Almost Surely Asymptotic Stability of Neutral Stochastic Differential Delay Equations with Markovian Switching

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Abstract

The main aim of this paper is to discuss the almost surely asymptotic stability of the neutral stochastic differential delay equations (NSDDEs) with Markovian switching. Linear NSDDEs with Markovian switching and nonlinear examples will be discussed to illustrate the theory.

Key words: Asymptotic stability, exponential stability, generalized Itô formula, Brownian motion, Markov chain.

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1 Introduction

Many dynamical systems not only depend on present and past states but also involve derivatives with delays. Hale and Lune [7] have studied deterministic neutral differential delay equations (NDDEs) and their stability. Taking the environmental disturbances into account, Kolmanovskii and Nosov [13] and Mao [15] discussed the neutral stochastic differential delay equations (NSDDEs)

$$d[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dB(t). \tag{1.1}$$

Kolmanovskii and Nosov [13] not only established the theory of existence and uniqueness of the solution to Eq. (1.1) but also investigated the stability and asymptotic stability of the equations, while Mao [15] studied the exponential stability of the equations.

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On the other hand, many practical systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems driven by continuous-time Markov chains have recently been developed to cope with such situation. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. Along the trajectories of the Markovian jump system, the mode switches from one value to another in a random way, governed by a Markov process with discrete state space; the evolution of this Markov process may also depend on the continuous state. The continuous state, on the other hand, flows along the solution of an ordinary or stochastic differential equation; the dynamics of this differential equation may depend on the value of the mode at the given time. In general, the continuous state may also display instantaneous jumps, concurrently or independently of the jumps of the mode. In the special case where the evolution of the continuous state does not display any jumps the resulting stochastic process is typically referred to as a switching diffusion process. An important class of hybrid systems is the jump linear systems

$$\dot{x}(t) = A(r(t))x(t) \tag{1.2}$$

where a part of the state $x(t)$ takes values in $\mathbb{R}^n$ while another part of the state $r(t)$ is a Markov chain taking values in $\mathbb{S} = \{1, 2, \cdots, N\}$. One of the important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the analysis of stability. For more detailed account on hybrid systems please see [1, 4, 5, 9, 10, 11, 19, 20, 22, 23, 24, 26, 27, 29].

Motivated by hybrid systems, Kolmanovskii et al [12] studied the NSDDEs with Markovian switching

$$\begin{aligned} & d[x(t) - D(x(t-\tau), r(t))] \\ & = f(x(t), x(t-\tau), t, r(t)) + g(x(t), x(t-\tau), t, r(t))dB(t). \end{aligned} \tag{1.3}$$

In [12], the existence and uniqueness of the solution to Eq. (1.3) are discussed and, moreover, both moment asymptotic boundedness and moment exponential stability are investigated. In this paper, we will mainly discuss the almost surely asymptotic stability of the equation.

2 NSDDEs with Markovian Switching

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm for vectors or the trace norm for matrices but $\|\cdot\|$ denote the operator norm for matrices. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its biggest and smallest eigenvalue respectively. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions on $[-\tau, 0]$. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$.

Let $r(t),\ t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j \\ 1 + \gamma_{ij}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is transition rate from i to j if $i \ne j$ while

$$\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(t)$ is a right continuous step function. It is useful to recall that a continuous-time Markov chain $r(t)$ with generator $\Gamma = \{\gamma_{ij}\}_{N\times N}$ can be represented as a stochastic integral with respect to a Poisson random measure (cf. [2, 6]):

$$dr(t) = \int_{\mathbb{R}} \bar{h}(r(t-), y)\nu(dt, dy), \quad t \ge 0 \tag{2.1}$$

with initial value $r(0) = i_0 \in \mathbb{S}$, where $\nu(dt, dy)$ is a Poisson random measure with intensity $dt \times m(dy)$ in which m is the Lebesgue measure on $\mathbb{R}$ while the explicit definition of $\bar{h} : S \times \mathbb{R} \to \mathbb{R}$ can be found in [2, 6] but we will not need it in this paper.

Consider an NSDDE with Markovian switching of the form

$$d[x(t) - D(x(t-\tau), r(t))] = f(x(t), x(t-\tau), t, r(t))dt + g(x(t), x(t-\tau), t, r(t))dB(t) \tag{2.2}$$

on $t \ge 0$ with initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0 \in \mathbb{S}$, where

$$D : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n, \quad f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n \text{ and } g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}.$$

In this paper the following assumption is imposed as a standing hypothesis.

Assumption 2.1 *Assume that both f and g satisfy the local Lipschitz condition. That is, for each $h > 0$, there is an $L_h > 0$ such that*

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)| \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)| \le L_h(|x - \bar{x}| + |y - \bar{y}|)$$

for all $(t, i) \in \mathbb{R}_+ \times \mathbb{S}$ and those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $x \vee y \vee \bar{x} \vee \bar{y} \le h$. Assume also that for each $i \in \mathbb{S}$, there is a constant $\kappa_i \in (0, 1)$ such that

$$|D(x, i) - D(y, i)| \le \kappa_i |x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Assume moreover that for all $(t, i) \in \mathbb{R}_+ \times \mathbb{S}$,

$$D(0, i) = 0, \quad f(0, 0, t, i) = 0, \quad g(0, 0, t, i) = 0.$$

In general, this Assumption will only guarantee a unique maximal local solution to Eq. (2.2) for any given initial data ξ and i_0. However, the additional conditions imposed in our main result, Theorem 3.1 below, will guarantee that this maximal local solution is in fact a unique global solution (see Theorem A.1 below), which is denoted by $x(t;\xi,i_0)$. To state our main result, we will need a few more notations. Let $C(\mathbb{R}^n;\mathbb{R}_+)$ and $C(\mathbb{R}^n \times [-\tau,\infty);\mathbb{R}_+)$ denote the families of all continuous non-negative functions defined on $\mathbb{R}^n$ and $\mathbb{R}^n \times [-\tau,\infty)$, respectively. Denote by $L^1(\mathbb{R}_+;\mathbb{R}_+)$ the family of all functions $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_0^\infty \gamma(t)dt<\infty$. If K is a subset of $\mathbb{R}^n$, denote by $d(x,K)$ the Haussdorff semi-distance between $x\in\mathbb{R}^n$ and the set K, namely $d(x,K)=\inf_{y\in K}|x-y|$. If W is a real-valued function defined on $\mathbb{R}^n$, then its kernel is denoted by $Ker(W)$, namely $Ker(W)=\{x\in\mathbb{R}^n : W(x)=0\}$. Let $C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$ denote the family of all non-negative functions $V(x,t,i)$ on $\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}$ that are continuously twice differentiable in x and once in t. If $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$, define an operator LV from $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned} LV(x,y,t,i) &= V_t(x-D(y,i),t,i)+V_x(x-D(y,i),t,i)f(x,y,t,i)\\ &\quad+\frac{1}{2}\text{trace}[g^T(x,y,t,i)V_{xx}(x-D(y,i),t,i)g(x,y,t,i)]\\ &\quad+\sum_{j=1}^{N}\gamma_{ij}V(x-D(y,i),t,j), \end{aligned} \tag{2.3}$$

where

$$V_t(x,t,i)=\frac{\partial V(x,t,i)}{\partial t},\quad V_x(x,t,i)=\left(\frac{\partial V(x,t,i)}{\partial x_1},\dots,\frac{\partial V(x,t,i)}{\partial x_n}\right)$$

and

$$V_{xx}(x,t,i)=\left(\frac{\partial^2 V(x,t,i)}{\partial x_i\partial x_j}\right)_{n\times n}.$$

For the convenience of the reader we cite the the generalized Itô's formula (cf. [25]): If $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S})$, then for any $t\ge 0$

$$\begin{aligned} &V(x(t)-D(x(t-\tau),r(t)),t,r(t)) = V(x(0)-D(\xi(-\tau),r(0)),0,r(0))\\ &+\int_0^t LV(x(s),x(s-\tau),s,r(s))ds+\int_0^t V_x(x(s)-D(x(s-\tau),r(s)),s,r(s))dB(s)\\ &+\int_0^t\int_R (V(x(s)-D(x(s-\tau),r(s)),s,i_0+\bar{h}(r(s-),l))\\ &\qquad -V(x(s)-D(x(s-\tau),r(s)),s,r(s)))\mu(ds,dl), \end{aligned} \tag{2.4}$$

where $\mu(ds,dl)=\nu(ds,dl)-m(dl)ds$ is a martingale measure.

