NUMERICAL SOLUTIONS OF NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper examines the numerical solutions of neutral stochastic functional differential equations (NSFDEs)

$$d[x(t) - u(x_t)] = f(x_t)dt + g(x_t)dw(t), \quad t \geq 0.$$

The key contribution is to establish the strong mean square convergence theory of the Euler–Maruyama approximate solution under the local Lipschitz condition, the linear growth condition and the contractive mapping. These conditions are generally imposed to guarantee the existence and uniqueness of the true solution, so the numerical results given here are obtained under quite general conditions. Although the way of analysis borrows from [16], to cope with $u(x_t)$, several new techniques have been developed.

Key words. Neutral stochastic functional differential equations, Strong convergence, Euler–Maruyama method, Local Lipschitz condition

AMS subject classifications. 65C20, 65C30

1. Introduction. The theory of stochastic functional differential equations (SFDEs) has received a great deal of attentions in the recent decades. More recently researchers have given special interests to the study of the equations in which the variable delay argument occurs in the derivative of the state variable, so called neutral stochastic functional differential equations (NSFDEs). Many well-known theorems in SFDEs are successfully extended to NSFDEs, for example, [9, 13, 14, 15, 16, 18].

As well as deterministic neutral functional differential equations and SFDEs, most NSFEDs can not be solved explicitly, so numerical methods become one of the powerful techniques. A number of papers study the numerical analysis of the deterministic neutral functional differential equations, for example, [1, 3, 5, 7, 10] and references therein. The numerical solutions of SFDEs have also been studied extensively by many authors. Here we mention some of them, for example, [2, 4, 6, 11, 12, 16, 17], and so on.

However, little is as yet known about numerical solutions for NSFDEs although it may be seen as a combination of deterministic neutral functional differential equations and SFDEs. This paper will fill the blank.

We study the Euler–Maruyama numerical solutions of the NSFDE

$$d[x(t) - u(x_t)] = f(x_t)dt + g(x_t)dw(t), \quad t \geq 0, \tag{1.1}$$

with initial data $x_0 = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$. Here,

$$f : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n, \quad g : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}, \quad u : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n,$$

$x(t) \in \mathbb{R}^n$ for each t,

$$x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n),$$

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and $w(t)$ is an m-dimensional Brownian motion. The initial data ξ is an $\mathcal{F}_0$-measurable $C([-\tau,0];\mathbb{R}^n)$-valued random variable such that $\mathbb{E}\|\xi\|^p < \infty$ for some $p \ge 2$. Our main aim is to extend the method developed by [16] and [17] to NSFDEs and study strong convergence for the Euler–Maruyama numerical approximations in the case where f and g satisfy both the local Lipschitz condition and the linear growth condition and u is a contractive mapping. These three conditions are standard for the existence and uniqueness of the true solutions.

Although the way of analysis borrows from [16], the existence of the neutral term $u(x_t)$ essentially changes the problem, and several new techniques have been developed to cope with the difficulties which have risen from the neutral term.

In section 2, we introduce some necessary assumptions and auxiliary results, define the Euler–Maruyama method for NSFDEs and state our main result that the approximate solutions strongly converge to the exact solution. The proof of the result is rather technical so we present several lemmas in section 3 and then complete the proof in section 4. In the final section, under the global Lipschitz condition, we reveal the order of convergence of approximate solutions.

2. The Euler–Maruyama method for NSFDEs. Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let $\mathbb{R}_+ = [0,\infty)$, and let $\tau > 0$. Denoted by $C([-\tau,0],\mathbb{R}^n)$ the family of continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0}|\varphi(\theta)|$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. Let $w(t) = (w_1(t),\cdots,w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $p > 0$ and $L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ be the family of $\mathcal{F}_0$-measurable $C([-\tau,0];\mathbb{R}^n)$-valued random variables ξ such that $\mathbb{E}\|\xi\|^p < \infty$. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau,\infty)$, we let $x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$ for $t \ge 0$.

Let $f : C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n$, $g : C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^{n\times m}$, and $u : C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n$. In this paper we impose the following hypotheses.

Assumption 2.1. *(Local Lipschitz condition). For each integer $j \ge 1$, there exists a positive constant C_j such that*

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \le C_j\|\varphi - \psi\|^2 \tag{2.1}$$

for $\varphi,\psi \in C([-\tau,0];\mathbb{R}^n)$ with $\|\varphi\| \vee \|\psi\| \le j$.

