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Stabilisation of Hybrid Stochastic Differential Equations by Delay Feedback Control

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Abstract

This paper is concerned with the exponential mean-square stabilisation of hybrid stochastic differential equations (also known as stochastic differential equations with Markovian switching) by delay feedback controls. Although the stabilisation by non-delay feedback controls for such equations has been discussed by several authors, there is so far little on the stabilisation by delay feedback controls and our aim here is mainly to close the gap. To make our theory more understandable as well as to avoid complicated notations, we will restrict our underlying hybrid stochastic differential equations to a relatively simple form. However our theory can certainly be developed to cope with much more general equations without any difficulty.

Key words: Brownian motion, Markov chain, exponential mean-square stability, linear Matrix inequality.

1 Introduction

The hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. An important class of hybrid systems is the hybrid stochastic differential equation (SDE),

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t),$$  \hspace{1cm} (1.1)

where a part of the state $x(t)$ takes values in $R^n$ while another part of the state $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, \ldots, N\}$. Such an SDE is also known as the SDE with Markovian switching. One of the important issues in the study of hybrid SDEs is the automatic control, with subsequent emphasis being placed on the analysis of stability. There is an intensive literature in this area, for example, \cite{1, 8, 14, 15, 16, 18, 19, 20, 22, 25, 26, 27, 28}, a few to name. In particular, we refer the reader to the recent book \cite{21}.\hfill \hfill \hfill
This paper is concerned with the exponential mean-square stabilisation of hybrid stochastic differential equations (also known as stochastic differential equations with Markovian switching) by delay feedback controls. The stabilisation by non-delay feedback controls for such equations has been discussed by several authors e.g. [20]. Here, given an unstable hybrid SDE in the form of (1.1), it is required to find a feedback control $u(x(t), r(t))$, based on the current state, so that the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t), r(t))]dt + g(x(t), r(t), t)dw(t)$$  \hspace{1cm} (1.2)

becomes stable. (It is possible to put the feedback control in the diffusion part but, in this paper, we will only consider the case where the control is put into the drift part.) However, it is more realistic in practice if the control depends on a past state, say $x(t - \tau)$, due to a time lag $\tau > 0$ between the time when the observation of the state is made and the time when the feedback control reaches the system. Accordingly, the control should be of the form $u(x(t - \tau), r(t))$. Hence, the stabilisation problem becomes to design a delay feedback control $u(x(t - \tau), r(t))$ in the drift part so that the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t - \tau), r(t))]dt + g(x(t), r(t), t)dw(t)$$  \hspace{1cm} (1.3)

becomes stable. There is so far little on this stabilisation problem by delay feedback controls and our aim here is mainly to close the gap.

Of course, one may consider to design a feedback control $u(x(t), x(t - \tau), r(t))$, based on both current and past state, so that the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t), x(t - \tau), r(t))]dt + g(x(t), r(t), t)dw(t)$$ \hspace{1cm} (1.4)

becomes stable. However, this is clearly easier than either (1.2) or (1.3) because (1.4) is possible if either (1.2) or (1.3) is possible.

To make our theory more understandable as well as to avoid complicated notations, we will restrict our underlying hybrid systems to a relatively simple form. However our theory can certainly be developed to cope with much more general equations without any difficulty. For example, the underlying system would be an unstable hybrid stochastic differential delay equation

$$dx(t) = f(x(t), x(t - \delta), r(t), t)dt + g(x(t), x(t - \delta), r(t), t)dw(t),$$  \hspace{1cm} (1.5)

and we could show that it is possible to design a delay feedback control $u(x(t - \tau), r(t))$ in the drift part so that the controlled system

$$dx(t) = [f(x(t), x(t - \delta), r(t), t) + u(x(t - \tau), r(t))]dt + g(x(t), x(t - \delta), r(t), t)dw(t)$$ \hspace{1cm} (1.6)

becomes stable.

2 Notation and Stabilisation Problem

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t)$ be a scalar Brownian motion defined on the probability space. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $A$
is a matrix, its operator norm is denoted by \( \|A\| = \sup\{|Ax| : |x| = 1\} \), where \(|\cdot|\) is the Euclidean norm.

Let \( r(t), t \geq 0 \), be a right-continuous Markov chain on the probability space taking values in a finite state space \( S = \{1, 2, \cdots, N\} \) with generator \( \Gamma = (\gamma_{ij})_{N \times N} \) given by

\[
\mathbb{P}\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} 
\gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\
1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j,
\end{cases}
\]

where \( \Delta > 0 \). Here \( \gamma_{ij} \geq 0 \) is the transition rate from \( i \) to \( j \) if \( i \neq j \) while

\[
\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.
\]

We assume that the Markov chain \( r(\cdot) \) is independent of the Brownian motion \( w(\cdot) \). It is known that almost all sample paths of \( r(t) \) are constant except for a finite number of simple jumps in any finite subinterval of \( R_+ := [0, \infty) \). We stress that almost all sample paths of \( r(t) \) are right continuous.

Consider an \( n \)-dimensional linear hybrid SDE

\[
dx(t) = A(r(t))x(t)dt + B(r(t))x(t)dw(t)
\]

on \( t \geq 0 \). Here \( A, B : S \to R^{n \times n} \) and we will often write \( A(i) = A_i \) and \( B(i) = B_i \).

Suppose that this given equation is unstable and we are required to design a feedback control \( u(x(t), r(t)) \) in the drift part so that the controlled SDE

\[
dx(t) = [A(r(t))x(t) + u(x(t), r(t))]dt + B(r(t))x(t)dw(t)
\]

will be mean-square exponentially stable, where \( u \) is a mapping from \( R^n \times S \) to \( R^n \).

We here note that the feedback control \( u(x(t), r(t)) \) depends on the current state \( x(t) \).

However, it is more realistic in practice if the control depends on a past state, say \( x(t-\tau) \), due to a time lag \( \tau > 0 \) between the time when the observation of the state is made and the time when the feedback control reaches the system. Accordingly, the control should be of the form \( u(x(t-\tau), r(t)) \). The stabilisation problem hence becomes to design a delay feedback control \( u(x(t-\tau), r(t)) \) in the drift part so that the controlled system

\[
dx(t) = [A(r(t))x(t) + u(x(t-\tau), r(t))]dt + B(r(t))x(t)dw(t)
\]

will be mean-square exponentially stable. As the given SDE (2.1) is linear, it is natural to use a linear feedback control. One of the most common linear feedback controls is the structure control of the form \( u(x, i) = F(i)G(i)x \), where \( F \) and \( G \) are mappings from \( S \) to \( R^{p \times 1} \) and \( R^{l \times n} \), respectively, and one of them is given while the other needs to be designed. These two cases are known as:

- State feedback: design \( F(\cdot) \) when \( G(\cdot) \) is given;
- Output injection: design \( G(\cdot) \) when \( F(\cdot) \) is given.

