(t) \le R(0) + \mu, \;\; \forall t \ge 0 \tag{3.1}$$

and

$$\limsup_{t\to\infty} \mathbb{E}R(t) \le \mu. \tag{3.2}$$

Proof Applying the Itô formula yields

$$d[e^{\lambda t}R(t)] = e^{\lambda t}[\lambda\mu dt + \sigma R(t)^{\gamma} dw(t)].$$

For any positive number n, define a stopping time

$$\tau_n = \inf\{t : R(t) > n\}.$$

Then

$$\begin{aligned}\mathbb{E}[e^{\lambda(t\wedge\tau_n)}R(t\wedge\tau_n)] &= R(0) + \lambda\mu\mathbb{E}\Big(\int_0^{t\wedge\tau_n} e^{\lambda s}ds\Big)\\ &\leq R(0) + \mu(e^{\lambda t} - 1).\end{aligned}$$

Letting $n \to \infty$, the Fatou theorem yields

$$e^{\lambda t}\mathbb{E}R(t) \leq R(0) + \mu(e^{\lambda t} - 1),$$

which implies the required assertions.

From the proof of Theorem 2.1, we observe that the average in time of the moment of the solutions will be bounded, which is described as follows.

Theorem 3.2 *For any $\theta \in (0, 1)$, there exists a positive constant K_θ such that for any initial value $R(0) > 0$, the solution of Eq. (2.1) has the property*

$$\limsup_{t\to\infty} \frac{1}{t}\Big[\int_0^t \mathbb{E}R^{\theta+2(\gamma-1)}(s)ds\Big] \leq K_\theta. \tag{3.3}$$

Proof It is obvious that there exists a constant K_2 such that

$$\lambda\theta(\mu R^{\theta-1} - R^{\theta} - \mu R^{-1} + 1) + \frac{\theta\sigma^2}{2}\Big[\frac{1}{2}(\theta - 1)R^{\theta+2(\gamma-1)} + R^{2(\gamma-1)}\Big] \leq K_2.$$

It then follows from (2.3) that

$$\frac{\theta(1-\theta)\sigma^2}{4}\mathbb{E}\int_0^{t\wedge\tau_k} R^{\theta+2(\gamma-1)}(s)ds + \mathbb{E}V(R(t\wedge\tau_k)) \leq V(R(0)) + K_2 t.$$

Letting $k \to \infty$ and applying the Fatou theorem, we have

$$\frac{\theta(1-\theta)\sigma^2}{4}\mathbb{E}\int_0^t R^{\theta+2(\gamma-1)}(s)ds \leq V(R(0)) + K_2 t,$$

which implies the required assertion.

Corollary 3.1 *If $\gamma > 3/2$, then there is a constant $K > 0$ such that for any $R(0) > 0$, the solution of Eq. (2.1) obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}R^2(s)ds \leq K. \tag{3.4}$$

Proof As $\gamma > 3/2$, we can choose $\theta \in (0, 1)$ for $\theta + 2(\gamma - 1) \geq 2$. By the Hölder inequality, we compute

$$\frac{1}{t}\int_0^t \mathbb{E}R^2(s)ds \leq \Big(\frac{1}{t}\int_0^t \mathbb{E}R^{\theta+2(\gamma-1)}(s)ds\Big)^{\frac{2}{\theta+2(\gamma-1)}}.$$

Letting $t \to \infty$ and applying Theorem 3.2 give the assertion.

3.2 Stochastic boundedness

The following result shows that $R(t)$ will stay in a belt area with a large probability.

Theorem 3.3 *If $\gamma \in (1,2)$, for any $\varepsilon \in (0,1)$ and $R(0) > 0$, there exists a pair of positive constants $H = H(\varepsilon, R(0))$ and $h = h(\varepsilon, R(0))$ such that*

$$\mathbb{P}(h \leq R(t) \leq H) \geq 1 - \varepsilon, \ \textit{for } \forall t \geq 0. \tag{3.5}$$

Proof For any $\varepsilon > 0$, let $H = 2(R(0) + \mu)/\varepsilon$. Then by the Chebyshev inequality and Theorem 3.1,

$$\mathbb{P}(R(t) > H) \leq \frac{R(0) + \mu}{H} = \frac{\varepsilon}{2}. \tag{3.6}$$

Set $y(t) = R^{-1}(t)$. By the Itô formula,

$$dy(t) = (-\lambda\mu y^2(t) + \lambda y(t) + \sigma^2 R^{2\gamma-3}(t))dt - \sigma y^{2-\gamma}(t)dw(t). \tag{3.7}$$

Fix any constant $\theta \in (0,1)$. Then

$$d[e^{\theta t}y(t)] = e^{\theta t}[-\lambda\mu y^2(t) + (\lambda + \theta)y(t) + \sigma^2 R^{2\gamma-3}(t)]dt - \sigma e^{\theta t}y^{2-\gamma}(t)dw(t).$$

Noting $\gamma \in (1,2)$, we have

$$R^{2\gamma-3} = 1_{\{\gamma=\frac{3}{2}\}} + y^{3-2\gamma}1_{\{1<\gamma<\frac{3}{2}\}} + R^{2\gamma-3}1_{\{\frac{3}{2}<\gamma<2\}}.$$

Noting $\lambda\mu > 0$ and $0 < 3 - 2\gamma < 1$ when $\gamma \in (1, 3/2)$, then we conclude that there exists a constant K_3 such that

$$-\lambda\mu y^2 + (\lambda + \theta)y + \sigma^2(1 + y^{3-2\gamma}1_{\{1<\gamma<\frac{3}{2}\}}) \leq K_3,$$

so, by the Jensen inequality and theorem 3.1, we may compute

$$\begin{aligned} e^{\theta t}\mathbb{E}y(t) &\leq \frac{1}{R(0)} + K_3\int_0^t e^{\theta s}ds + \sigma^2\mathbb{E}\int_0^t e^{\theta s}R^{2\gamma-3}(s)1_{\{\frac{3}{2}<\gamma<2\}}(s)ds \\ &\leq \frac{1}{R(0)} + \frac{K_3}{\theta}(e^{\theta t} - 1) + \sigma^2\int_0^t e^{\theta s}(\mathbb{E}R(s))^{2\gamma-3}1_{\{\frac{3}{2}<\gamma<2\}}ds \\ &\leq \frac{1}{R(0)} + \frac{K_3}{\theta}(e^{\theta t} - 1) + \frac{\sigma^2}{\theta}(R(0) + \mu)^{2\gamma-3}(e^{\theta t} - 1). \end{aligned}$$