Before we state our main result, let us cite the useful convergence theorem of non-negative semimartingales (see [14, Theorem 7 on p.139]) as a lemma.

Lemma 2.2 *Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing processes on $t\ge 0$ with $A_1(0)=A_2(0)=0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0)=0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $E\zeta<\infty$. Define*

$$X(t)=\zeta+A_1(t)-A_2(t)+M(t) \text{ for } t\ge 0.$$

If $X(t)$ is nonnegative, then

$$\left\{\lim_{t\to\infty} A_1(t) < \infty\right\} \subset \left\{\lim_{t\to\infty} X(t) < \infty\right\} \cap \left\{\lim_{t\to\infty} A_2(t) < \infty\right\} \quad a.s.$$

where $C \subset D$ a.s. means $\mathbb{P}(C \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A_1(t) < \infty$ a.s., then, with probability one,

$$\lim_{t\to\infty} X(t) < \infty, \quad \lim_{t\to\infty} A_2(t) < \infty$$

and

$$-\infty < \lim_{t\to\infty} M(t) < \infty.$$

That is, all of the three processes $X(t)$, $A_2(t)$ and $M(t)$ converge to finite random variables.

3 Almost Surely Asymptotic Stability

With the notations above, we can now state our main result in this paper.

Theorem 3.1 *Let Assumption 2.1 hold. Assume that there are functions $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $U \in C(\mathbb{R}^n \times [-\tau, \infty); \mathbb{R}_+)$ and $W \in C(\mathbb{R}^n; \mathbb{R}_+)$ such that*

$$LV(x, y, t, i) \le \gamma(t) - U(x, t) + U(y, t-\tau) - W(x - D(y, i)) \tag{3.1}$$

for $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, and

$$\lim_{|x|\to\infty} \left[\inf_{(t,i)\in\mathbb{R}_+\times\mathbb{S}} V(x, t, i)\right] = \infty. \tag{3.2}$$

Then for any initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0 \in \mathbb{S}$, Eq. (2.2) has a unique global solution which is denoted by $x(t; \xi, i_0)$. Moreover, the solution obeys that

$$\limsup_{t\to\infty} V(x(t; \xi, i_0) - D(x(t-\tau; \xi, i_0), r(t)), t, r(t)) < \infty \quad a.s. \tag{3.3}$$

and $Ker(W) \neq \emptyset$ and

$$\lim_{t\to\infty} d(x(t; \xi, i_0) - D(x(t-\tau; \xi, i_0), r(t)), Ker(W)) = 0 \quad a.s. \tag{3.4}$$

In particular, if W moreover has the property that

$$W(x) = 0 \text{ if and only if } x = 0, \tag{3.5}$$

then the solution further obeys that

$$\lim_{t\to\infty} x(t; \xi, i_0) = 0 \quad a.s. \tag{3.6}$$

Proof. The existence and uniqueness of the solution follows from Theorem A.1 we therefore need only to prove the other assertions here. As the whole proof is very technical, we will divide it into five steps.

Step 1. Let us first show assertion (3.3). Fix any initial data ξ and i_0 and write $x(t;\xi,i_0) = x(t)$ for simplicity. By the generalised Itô formula (2.4) and condition (3.1) we have

$$
\begin{aligned}
&V(x(t)-D(x(t-\tau),r(t)),t,r(t))\\
&\le V(x(0)-D(\xi(-\tau),r(0)),0,r(0)) + M(t)\\
&+\int_0^t \Big[\gamma(s) - U(x(s),s) + U(x(s-\tau),s-\tau) - W(x(s)-D(x(s-\tau),r(s)))\Big]ds\\
&\le V(x(0)-D(\xi(-\tau),i_0),0,i_0) + \int_{-\tau}^0 U(\xi(s),s)ds\\
&+\int_0^t \gamma(s)ds - \int_0^t W(x(s)-D(x(s-\tau),r(s)))ds + M(t),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
M(t) = &\int_0^t V_x(x(s)-D(x(s-\tau),r(s)),s,r(s))g(x(s),x(s-\tau),s,r(s))dB(s)\\
&+\int_0^t\int_R (V(x(s)-D(x(s-\tau),r(s)),s,i_0+\bar{h}(r(s-),l))\\
&\qquad - V(x(s)-D(x(s-\tau),r(s)),s,r(s)))\mu(ds,dl),
\end{aligned}
$$

which is a continuous local martingale with $M(0) = 0$ a.s. Applying Lemma 2.2 we immediately obtain that

$$
\limsup_{t\to\infty} V(x(t)-D(x(t-\tau),r(t)),t,r(t)) < \infty \quad a.s. \tag{3.8}
$$

which is the required assertion (3.3). It then follows easily that

$$
\sup_{0\le t<\infty} V(x(t)-D(x(t-\tau),r(t)),t,r(t)) < \infty \quad a.s.
$$

This, together with (3.2), yields

$$
\sup_{0\le t<\infty} |x(t)-D(x(t-\tau),r(t))| < \infty. \tag{3.9}
$$

But for any $T > 0$, by Assumption 2.1, we have, if $0 \le t \le T$,

$$
\begin{aligned}
|x(t)| &\le |D(x(t-\tau),r(t))| + |x(t)-D(x(t-\tau),r(t))|\\
&\le \kappa|x(t-\tau)| + |x(t)-D(x(t-\tau),r(t))|,
\end{aligned}
$$

where $\kappa = \max_{i\in\mathbb{S}}\kappa_i < 1$. This implies

$$
\begin{aligned}
\sup_{0\le t\le T}|x(t)| &\le \kappa \sup_{0\le t\le T}|x(t-\tau)| + \sup_{0\le t\le T}|x(t)-D(x(t-\tau),r(t))|\\
&\le \kappa\beta + \kappa\sup_{0\le t\le T}|x(t)| + \sup_{0\le t\le T}|x(t)-D(x(t-\tau),r(t))|,
\end{aligned}
$$

where β is the bound for the initial data ξ. Hence

$$\sup_{0\le t\le T}|x(t)| \le \frac{1}{1-\kappa}\Big(\kappa\beta + \sup_{0\le t\le T}|x(t)-D(x(t-\tau),r(t))|\Big).$$