Again, we will often write \( F(i) = F_i \) and \( G(i) = G_i \). As a result, the controlled system (2.2) becomes

\[
dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(t-\tau)]dt + B(r(t))x(t)dw(t).
\]
This controlled system is a hybrid stochastic differential delay equation (SDDE). For an SDDE, it is required to know some initial data, for example, $x(t)$ on $t \in [-\tau, 0]$ in order for its solution to be well defined. Given that our underlying equation (2.1) is non-delay and it only requires the initial value $x(0) \in \mathbb{R}^n$, it is more natural to assume that for our controlled system (2.3) we know the initial data $x(t)$ on $t \in [0, \tau]$. This can be interpreted as follows: Let the underlying equation (2.1) evolve from time 0 to $\tau$ and observe the whole segment $\{x(t) : 0 \leq t \leq \tau\}$. Starting from $\tau$, design the feedback control $F(r(t))G(r(t))x(t-\tau)$ based on the past observation $\{x(t) : 0 \leq t \leq \tau\}$ as well as furthermore observation as time evolves. In other words, we shall regard the controlled system (2.3) as an SDDE on $t \geq \tau$ with the initial data $\{x(t) : 0 \leq t \leq \tau\}$ which are generated by the SDE (2.1) given the initial value $x(0) \in \mathbb{R}^n$. By the theory of hybrid SDEs (see e.g. [14]), we know

$$\mathbb{E}|x(t)|^2 < \infty \quad \text{on} \quad t \in [0, \tau],$$

which in turn implies, by the theory of hybrid SDDEs (see e.g. [21]), that

$$\mathbb{E}|x(t)|^2 < \infty \quad \text{for} \quad t \geq \tau.$$

Our aim is to design either $G(\cdot)$ given $F(\cdot)$ or $F(\cdot)$ given $G(\cdot)$ so that $\mathbb{E}|x(t)|^2$ will tend to zero exponentially. We shall discuss the former case in the next section while leave the later case to Section 4.

3 State Feedback: Design $F(\cdot)$ when $G(\cdot)$ is given

One technique used frequently in the study of stability of SDDEs is the method of linear matrix inequalities (LMIs) (see e.g. [4, 5, 7, 24, 29, 30, 31, 32]), although there are other methods (see e.g. the recent survey paper [17]). The principal procedure of the LMI method is: (i) Design a positive-definite quadratic Lyapunov function or functional $V$. (ii) Apply the Itô formula to compute the Itô differential $dV$. (iii) Arrange the drift part of $dV$ in the form of LMIs. For our stabilisation purpose related to the controlled SDDE (2.3) we shall use a positive-definite quadratic Lyapunov functional on the segment $\hat{x}_t := \{x(t+s) : -2\tau \leq s \leq 0\}$ for $t \geq 2\tau$. More precisely, the Lyapunov functional used in this paper will be of the form

$$V(\hat{x}_t, r(t), t) = x^T(t)Q(r(t))x(t) + \int_{t-\tau}^{t} \int_{s}^{t} [\alpha_1|x(u)|^2 + \alpha_2|x(u-\tau)|^2]duds \quad (3.1)$$

for $t \geq 2\tau$. Here $\alpha_1$ and $\alpha_2$ are two positive numbers while $Q$ is defined on $S$ and takes its values of symmetric positive-definite $n \times n$-matrices. Of course, we shall write $Q(i) = Q_i$. Accordingly we shall regard the controlled system (2.3) as an SDDE on $t \geq 2\tau$ with initial data $\{x(s) : 0 \leq s \leq 2\tau\}$. Applying the Itô formula (see e.g. [16, 21]) to the Lyapunov functional defined by (3.1) yields

$$dV(\hat{x}_t, r(t), t) = LV(\hat{x}_t, r(t), t)dt + 2x^T(t)Q(r(t))B(r(t))x(t)dw(t), \quad (3.2)$$
for $t \geq 2\tau$, where, when $r(t) = i$,

$$LV(\dot{x}_t, i, t) = 2x^T(t)Q_i[A_i x(t) + F_i G_i x(t - \tau)]$$

$$+ x^T(t)B_i Q_i B_i x(t) + \sum_{j=1}^{N} \gamma_{ij} x^T(t)Q_j x(t)$$

$$+ \alpha_1 \tau |x(t)|^2 - \alpha_1 \int_{t-\tau}^{t} |x(s)|^2 ds$$

$$+ \alpha_2 \tau |x(t - \tau)|^2 - \alpha_2 \int_{t-\tau}^{t} |x(s - \tau)|^2 ds. \quad (3.3)$$

To see why (3.2) holds, we regard the solution $x(t)$ of equation (2.3) as an Itô process and apply the Itô formula (see e.g. [16, 21]) to $x^T(t)Q(r(t))x(t)$ to get

$$d[x^T(t)Q(r(t))x(t)] = \left(2x^T(t)Q(r(t))[A(r(t))x(t) + F(r(t))G(r(t))x(t - \tau)]$$

$$+ x^T(t)B(r(t))Q(r(t))B(r(t))x(t) + \sum_{j=1}^{N} \gamma_{r(t), j} x^T(t)Q_j x(t)\right)dt$$

$$+ 2x^T(t)Q(r(t))B(r(t))x(t)dw(t).$$

On the other hand, the fundamental theory of calculus shows

$$d\left(\int_{t-\tau}^{t} \int_{s}^{t} [\alpha_1 |x(u)|^2 + \alpha_2 |x(u - \tau)|^2] du ds\right)$$

$$= (\alpha_1 \tau |x(t)|^2 - \alpha_1 \int_{t-\tau}^{t} |x(s)|^2 ds + \alpha_2 \tau |x(t - \tau)|^2 - \alpha_2 \int_{t-\tau}^{t} |x(s - \tau)|^2 ds)dt.$$

Combining these two equalities gives (3.2). Let us now present a useful lemma.

**Lemma 3.1** If there are numbers $\lambda_1 > \lambda_2 \geq 0$ and $\lambda_3 > 0$ such that

$$\mathbb{E}(LV(\dot{x}_t, r(t), t)) \leq -\lambda_1 \mathbb{E}|x(t)|^2 + \lambda_2 \mathbb{E}|x(t - \tau)|^2 - \lambda_3 \mathbb{E}\int_{t-2\tau}^{t} |x(s)|^2 ds \quad (3.4)$$

for all $t \geq 2\tau$, then

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq -\gamma, \quad (3.5)$$

where $\gamma > 0$ obeys

$$\gamma \leq \frac{\lambda_3}{\tau (\alpha_1 \vee \alpha_2)} \quad \text{and} \quad \lambda_1 \geq \lambda_2 e^{\gamma \tau} + \gamma \hat{q},$$

in which $\hat{q} = \max_{i \in S} \|Q_i\|.$

**Proof.** By the Itô formula (see e.g. [16, 21]), we have

$$e^{\gamma t}\mathbb{E}(\dot{x}_t, r(t), t) = C + \int_{2\tau}^{t} e^{\gamma s} \left[\gamma \mathbb{E}V(\dot{x}_s, r(s), s) + \mathbb{E}(LV(\dot{x}_s, r(s), s))\right] ds, \quad (3.6)$$