Therefore, there exists a constant K_4 such that

$$\mathbb{E}y(t) \leq K_4.$$

Similarly, by the Chebyshev inequality, there exists a positive constant $h = h(\varepsilon, R(0))$ such that

$$\begin{aligned} \mathbb{P}(R(t) \geq h) &= P(y(t) \leq h^{-1}) \\ &\geq 1 - \frac{\varepsilon}{2}. \end{aligned}$$

This, together with (3.6), implies

$$\begin{aligned}\mathbb{P}(h \le R(t) \le H) &= \mathbb{P}(R(t) \ge h) - \mathbb{P}(R(t) > H) \\ &\ge 1 - \varepsilon,\end{aligned}$$

as required.

3.3 Pathwise estimations

Theorem 3.4 *If $\gamma \in (1,2)$, then for any initial value $R(0) > 0$,*

$$\liminf_{t\to\infty} \frac{\log R(t)}{\log t} \ge -1 \quad a.s. \tag{3.8}$$

Proof From Eq. (3.7),

$$\mathbb{E}y(t+1) + \frac{\lambda\mu}{2}\mathbb{E}\int_t^{t+1} y^2(s)ds = \mathbb{E}y(t) + \mathbb{E}\int_t^{t+1}\Big[-\frac{\lambda\mu}{2}y^2(s) + \lambda y(s) + \sigma^2 R^{2\gamma-3}(s)\Big]ds. \tag{3.9}$$

There exists a constant K_5 such that

$$-\frac{\lambda\mu}{2}y^2 + \lambda y + \sigma^2 + \sigma^2 y^{3-2\gamma}1_{\{1<\gamma<\frac{3}{2}\}} < K_5,$$

so we have

$$\begin{aligned}\frac{\lambda\mu}{2}\mathbb{E}\int_t^{t+1} y^2(s)ds &\le \mathbb{E}y(t) + K_5 + \sigma^2\mathbb{E}\int_t^{t+1} R^{2\gamma-3}(s)1_{\{\frac{3}{2}<\gamma<2\}}ds \\ &\le K_4 + K_5 + \sigma^2[R(0)+\mu]^{2\gamma-3}.\end{aligned} \tag{3.10}$$

By Eq. (3.7), for any $u \in [t, t+1]$,

$$y(u) = y(t) - \lambda\mu\int_t^u y^2(s)ds + \lambda\int_t^u y(s)ds + \sigma^2\int_t^u R^{2\gamma-3}(s)ds - \sigma\int_t^u y^{2-\gamma}(s)dw(s),$$

so we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le u\le t+1} y(u)\Big) &\le \mathbb{E}y(t) + \lambda\int_t^{t+1}\mathbb{E}y(s)ds + \sigma^2\int_t^{t+1}\mathbb{E}R^{2\gamma-3}(s)ds \\ &\quad +\sigma\mathbb{E}\Big(\sup_{t\le u\le t+1}\Big|\int_t^u y^{2-\gamma}(s)dw(s)\Big|\Big).\end{aligned} \tag{3.11}$$

By the Lyapunov inequality, we have

$$\mathbb{E}R^{2\gamma-3}(t) \le 1 + (\mathbb{E}y(t))^{3-2\gamma}1_{\{1<\gamma<\frac{3}{2}\}} + (\mathbb{E}R(t))^{2\gamma-3}1_{\{\frac{3}{2}<\gamma<2\}}.$$

Applying the Burkholder–Davis–Gundy inequality and the Jensen inequality etc. yields

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le u\le t+1}\Big|\int_t^u y^{2-\gamma}(s)dw(s)\Big|\Big) &\le 6\mathbb{E}\Big(\int_t^{t+1} y^{2(2-\gamma)}(s)ds\Big)^{\frac{1}{2}}\\ &\le 6\Big(\int_t^{t+1}(\mathbb{E}(\gamma-1+(2-\gamma)y^2(s))ds\Big)^{\frac{1}{2}}\\ &\le 6\Big(\int_t^{t+1}(\gamma-1)+(2-\gamma)\mathbb{E}y^2(s)ds\Big)^{\frac{1}{2}}.\end{aligned}$$

By the boundedness of $\mathbb{E}\int_t^{t+1} y^2(s)ds$ and $\mathbb{E}y(t)$, we therefore see from (3.11) there exists a constant K_6 such that

$$\mathbb{E}\Big(\sup_{t\le u\le t+1} y(u)\Big)\le K_6. \tag{3.12}$$

Let $\varepsilon>0$ be arbitrary. By the Chebyshev inequality, we have

$$\mathbb{P}\{\sup_{k\le t\le k+1} y(t)>k^{1+\varepsilon}\}\le\frac{K_6}{k^{1+\varepsilon}},\quad k=1,2,\cdots.$$

Applying the Borel-Cantelli lemma yields, for almost all $\omega\in\Omega$,

$$\sup_{k\le t\le k+1} y(t)\le k^{1+\varepsilon} \tag{3.13}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega\in\Omega$, for which (3.13) holds whenever $k\ge k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\ge k_0$ and $k\le t\le k+1$,

$$\frac{\log y(t)}{\log t}\le\frac{(1+\varepsilon)\log k}{\log k}=1+\varepsilon. \tag{3.14}$$

That is,

$$\liminf_{t\to\infty}\frac{\log R(t)}{\log t}\ge-(1+\varepsilon). \tag{3.15}$$

Letting $\varepsilon\to0$, we obtain the desired assertion (3.8). The proof is therefore complete.