Letting $T\to\infty$ and using (3.9) we obtain that

$$\sup_{0\le t<\infty}|x(t)|<\infty \quad a.s. \tag{3.10}$$

Step 2. Taking the expectations on both sides of (3.7) and letting $t\to\infty$ (if necessary, using the procedure of stopping times), we obtain that

$$\mathbb{E}\int_0^\infty W(x(s)-D(x(s-\tau),r(s)))ds<\infty. \tag{3.11}$$

This of course implies

$$\int_0^\infty W(x(s)-D(x(s-\tau),r(s)))ds<\infty \quad a.s. \tag{3.12}$$

Set $z(t)=x(t)-D(x(t-\tau),r(t))$ for $t\ge 0$. It is straightforward to see from (3.12) that

$$\liminf_{t\to\infty} W(z(t))=0 \quad a.s. \tag{3.13}$$

We now claim that

$$\lim_{t\to\infty} W(z(t))=0 \quad a.s. \tag{3.14}$$

If this is false, then

$$\mathbb{P}\left\{\limsup_{t\to\infty} W(z(t))>0\right\}>0.$$

Hence there is a number $\varepsilon>0$ such that

$$\mathbb{P}(\Omega_1)\ge 3\varepsilon, \tag{3.15}$$

where

$$\Omega_1=\left\{\limsup_{t\to\infty} W(z(t))>2\varepsilon\right\}.$$

Recalling (3.10) as well as the boundedness of the initial data ξ, we can find a positive number h, which depends on ε, sufficiently large for

$$\mathbb{P}(\Omega_2)\ge 1-\varepsilon, \tag{3.16}$$

where

$$\Omega_2=\left\{\sup_{-\tau\le t<\infty}|z(t)|<h\right\}.$$

It is easy to see from (3.15) and (3.16) that

$$\mathbb{P}(\Omega_1\cap\Omega_2)\ge 2\varepsilon. \tag{3.17}$$

We now define a sequence of stopping times,

$$\begin{aligned}\tau_h &= \inf\{t \ge 0 : |z(t)| \ge h\},\\ \sigma_1 &= \inf\{t \ge 0 : W(z(t)) \ge 2\varepsilon\},\\ \sigma_{2k} &= \inf\{t \ge \sigma_{2k-1} : W(z(t)) \le \varepsilon\}, \quad k = 1, 2, \dots,\\ \sigma_{2k+1} &= \inf\{t \ge \sigma_{2k} : W(z(t)) \ge 2\varepsilon\}, \quad k = 1, 2, \dots,\end{aligned}$$

where throughout this paper we set $\inf \emptyset = \infty$. From (3.13) and the definitions of Ω_1 and Ω_2, we observe that if $\omega \in \Omega_1 \cap \Omega_2$, then

$$\tau_h = \infty \quad \text{and} \quad \sigma_k < \infty, \quad \forall k \ge 1. \tag{3.18}$$

Let I_A denote the indicator function of set A. Noting the fact that $\sigma_{2k} < \infty$ whenever $\sigma_{2k-1} < \infty$, we derive from (3.11) that

$$\begin{aligned}\infty >& \mathbb{E}\int_0^\infty W(z(t))dt\\ \ge& \sum_{k=1}^\infty \mathbb{E}\left[I_{\{\sigma_{2k-1}<\infty, \sigma_{2k}<\infty, \tau_h=\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k}} W(z(t))dt \right]\\ \ge& \varepsilon \sum_{k=1}^\infty \mathbb{E}[I_{\{\sigma_{2k-1}<\infty, \tau_h=\infty\}} (\sigma_{2k} - \sigma_{2k-1})].\end{aligned} \tag{3.19}$$

On the other hand, by Assumption 2.1, there exists a constant $K_h > 0$ such that

$$|f(x, y, t, i)|^2 \vee |g(x, y, t, i)|^2 \le K_h$$

whenever $|x| \vee |y| \le h$ and $(t, i) \in \mathbb{R}_+ \times \mathbb{S}$. By the Hölder inequality and the Doob martingale inequality, we compute that, for any $T > 0$ and $k = 1, 2, \cdots$,

$$\begin{aligned}&\mathbb{E}\Big[I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \sup_{0 \le t \le T} |z(\tau_h \wedge (\sigma_{2k-1} + t)) - z(\tau_h \wedge \sigma_{2k-1})|^2 \Big]\\ &\le 2\mathbb{E}\Big[I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \sup_{0 \le t \le T} \Big| \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1}+t)} f(x(s), x(s-\tau), s, r(s)) ds \Big|^2 \Big]\\ &+ 2\mathbb{E}\Big[I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \sup_{0 \le t \le T} \Big| \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1}+t)} g(x(s), x(s-\tau), s, r(s)) dB(s) \Big|^2 \Big]\\ &\le 2T\mathbb{E}\Big[I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1}+T)} |f(x(s), x(s-\tau), s, r(s))|^2 ds \Big]\\ &+ 8\mathbb{E}\Big[I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1}+T)} |g(x(s), x(s-\tau), s, r(s))|^2 ds \Big]\\ &\le 2K_h T(T+4).\end{aligned} \tag{3.20}$$

Since $W(\cdot)$ is continuous in $\mathbb{R}^n$, it must be uniformly continuous in the closed ball $\bar{S}_h = \{x \in \mathbb{R}^n : |x| \le h\}$. We can therefore choose $\delta = \delta(\varepsilon) > 0$ so small such that

$$|W(x) - W(y)| < \varepsilon/2 \quad \text{whenever } x, y \in \bar{S}_h, \ |x - y| < \delta. \tag{3.21}$$

We furthermore choose $T = T(\varepsilon, \delta, h) > 0$ sufficiently small for

$$\frac{2K_h T(T+4)}{\delta^2} < \varepsilon.$$

It then follows from (3.20) that

$$\mathbb{P}\left(\{\sigma_{2k-1} \wedge \tau_h < \infty\} \cap \left\{\sup_{0\leq t\leq T} |z(\tau_h \wedge (\sigma_{2k-1}+t)) - z(\tau_h \wedge \sigma_{2k-1})| \geq \delta\right\}\right) < \varepsilon.$$

Noting that

$$\{\sigma_{2k-1} < \infty, \tau_h = \infty\} = \{\tau_h \wedge \sigma_{2k-1} < \infty, \tau_h = \infty\} \subset \{\tau_h \wedge \sigma_{2k-1} < \infty\},$$

we hence have

$$\mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \left\{\sup_{0\leq t\leq T} |z(\sigma_{2k-1}+t) - z(\sigma_{2k-1})| \geq \delta\right\}\right) < \varepsilon.$$

By (3.17) and (3.18), we further compute

$$\begin{aligned}
&\mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \left\{\sup_{0\leq t\leq T} |z(\sigma_{2k-1}+t) - z(\sigma_{2k-1})| < \delta\right\}\right)\\
&= \mathbb{P}(\{\sigma_{2k-1} < \infty, \tau_h = \infty\})\\
&- \mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \left\{\sup_{0\leq t\leq T} |z(\sigma_{2k-1}+t) - z(\sigma_{2k-1})| \geq \delta\right\}\right)\\
&> 2\varepsilon - \varepsilon = \varepsilon. \qquad (3.22)
\end{aligned}$$

By (3.21) we hence obtain that

$$\mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \left\{\sup_{0\leq t\leq T} |W(z(\sigma_{2k-1}+t)) - W(z(\sigma_{2k-1}))| < \varepsilon\right\}\right) > \varepsilon. \quad (3.23)$$

Set

$$\bar{\Omega}_k = \left\{\sup_{0\leq t\leq T} |W(z(\sigma_{2k-1}+t)) - W(z(\sigma_{2k-1}))| < \varepsilon\right\}.$$

Noting that

$$\sigma_{2k}(\omega) - \sigma_{2k-1}(\omega) \geq T \quad \text{ if } \omega \in \{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \bar{\Omega}_k,$$

we derive from (3.19) and (3.23) that

$$\begin{aligned}
\infty &> \varepsilon \sum_{k=1}^{\infty} \mathbb{E}\left[I_{\{\sigma_{2k-1}<\infty,\tau_h=\infty\}}(\sigma_{2k} - \sigma_{2k-1})\right]\\
&\geq \varepsilon \sum_{k=1}^{\infty} \mathbb{E}\left[I_{\{\sigma_{2k-1}<\infty,\tau_h=\infty\}\cap\bar{\Omega}_k}(\sigma_{2k} - \sigma_{2k-1})\right]\\
&\geq \varepsilon T \sum_{k=1}^{\infty} \mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \bar{\Omega}_k\right)\\
&\geq \varepsilon T \sum_{k=1}^{\infty} \varepsilon = \infty,
\end{aligned}$$

which is a contradiction. So (3.14) must hold.