where $C$ is a constant. Then, using this inequality and the initial condition (3.3), we have
where \( C = e^{2\gamma r} \mathbb{E} V(\hat{x}_{2r}, r(2\tau), 2\tau) \). We compute
\[
\begin{align*}
\mathbb{E} V(\hat{x}_t, r(t), t) & \leq \hat{q} \mathbb{E} |x(t)|^2 + \int_{t-\tau}^{t} \int_s^{t} [\alpha_1 \mathbb{E} |x(u)|^2 + \alpha_2 \mathbb{E} |x(u - \tau)|^2] \, du \, ds \\
& \leq \hat{q} \mathbb{E} |x(t)|^2 + \int_{t-\tau}^{t} [\alpha_1 \tau \mathbb{E} |x(u)|^2 + \alpha_2 \tau \mathbb{E} |x(u - \tau)|^2] \, du \\
& \leq \hat{q} \mathbb{E} |x(t)|^2 + \tau (\alpha_1 + \alpha_2) \int_{t-2\tau}^{t} \mathbb{E} |x(u)|^2 \, du.
\end{align*}
\]
Substituting this and (3.4) into (3.6) and noting that \( \lambda_3 \geq \tau \gamma (\alpha_1 + \alpha_2) \), we get
\[
e^{\gamma t} \mathbb{E} V(\hat{x}_t, r(t), t) \leq C + \int_{2\tau}^{t} e^{\gamma s} \left[ (-\lambda_1 + \gamma \hat{q}) \mathbb{E} |x(s)|^2 + \lambda_2 \mathbb{E} |x(s - \tau)|^2 \right] ds.
\]
But
\[
\int_{2\tau}^{t} e^{\gamma s} \mathbb{E} |x(s - \tau)|^2 \, ds = \int_{\tau}^{t} e^{\gamma (s + \tau)} \mathbb{E} |x(s)|^2 \, ds
\]
\[
\leq \int_{\tau}^{2\tau} e^{\gamma (s + \tau)} \mathbb{E} |x(s)|^2 \, ds + \int_{t-\tau}^{t} e^{\gamma (s + \tau)} \mathbb{E} |x(s)|^2 \, ds.
\]
Hence, recalling that \( \lambda_1 - \gamma \hat{q} \geq \lambda_2 e^{\gamma r} \),
\[
e^{\gamma t} \mathbb{E} V(\hat{x}_t, r(t), t) \leq C + \lambda_2 \int_{\tau}^{2\tau} e^{\gamma (s + \tau)} \mathbb{E} |x(s)|^2 \, ds.
\]
But we clearly have
\[
\mathbb{E} V(\hat{x}_t, r(t), t) \geq \hat{q} \mathbb{E} |x(t)|^2,
\]
where \( \hat{q} = \min_{i \in S} \lambda_{\min}(Q_i) > 0 \). We therefore obtain
\[
\hat{q} e^{\gamma t} \mathbb{E} |x(t)|^2 \leq C + \lambda_2 \int_{\tau}^{2\tau} e^{\gamma (s + \tau)} \mathbb{E} |x(s)|^2 \, ds,
\]
which implies the desired assertion (3.5) immediately. \( \square \)

**Theorem 3.2** Choose five positive numbers \( \alpha_1, \alpha_2, \alpha_3 \) and \( \beta_1, \beta_2 \) such that
\[
\beta_1 > \max_{i \in S} (2\|B_i\|^2), \quad \alpha_1 > \alpha_3 \beta_1, \quad \alpha_2 > \alpha_3 \beta_2. \tag{3.7}
\]
Assume that for these chosen numbers, the following LMIs
\[
\begin{bmatrix}
\bar{K}_i & Y_i G_i \\
G_i^T Y_i^T & -\alpha_3 I
\end{bmatrix} < 0, \quad i \in S \tag{3.8}
\]
have solutions \( \bar{r} > 0 \) and \( Q_i, Y_i \in R^{n \times n} \) with \( Q_i = Q_i^T > 0 \), where \( I \) is the \( n \times n \) identity matrix and
\[
\bar{K}_i = Q_i A_i + Y_i G_i + A_i^T Q_i + G_i^T Y_i + B_i^T Q_i B_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + (\alpha_1 + \alpha_2) \bar{r} I.
\]
Let \( \tau^* \) be the largest number in \( (0, \bar{r}] \) which obeys
\[
\max_{i \in S} (4\tau^* \|A_i\|^2 + 2\|B_i\|^2) \leq \beta_1 \quad \text{and} \quad \max_{i \in S} (4\tau^* \|Q_i^{-1} Y_i G_i\|^2) \leq \beta_2. \tag{3.9}
\]
Then, if \( \tau \leq \tau^* \), by setting
\[
F_i = Q_i^{-1} Y_i, \quad i \in S, \tag{3.10}
\]
the controlled system (2.3) is exponentially stable in mean square.
Proof. We first note that by the well-known Schur complements, the LMIs (3.8) are equivalent to the following matrix inequalities (MIs)

\[ \bar{H}_i := \bar{K}_i + \alpha_3^{-1}Y_iG_i(Y_iG_i)^T < 0, \quad i \in S. \]  

(3.11)

With \( F_i \) defined by (3.10), we compute

\[
2x^T(t)Q_i[A_i x(t) + F_i x(t - \tau)] = 2x^T(t)Q_i[(A_i + F_i x(t) - F_i x(t - \tau))]
\leq x^T(t)[Q_i(A_i + F_i) + (A_i + F_i)^T Q_i] x(t)
+ \alpha_3^{-1}|x(t)|Q_i F_i|^2 + \alpha_3|x(t) - x(t - \tau)|^2.
\]

Hence, we see from (3.3) that

\[
LV(\dot{x}_t, i, t) \leq x^T(t)H_i x(t) - \alpha_2|\tau||x(t)|^2 - |x(t - \tau)|^2 + \alpha_3|x(t) - x(t - \tau)|^2
- \alpha_1 \int_{\tau}^t |x(s)|^2 ds - \alpha_2 \int_{\tau}^t |x(s) - \tau|^2 ds,
\]

where

\[
H_i = K_i + \alpha_3^{-1}Y_i G_i (Y_iG_i)^T,
\]

in which

\[
K_i = Q_i A_i + Y_i G_i + A_i^T Q_i + G_i^T Y_i + B_i^T Q_i B_i + \sum_{j=1}^N \gamma_{ij} Q_j + (\alpha_1 + \alpha_2) \tau I.
\]

As \( \tau \leq \bar{\tau} \), we see from (3.11) that

\[
H_i < 0, \quad i \in S.
\]

(3.13)