Assumption 2.2. *(Linear growth condition). There is a constant $K > 0$ such that*

$$|f(\varphi)|^2 \vee |g(\varphi)|^2 \le K(1 + \|\varphi\|^2) \tag{2.2}$$

for $\varphi \in C([-\tau,0];\mathbb{R}^n)$.

Assumption 2.3. *(Contractive mapping). There exists a constant $\kappa \in (0,1)$ such that for all $\varphi,\psi \in C([-\tau,0];\mathbb{R}^n)$,*

$$|u(\varphi) - u(\psi)| \le \kappa\|\varphi - \psi\| \tag{2.3}$$

for $\varphi,\psi \in C([-\tau,0];\mathbb{R}^n)$ and $u(0) = 0$.

Consider the n-dimensional NSFDE:

$$d[x(t) - u(x_t)] = f(x_t)dt + g(x_t)dw(t), \quad t \ge 0, \tag{2.4}$$

with initial data $x_0 = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

We know that Assumptions 2.1, 2.2 and 2.3 are standard conditions for the existence and uniqueness of the solution of Eq. (2.4) except that we impose $u(0) = 0$. Actually, $u(0) = 0$ is also standard for the boundedness of the solution's moments (for example, see [14]).

We impose the following condition on the initial data.

Assumption 2.4. *$\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ for some $p \geq 2$ and there exists a nondecreasing function $\alpha(\cdot)$ such that*

$$\mathbb{E}\Big(\sup_{-\tau \leq s \leq t \leq 0} |\xi(t) - \xi(s)|^2\Big) \leq \alpha(t - s) \tag{2.5}$$

with the property $\alpha(s) \to 0$ as $s \to 0$.

From Mao ([14, p.211]), we may therefore state the following theorem.

Theorem 2.1. *Let $p \geq 2$. If Assumptions 2.2–2.4 are satisfied, then*

$$\mathbb{E}\Big(\sup_{-\tau \leq t \leq T} |x(t)|^p\Big) \leq (1 + \bar{C}\mathbb{E}\|\xi\|^p)e^{\widetilde{C}T} \tag{2.6}$$

for any $T > 0$, where

$$\widetilde{C} = \frac{2p(1+\kappa)^{p-2}}{(1-p)^p}[\sqrt{2K}(1+\kappa) + K(33p-1)] \text{ and } \bar{C} = \frac{1}{1-\kappa} + \frac{2(1+\kappa)^p}{(1-\kappa)^p}.$$

From now on, we will consider Eq. (2.4) on the finite time interval $[0, T]$ as our aim is to discuss the finite time convergence of the Euler–Maruyama method. Without loss of any generality, we may assume that T/τ is a rational number, otherwise we may replace T by a larger number. Let the step size $\triangle \in (0, 1)$ be a fraction of τ and T, namely $\triangle = \tau/N = T/M$ for some integers $N > \tau$ and $M > T$. The explicit discrete Euler–Maruyama approximate solution $\bar{y}(k\triangle)$, $k \geq -N$ is defined as follows:

$$\begin{cases} \bar{y}(k\triangle) = \xi(k\triangle), & -N \leq k \leq 0, \\ \bar{y}((k+1)\triangle) = \bar{y}(k\triangle) + u(\bar{y}_{k\triangle}) - u(\bar{y}_{(k-1)\triangle}) + f(\bar{y}_{k\triangle})\triangle + g(\bar{y}_{k\triangle})\triangle w_k, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \leq k \leq M-1, \end{cases} \tag{2.7}$$

where $\triangle w_k = w((k+1)\triangle) - w(k\triangle)$ and $\bar{y}_{k\triangle} = \{\bar{y}_{k\triangle}(\theta) : -\tau \leq \theta \leq 0\}$ is a $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable defined by

$$\begin{aligned} \bar{y}_{k\triangle}(\theta) &= \bar{y}((k+i)\triangle) + \frac{\theta - i\triangle}{\triangle}[\bar{y}((k+i+1)\triangle) - \bar{y}((k+i)\triangle)] \\ &\quad \text{for } i\triangle \leq \theta \leq (i+1)\triangle,\ i = -N, -(N-1), \cdots, -1, \end{aligned} \tag{2.8}$$

where in order for $\bar{y}_{-\triangle}$ to be well defined, we set $\bar{y}(-(N+1)\triangle) = \xi(-N\triangle)$.