Step 3. Let us now show that $Ker(W) \neq \emptyset$. From (3.14) and (3.9) we see that there is an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{t\to\infty} W(z(t,\omega)) = 0 \text{ and } \sup_{0\le t<\infty} |z(t,\omega)| < \infty \quad \text{for all } \omega \in \Omega_0. \tag{3.24}$$

Choose any $\omega \in \Omega_0$. Then $\{z(t,\omega)\}_{t\ge0}$ is bounded in $\mathbb{R}^n$ so there must be an increasing sequence $\{t_k\}_{k\ge1}$ such that $t_k \to \infty$ and $\{z(t_k,\omega)\}_{k\ge1}$ converges to some $\bar{z} \in \mathbb{R}^n$. Thus

$$W(\bar{z}) = \lim_{k\to\infty} W(z(t_k,\omega)) = 0,$$

which implies that $\bar{z} \in Ker(W)$ whence $Ker(W) \neq \emptyset$.

Step 4. We can now show assertion (3.4). It is clearly sufficient if we could show that

$$\lim_{t\to\infty} d(z(t,\omega), Ker(W)) = 0 \quad \text{for all } \omega \in \Omega_0. \tag{3.25}$$

If this is false, then there is some $\bar{\omega} \in \Omega_0$ such that

$$\limsup_{t\to\infty} d(z(t,\bar{\omega}), Ker(W)) > 0.$$

Hence there is a subsequence $\{z(t_k,\bar{\omega})\}_{k\ge0}$ of $\{z(t,\bar{\omega})\}_{t\ge0}$ such that

$$\lim_{k\to\infty} d(z(t_k,\bar{\omega}), Ker(W)) > \bar{\varepsilon}$$

for some $\bar{\varepsilon} > 0$. Since $\{z(t_k,\bar{\omega})\}_{k\ge0}$ is bounded, we can find its subsequence $\{z(\bar{t}_k,\bar{\omega})\}_{k\ge0}$ which converges to some $\hat{z} \in \mathbb{R}^n$. Clearly, $\hat{z} \notin Ker(W)$ so $W(\hat{z}) > 0$. But, by (3.24),

$$W(\hat{z}) = \lim_{k\to\infty} W(z(\bar{t}_k,\bar{\omega})) = 0,$$

a contradiction. Hence (3.25) must hold.

Step 5. Finally, let us show assertion (3.6) under the additional condition (3.5). Clearly, (3.5) implies that $Ker(W) = \{0\}$. It then follows from (3.4) that

$$\lim_{t\to0}[x(t) - D(x(t-\tau), r(t))] = \lim_{t\to0} z(t) = 0 \quad a.s.$$

But, by Assumption 2.1,

$$\begin{aligned}|x(t)| &\le |D(x(t-\tau), r(t))| + |x(t) - D(x(t-\tau), r(t))| \\ &\le \kappa|x(t-\tau)| + |x(t) - D(x(t-\tau), r(t))|,\end{aligned}$$

where $\kappa \in (0,1)$ has been defined above. Letting $t \to \infty$ we obtain that

$$\limsup_{t\to\infty} |x(t)| \le \kappa \limsup_{t\to\infty} |x(t)| \quad a.s.$$

This, together with (3.10), yields

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s.$$

which is the required assertion (3.6). The proof is therefore complete. □

4 Rate of Decay

Although Theorem 3.1 shows that the solution will tends to zero asymptotically with probability 1, it does not give a rate of decay. To reveal the rate of decay, we will slightly strengthen the condition on function V while, in return, we will not need to use function W. To state our new theorem, let us introduce one more new notation. Denote by $\mathcal{K}_\infty$ the family of nondecreasing functions $\mu : \mathbb{R}_+ \to (0, \infty)$ such that $\lim_{t\to\infty} \mu(t) = \infty$.

Theorem 4.1 *Let Assumption 2.1 hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $U \in C(\mathbb{R}^n \times [-\tau, \infty); \mathbb{R}_+)$ *such that*

$$LV(x, y, t, i) \le \gamma(t) - U(x, t) + U(y, t - \tau) \tag{4.1}$$

for $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$. *Then for any initial data* $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ *and* $r(0) = i_0 \in \mathbb{S}$, *the solution of (2.2) obeys the following properties:*

(i) $\lim_{t\to\infty} x(t; \xi, i_0) = 0$ *a.s. if there is a function* $\mu \in \mathcal{K}_\infty$ *such that*

$$\mu(t)\mu(|x|) \le V(x, t, i) \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}. \tag{4.2}$$

(ii) $\limsup_{t\to\infty}[\log(|x(t; \xi, i_0)|)/\log(t)] \le -\alpha/p$ *a.s. if there are two positive constants* p *and* α *such that*

$$(1 + t)^\alpha |x|^p \le V(x, t, i) \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}. \tag{4.3}$$

(iii) $\limsup_{t\to\infty}[t^{-1} \log(|x(t; \xi, i_0)|)] \le -\alpha/p$ *a.s. if there are two positive constants* p *and* α *such that* $\alpha < \frac{p}{\tau} \log(\frac{1}{\kappa})$ *and*

$$e^{\alpha t}|x|^p \le V(x, t, i) \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}, \tag{4.4}$$

where $\kappa = \max_{i\in\mathbb{S}} \kappa_i \in (0, 1)$.

To prove this theorem, let us present a lemma.

Lemma 4.2 *Assume that there is a constant* $\kappa \in (0, 1)$ *such that*

$$|D(x, i)| \le \kappa|x| \quad \forall (x, i) \in \mathbb{R}^n \times \mathbb{S}.$$

Let $\rho : R_+ \to (0, \infty)$ *be a continuous function and* $x(t)$ *be the solution of equation (2.2) with initial data* $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ *and* $r(0) = i_0 \in \mathbb{S}$. *Assume that*

$$\sigma_1 := \limsup_{t\to\infty} \frac{\rho(t)}{\rho(t - \tau)} < \frac{1}{\kappa}, \tag{4.5}$$

and

$$\sigma_2 := \limsup_{t\to\infty} \left[\rho(t)|x(t) - D(x(t - \tau), r(t))|\right] < \infty \quad a.s. \tag{4.6}$$

Then

$$\limsup_{t\to\infty}[\rho(t)|x(t)|] \le \frac{\sigma_2}{1 - \kappa\sigma_1} \quad a.s. \tag{4.7}$$

The proof is similar to that of Lemma 3.1 in Mao [18] and is hence omitted. Using this lemma and Theorem 3.1 we can prove Theorem 4.1 quite easily.