Noting

\[
x(t) - x(t - \tau) = \int_{t-\tau}^t [A(r(s)) x(s) + F(r(s)) G(r(s)) x(s-\tau)] ds
+ \int_{t-\tau}^t B(r(s)) x(s) dw(s),
\]

we estimate

\[
E|x(t) - x(t - \tau)|^2 \leq 2 \tau E \int_{t-\tau}^t |A(r(s)) x(s) + F(r(s)) G(r(s)) x(s-\tau)|^2 ds
+ 2 \bar{E} \int_{t-\tau}^t |B(r(s)) x(s)|^2 ds
\leq \bar{\beta}_1 E \int_{t-\tau}^t |x(s)|^2 ds + \bar{\beta}_2 E \int_{t-\tau}^t |x(s-\tau)|^2 ds,
\]

where

\[
\bar{\beta}_1 = \max_{i \in S}(4 \tau \|A_i\|^2 + 2 \|B_i\|^2) \quad \text{and} \quad \bar{\beta}_2 = \max_{i \in S}(4 \tau \|F_i G_i\|^2).
\]

Recalling (3.9) we observe that \( \bar{\beta}_1 \leq \beta_1 \) and \( \bar{\beta}_2 \leq \beta_2 \). Set

\[
-\lambda = \max_{i \in S} \lambda_{\max}(H_i).
\]
By (3.13), \(\lambda > 0\). Now, replacing \(i\) by \(r(t)\) in (3.12), taking the expectation on both sides of (3.12) and then making use of the above estimations we obtain

\[
\mathbb{E}(LV(\hat{x}_t, r(t), t)) \leq - (\lambda + \alpha_2 \tau)\mathbb{E}|x(t)|^2 + \alpha_2 \tau \mathbb{E}|x(t - \tau)|^2 \\
- (\alpha_1 - \alpha_3 \beta_1)\mathbb{E} \int_{t-\tau}^{t} |x(s)|^2 ds \\
- (\alpha_2 - \alpha_3 \beta_2)\mathbb{E} \int_{t-\tau}^{t} |x(s - \tau)|^2 ds \\
\leq - \lambda_1 \mathbb{E}|x(t)|^2 + \lambda_2 \mathbb{E}|x(t - \tau)|^2 - \lambda_3 \mathbb{E} \int_{t-2\tau}^{t} |x(s)|^2 ds, 
\]

(3.14)

where

\[
\lambda_1 := (\lambda + \alpha_2 \tau) > \lambda_2 := \alpha_2 \tau \quad \text{and} \quad \lambda_3 := (\alpha_1 - \alpha_3 \beta_1) \land (\alpha_1 - \alpha_3 \beta_1) > 0.
\]

Hence the conclusion follows from Lemma 3.1. \(\Box\)

4 Output Injection: Design \(G(\cdot)\) when \(F(\cdot)\) is given

Let us now discuss the case where we are given the mapping \(F : S \to R^{n \times l}\) but are required to design the mapping \(G : S \to R^{l \times n}\). The following theorem gives an answer.

**Theorem 4.1** Choose five positive numbers \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta_1, \beta_2\) such that

\[
\beta_1 > \max_{i \in S}(2\|B_i\|^2), \quad \alpha_1 > \alpha_3 \beta_1, \quad \alpha_2 > \alpha_3 \beta_2. 
\]

(4.1)

Choose furthermore \(N\) positive numbers \(\rho_i, \ i \in S\). Assume that for these chosen numbers, the following LMI

\[
\begin{bmatrix}
M_{i1} & F_i Y_i & \sqrt{\alpha_1 + \alpha_2 X_i} & M_{i2} \\
Y_i F_i^T & -\alpha_3 \rho_i I & 0 & 0 \\
\sqrt{\alpha_1 + \alpha_2 X_i} & 0 & -\tau I & 0 \\
M_{i2} & 0 & 0 & -M_{i3}
\end{bmatrix} < 0, \quad i \in S 
\]

(4.2)

and

\[-2X_i + (1 + \rho_i)I < 0, \quad i \in S \]

(4.3)

have solutions \(\bar{\tau} > 0\) and \(X_i, Y_i \in R^{n \times n}\) with \(X_i = X_i^T > 0\), where \(I\) is the \(n \times n\) identity matrix and

\[
M_{i1} = A_i X_i + F_i Y_i + X_i A_i^T + Y_i^T F_i^T + \gamma_i X_i, \\
M_{i2} = \begin{bmatrix}
\sqrt{\gamma_i} X_i, \cdots, \sqrt{\gamma_i} X_i, \\
\sqrt{\gamma_i} X_i, \cdots, \sqrt{\gamma_i} X_i,
\end{bmatrix}, \\
M_{i3} = \text{diag}(X_1, \cdots, X_N).
\]

Let \(\tau^*\) be the largest number in \((0, \bar{\tau})\) which obeys

\[
\max_{i \in S}(4\tau^* \|A_i\|^2 + 2\|B_i\|^2) \leq \beta_1 \quad \text{and} \quad \max_{i \in S}(4\tau^* \|F_i X_i^{-1} Y_i\|^2) \leq \beta_2. 
\]

(4.4)

Then, if \(\tau \leq \tau^*\), by setting

\[
G_i = X_i^{-1} Y_i, \quad i \in S, 
\]

(4.5)

the controlled system (2.3) is exponentially stable in mean square.
Proof. Let $Q_i = X_i^{-1}$ and $V$ be the same as defined by (3.1). We still have (3.12) with
\[ H_i = K_i + \alpha_3^{-1} Q_i F_i G_i G_i^T F_i^T Q_i, \]
in which
\[ K_i = Q_i A_i + Q_i F_i G_i + A_i^T Q_i + G_i^T F_i^T Q_i + B_i^T Q_i B_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + (\alpha_1 + \alpha_2) \tau I. \]
Hence, if we can show that
\[ H_i < 0, \quad i \in S, \tag{4.6} \]
we can still imply (3.14) from (3.12) and hence obtain the assertion by Lemma 3.1 in the same way as Theorem 3.2 was proved. In other words, to complete the proof, all we need to do is to show (4.6).