This theorem shows that for any $\varepsilon>0$, there exists a positive random variable T_ε such that, with probability one,

$$R(t)\ge t^{-(1+\varepsilon)},\quad \text{for } \forall t\ge T_\varepsilon. \tag{3.16}$$

In other words, with probability one, the solution will not decay faster than $t^{-(1+\varepsilon)}$. The following theorem describes the growth constraint.

Theorem 3.5 *If $\gamma\in(1,2)$, then for any initial value $R(0)>0$,*

$$\limsup_{t\to\infty}\frac{\log R(t)}{t^{1+\varepsilon}}\le 0\quad a.s., \tag{3.17}$$

where ε is an arbitrary positive constant.

Proof For any positive constant θ, applying the Itô formula to $e^{\theta t}\log R(t)$ results in

$$e^{\theta t}\log R(t) = \log R(0) + \int_0^t e^{\theta s}\Big[\theta \log R(s) - \lambda - \frac{\sigma^2}{2}R^{2(\gamma-1)}(s) + \lambda\mu y(s)\Big]ds + M(t), \quad (3.18)$$

where

$$M(t) = \sigma\int_0^t e^{\theta s}R^{\gamma-1}(s)dw(s)$$

is a real-valued continuous local martingale vanishing at $t=0$ with quadratic variation

$$\langle M(t), M(t)\rangle = \sigma^2\int_0^t e^{2\theta s}R^{2(\gamma-1)}(s)ds.$$

Fix any $\epsilon \in (0,1)$ and $\xi > 1$. For every integer $k \geq 1$, using the exponential martingale inequality we have

$$\mathbb{P}\Big(\sup_{0\leq t\leq k}\Big[M(t) - \frac{\epsilon}{2}\langle M(t), M(t)\rangle > \frac{\xi e^{\theta t}}{\epsilon}\log k\Big]\Big) \leq \frac{1}{k^{\xi}}.$$

By the Borel-Cantelli lemma we observe that there exists an integer $k = k(\omega)$ such that

$$M(t) \leq \frac{\epsilon}{2}e^{-\theta k}\langle M(t), M(t)\rangle + \frac{\xi e^{\theta t}}{\epsilon}\log k$$

for almost all $\omega \in \Omega$. Thus Eq. (3.18) leads to

$$\begin{aligned} e^{\theta t}\log R(t) &\leq \log R(0) + \int_0^t e^{\theta s}\Big[\theta\log R(s) - \lambda - \frac{\sigma^2(1-\epsilon e^{-\gamma(k-s)})}{2}R^{2(\gamma-1)}(s)\Big]ds \\ &\quad + \lambda\mu\int_0^t e^{\theta s}y(s)ds + \frac{\xi e^{\theta t}}{\epsilon}\log k. \end{aligned} \quad (3.19)$$

for $0 \leq t \leq k$ and $k \geq k(\omega)$. We may easily observe that there exists a constant K_7 such that

$$\theta\log R - \lambda - \frac{\sigma^2(1-\epsilon e^{-\gamma(k-s)})}{2}R^{2(\gamma-1)} \leq K_7,$$

so we have, for $t > 1$,

$$\begin{aligned} e^{\theta t}\log R(t) &\leq \log R(0) + \frac{K_7}{\theta}(e^{\theta t}-1) + \lambda\mu\int_0^t e^{\theta s}y(s)ds + + \frac{\xi e^{\theta t}}{\epsilon}\log k \\ &\leq \log R(0) + \frac{K_7}{\theta}(e^{\theta t}-1) + \lambda\mu\int_0^{T_\varepsilon\vee 1} e^{\theta s}y(s)ds \\ &\quad + \lambda\mu\int_{T_\varepsilon\vee 1}^t e^{\theta s}y(s)ds + \frac{\xi e^{\theta t}}{\epsilon}\log k \end{aligned} \quad (3.20)$$

where T_ε is defined in (3.16). Compute

$$\begin{aligned} \int_{T_\varepsilon\vee 1}^t e^{\theta s}y(s)ds &\leq \int_{T_\varepsilon\wedge 1}^t e^{\theta s}s^{1+\varepsilon}ds \\ &\leq \frac{1}{\theta}s^{1+\varepsilon}e^{\theta s}\Big|_{T_\varepsilon\wedge 1}^t - \frac{1+\varepsilon}{\theta}\int_{T_\varepsilon\wedge 1}^t e^{\theta s}s^{\varepsilon}ds \\ &\leq \frac{1}{\theta}t^{1+\varepsilon}e^{\theta t}. \end{aligned}$$

Hence,

$$e^{\theta t}\log R(t) \le \log R(0) + \frac{K_7}{\theta}(e^{\theta t}-1) + \lambda\mu\int_0^{T_\varepsilon\vee 1} e^{\theta s}y(s)ds + \frac{\lambda\mu}{\theta}t^{1+\varepsilon}e^{\theta t} + \frac{\xi e^{\theta t}}{\epsilon}\log k.$$

Thus

$$\log R(t) \le |\log R(0)| + \frac{K_7}{\theta} + \frac{\lambda\mu}{\theta}t^{1+\varepsilon} + \frac{\xi}{\epsilon}\log k + \lambda\mu e^{-\theta t}\int_0^{T_\varepsilon\vee 1} e^{\theta s}y(s)ds \tag{3.21}$$

for $1 \le t \le k$ and $k \ge k(\omega)$. If $k-1 \le t \le k$, it follows that

$$\frac{\log R(t)}{t^{1+\varepsilon}} \le \frac{1}{t^{1+\varepsilon}}\Big[|\log R(0)| + \frac{K_7}{\theta}\Big] + \frac{\lambda\mu}{\theta} + \frac{\xi}{\epsilon}\frac{\log k}{(k-1)^{1+\varepsilon}} + \frac{\lambda\mu}{e^{\theta t}t^{1+\varepsilon}}\int_0^{T_\varepsilon\vee 1} e^{\theta s}y(s)ds, \tag{3.22}$$

then we obtain

$$\limsup_{t\to\infty}\frac{\log R(t)}{t^{1+\varepsilon}} \le \frac{\lambda\mu}{\theta} \quad a.s.$$

Letting $\theta \to \infty$ yields the required assertion (3.20).