It is easy to see from (2.8) that $\bar{y}_{k\triangle}(\cdot)$ is the linear interpolation of $\bar{y}((k-N)\triangle), \cdots, \bar{y}(k\triangle)$ and (2.8) can be rewritten as

$$\bar{y}_{k\triangle}(\theta) = \frac{(i+1)\triangle - \theta}{\triangle}\bar{y}((k+i)\triangle) + \frac{\theta - i\triangle}{\triangle}\bar{y}((k+i+1)\triangle), \tag{2.9}$$

which yields

$$\begin{aligned} |\bar{y}_{k\triangle}(\theta)| &\leq \frac{(i+1)\triangle - \theta}{\triangle}|\bar{y}((k+i)\triangle)| + \frac{\theta - i\triangle}{\triangle}|\bar{y}((k+i+1)\triangle)| \\ &\leq |\bar{y}((k+i)\triangle)| \vee |\bar{y}((k+i+1)\triangle)| \end{aligned}$$

because $(i+1)\triangle - \theta$ and $\theta - i\triangle \in [0, \triangle]$. We therefore have

$$\|\bar{y}_{k\triangle}\| = \max_{-N \le i \le 0} |\bar{y}((k+i)\triangle)|, \quad \forall k = -1, 0, 1, 2, \cdots, M-1. \tag{2.10}$$

It is obvious that $\|\bar{y}_{-\triangle}\| \le \|\bar{y}_0\|$.

In our analysis it will be more convenient to use continuous-time approximations. We hence introduce the $C([-\tau, 0]; \mathbb{R}^n)$-valued step process

$$\bar{y}_t = \sum_{k=0}^{M-2} \bar{y}_{k\triangle} 1_{[k\triangle, (k+1)\triangle)}(t) + \bar{y}_{(M-1)\triangle} 1_{[(M-1)\triangle, M\triangle]}(t), \tag{2.11}$$

and we define the continuous Euler–Maruyama approximate solution as follows: let $y(t) = \xi(t)$ for $-\tau \le t \le 0$, while for $t \in [k\triangle, (k+1)\triangle]$, $k = 0, 1, \cdots, M-1$,

$$\begin{aligned} y(t) = \xi(0) &+ u\Big(\bar{y}_{(k-1)\triangle} + \frac{t - k\triangle}{\triangle}(\bar{y}_{k\triangle} - \bar{y}_{(k-1)\triangle})\Big) - u(\bar{y}_{-\triangle}) \\ &+ \int_0^t f(\bar{y}_s) ds + \int_0^t g(\bar{y}_s) dw(s). \end{aligned} \tag{2.12}$$

Clearly, (2.12) can also be written as

$$\begin{aligned} y(t) = \bar{y}(k\triangle) &+ u\Big(\bar{y}_{(k-1)\triangle} + \frac{t - k\triangle}{\triangle}(\bar{y}_{k\triangle} - \bar{y}_{(k-1)\triangle})\Big) - u(\bar{y}_{(k-1)\triangle}) \\ &+ \int_{k\triangle}^t f(\bar{y}_s) ds + \int_{k\triangle}^t g(\bar{y}_s) dw(s). \end{aligned} \tag{2.13}$$

In particular, this shows that $y(k\triangle) = \bar{y}(k\triangle)$, that is, the discrete and continuous Euler–Maruyama approximate solutions coincide at the gridpoints. We know that $y(t)$ is not computable because it requires knowledge of the entire Brownian path, not just its $\triangle$-increments. However, $y(k\triangle) = \bar{y}(k\triangle)$, so the error bound for $y(t)$ will automatically imply the error bound for $\bar{y}(k\triangle)$. It is then obvious that

$$\|\bar{y}_{k\triangle}\| \le \|y_{k\triangle}\|, \quad \forall k = 0, 1, 2, \cdots, M-1. \tag{2.14}$$

Moreover, for any $t \in [0, T]$,

$$\begin{aligned} \sup_{0 \le t \le T} \|\bar{y}_t\| &= \sup_{0 \le k \le M-1} \|\bar{y}_{k\triangle}\| \\ &\le \sup_{0 \le k \le M-1} \|y_{k\triangle}\| \\ &= \sup_{0 \le k \le M-1} \sup_{-\tau \le \theta \le 0} |y(k\triangle + \theta)| \\ &\le \sup_{0 \le t \le T} \sup_{-\tau \le \theta \le 0} |y(t + \theta)| \\ &\le \sup_{-\tau \le s \le T} |y(s)| \end{aligned} \tag{2.15}$$

and letting $[t/\triangle]$ be the integer part of $t/\triangle$, then

$$\|\bar{y}_t\| = \|\bar{y}_{[t/\triangle]\triangle}\| \le \|y_{[t/\triangle]\triangle}\| \le \sup_{-\tau \le s \le t} |y(s)|. \tag{2.16}$$

These properties will be used frequently in what follows, without further explanation.