By the Schur complements, (4.6) holds if and only if
\[
\begin{bmatrix}
U_i & \sqrt{\alpha_1 + \alpha_2 Q_i} & \sqrt{\alpha_1 + \alpha_2 Q_i} \\
G_i^T F_i^T Q_i & -\alpha_3 I & 0 \\
\sqrt{\alpha_1 + \alpha_2 Q_i} & 0 & -\tau^{-1} Q_i^2 
\end{bmatrix} < 0, \quad i \in S, \tag{4.7}
\]
where
\[ U_i = Q_i A_i + Q_i F_i G_i + A_i^T Q_i + G_i^T F_i^T Q_i + B_i^T Q_i B_i + \sum_{j=1}^{N} \gamma_{ij} Q_j. \]
Noting that $X_i = Q_i^{-1}$ and $Y_i = G_i X_i$, we can pre- and post-multiply (4.7) to see that (4.7) is equivalent to
\[
\begin{bmatrix}
X_i U_i X_i & F_i Y_i & \sqrt{\alpha_1 + \alpha_2 X_i} \\
Y_i^T F_i^T & -\alpha_3 X_i^2 & 0 \\
\sqrt{\alpha_1 + \alpha_2 X_i} & 0 & -\tau^{-1} I 
\end{bmatrix} < 0, \quad i \in S, \tag{4.8}
\]
where
\[
X_i U_i X_i = A_i X_i + F_i Y_i + X_i A_i^T + Y_i^T F_i^T + X_i B_i^T X_i^{-1} B_i X_i \\
+ \gamma_i X_i + \sum_{j \neq i} \gamma_{ij} X_i X_j^{-1} X_i. \tag{4.9}
\]
Now, noting that
\[ 0 \leq (X_i - \rho_i I)^2 = X_i^2 - 2 \rho_i X_i + \rho_i^2 I = X_i^2 + \rho_i (-2 X_i + \rho_i) I, \]
and recalling condition (4.3), we observe that
\[ -X_i^2 \leq \rho_i (-2 X_i + \rho_i) I = \rho_i [-2 X_i + (1 + \rho_i) I] - \rho_i I < -\rho_i I. \]
Also, $\tau \leq \tau^* \leq \bar{\tau}^{-1}$ implies $\tau^{-1} \geq \bar{\tau}$. Hence, under (4.3), the matrix inequalities (4.8) are guaranteed by
\[
\begin{bmatrix}
X_i U_i X_i & F_i Y_i & \sqrt{\alpha_1 + \alpha_2 X_i} \\
Y_i^T F_i^T & -\alpha_3 \rho_i I & 0 \\
\sqrt{\alpha_1 + \alpha_2 X_i} & 0 & -\bar{\tau} I 
\end{bmatrix} < 0, \quad i \in S. \tag{4.10}
\]
But, by the Schur complements, these MIs are equivalent to those LMI$s in (4.2). The proof is therefore complete. \qed
5 Stabilisation of Nonlinear Hybrid SDEs

Let us now discuss a more general nonlinear problem. Assume that the underlying system is now described by a nonlinear hybrid SDE
\[ dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t). \] (5.1)

Here, \( f \) and \( g \) are both mappings from \( R^n \times S \times R_+ \) to \( R^n \). Assume that both \( f \) and \( g \) are locally Lipschitz continuous and obey the linear growth condition (see e.g. [21]).

Suppose that the given SDE \( (5.1) \) is unstable and we are required to design a delay feedback control \( u(x(t) - \tau, r(t)) \) in the drift part so that the controlled system
\[ dx(t) = [f(x(t), r(t), t) + u(x(t) - \tau, r(t))]dt + g(x(t), r(t), t)dw(t) \] (5.2)
will be mean-square exponentially stable. As the given SDE \( (5.1) \) is nonlinear, we may need to design nonlinear controls. However, we here consider only a class of SDEs which can be stabilised by linear feedback controls. As in the linear case, we therefore consider the structure control of the form \( u(x, i) = F(i)G(i)x \), where \( F \) and \( G \) are mappings from \( S \) to \( R^{m \times l} \) and \( R^{l \times n} \), respectively, and one of them is given while the other needs to be designed.

As a result, the controlled system \( (5.2) \) becomes
\[ dx(t) = [f(x(t), r(t), t) + F(r(t))G(r(t))x(t - \tau)]dt + g(x(t), r(t), t)dw(t). \] (5.3)

Again, this controlled system is a hybrid SDDE. Given that our underlying equation \( (5.1) \) is non-delay and it only requires the initial value \( x(0) \in R^n \), it is more natural to assume that for our controlled system \( (5.3) \) we know the initial data \( x(t) \) on \( t \in [0, \tau] \). This can be interpreted as follows: Let the underlying equation \( (5.1) \) evolve from time 0 to \( \tau \) and observe the whole segment \( \{x(t) : 0 \leq t \leq \tau\} \). Starting from \( \tau \) on, design the feedback control \( F(r(t))G(r(t))x(t - \tau) \) based on the past observation \( \{x(t) : 0 \leq t \leq \tau\} \) as well as furthermore observation as time evolves. In other words, we shall regard the controlled system \( (5.3) \) as an SDDE on \( t \geq \tau \) with the initial data \( \{x(t) : 0 \leq t \leq \tau\} \) which are generated by the SDE \( (5.1) \) given the initial value \( x(0) \in R^n \). By the theory of hybrid SDEs (see e.g. [14]), we know
\[ \mathbb{E}|x(t)|^2 < \infty \quad \text{on} \quad t \in [0, \tau], \]
which in turn implies, by the theory of hybrid SDDEs (see e.g. [21]), that
\[ \mathbb{E}|x(t)|^2 < \infty \quad \text{for} \quad t \geq \tau. \]

Our aim is to design either \( G(\cdot) \) given \( F(\cdot) \) or \( F(\cdot) \) given \( G(\cdot) \) so that \( \mathbb{E}|x(t)|^2 \) will tend to zero exponentially.

We still use the Lyapunov functional defined by \( (3.1) \). By the Itô formula (see e.g. [16, 21]), we have
\[ dV(\hat{x}_t, r(t), t) = LV(\hat{x}_t, r(t), t)dt + 2x^T(t)Q(r(t))g(x(t), r(t), t)dt, \] (5.4)
for \( t \geq 2\tau \), where, when \( r(t) = i \),

\[
LV(\dot{x}_i, i, t) = 2x^T(t)Q_i[f(x(t), i, t) + F_iG_i x(t - \tau)] + g^T(x(t), i, t)Q_i g(x(t), i, t) + \sum_{j=1}^{N} \gamma_{ij}x^T(t)Q_j x(t)
\]

\[
+ \alpha_1 \tau|x(t)|^2 - \alpha_1 \int_{t-\tau}^{t} |x(s)|^2 ds
\]

\[
+ \alpha_2 \tau|x(t - \tau)|^2 - \alpha_2 \int_{t-2\tau}^{t} |x(s - \tau)|^2 ds.
\]

(5.5)

Clearly, Lemma 3.1 still holds for the non-linear controlled SDDE (5.3) with \( LV \) being defined by (5.5).

Given that we use a linear control to stabilise a nonlinear system, it is natural to impose some conditions on the nonlinear coefficients \( f \) and \( g \). More precisely, we observe from (5.5) that we need to use the linear term \( 2x^T(t)Q_iF_iG_i x(t - \tau) \) to control the nonlinear terms \( 2x^T(t)Q_i f(x, i, t) \) and \( g^T(x(t), i, t)Q_i g(x(t), i, t) \). This observation leads us to impose the following assumption.

**Assumption 5.1** For each \( i \in S \), there is a pair of symmetric \( n \times n \)-matrices \( Q_i \) and \( \bar{Q}_i \) with \( Q_i \) being positive-definite such that

\[
2x^T(t)Q_i f(x, i, t) + g^T(x(t), i, t)Q_i g(x(t), i, t) \leq x^T \bar{Q}_i x
\]

for all \((x, i, t) \in R^n \times S \times R_+ \).

Moreover, we will write

\[
2x^T(t)Q_iF_iG_i x(t - \tau) = 2x^T(t)Q_iF_iG_i \left( x(t) - [x(t) - x(t - \tau)] \right)
\]

\[
\leq 2x^T(t)Q_iF_iG_i x(t) + \alpha_2^{-1}|x^T(t)Q_iF_iG_i|^2 + \alpha_3|x(t) - x(t - \tau)|^2,
\]

(5.6)

whence we need to estimate \( \mathbb{E}|x(t) - x(t - \tau)|^2 \). For this purpose we impose one more assumption.