4 The Euler–Maruyama method

This section deals with the regime where the time interval, $[0,T]$, is fixed. There is so far no explicit solution to Eq. (2.1) so we consider its numerical solution. We refer to it as the Euler–Maruyama (EM) method. Now we define the discrete EM approximate solution to (2.1) for a given fixed timestep $\triangle \in (0,1)$ and $r_0 = R(0)$,

$$r_{k+1} = r_k + \lambda(\mu - r_k)\triangle + \sigma|r_k|^\gamma \triangle w_k, \tag{4.1}$$

where $\triangle w_k = w(t_{k+1}) - w(t_k)$ is a Brownian motion increment.

In our analysis, it will be more convenient to use continuous-time approximation. Letting $[t/\triangle]$ be the integer part of $t/\triangle$, we hence introduce the step process

$$\bar{r}(t) = \sum_{k=0}^{[T/\triangle]-1} r_k 1_{[k\triangle,(k+1)\triangle)}(t) \tag{4.2}$$

and define the continuous approximation

$$r(t) = r_0 + \lambda\int_0^t \mu - \bar{r}(s)ds + \sigma\int_0^t |\bar{r}(s)|^\gamma dw(s). \tag{4.3}$$

We know that $r(t)$ is not computable because it requires knowledge of the entire Brownian path, not just its $\triangle$-increments. However, since $r_k = r(t_k)$, an error bound for $r(t)$ will automatically implies the error bound for $\{r_k\}_{k\ge 0}$. Therefore, we mainly investigate the error bound for $r(t)$. For this error bound, we have the following theorem.

Theorem 4.1 *For $R(t)$ in (2.1) and $r(t)$ in (4.3),*

$$\lim_{\triangle\to 0}\Big(\sup_{0\le t\le T}|R(t)-r(t)|^2\Big)=0, \quad \textit{in probability.} \tag{4.4}$$

Proof We divide the whole proof into four steps.

Step 1. We use the same notation as in the proof of Theorem 2.1, where we have shown that for the stopping time τ_k,

$$\mathbb{P}(\tau_k\le T)\le \frac{1}{V(1/k)\wedge V(k)}[V(R(0))+K_1T]. \tag{4.5}$$

Step 2. We define the similar stopping time,

$$\rho_k=\inf\{t\in[0,T]:\ r(t)\notin[1/k,k]\}.$$

We choose $\theta=1/2$ in (2.2), namely in the definition of the function $V(\cdot)$. Applying the Itô formula to (4.3) yields

$$\begin{aligned}\mathbb{E}V(r(t\wedge\rho_k)) &= V(R(0))+\mathbb{E}\int_0^{t\wedge\rho_k}\Big[\frac{\lambda\mu}{2}(r^{-\frac12}(s)-r^{-1}(s))-\frac{\lambda}{2}(r^{-\frac12}(s)\bar r(s)-r^{-1}(s)\bar r(s))\\ &\quad+\frac{\sigma^2}{4}r^{-2}(s)\bar r^{2\gamma}(s)-\frac{\sigma^2}{8}r^{-\frac32}(s)\bar r^{2\gamma}(s)\Big]ds. \end{aligned}\tag{4.6}$$

Rearranging the terms on the right-hand side gives

$$\begin{aligned}\mathbb{E}V(r(t\wedge\rho_k)) &= V(R(0))+\mathbb{E}\int_0^{t\wedge\rho_k}\Big[\frac{\lambda\mu}{2}(r^{-\frac12}(s)-r^{-1}(s))-\frac{\lambda}{2}(r^{-\frac12}(s)-1))\\ &\quad+\frac{\sigma^2}{4}r^{2(\gamma-1)}(s)-\frac{\sigma^2}{8}r^{2\gamma-\frac32}(s)+\frac{\lambda}{2}(r^{-\frac12}(s)-r^{-1}(s))(r(s)-\bar r(s))\\ &\quad+\frac{\sigma^2}{8}(r^{-\frac32}(s)-2r^{-2}(s))(r^{2\gamma}(s)-\bar r^{2\gamma}(s))\Big]ds. \end{aligned}\tag{4.7}$$

Note that there exists a constant c_1 such that

$$\frac{\lambda\mu}{2}(r^{-\frac12}-r^{-1})-\frac{\lambda}{2}(r^{-\frac12}-1)+\frac{\sigma^2}{4}r^{2(\gamma-1)}-\frac{\sigma^2}{8}r^{2\gamma-\frac32}\le c_1$$

and a constant $c_2(k)$ such that

$$|r^{2\gamma}-\bar r^{2\gamma}|\le c_2(k)|r-\bar r|^{2\gamma}.$$

When $r\in[1/k,k]$, which implies $\bar r\in[1/k,k]$, we have

$$\begin{aligned}\mathbb{E}V(r(t\wedge\rho_k)) &\le V(R(0))+c_1T+\mathbb{E}\int_0^{t\wedge\rho_k}\Big[\frac{\lambda}{2}|r^{-\frac12}(s)+r^{-1}(s)||r(s)-\bar r(s)|\\ &\quad+\frac{\sigma^2c_2(k)}{8}|r^{-\frac32}(s)-2r^{-2}(s)||r(s)-\bar r(s)|^{2\gamma}\Big]ds\\ &\le V(R(0))+c_1T++\frac{\lambda}{2}(k^{\frac12}+k)\mathbb{E}\int_0^{t\wedge\rho_k}|r(s)-\bar r(s)|ds\\ &\quad+\frac{\sigma^2c_2(k)}{8}(k^{\frac32}+2k^2)\mathbb{E}\int_0^{t\wedge\rho_k}|r(s)-\bar r(s)|^{2\gamma}ds. \end{aligned}\tag{4.8}$$