Proof of Theorem 4.1. We first observe that either condition (4.2) or (4.3) or (4.4) implies that

$$\lim_{|x|\to\infty}\Big(\inf_{(t,i)\in\mathbb{R}_+\times\mathbb{S}} V(x,t,i)\Big)=\infty.$$

So the unique global solution of equation (2.2) for the given initial data follows from Theorem A.1. Again let us write the solution $x(t;\xi,i_0)=x(t)$ for simplicity. Applying Theorem 3.1 with $W=0$ we see that

$$\limsup_{t\to\infty} V(x(t)-D(x(t-\tau),r(t)),t,r(t))<\infty \quad a.s. \tag{4.8}$$

(i) By condition (4.2), we then have

$$\lim_{t\to\infty}\mu(|x(t)-D(x(t-\tau),r(t))|)=0 \quad a.s.$$

Since $\mu\in\mathcal{K}_\infty$, we must have

$$\lim_{t\to\infty}|x(t)-D(x(t-\tau),r(t))|=0 \quad a.s.$$

Applying Lemma 4.2 with $\rho\equiv 1$ (so $\sigma_1=1$ and $\sigma_2=0$), we obtain the required assertion that $\lim_{t\to\infty}x(t)=0$ a.s.

(ii) By condition (4.3), we see from (4.8) that

$$\sigma_2:=\limsup_{t\to\infty}[(1+t)^{\frac{\alpha}{p}}|x(t)-D(x(t-\tau),r(t))|]<\infty \quad a.s.$$

To apply Lemma 4.2, let $\rho(t)=(1+t)^{\frac{\alpha}{p}}$, and then

$$\sigma_1:=\limsup_{t\to\infty}\frac{(1+t)^{\frac{\alpha}{p}}}{(1+t-\tau)^{\frac{\alpha}{p}}}=1<\frac{1}{\kappa}.$$

Hence, by Lemma 4.2, we have

$$\limsup_{t\to\infty}\Big[(1+t)^{\frac{\alpha}{p}}|x(t)|\Big]<\frac{\sigma_2}{1-\kappa} \quad a.s.$$

which implies

$$\limsup_{t\to\infty}\frac{\log(|x(t)|)}{\log(t)}\le-\frac{\alpha}{p} \quad a.s.$$

as required.

Since the proof of (iii) is similar to that of (ii), it is omitted here. The proof is therefore complete. □

Let us now establish a new result on the almost surely exponential stability which will be used in the following section when we discuss the linear NSDDEs.

Theorem 4.3 *Let Assumption 2.1 hold. Assume that there are functions* $\bar{V} \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ *and positive constants* p *and* β_j $(1 \le j \le 4)$ *such that* $\beta_3 > \beta_4$,

$$\beta_1 |x|^p \le \bar{V}(x,t,i) \le \beta_2 |x|^p \tag{4.9}$$

for $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ *and*

$$L\bar{V}(x,y,t,i) \le -\beta_3 |x|^p + \beta_4 |y|^p \tag{4.10}$$

for $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$. *Then for any initial data* $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ *and* $r(0) = i_0 \in \mathbb{S}$, *the solution of (2.2) obeys the following property that*

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t;\xi,i_0)|)] \le -\frac{\bar{\alpha}}{p} \quad a.s. \tag{4.11}$$

namely, the trivial solution of equation (2.2) is almost surely exponentially stable, where

$$\bar{\alpha} = \sup\big\{\alpha \in (0, p\tau^{-1}\log(1/\kappa)] : \beta_3 - \alpha\beta_2 \ge \beta_4 e^{\alpha\tau}\big\}. \tag{4.12}$$

Proof. Fix any $\alpha \in (0, \bar{\alpha})$. By (4.12), we have that

$$\alpha < \frac{p}{\tau}\log(1/\kappa) \quad \text{and} \quad \beta_3 - \alpha\beta_2 \ge \beta_4 e^{\alpha\tau}. \tag{4.13}$$

Define $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ by

$$V(x,t,i) = \frac{e^{\alpha t}}{\beta_1}\bar{V}(x,t,i).$$

It is then clear that

$$e^{\alpha t}|x|^p \le V(x,t,i) \quad \forall (x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}.$$

Moreover, for $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$,

$$\begin{aligned} LV(x,y,t,i) &= \frac{e^{\alpha t}}{\beta_1}\Big[\alpha\bar{V}(x,t,i) + L\bar{V}(x,y,t,i)\Big] \\ &\le \frac{e^{\alpha t}}{\beta_1}\Big[-(\beta_3 - \alpha\beta_2)|x|^p + \beta_4|y|^p\Big] \\ &\le -U(x,t) + U(y,t-\tau), \end{aligned}$$

where $U \in C(\mathbb{R}^n \times [-\tau, \infty); \mathbb{R}_+)$ is defined by

$$U(x,t) = \frac{\beta_3 - \alpha\beta_2}{\beta_1} e^{\alpha t}|x|^p.$$

An application of Theorem 4.1 shows that the solution of (2.2) obeys

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t;\xi,i_0)|)] \le -\frac{\alpha}{p} \quad a.s. \tag{4.14}$$

Letting $\alpha \to \bar{\alpha}$ yields the required assertion (4.11). □

The following criterion is very convenient in applications as the main condition is in terms of an M-matrix.

Theorem 4.4 *Let Assumption 2.1 hold. Assume that*

$$(x - D(y,i))^T f(x,y,t,i) + \tfrac{1}{2}|g(x,y,t,i)|^2 \le \alpha_i |x|^2 + \sigma_i |y|^2 \tag{4.15}$$

for $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, *where* α_i, σ_i *are all real numbers. Define the* $N \times N$ *matrix* $\bar{\Gamma} = (|\gamma_{ij}|\kappa_i)_{N\times N}$ *and assume that*

$$\mathcal{A} := -\mathrm{diag}(2\alpha_1, 2\alpha_2, \cdots, 2\alpha_N) - \Gamma - \bar{\Gamma} \tag{4.16}$$

is a nonsingular M-matrix. Set

$$(q_1, \cdots, q_N)^T := \mathcal{A}^{-1}\vec{1}, \tag{4.17}$$

where $\vec{1} = (1, \cdots, 1)^T$. *Then* $q_i > 0$ *for all* $i \in \mathbb{S}$. *If, moreover,*

$$2q_i\sigma_i + \sum_{j=1}^N |\gamma_{ij}| q_j \kappa_i + \sum_{j=1, j\neq i}^N \gamma_{ij} q_j \kappa_i^2 < 1, \quad \forall i \in \mathbb{S}, \tag{4.18}$$

then the trivial solution of equation (2.2) is almost surely exponentially stable.

Proof. By the theory of M-matrices (see e.g. [3, 16]), every element of $\mathcal{A}^{-1}$ is nonnegative. As it is nonsingular, each row of $\mathcal{A}^{-1}$ must have at least one positive element. We hence see that $q_i > 0$ for all $i \in \mathbb{S}$.