**Assumption 5.2** There is a pair of positive constants \( \delta_1 \) and \( \delta_2 \) such that

\[
|f(x, i, t)|^2 \leq \delta_1|x|^2 \quad \text{and} \quad |g(x, i, t)|^2 \leq \delta_2|x|^2
\]

for all \((x, i, t) \in R^n \times S \times R_+ \).

### 5.1 Design \( F(\cdot) \) given \( G(\cdot) \)

Let us first consider the case when \( G(\cdot) \) is given so we need to design \( F(\cdot) \).

**Theorem 5.3** Let Assumptions 5.1 and 5.2 hold. Choose five positive numbers \( \alpha_1, \alpha_2, \alpha_3 \) and \( \beta_1, \beta_2 \) such that

\[
\beta_1 > 2\delta_2, \quad \alpha_1 > \alpha_3\beta_1, \quad \alpha_2 > \alpha_3\beta_2.
\]

(5.7)
Assume that for these chosen numbers, the following LMIs
\[
\begin{bmatrix}
\bar{K}_i & Q_i F_i G_i \\
G_i^T F_i^T Q_i & -\alpha_3 I
\end{bmatrix} < 0, \quad i \in S
\]  
(5.8)
have solutions \(\bar{\tau} > 0\) and \(F_i \in \mathbb{R}^{n \times 1}\), where \(I\) is the \(n \times n\) identity matrix and
\[
\bar{K}_i = \bar{Q}_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + Q_i F_i G_i + G_i^T F_i^T Q_i + (\alpha_1 + \alpha_2) \bar{\tau} I.
\]

Let \(\tau^*\) be the largest number in \((0, \bar{\tau})\) which obeys
\[
4\tau^* \delta_1 + 2\delta_2 \leq \beta_1 \quad \text{and} \quad 4\tau^* \max_{t \in \mathbb{S}}(\| F_i G_i \|) \leq \beta_2.
\]  
(5.9)
Then, if \(\tau \leq \tau^*\), the controlled system (5.3) is exponentially stable in mean square.

**Proof.** By Assumption 5.1, we derive from (5.5) that
\[
LV(\hat{x}_t, i, t) \leq 2x^T(t)\bar{Q}_i x(t) + 2x^T(t)Q_i F_i G_i x(t - \tau) + \sum_{j=1}^{N} \gamma_{ij} x^T(t)Q_j x(t)
\]
\[
+ \alpha_1 \tau |x(t)|^2 - \alpha_1 \int_{t-\tau}^{t} |x(s)|^2 ds
\]
\[
+ \alpha_2 \tau |x(t - \tau)|^2 - \alpha_2 \int_{t-\tau}^{t} |x(s - \tau)|^2 ds.
\]  
(5.10)
Using (5.6) we then have
\[
LV(\hat{x}_t, i, t) \leq x^T(t)H_i x(t) - \alpha_2 \tau (|x(t)|^2 - |x(t - \tau)|^2) + \alpha_3 |x(t) - x(t - \tau)|^2
\]
\[
- \alpha_1 \int_{t-\tau}^{t} |x(s)|^2 ds - \alpha_2 \int_{t-\tau}^{t} |x(s - \tau)|^2 ds,
\]  
(5.11)
where
\[H_i = K_i + \alpha_3^{-1} Q_i F_i G_i (Q_i F_i G_i)^T,\]
in which
\[K_i = \bar{Q}_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + Q_i F_i G_i + G_i^T F_i^T Q_i + (\alpha_1 + \alpha_2) \tau I.\]

However, by the well-known Schur complements, the LMIs (5.8) are equivalent to the following MIs
\[
\bar{H}_i := \bar{K}_i + \alpha_3^{-1} Q_i F_i G_i (Q_i F_i G_i)^T < 0, \quad i \in S.
\]  
(5.12)
As \(\tau \leq \bar{\tau}\), we then have
\[
H_i < 0, \quad i \in S.
\]  
(5.13)
Noting
\[
 x(t) - x(t - \tau) = \int_{t-\tau}^{t} f(x(s), r(s), s) + F(r(s))G(r(s))x(s - \tau)ds
\]
\[
+ \int_{t-\tau}^{t} g(x(s), r(s), s)dw(s),
\]
we estimate, by Assumption 5.2, that

\[
\mathbb{E}|x(t) - x(t - \tau)|^2 \leq 2\tau \mathbb{E} \int_{t-\tau}^{t} |f(x(s), r(s), s) + F(r(s))G(r(s))x(s - \tau)|^2 ds + 2\mathbb{E} \int_{t-\tau}^{t} |g(x(s), r(s), s)|^2 ds
\]

\[
\leq \beta_1 \mathbb{E} \int_{t-\tau}^{t} |x(s)|^2 ds + \beta_2 \mathbb{E} \int_{t-\tau}^{t} |x(s - \tau)|^2 ds,
\]

where

\[
\beta_1 = 4\tau \delta_1 + 2\delta_2 \quad \text{and} \quad \beta_2 = 4\tau \max_{i \in S} (\|F_i G_i\|^2).
\]

Recalling (5.9) we observe that \(\beta_1 \leq \beta_1\) and \(\beta_2 \leq \beta_2\). Set

\[
-\lambda = \max_{i \in S} \lambda_{\max}(H_i).
\]

By (5.13), \(\lambda > 0\). Now, replacing \(i\) by \(r(t)\) in (5.11), taking the expectation on both sides of (5.11) and then making use the above estimations we obtain

\[
\mathbb{E}(LV(x_i, r(t), t)) \leq -(\lambda + \alpha_2 \tau) \mathbb{E}|x(t)|^2 + \alpha_2 \tau \mathbb{E}|x(t - \tau)|^2
\]

\[
- (\alpha_1 - \alpha_3 \beta_1) \mathbb{E} \int_{t-\tau}^{t} |x(s)|^2 ds
\]

\[
- (\alpha_2 - \alpha_3 \beta_2) \mathbb{E} \int_{t-\tau}^{t} |x(s - \tau)|^2 ds
\]

\[
\leq -\lambda_1 \mathbb{E}|x(t)|^2 + \lambda_2 \mathbb{E}|x(t - \tau)|^2 - \lambda_3 \mathbb{E} \int_{t-2\tau}^{t} |x(s)|^2 ds, \tag{5.14}
\]

where

\[
\lambda_1 := (\lambda + \alpha_2 \tau) > \lambda_2 := \alpha_2 \tau \quad \text{and} \quad \lambda_3 := (\alpha_1 - \alpha_3 \beta_1) \wedge (\alpha_1 - \alpha_3 \beta_1) > 0.
\]

Hence the conclusion follows from Lemma 3.1. \(\square\)

Theorem 5.3 depends on the choices of \(2N\) matrices \(Q_i\) and \(Q_i\). In theory, it is flexible, but in practice, it means more work to be done in finding these \(2N\) matrices. It is in this spirit that we introduce a stronger assumption.