For $\gamma > 1$,

$$\begin{aligned} |r(s)-\bar{r}(s)|^{2\gamma} &\le |r(s)-\bar{r}(s)||(r(s)|+|\bar{r}(s)|)^{2\gamma-1} \\ &\le (2k)^{2\gamma-1}|r(s)-\bar{r}(s)|. \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8) yields that there exists a constant $c_3(k)$ such that

$$\mathbb{E}V(r(t\wedge\rho_k)) \le V(R(0)) + c_1T + c_3(k)\mathbb{E}\int_0^{t\wedge\rho_k}|r(s)-\bar{r}(s)|ds. \tag{4.10}$$

Now we compute $\mathbb{E}\int_0^{t\wedge\rho_k}|r(s)-\bar{r}(s)|ds$. For $s\in[0,t\wedge\rho_k]$, by definition (4.3),

$$\begin{aligned} r(s)-\bar{r}(s) &= \lambda(\mu - r_{[s/\triangle]})(s-[s/\triangle]\triangle) + \sigma|r_{[s/\triangle]}|^{\gamma}(w(s)-w([s/\triangle]\triangle)) \\ &\le \lambda(\mu+k)\triangle + \sigma k^{\gamma}|w(s)-w([s/\triangle]\triangle)|, \end{aligned} \tag{4.11}$$

so, noting $\triangle\in(0,1)$, we have

$$\begin{aligned} \mathbb{E}\int_0^{t\wedge\rho_k}|r(s)-\bar{r}(s)|dr &\le \lambda(\mu+k)T\triangle + \sigma k^{\gamma}\mathbb{E}\int_0^{t\wedge\rho_k}|w(s)-w([s/\triangle]\triangle)|ds \\ &\le \lambda(\mu+k)T\triangle + \sigma k^{\gamma}\mathbb{E}\int_0^{T}|w(s)-w([s/\triangle]\triangle)|ds \\ &\le \lambda(\mu+k)T\triangle + \sigma k^{\gamma}\int_0^{T}\mathbb{E}|w(s)-w([s/\triangle]\triangle)|ds \\ &\le \lambda(\mu+k)T\triangle + \sigma k^{\gamma}T\triangle^{\frac{1}{2}} \\ &\le [\lambda(\mu+k)+\sigma k^{\gamma}]T\triangle^{\frac{1}{2}} \\ &=: D\triangle^{\frac{1}{2}}. \end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.10) yields

$$\mathbb{E}V(r(t\wedge\rho_k)) \le V(R(0)) + c_1T + c_3(k)D\triangle^{\frac{1}{2}}. \tag{4.13}$$

Therefore, we can show

$$\mathbb{P}(\rho_k\le T) \le \frac{1}{V(1/k)\wedge V(k)}[V(R(0)) + c_1T + c_3(k)D\triangle^{\frac{1}{2}}]. \tag{4.14}$$

Step 3. Let $\theta_k = \tau_k\wedge\rho_k$. We claim that there exists a constant $c_4(k)$ such that

$$\mathbb{E}\Big[\sup_{0\le t\le\theta_k\wedge T}|R(t)-r(t)|^2\Big] \le c_4(k)\triangle. \tag{4.15}$$

For any $0\le t_1\le T$, from (4.3) and (2.1), we have

$$R(t_1\wedge\theta_k) - r(t_1\wedge\theta_k) = -\lambda\int_0^{t_1\wedge\theta_k}(R(s)-\bar{r}(s))ds + \sigma\int_0^{t_1\wedge\theta_k}(R^{\gamma}(s)-|\bar{r}(s)|^{\gamma})ds.$$

Therefore, for any $t \in [0,T]$, by the Hölder inequality and the Doob martingale inequality (cf. Mao [9]), we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t_1\le t}|R(t_1\wedge\theta_k)-r(t_1\wedge\theta_k)|^2\Big)\\
\le\ & 2\lambda^2 t\mathbb{E}\int_0^{t\wedge\theta_k}|R(s)-\bar r(s)|^2ds+8\sigma^2\mathbb{E}\int_0^{t\wedge\theta_k}|R^\gamma(s)-\bar r^\gamma(s)|^2ds. \qquad (4.16)
\end{aligned}$$

Note that the function $h(x)=x^\gamma$ for any $x>0$ and $\gamma\ge 1$ is locally Lipschitz continuous, which implies that for $R(s),\bar r(s)\in[1/k,k]$, there exists a constant $c_5(k)$ such that

$$\begin{aligned}
|R^\gamma(s)-\bar r^\gamma(s)|^2 &\le c_5(k)|R(s)-\bar r(s)|^{2\gamma}\\
&\le c_5(k)|R(s)-\bar r(s)|^2|R(s)-\bar r(s)|^{2\gamma-2}\\
&\le c_5(k)(2k)^{2(\gamma-1)}|R(s)-\bar r(s)|^2\\
&\le 2c_5(k)(2k)^{2(\gamma-1)}\{|R(s)-r(s)|^2+|r(s)-\bar r(s)|^2\}. \qquad (4.17)
\end{aligned}$$

Substituting (4.17) into (4.16) yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t_1\le t}|R(t_1\wedge\theta_k)-r(t_1\wedge\theta_k)|^2\Big)\\
\le\ & 4[\lambda^2 t+4\sigma^2c_5(k)(2k)^{2(\gamma-1)}]\Big(\mathbb{E}\int_0^{t\wedge\theta_k}|R(s)-r(s)|^2ds+\mathbb{E}\int_0^{t\wedge\theta_k}|r(s)-\bar r(s)|^2ds\Big)\\
\le\ & 4[\lambda^2 t+4\sigma^2c_5(k)(2k)^{2(\gamma-1)}]\Big(\int_0^{t}\mathbb{E}|R(s\wedge\theta_k)-r(s\wedge\theta_k)|^2ds+\mathbb{E}\int_0^{t\wedge\theta_k}|r(s)-\bar r(s)|^2ds\Big).
\end{aligned}$$
(4.18)

In the same way as the computation of (4.12), there exists a constant $\bar D$ such that

$$\mathbb{E}\int_0^{t\wedge\theta_k}|r(s)-\bar r(s)|^2ds\le \bar D\triangle. \qquad (4.19)$$

Substituting (4.19) into (4.18) gives,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t_1\le t}(R(t_1\wedge\theta_k)-r(t_1\wedge\theta_k))^2\Big)\\
\le\ & 4[\lambda^2 t+4\sigma^2c_5(k)(2k)^{2(\gamma-1)}]\int_0^t\mathbb{E}[R(s\wedge\theta_k)-r(s\wedge\theta_k)]^2ds\\
&+4T[\lambda^2 t+4\sigma^2c_5(k)(2k)^{2(\gamma-1)}]\bar D\triangle.
\end{aligned}$$

Using the Gronwell inequality yields (4.15).