To apply Theorem 4.3, we define $\bar{V}(x,t,i) = q_i|x|^2$. Clearly, V obeys (4.9) with $\beta_1 = \min_{i\in\mathbb{S}} q_i$ and $\beta_2 = \max_{i\in\mathbb{S}} q_i$. Moreover, using (4.15), (4.17), Assumption 2.1 and noting that $\gamma_{ii}, i \in \mathbb{S}$ are non-positive, we compute the operator

$$\begin{aligned}
LV(x,y,t,i) &= 2q_i\Big[(x - D(y,i))^T f(x,y,t,i) + \tfrac{1}{2}|g(x,y,t,i)|^2\Big] \\
&\quad + \sum_{j=1}^N \gamma_{ij} q_j (x - D(y,i))^T (x - D(y,i)) \\
&\le 2q_i\Big[(x - D(y,i))^T f(x,y,t,i) + \tfrac{1}{2}|g(x,y,t,i)|^2\Big] \\
&\quad + \sum_{j=1}^N \gamma_{ij} q_j |x|^2 - 2\sum_{j=1}^N \gamma_{ij} q_j x^T D(y,i) + \sum_{j=1}^N \gamma_{ij} q_j |D(y,i)|^2 \\
&\le \left[2q_i\alpha_i + \sum_{j=1}^N \gamma_{ij} q_j + \sum_{j=1}^N |\gamma_{ij}| q_j \kappa_i\right] |x|^2 \\
&\quad + \left[2q_i\sigma_i + \sum_{j=1}^N |\gamma_{ij}| q_j \kappa_i + \sum_{j=1, j\neq i}^N \gamma_{ij} q_j \kappa_i^2\right] |y|^2 \\
&\le -|x|^2 + \beta_4 |y|^2,
\end{aligned}$$

where

$$\beta_4 = \max_{i\in\mathbb{S}} \left[2q_i\sigma_i + \sum_{j=1}^N |\gamma_{ij}| q_j \kappa_i + \sum_{j=1, j\neq i}^N \gamma_{ij} q_j \kappa_i^2\right].$$

By (4.18), $\beta_4 < 1$, the assertion follows hence from Theorem 4.3. □

5 Linear NSDDEs with Markovian Switching

Let us now consider the linear autonomous NSDDE with Markovian switching of the form

$$d[x(t) - D(r(t))x(t-\tau)] = [A(r(t))x(t) + F(r(t))x(t-\tau)]\,dt + \sum_{k=1}^{m}[C_k(r(t))x(t) + G_k(r(t))x(t-\tau)]dB_k(t) \tag{5.1}$$

on $t \geq 0$ with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. For $i \in \mathbb{S}$, we will write $A(i) = A_i$, $C_k(i) = C_{ki}$ etc. for simplicity, and they are all $n \times n$-matrices. Let Q_i, $i \in \mathbb{S}$, be symmetric positive-definite $n \times n$-matrices and let $V(x, t, i) = x^T Q_i x$. As V is independent of t, we will write it as $V(x, i)$. Then the operator $LV : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}$ associated with equation (5.1) has the form

$$LV(x, y, i) = 2(x - D_i y)^T Q_i (A_i x + F_i y) + \sum_{k=1}^{m}(C_{ki}x + G_{ki}y)^T Q_i (C_{ki}x + G_{ki}y) + (x - D_i y)^T \sum_{j=1}^{N} \gamma_{ij} Q_j (x - D_i y),$$

where once again we have dropped $t \in \mathbb{R}_+$ from LV as it is independent of t. It is easy to show that

$$LV(x, y, i) = (x^T, y^T) H_i \begin{pmatrix} x \\ y \end{pmatrix} + |y|^2, \tag{5.2}$$

where the symmetric matrix $H_i \in \mathbb{R}^{2n \times 2n}$ is defined by

$$H_i = \begin{pmatrix} Q_i A + A^T Q_i, & -A_i^T Q_i D_i^T + Q_i F_i \\ -D_i Q_i A_i + F_i Q_i, & -I_n - D_i^T Q_i F_i - F_i Q_i D_i^T \end{pmatrix} + (C_{ki}, G_{ki})^T Q_i (C_{ki}, G_{ki}) + (I_n, -D_i)^T \Big(\sum_{j=1}^{N} \gamma_{ij} Q_j \Big) (I_n, -D_i), \tag{5.3}$$

in which I_n is the $n \times n$ identity matrix.

Theorem 5.1 *Assume that there are symmetric positive-definite matrices Q_i, $i \in \mathbb{S}$ such that the matrices H_i defined by (5.3) are all negative-definite and*

$$\lambda := -\max_{i \in \mathbb{S}} \lambda_{\max}(H_i) > \tfrac{1}{2}. \tag{5.4}$$

Assume moreover that

$$\kappa := \max_{i \in \mathbb{S}} \|D_i\| < 1. \tag{5.5}$$

Then for any initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0 \in \mathbb{S}$, the solution of (5.1) obeys the property that

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t; \xi, i_0)|)] \leq -\frac{\bar{\alpha}}{2} \quad a.s. \tag{5.6}$$

where

$$\bar{\alpha} = \sup\big\{\alpha \in (0, p\tau^{-1}\log(1/\kappa)] : \lambda - \alpha\big[\max_{i \in \mathbb{S}} \lambda_{\max}(Q_i)\big] \geq [0 \vee (1 - \lambda)]e^{\alpha\tau}\big\}. \tag{5.7}$$

Proof. It follows from (5.2) that

$$LV(x,y,i) \le \lambda_{\max}(H_i)(|x|^2+|y|^2)+|y|^2 \le -\lambda|x|^2+[0\vee(1-\lambda)]|y|^2.$$

But, we have clearly that

$$\big[\min_{i\in\mathbb{S}}\lambda_{\min}(Q_i)\big]|x|^2 \le V(x,i) \le \big[\max_{i\in\mathbb{S}}\lambda_{\max}(Q_i)\big]|x|^2.$$

The assertion (5.6) now follows immediately from Theorem 4.3. □

Example 5.2 Let $B(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S}=\{1,2\}$ with generator

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix}-0.1 & 0.1\\ 10 & -10\end{pmatrix}.$$

Assume that $B(t)$ and $r(t)$ are independent. Consider a two-dimensional linear NSDDE with Markovian switching of the form

$$d[x(t)-Dx(t-\tau)]=A(r(t))x(t)dt+G(r(t))x(t-\tau)dB(t) \tag{5.8}$$

on $t\ge 0$, where

$$D=\begin{pmatrix}0.2 & 0\\ 0 & 0.2\end{pmatrix},\quad A(1)=A_1=\begin{pmatrix}-8 & 0\\ 0 & -7\end{pmatrix},\quad A(2)=A_2=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix},$$

$$G(1)=G_1=\begin{pmatrix}0.1 & 0.2\\ -0.3 & 0.4\end{pmatrix},\quad G(2)=G_2=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix}.$$

By choosing

$$Q_1=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix},\quad Q_2=\begin{pmatrix}4 & 0\\ 0 & 5\end{pmatrix},$$

the matrices defined by (5.3) become

$$H_i=\begin{pmatrix}Q_iA_i+A_i^TQ_i, & -A_i^TQ_iD_i^T\\ -D_iQ_iA_i, & -I_2\end{pmatrix}$$
$$+(0,G_i)^TQ_i(0,G_i)+(I_2,-D_i)^T\Big(\sum_{j=1}^N\gamma_{ij}Q_j\Big)(I_2,-D_i).$$

More precisely,

$$H_1=\begin{pmatrix}-1.1100 & 0 & 0.0820 & 0\\ 0 & -0.9100 & 0 & 0.042\\ 0.0820 & 0 & -0.8844 & -0.1\\ 0 & 0.042 & -0.1 & -0.7804\end{pmatrix},\quad H_2=\begin{pmatrix}-23 & 0 & 6.2 & 0\\ 0 & -39 & 0 & 8.8\\ 6.2 & 0 & -2.52 & 0\\ 0 & 8.8 & 0 & -2.91\end{pmatrix}.$$

Simple computations give that the eigenvalues of H_1 and H_2 are

$$(-1.1396,-0.9410,-0.8954,-0.7087) \text{ and } (-41.0341,-24.7307,-0.8786,-0.7893),$$

respectively. Hence λ defined by (5.4) is: $\lambda=0.7087>\frac{1}{2}$. By Theorem 5.1, we can conclude that equation (5.8) is almost surely exponentially stable.

6 Nonlinear Examples

Let us now discuss a couple of nonlinear examples to illustrate our theory.