**Assumption 5.4** There are \(N + 1\) symmetric \(n \times n\) matrices \(Z\) and \(Z_i (i \in S)\) with \(Z\) being positive-definite such that

\[
2x^T Z f(x, i, t) + g^T(x, i, t) Z g(x, i, t) \leq x^T Z_i x
\]

for all \((x, i, t) \in R^n \times S \times R_+\).

**Corollary 5.5** Let Assumptions 5.2 and 5.4 hold. Choose five positive numbers \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta_1, \beta_2\) which obey (5.7). Assume that for these chosen numbers, the following LMIs

\[
\begin{bmatrix}
\bar{K}_i & Y_i G_i \\
G_i^T Y_i^T & -\alpha_3 I
\end{bmatrix} < 0, \quad i \in S \tag{5.15}
\]

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have solutions $\bar{\tau} > 0$, $q_i > 0$ and $Y_i \in R^{n \times l}$, where $I$ is the $n \times n$ identity matrix and

$$
\bar{K}_i = q_i Z_i + \sum_{j=1}^{N} \gamma_{ij} q_j Z_i + Y_i G_i + G_i^T Y_i^T + (\alpha_1 + \alpha_2) \bar{\tau} I.
$$

Let $\tau^*$ be the largest number in $(0, \bar{\tau}]$ which obeys

$$
4\tau^* \delta_1 + 2 \delta_2 \leq \beta_1 \quad \text{and} \quad 4\tau^* \max_{i \in S} (\|q_i Z_i^{-1} Y_i G_i\|^2) \leq \beta_2.
$$

(5.16)

Then, if $\tau \leq \tau^*$, by setting

$$
F_i = (q_i Z_i)^{-1} Y_i, \quad i \in S,
$$

the controlled system (5.3) is exponentially stable in mean square.

Proof. Assumption 5.4 implies

$$
2x^T(q_i Z_i) f(x, i, t) + g^T(x, i, t)(q_i Z_i) g(x, i, t) \leq x^T(q_i Z_i) x.
$$

This means that Assumption 5.1 holds with $Q_i = q_i Z_i$ and $\bar{Q}_i = q_i Z_i$. Hence the corollary follows immediately from Theorem 5.3.

An even simpler (but in fact stronger) condition is:

**Assumption 5.6** There are constants $z_i$ ($i \in S$) such that

$$
2x^T f(x, i, t) + |g(x, i, t)|^2 \leq z_i |x|^2
$$

for all $(x, i, t) \in R^n \times S \times R_+$.

This assumption implies Assumption 5.4 with $Z = I$ and $Z_i = z_i I$. Hence, under Assumptions 5.2 and 5.6, Corollary 5.5 holds with $Z = I$ and $Z_i = z_i I$.

### 5.2 Design $G(\cdot)$ given $F(\cdot)$

Let us now consider the case when $F(\cdot)$ is given so we need to design $G(\cdot)$. The results established in the previous subsection work for this case as long as we treat $F$ as given and seek for $G$. To be precise, let us state:

**Theorem 5.7** Theorem 5.3 holds for this case, if the LMIs (5.8) have solutions $\bar{\tau} > 0$ and $G_i \in R^{l \times n}$.

**Corollary 5.8** Let Assumptions 5.2 and 5.4 hold. Choose five positive numbers $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2$ which obey (5.7). Assume that for these chosen numbers, the following LMIs

$$
\begin{bmatrix}
\bar{K}_i & Z Y_i^T \\
Y_i^T F_i^T Z & -\alpha_3 I
\end{bmatrix} < 0, \quad i \in S
$$

(5.17)

have solutions $\bar{\tau} > 0$, $q_i > 0$ and $Y_i \in R^{n \times n}$, where $I$ is the $n \times n$ identity matrix and

$$
\bar{K}_i = q_i Z_i + \sum_{j=1}^{N} \gamma_{ij} q_j Z_i + Z F_i Y_i + Y_i^T F_i^T Z + (\alpha_1 + \alpha_2) \bar{\tau} I.
$$

Let $\tau^*$ be the largest number in $(0, \bar{\tau}]$ which obeys

$$
4\tau^* \delta_1 + 2 \delta_2 \leq \beta_1 \quad \text{and} \quad 4\tau^* \max_{i \in S} (\|q_i^{-1} F_i Y_i\|^2) \leq \beta_2.
$$

(5.18)

Then, if $\tau \leq \tau^*$, by setting

$$
G_i = q_i^{-1} Y_i, \quad i \in S,
$$

the controlled system (5.3) is exponentially stable in mean square.
6 Examples

Let us now discuss some examples to illustrate our theory.

Example 6.1 Consider the controlled SDDE (2.3) with the system matrices given below:

\[ S = \{1, 2\}, \quad \Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}; \]

\[ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}; \quad G_1 = (1, 0), \quad G_2 = (0, 1). \]

Our aim here is to seek for a mapping \( F : S \rightarrow \mathbb{R}^{2 \times 1} \) and \( \tau^* > 0 \) such that if \( \tau \leq \tau^* \), then the controlled SDDE (2.3) is exponentially stable in mean square. To apply Theorem 3.2, we choose

\[ \alpha_1 = 61, \quad \alpha_2 = 101, \quad \alpha_3 = 10, \quad \beta_1 = 6, \quad \beta_2 = 10. \]

Noting \( \| B_1 \|^2 = \| B_2 \|^2 = 2 \), we see that these positive numbers satisfy (3.7). It is also not difficult to verify that the LMIs (3.8) have solutions

\[ \tau = 0.02, \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix}. \]

It is also easy to compute

\[ \| A_1 \|^2 = \| A_2 \|^2 = 27.42, \quad \| Q_1^{-1} Y_1 G_1 \|^2 = \| Q_1^{-1} Y_1 G_1 \|^2 = 100. \]

By (3.9), \( \tau^* \) is the largest number in \((0, 0.02]\) which obeys

\[ 4 \times 27.42 \tau^* + 4 \leq 6 \quad \text{and} \quad 400 \tau^* \leq 10, \]

whence \( \tau^* = 0.0182 \). By Theorem 3.2, if \( \tau \leq 0.0182 \), by setting

\[ F_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix}, \]

the corresponding controlled SDDE will be exponentially stable in mean square.

Example 6.2 Let us now discuss one more example, where we will not only illustrate our theory but also explain a new concept which may motivate a further research.