Step 4. Let $\varepsilon,\delta\in(0,1)$ be arbitrarily small. Set

$$\bar\Omega=\{\omega:\ \sup_{0\le t\le T}|R(t)-r(t)|^2\ge\delta\}.$$

Using (4.15),

$$\begin{aligned}\delta\mathbb{P}(\bar{\Omega}\cap\{\theta_k\geq T\}) &= \mathbb{E}[1_{\{\theta_k\geq T\}}1_{\{\bar{\Omega}\}}]\\ &\leq \mathbb{E}\Big[1_{\{\theta_k\geq T\}}\sup_{0\leq t\leq T\wedge\theta_k}|R(t)-r(t)|^2\Big]\\ &\leq \mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\theta_k}|R(t)-r(t)|^2\Big]\\ &\leq c_4(k)\triangle.\end{aligned}$$

This, together with (4.5) and (4.14), implies

$$\begin{aligned}\mathbb{P}(\bar{\Omega}) &\leq \mathbb{P}(\bar{\Omega}\cap\{\theta_k\geq T\})+\mathbb{P}(\theta_k\leq T)\\ &\leq \mathbb{P}(\bar{\Omega}\cap\{\theta_k\geq T\})+\mathbb{P}(\tau_k\leq T)+\mathbb{P}(\rho_k\leq T)\\ &\leq \frac{c_4(k)}{\delta}\triangle+\frac{2V(R(0))+K_1T+c_1T+Dc_3(k)\triangle^{\frac{1}{2}}}{V(1/k)\wedge V(k)}.\end{aligned}\tag{4.20}$$

Recalling that $k\to\infty$, $V(1/k)\wedge V(k)\to\infty$, we can choose k sufficiently large for

$$\frac{2V(R(0))+K_1T+c_1T}{V(1/k)\wedge V(k)}<\frac{\varepsilon}{2}$$

and then choose $\triangle$ sufficiently small for

$$\frac{c_4(k)}{\delta}\triangle+\frac{Dc_3(k)\triangle^{\frac{1}{2}}}{V(1/k)\wedge V(k)}<\frac{\varepsilon}{2}$$

to obtain

$$\mathbb{P}(\bar{\Omega})=\mathbb{P}\Big(\sup_{0\leq t\leq T}|R(t)-r(t)|^2\geq\delta\Big)<\varepsilon,\tag{4.21}$$

which is the desired assertion (4.4).

5 Valuation of Bonds and Options

In this section we will show that the EM method can be used to compute financial quantities. Typically, we choose bonds and barrier options to demonstrate our theory.

5.1 Bonds. In the case where $R(t)$ in (2.1) models the short-term interest rate dynamics, the price of a bond at the end of period is given by

$$B(T)=\mathbb{E}\Big[\exp\Big(-\int_0^T R(t)dt\Big)\Big].\tag{5.1}$$

Using the step function $\bar{r}(t)$ in (4.2), a natural approximation to $B(T)$ is

$$\bar{B}_\triangle(T)=\mathbb{E}\Big[\exp\Big(-\int_0^T|\bar{r}(t)|dt\Big)\Big].\tag{5.2}$$

The following result shows the convergence of this approximation.

Theorem 5.1 *In the notations above,*

$$\lim_{\triangle\to 0}|B(T)-\bar{B}_{\triangle}(T)|=0. \tag{5.3}$$

To prove this assertion, we need the following lemmas.

Lemma 5.1 *For $r(t)$ in (4.3) and $\bar{r}(t)$ in (4.2),*

$$\lim_{\triangle\to 0}\Big(\sup_{0\le t\le T}|r(t)-\bar{r}(t)|^2\Big)=0 \quad \textit{in probability.} \tag{5.4}$$

Proof We first prove

$$\lim_{\triangle\to 0}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\rho_k}|r(t)-\bar{r}(t)|\Big)=0.$$

By (4.11), for any $t\in[0,T\wedge\rho_k]$, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\rho_k}|r(t)-\bar{r}(t)|^2\Big)\\
\le\;& 2\lambda^2(\mu+k)^2\triangle^2+2\sigma^2k^{2\gamma}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\rho_k}|w(t)-w([t/\triangle]\triangle)|^2\Big)\\
\le\;& 2\lambda^2(\mu+k)^2\triangle^2+2\sigma^2k^{2\gamma}\mathbb{E}\Big(\sup_{0\le t\le T}|w(t)-w([t/\triangle]\triangle)|^2\Big).
\end{aligned} \tag{5.5}$$

By the Doob martingale inequality,

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T}|w(t)-w([t/\triangle]\triangle)|^4\Big) &= \mathbb{E}\Big(\sup_{0\le k\le[T/\triangle]-1}\ \sup_{k\triangle\le t\le(k+1)\triangle}|w(t)-w(k\triangle)|^4\Big)\\
&\le \sum_{k=0}^{[T/\triangle]-1}\mathbb{E}\Big(\sup_{k\triangle\le t\le(k+1)\triangle}|w(t)-w(k\triangle)|^4\Big)\\
&\le \sum_{k=0}^{[T/\triangle]-1}\mathbb{E}|w((k+1)\triangle)-w(k\triangle)|^4\\
&\le 3\sum_{k=0}^{[T/\triangle]-1}\triangle^2\\
&= 3T\triangle.
\end{aligned}$$