Example 6.1 Consider a scalar nonlinear NSDDE

$$d[x(t) - D(x(t-\tau), r(t))] = f(x(t), t, r(t)) + g(x(t-\tau), t, r(t))dB(t). \tag{6.1}$$

Here $B(t)$ is a scalar Brownian and $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

Of course, they are independent. Moreover, for $(x, y, t, i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}$,

$$D(y, i) = \begin{cases} 0.1y, & \text{if } i = 1, \\ -0.1y, & \text{if } i = 2; \end{cases} \qquad f(x, t, i) = \begin{cases} -x^3 + x/(1+t), & \text{if } i = 1, \\ -x^3 - 2x, & \text{if } i = 2; \end{cases}$$

and

$$g(y, t, i) = \begin{cases} |y|^{1.5}/(1+t), & \text{if } i = 1, \\ y^2 \sin t, & \text{if } i = 2. \end{cases}$$

Define $V(x, i) = 2x^2$ for $i = 1$ but x^2 for $i = 2$. Then the operator $LV : \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{S} \to \mathbb{R}$ takes the forms

$$\begin{aligned} LV(x, y, t, 1) = {} & 4(x - 0.1y)(-x^3 + x/(1+t)) + 2|y|^3/(1+t)^2 \\ & - 4(x - 0.1y)^2 + 2(x + 0.1y)^2 \end{aligned}$$

and

$$LV(x, y, t, 2) = 2(x + 0.1y)(-x^3 - 2x) + y^4 \sin^2 t + 2(x - 0.1y)^2 - (x + 0.1y)^2.$$

By the elementary inequality

$$a^u b^{(1-u)} \le ua + (1-u)b, \quad \forall a, b \ge 0,\ 0 \le u \le 1,$$

it is not very difficult to show that

$$LV(x, y, t, 1) \le \frac{c}{(1+t)^2} - 2.7x^4 - 1.4x^2 + 1.1y^4 + 0.58y^2$$

and

$$LV(x, y, t, 2) \le -1.85x^4 - 2.5x^2 + 1.05y^4 + 0.51y^2,$$

where c is a positive constant. These imply that

$$LV(x, y, t, 1) \le \frac{c}{(1+t)^2} - 2.7x^4 - 1.2x^2 + 1.1y^4 + 0.6y^2 - 0.1(x - 0.1y)^2$$

and

$$LV(x, y, t, 2) \le -1.85x^4 - 2.3x^2 + 1.05y^4 + 0.53y^2 - 0.1(x + 0.1y)^2.$$

By defining

$$\gamma(t) = \frac{c}{(1+t)^2}, \quad U(x,t) = U(x) = 1.85x^4 + 1.2x^2, \quad W(x) = 0.1x^2,$$

we hence have

$$LV(x,y,t,i) \le \gamma(t) - U(x) + U(y) - W(x - D(y,i)).$$

By Theorem 3.1 we can therefore conclude that the solution of equation (5.1) obeys

$$\lim_{t\to 0} x(t) = 0 \quad a.s.$$

Example 6.2 Let $B(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1,2,3\}$ with generator

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

Assume that $B(t)$ and $r(t)$ are independent. Consider a three-dimensional semi-linear NSDDE with Markovian switching of the form

$$d[x(t) - 0.01x(t-\tau)] = A(r(t))x(t)dt + g(x(t-\tau), r(t))dB(t), \tag{6.2}$$

on $t \ge 0$. Here

$$A(1) = \begin{pmatrix} -2 & -1 & -2 \\ 2 & -2 & 1 \\ 1 & -2 & -3 \end{pmatrix}, A(2) = \begin{pmatrix} 0.5 & 1 & 0.5 \\ -0.8 & 0.5 & 1 \\ -0.7 & -0.9 & 0.2 \end{pmatrix}, A(3) = \begin{pmatrix} -0.5 & -0.9 & -1 \\ 1 & -0.6 & -0.7 \\ 0.8 & 1 & -1 \end{pmatrix}.$$

Moreover, we assume that $|g(y,i)|^2 \le \rho_i |y|^2$ for $(y,i) \in \mathbb{R}^3 \times \mathbb{S}$. Set $\beta_i = \frac{1}{2}\lambda_{\max}(A(i) + A^T(i))$ and $\theta_i = \sqrt{\lambda_{\max}(A(i)A^T(i))}$. Compute

$$\beta_1 = -1.2192, \quad \beta_2 = 0.6035, \quad \beta_3 = -0.4753,$$

and

$$\theta_1 = 4.2349, \quad \theta_2 = 1.6306, \quad \theta_3 = 1.8040.$$

Compute furthermore that

$$\begin{aligned} &(x - D(y,i))^T A(i)x + \frac{1}{2}|g(y,i)|^2 \\ &\le \frac{1}{2}\lambda_{\max}(A(i) + A^T(i))|x|^2 + \frac{1}{10}|\sqrt{\lambda_{\max}(A(i)A^T(i))}|x||y| + \frac{1}{2}|g(y,i)|^2 \\ &\le \beta_i|x|^2 + \frac{\theta_i}{20}|x|^2 + \frac{\theta_i}{20}|y|^2 + \frac{\rho_i}{2}|y|^2. \end{aligned}$$

So the parameters used in (4.15) are $\alpha_i = \beta_i + \frac{\theta_i}{20}$ and $\sigma_i = \theta_i/20 + \rho_i/2$. The matrix defined by (4.16) becomes

$$\mathcal{A} = -\text{diag}(2\alpha_1, 2\alpha_2, 2\alpha_3) - \Gamma - \bar{\Gamma} = \begin{pmatrix} 4.3347 & -1.01 & -1.01 \\ -3.03 & 2.6715 & -1.01 \\ -1.01 & -1.01 & 2.8404 \end{pmatrix}.$$

Compute

$$\mathcal{A}^{-1} = \begin{pmatrix} 0.5079 & 0.3007 & 0.2875 \\ 0.7444 & 0.8732 & 0.5752 \\ 0.4453 & 0.4174 & 0.6588 \end{pmatrix}.$$

Therefore $\mathcal{A}$ is a nonsingular M-matrix and

$$(q_1, q_2, q_3)^T := \mathcal{A}^{-1}\vec{1} = (1.0961, 2.1928, 1.5215)^T.$$

Condition (4.18) becomes

$$\begin{aligned} &2q_1(\theta_1/20 + \rho_1/2) + 2q_1\kappa_1 + q_2\kappa_1 + q_3\kappa_1 + q_2\kappa_1^2 + q_3\kappa_1^2 < 1, \\ &2q_2(\theta_2/20 + \rho_2/2) + 3q_1\kappa_2 + 4q_2\kappa_2 + q_3\kappa_2 + 3q_1\kappa_2^2 + q_3\kappa_2^2 < 1, \\ &2q_3(\theta_3/20 + \rho_3/2) + 2q_1\kappa_3 + q_2\kappa_3 + 2q_3\kappa_3 + q_1\kappa_3^2 + q_2\kappa_3^2 < 1. \end{aligned}$$

Noting $\kappa_1 = \kappa_2 = \kappa_3 = 0.01$, that is

$$\rho_1 < 0.4351, \quad \rho_2 < 0.1777, \quad \rho_3 < 0.4281. \tag{6.3}$$

By Theorem 4.4, we can therefore conclude that the trivial solution of equation (6.2) is almost surely exponentially stable provided (6.3) holds.

A Appendix

In this appendix we shall establish an existence-and-uniqueness theorem of the global solution to equation (2.2). The following theorem covers all the NSDDEs discussed in this paper.