Let \( r(t), t \geq 0 \), be a right-continuous Markov chain on the probability space taking values in the state space \( S = \{1, 2\} \) with generator

\[ \Gamma = \begin{bmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{bmatrix}. \]

Consider an unstable nonlinear hybrid SDE

\[ dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t). \quad (6.1) \]

Here, \( f \) and \( g \) are both mappings from \( \mathbb{R}^n \times S \times R_+ \) to \( \mathbb{R}^n \). This SDE may be regarded as a system which switches between two operation modes, say mode 1 and mode 2, and
the switching obeys the law of the Markov chain, where in mode 1, the system evolves according to the SDE

$$dx(t) = f(x(t), 1, t)dt + g(x(t), 1, t)dw(t),$$

while in mode 2, according to the other SDE

$$dx(t) = f(x(t), 2, t)dt + g(x(t), 2, t)dw(t).$$

Assume that in mode 1, the state $x(t)$ can be observed completely but in mode 2, it is not observable. Therefore, we can design a feedback control in mode 1, with some time lag of course, but we cannot have a feedback control in mode 2. In terms of mathematics, the controlled SDE is

$$dx(t) = [f(x(t), r(t), t) + F(r(t))G(r(t))x(t - \tau)]dt + g(x(t), r(t), t)dw(t),$$

where $G_1 = I$, the $n \times n$ identity matrix but $G_2 = 0$. Given $G_2 = 0$ we can simply set $F_2 = 0$. Hence, the stabilisation problem becomes: can we find a matrix $F_1 \in \mathbb{R}^{n \times n}$ so that the controlled SDE (6.2) becomes exponentially stable in mean square?

To give a positive answer to the question, we assume that $f$ and $g$ obey Assumptions 5.2 and 5.6. Moreover, we assume that

$$\gamma_{21} - z_2 \geq 3\gamma_{12}. \tag{6.3}$$

This means that the rate at which the system switches from the unobservable mode 2 to the observable mode 1 should be sufficiently larger than the rate from mode 1 to mode 2. This is reasonable because the system in mode 2 is not controllable while it is controllable (hence stabilisable) in mode 1.

To apply Corollary 5.5, we need to find five positive numbers $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2$ that obey

$$\beta_1 > 2\delta_2, \quad \alpha_1 > \alpha_3\beta_1, \quad \alpha_2 > \alpha_3\beta_2, \tag{6.4}$$

as well as to find positive numbers $\bar{\tau}, q_1, q_2$ and a matrix $Y_1 \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} \bar{K}_1 & Y_1 \\ Y_1^T & -\alpha_3I \end{bmatrix} < 0 \quad \text{and} \quad \bar{K}_2 < 0, \tag{6.5}$$

where

$$\bar{K}_1 = q_1z_1I + \gamma_{12}(q_2 - q_1)I + Y_1 + Y_1^T + (\alpha_1 + \alpha_2)\tau I,$$

and

$$\bar{K}_2 = q_2z_2I + \gamma_{21}(q_1 - q_2)I + (\alpha_1 + \alpha_2)\tau I.$$ 

We first choose $\beta_1 > 2\delta_2$ and $\beta_2 > 0$. Let $\alpha_3 > 0$ be a free parameter to be determined and set

$$\alpha_1 = \alpha_3\beta_1 + 1, \quad \alpha_2 = \alpha_3\beta_2 + 1. \tag{6.6}$$

Clearly, (6.6) is therefore satisfied. Choose furthermore that

$$\bar{\tau} = \frac{1}{3(\beta_1 + \beta_2)}. \tag{6.7}$$
Now, set
\[ q_1 = 1, \quad q_2 = \frac{\alpha_3}{\gamma_{21} - z_2}, \quad Y_1 = -\alpha_3 I. \] (6.8)

We then have
\[ \bar{K}_1 = \left( z_1 - \gamma_{12} + \frac{\gamma_{12}\alpha_3}{\gamma_{21} - z_2} - 2\alpha_3 + [\alpha_3(\beta_1 + \beta_2) + 2|\bar{\tau}|]I \right) \]
\[ \leq \left( z_1 - \gamma_{12} - 4\frac{3}{3}\alpha_3 + 2|\bar{\tau}| \right)I \] (6.9)

and
\[ \bar{K}_2 = \left( \gamma_{21} - \alpha_3 + [\alpha_3(\beta_1 + \beta_2) + 2|\bar{\tau}|]I \right) = \left( \gamma_{21} - \frac{2}{3}\alpha_3 + 2|\bar{\tau}| \right)I. \] (6.10)

We now set
\[ \alpha_3 = [3(z_1 - \gamma_{12} + 2|\bar{\tau} + 1|)] \lor [1.5(\gamma_{21} + 2|\bar{\tau} + 1|)]. \] (6.11)

Then
\[ \bar{K}_1 \leq - (\alpha_3 + 1)I \quad \text{and} \quad \bar{K}_2 \leq - I, \]
whence
\[ \begin{bmatrix} \bar{K}_1 & Y_1 \\ Y_1^T & -\alpha_3 I \end{bmatrix} \leq \begin{bmatrix} - (\alpha_3 + 1)I & -\alpha_3 I \\ -\alpha_3 I & -\alpha_3 I \end{bmatrix} < 0, \]

namely (6.5) is satisfied. In summary, we choose \( \beta_1 > 2\delta_2, \beta_2 > 0, \) and set \( \bar{\tau}, \alpha_3, \alpha_1 \) and \( \alpha_2 \) by (6.7), (6.11) and (6.6), respectively, then \( \bar{\tau} \) and \( q_1, q_2, Y_1 \) specified by (6.8) obey the LMI (6.5).

Finally, let \( \tau^* \) be the largest number in \( (0, \bar{\tau}] \) that obeys
\[ 4\tau^*\delta_1 + 2\delta_2 \leq \beta_1 \quad \text{and} \quad 4\tau^*(\gamma_{21} - z_2) \leq \beta_2, \]

namely
\[ \tau^* = \left( \frac{1}{3(\beta_1 + \beta_2)} \right) \land \left( \frac{\beta_1 - 2\delta_2}{4\delta_2} \right) \land \left( \frac{\beta_2}{4(\gamma_{21} - z_2)} \right). \] (6.12)

Then, by Corollary 5.5, we can conclude that if \( \tau \leq \tau^* \), by setting
\[ F_1 = -\alpha_3 I \]
the controlled system (6.2) is exponentially stable in mean square.

## 7 Further Comments

In this paper we have shown clearly that unstable hybrid SDEs can be stabilised by delay state feedback and output injection. Let us make a few comments to close our paper.

First of all, we emphasise once again that to make our theory more understandable as well as to avoid complicated notations, we have restricted our underlying hybrid systems to a relatively simple form, namely the hybrid SDE (1.1) driven by a scalar Brownian motion. Our theory can certainly be generalised to cope with more general hybrid SDEs driven by multi-dimensional Brownian motions as well as SDDEs.

Mathematically speaking, our stability analysis is based on the Lyapunov functional defined by (3.1). It is certainly possible to design more general Lyapunov functionals to obtain more general stabilisation criteria, for example
\[ V(\dot{x}, r(t), t) = x^T(t)Q(r(t))x(t) + \int_{t-\tau}^{t} \left[ x^T(u)U_1x(u) + x^T(u-\tau)U_2x(u-\tau) \right] duds, \]
with $U_1 = U_1^T \geq 0$ and $U_2 = U_2^T \geq 0$.

Moreover, Example 6.2 demonstrates that it is possible to stabilise an unstable hybrid SDE even though we can only control the system in some modes. The idea illustrated there can be developed into a general partial control problem but we will report elsewhere due to the page limit of this paper.

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