Hence, by the Lyapunov inequality, we have

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T}|w(t)-w([t/\triangle]\triangle)|^2\Big) &\le \Big[\mathbb{E}\Big(\sup_{0\le t\le T}|w(t)-w([t/\triangle]\triangle)|^4\Big)\Big]^{\frac{1}{2}}\\
&\le (3T)^{\frac{1}{2}}\triangle^{\frac{1}{2}}.
\end{aligned} \tag{5.6}$$

Substituting (5.6) into (5.5) and noting $\triangle\in(0,1)$ yield that there exists a constant $c_5(k)$ such that

$$\mathbb{E}\Big(\sup_{0\le t\le T\wedge\rho_k}|r(t)-\bar{r}(t)|^2\Big)\le c_5(k)\triangle^{\frac{1}{2}}. \tag{5.7}$$

For arbitrarily small constants $\delta, \varepsilon \in (0,1)$, set

$$\tilde{\Omega} = \{\omega : \sup_{0\le t\le T} |r(t) - \bar{r}(t)|^2 \ge \delta\}.$$

Then,

$$\begin{aligned}
\delta \mathbb{P}(\tilde{\Omega} \cap \{\rho_k \ge T\}) &= \delta \mathbb{E}(1_{\{\rho_k \ge T\}} 1_{\{\tilde{\Omega}\}}) \\
&\le \mathbb{E}\Big(1_{\{\rho_k \ge T\}} \sup_{0\le t\le T\wedge\rho_k} |r(t) - \bar{r}(t)|^2\Big) \\
&\le \mathbb{E}\Big(\sup_{0\le t\le T\wedge\rho_k} |r(t) - \bar{r}(t)|^2\Big) \\
&\le c_5(k)\triangle^{\frac{1}{2}}.
\end{aligned} \tag{5.8}$$

This, together with (4.14), yields that

$$\begin{aligned}
\mathbb{P}(\tilde{\Omega}) &\le \mathbb{P}(\tilde{\Omega} \cap \{\rho_k \ge T\}) + \mathbb{P}(\rho_k \le T) \\
&\le \frac{c_5(k)}{\delta}\triangle^{\frac{1}{2}} + \frac{V(R(0)) + c_1 T + Dc_3(k)\triangle^{\frac{1}{2}}}{V(1/k) \wedge V(k)}.
\end{aligned} \tag{5.9}$$

Choose k sufficiently large such that

$$\frac{V(r(0)) + c_1 T}{V(1/k) \wedge V(k)} < \frac{\varepsilon}{2}$$

and then choose $\triangle$ sufficiently small such that

$$\frac{c_5(k)}{\delta}\triangle^{\frac{1}{2}} + \frac{Dc_3(k)\triangle^{\frac{1}{2}}}{V(1/k) \wedge V(k)} < \frac{\varepsilon}{2}.$$

Hence we have

$$\mathbb{P}\Big(\sup_{0\le t\le T} |r(t) - \bar{r}(t)|^2 \ge \delta\Big) < \varepsilon, \tag{5.10}$$

which is the desired assertion.

Lemma 5.2 *For $R(t)$ in (2.1) and $\bar{r}(t)$ in (4.2),*

$$\lim_{\triangle\to 0}\Big(\sup_{0\le t\le T} |R(t) - \bar{r}(t)|\Big) = 0 \quad \textit{in probability.} \tag{5.11}$$

Proof For sufficiently small $\varepsilon, \delta \in (0,1)$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le t\le T} |R(t) - \bar{r}(t)| \ge \delta\Big) \\
\le\ &\mathbb{P}\Big(\sup_{0\le t\le T} |R(t) - r(t)| + \sup_{0\le t\le T} |r(t) - \bar{r}(t)| \ge \delta\Big) \\
\le\ &\mathbb{P}\Big(\sup_{0\le t\le T} |R(t) - r(t)| + \sup_{0\le t\le T} |r(t) - \bar{r}(t)| \ge \delta, \quad \sup_{0\le t\le T} |r(t) - \bar{r}(t)| \ge \frac{\delta}{2}\Big) \\
&+\mathbb{P}\Big(\sup_{0\le t\le T} |R(t) - r(t)| + \sup_{0\le t\le T} |r(t) - \bar{r}(t)| \ge \delta, \quad \sup_{0\le t\le T} |r(t) - \bar{r}(t)| \le \frac{\delta}{2}\Big) \\
\le\ &\mathbb{P}\Big(\sup_{0\le t\le T} |R(t) - r(t)| \ge \frac{\delta}{2}\Big) + \mathbb{P}\Big(\sup_{0\le t\le T} |r(t) - \bar{r}(t)| \ge \frac{\delta}{2}\Big).
\end{aligned}$$

Theorem 4.1 and Lemma 5.1 therefore yield the desired assertion.

Proof of Theorem 5.1 It is sufficient if we can prove

$$\exp\Big(-\int_0^T |\bar{r}(t)|\Big) \xrightarrow{\mathbb{P}} \exp\Big(-\int_0^T |R(t)|\Big).$$

In other words, we need to prove that, for arbitrarily small constants $\delta, \varepsilon \in (0,1)$,

$$\mathbb{P}\Big[\Big|\exp\Big(-\int_0^T R(t)dt\Big) - \exp\Big(-\int_0^T |\bar{r}(t)|dt\Big)\Big| \geq \delta\Big] < \varepsilon. \tag{5.12}$$

Using $e^{-|x|} - e^{-|y|} \leq |x-y|$ and the nonnegativity of $R(t)$, we have

$$\begin{aligned} \Big|\exp\Big(-\int_0^T R(t)dt\Big) - \exp\Big(-\int_0^T |\bar{r}(t)|dt\Big)\Big| &\leq \Big|\int_0^T [R(t) - \bar{r}(t)]dt\Big| \\ &\leq T \sup_{0\leq t\leq T} |R(t) - \bar{r}(t)|. \end{aligned} \tag{5.13}$$

Applying Lemma 5.2 yields the desired assertion.