Theorem A.1 *Let Assumption 2.1 hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $U \in C(\mathbb{R}^n \times [-\tau, \infty); \mathbb{R}_+)$ *such that*

$$LV(x, y, t, i) \le \gamma(t) - U(x, t) + U(y, t - \tau) \tag{A.1}$$

for $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, *and*

$$\lim_{|x|\to\infty} \left[\inf_{(t,i)\in\mathbb{R}_+\times\mathbb{S}} V(x, t, i) \right] = \infty. \tag{A.2}$$

Then for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ *and* $r(0) = i_0 \in \mathbb{S}$, *equation (2.2) has a unique global solution* $x(t)$ *on* $t \ge -\tau$.

Proof. Let β be the bound for ξ. For each integer $k \ge \beta$, define

$$f^{(k)}(x, y, t, i) = f\left(\frac{|x| \wedge k}{|x|} x, \frac{|y| \wedge k}{|y|} y, t, i \right),$$

where we set $(|x| \wedge k/|x|)x = 0$ when $x = 0$. Define $g^{(k)}(x, y, t, i)$ similarly. Consider the NSDDE

$$\begin{aligned} d[x_k(t) - D(x_k(t-\tau), r(t))] &= f^{(k)}(x_k(t), x_k(t-\tau), t, r(t))dt \\ &\quad + g^{(k)}(x_k(t), x_k(t-\tau), t, r(t))dB(t) \end{aligned} \tag{A.3}$$

on $t \geq 0$ with initial data ξ and i_0. By Assumption 2.1, we observe that $f^{(k)}$ and $g^{(k)}$ satisfy the global Lipschitz condition and the linear growth condition. By the known existence-and-uniqueness theorem (see e.g. [17], [25]), there exists a unique global solution $x_k(t)$ on $t \in [0, \tau]$ to the equation

$$\begin{aligned} x_k(t) &= \xi(0) + D(\xi(t-\tau), r(t)) - D(\xi(-\tau), i_0) \\ &+ \int_0^t f^{(k)}(x_k(s), x_k(s-\tau), s, r(s))ds + \int_0^t g^{(k)}(x_k(s), x_k(s-\tau), s, r(s))dB(s). \end{aligned} \tag{A.4}$$

Once we obtain the unique solution on $[0, \tau]$ we can regard them as the initial data and consider equation (A.3) on $t \in [\tau, 2\tau]$. In this case, equation (A.3) can be written as

$$\begin{aligned} x_k(t) &= \xi(\tau) + D(x_k(t-\tau), r(t)) - D(x_k(0), r(\tau)) \\ &+ \int_\tau^t f^{(k)}(x_k(s), x_k(s-\tau), s, r(s))ds + \int_\tau^t g^{(k)}(x_k(s), x_k(s-\tau), s, r(s))dB(s). \end{aligned} \tag{A.5}$$

Again, by the known existence-and-uniqueness theorem (see e.g. [17], [25]), equation (A.3) has a unique solution $x_k(t)$ on $[\tau, 2\tau]$. Repeating this procedure on intervals $[2\tau, 3\tau]$, $[3\tau, 4\tau]$ and so on we obtain the unique solution $x_k(t)$ to equation (A.3) on $t \geq -\tau$.

Let us now define the stopping time

$$\sigma_k = \inf\{t \geq 0 : |x_k(t)| \geq k\}.$$

Clearly, $|x_k(s)| \vee |x_k(s-\tau)| \leq k$ for $0 \leq s \leq \sigma_k$. Therefore

$$f^{(k)}(x_k(s), x_k(s-\tau), s, r(s)) = f^{(k+1)}(x_k(s), x_k(s-\tau), s, r(s))$$

and

$$g^{(k)}(x_k(s), x_k(s-\tau), s, r(s)) = g^{(k+1)}(x_k(s), x_k(s-\tau), s, r(s)),$$

on $\tau \leq s \leq \sigma_k$. These imply

$$\begin{aligned} x_k(t) &= x_k(t) = \xi(0) + D(\xi(t-\tau), r(t)) - D(\xi(-\tau), i_0) \\ &+ \int_0^{t\wedge\sigma_k} f^{(k+1)}(x_k(s), x_k(s-\tau), s, r(s))ds + \int_0^{t\wedge\sigma_k} g^{(k+1)}(x_k(s), x_k(s-\tau), s, r(s))dB(s). \end{aligned}$$

So we have

$$x_k(t) = x_{k+1}(t) \quad \text{if } 0 \leq t \leq \sigma_k.$$

This implies that σ_k is increasing in k. Let $\sigma = \lim_{k\to\infty} \sigma_k$. The property above also enables us to define $x(t)$ for $t \in [-\tau, \sigma)$ as follows

$$x(t) = x_k(t) \quad \text{if} -\tau \leq t \leq \sigma_k.$$

It is clear that $x(t)$ is a unique solution to equation (2.2) for $t \in [-\tau, \sigma)$. To complete the proof, we need to show that $\mathbb{P}\{\sigma = \infty\} = 1$. By the generalized Itô formula, we have that for any $t > 0$,

$$\begin{aligned} &\mathbb{E}V(x_k(t\wedge\sigma_k) - D(x_k(t\wedge\sigma_k - \tau), r(t\wedge\sigma_k)), t\wedge\sigma_k, r(t\wedge\sigma_k)) \\ =&\mathbb{E}V(x_k(0) - D(\xi(-\tau), 0, r(0)) + \mathbb{E}\int_0^{t\wedge\sigma_k} L^{(k)}V(x_k(s), x_k(s-\tau), s, r(s))ds, \end{aligned} \tag{A.6}$$

where operator $L^{(k)}V$ is defined similarly as LV was defined by (2.3) except f and g there are replaced by $f^{(k)}$ and $g^{(k)}$, respectively. By the definitions of $f^{(k)}$ and $g^{(k)}$, we hence observe that

$$L^{(k)}V(x_k(s), x_k(s-\tau), s, r(s)) = LV(x_k(s), x_k(s-\tau), s, r(s)) \quad \text{if } 0 \le s \le t \wedge \sigma_k.$$

Using (A.1), we can then derive easily from (A.6) that

$$\begin{aligned} &\mathbb{E}V(x_k(t\wedge\sigma_k) - D(x_k(t\wedge\sigma_k - \tau), r(t\wedge\sigma_k)), t\wedge\sigma_k, r(t\wedge\sigma_k)) \\ &\le \zeta := \mathbb{E}V(\xi(0) - D(\xi(-\tau), i_0), 0, i_0) + \mathbb{E}\int_{-\tau}^{0} U(\xi(\theta), \theta)d\theta + \int_0^\infty \gamma(s)ds. \end{aligned} \quad \text{(A.7)}$$

Define, for each $u \ge 0$,

$$\mu(u) = \inf\big\{V(x,t,i) : (x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \text{ with } |x| \ge u\big\}.$$

By condition (A.2), we note that $\lim_{u\to\infty}\mu(u) = \infty$. On the other hand, for any $\omega \in \{\sigma_k \le t\}$, we have $|x(\sigma_k)| = k$ and $|x(\sigma_k - \tau)| \le k$ so

$$|x_k(t\wedge\sigma_k) - D(x_k(t\wedge\sigma_k - \tau)| \ge (1-\kappa)k,$$

where $\kappa = \max_{i\in\mathbb{S}}\kappa_i \in (0,1)$. It then follows from (A.7) that

$$\mathbb{P}\{\sigma_k \le t\} \le \frac{\zeta}{\mu((1-\kappa)k)}.$$

Letting $k \to \infty$, we obtain that $\mathbb{P}\{\sigma \le t\} = 0$. Since t is arbitrary, we must have

$$\mathbb{P}\{\sigma = \infty\} = 1$$

as desired. □

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