5.2 A path-dependent option. We now consider the case where Eq. (2.1) models a single barrier call option, which, at expiry time T, pays the European call value if $R(t)$ never exceeded the fixed barrier B, and pays zero otherwise. We suppose that the expected payoff is computed from a Monte–Carlo simulation based on the step function method (4.2). The following theorem shows that the expected payoff computed by numerical method will converge to the real expected payoff as $\triangle \to 0$.

Theorem 5.2 *Let $R(t)$ and $\bar{r}(t)$ be defined by (2.1) and (4.2) respectively. Let E be the exercise price and let B be a barrier. Define*

$$V := \mathbb{E}[(R(T) - E)^+ 1_{\{0\leq R(t)\leq B, 0\leq t\leq T\}}]; \tag{5.14}$$

$$\overline{V}_\triangle := \mathbb{E}[(\bar{r}(T) - E)^+ 1_{\{0\leq \bar{r}(t)\leq B, 0\leq t\leq T\}}]. \tag{5.15}$$

Then

$$\lim_{\triangle\to 0} |V - \overline{V}_\triangle| = 0. \tag{5.16}$$

Proof Set $A = \{0 \leq R(t) \leq B, 0 \leq t \leq T\}, \bar{A}_\triangle = \{0 \leq \bar{r}(t) \leq B, 0 \leq t \leq T\}$. We will complete the proof if we can prove

$$(\bar{r}(T) - E)^+ 1_{\bar{A}_\triangle} \xrightarrow{\mathbb{P}} (R(T) - E)^+ 1_A,$$

which equivalent to that, for any arbitrarily small constants $\varepsilon, \delta \in (0,1)$,

$$\mathbb{P}(|(\bar{r}(T) - E)^+ 1_{\bar{A}_\triangle} - (R(t) - E)^+ 1_A| \geq \delta) < \varepsilon. \tag{5.17}$$

Making use of the inequality

$$|(\bar{r}(T)-E)^+ - (R(t)-E)^+| \le |R(t)-\bar{r}(t)|,$$

we have

$$\begin{aligned}
& \mathbb{P}(|(\bar{r}(T)-E)^+ 1_{\bar{A}_\triangle} - (R(t)-E)^+ 1_A| \ge \delta) \\
= & \mathbb{P}(\{|(\bar{r}(T)-E)^+ 1_{\bar{A}_\triangle} - (R(t)-E)^+ 1_A| \ge \delta\} \cap (A \cap \bar{A}_\triangle)) \\
& + \mathbb{P}(\{|(\bar{r}(T)-E)^+ 1_{\bar{A}_\triangle} - (R(t)-E)^+ 1_A| \ge \delta\} \cap (A \cap \bar{A}^c_\triangle)) \\
& + \mathbb{P}(\{|(\bar{r}(T)-E)^+ 1_{\bar{A}_\triangle} - (R(t)-E)^+ 1_A| \ge \delta\} \cap (A^c \cap \bar{A}_\triangle)) \\
\le & \mathbb{P}(|R(t)-\bar{r}(t)| \ge \delta) + \mathbb{P}(A \cap \bar{A}^c_\triangle) + \mathbb{P}(A^c \cap \bar{A}_\triangle).
\end{aligned}$$

By Lemma 5.2, we have $\mathbb{P}(|R(t)-\bar{r}(t)| \ge \delta) \le \varepsilon/3$. We will then complete the proof if we can prove

$$\mathbb{P}(A \cap \bar{A}^c_\triangle) \le \frac{\varepsilon}{3} \tag{5.18}$$

and

$$\mathbb{P}(A^c \cap \bar{A}_\triangle) \le \frac{\varepsilon}{3}. \tag{5.19}$$

For any sufficient small κ, we have

$$\begin{aligned}
A &= \{\sup_{0\le t\le T} R(t) \le B\} \\
&= \{\sup_{0\le t\le T} R(t) \le B-\kappa\} \cup \{B-\kappa \le \sup_{0\le t\le T} R(t) \le B\} \\
&=: A_1 \cup A_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
A \cap \bar{A}^c_\triangle &= (A_1 \cap \bar{A}^c_\triangle) \cup (A_2 \cap \bar{A}^c_\triangle) \\
&\subseteq \{\sup_{0\le t\le T} |R(t)-\bar{r}(t)| \ge \kappa\} \cup A_2.
\end{aligned}$$

So

$$\mathbb{P}(A \cap \bar{A}^c_\triangle) \le \mathbb{P}\Big(\sup_{0\le t\le T} |R(t)-\bar{r}(t)| \ge \kappa\Big) + \mathbb{P}(A_2).$$

Now we may choose κ so small that $\mathbb{P}(A_2) < \varepsilon/6$, then by Lemma 5.2, choose $\triangle$ so small that $\mathbb{P}\Big(\sup_{0\le t\le T} |R(t)-\bar{r}(t)| \ge \delta\Big) < \varepsilon/6$, so (5.18) holds.

Now, for any $\kappa > 0$, we write

$$\begin{aligned}
A^c &= \{\sup_{0\le t\le T} R(t) > B\} \\
&= \{\sup_{0\le t\le T} R(t) > B+\kappa\} \cup \{B < \sup_{0\le t\le T} R(t) \le B+\kappa\} \\
&=: A^c_1 \cup A^c_2.
\end{aligned}$$

So

$$\begin{aligned}\mathbb{P}(A^c \cap \bar{A}_\triangle) &\leq \mathbb{P}(A_1^c \cap \bar{A}_\triangle) + \mathbb{P}(A_2^c) \\ &\leq \mathbb{P}\Big(\sup_{0\leq t\leq T} |R(t) - \bar{r}(t)| > \kappa\Big) + \mathbb{P}(A_2^c).\end{aligned}$$

Repeating the above process, (5.19) also holds. The proof is complete.

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