The primary aim of this paper is to establish the following strong mean square convergence theorem for the Euler–Maruyama approximations.

THEOREM 2.2. *Under Assumptions 2.1–2.4,*

$$\lim_{\triangle\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|x(t)-y(t)|^2\Big)=0. \tag{2.17}$$

The proof of this theorem is very technical, so we present some lemmas in the next section, and then complete the proof in the sequent section.

3. Lemmas. LEMMA 3.1. *Under Assumptions 2.2–2.4, for any $p\ge 2$, there exists a constant $H(p)$ such that*

$$\mathbb{E}\Big(\sup_{-\tau\le t\le T}|y(t)|^p\Big)\le H(p), \tag{3.1}$$

where $H(p)$ is independent of $\triangle$.

Proof. For $t\in[k\triangle,(k+1)\triangle]$, $k=0,1,2,\cdots,M-1$, set $\tilde{y}(t):=y(t)-u(\bar{y}_{(k-1)\triangle}+(t-k\triangle)(\bar{y}_{k\triangle}-\bar{y}_{(k-1)\triangle})/\triangle)$ and

$$h(t):=\mathbb{E}\Big(\sup_{-\tau\le s\le t}|y(s)|^p\Big);$$
$$\tilde{h}(t):=\mathbb{E}\Big(\sup_{0\le s\le t}|\tilde{y}(s)|^p\Big).$$

Recall the inequality that for $p\ge 1$ and any $\varepsilon>0$,

$$|x+y|^p\le(1+\varepsilon)^{p-1}(|x|^p+\varepsilon^{1-p}|y|^p).$$

This, together with Assumption 2.3, yields,

$$\begin{aligned}|y(t)|^p&\le(1+\varepsilon)^{p-1}(|\tilde{y}(t)|^p+\varepsilon^{1-p}|u(\bar{y}_{(k-1)\triangle}+(t-k\triangle)(\bar{y}_{k\triangle}-\bar{y}_{(k-1)\triangle})/\triangle)|^p)\\&\le(1+\varepsilon)^{p-1}\Big(|\tilde{y}(t)|^p+\varepsilon^{1-p}\kappa^p\Big\|\bar{y}_{(k-1)\triangle}+\frac{t-k\triangle}{\triangle}(\bar{y}_{k\triangle}-\bar{y}_{(k-1)\triangle})\Big\|^p\Big).\end{aligned}$$

By (2.9) and $\|\bar{y}_{-\triangle}\|\le\|\bar{y}_0\|$, noting $k=0,1,2,\cdots,M-1$,

$$\begin{aligned}&\Big\|\bar{y}_{(k-1)\triangle}+\frac{t-k\triangle}{\triangle}(\bar{y}_{k\triangle}-\bar{y}_{(k-1)\triangle})\Big\|^p\\&\le\Big|\frac{(k+1)\triangle-t}{\triangle}\|\bar{y}_{(k-1)\triangle}\|+\frac{t-k\triangle}{\triangle}\|\bar{y}_{k\triangle}\|\Big|^p\\&\le\Big[\frac{(k+1)\triangle-t}{\triangle}\Big(\sup_{-\tau\le s\le t}|y(s)|\Big)+\frac{t-k\triangle}{\triangle}\Big(\sup_{-\tau\le s\le t}|y(s)|\Big)\Big]^p\\&\le\sup_{-\tau\le s\le t}|y(s)|^p.\end{aligned}$$

Consequently,

$$|y(t)|^p\le(1+\varepsilon)^{p-1}\Big[|\tilde{y}(t)|^p+\varepsilon^{1-p}\kappa^p\Big(\sup_{-\tau\le s\le t}|y(s)|^p\Big)\Big].$$

Choose $\varepsilon=\kappa/(1-\kappa)$, then

$$|y(t)|^p\le(1-\kappa)^{1-p}|\tilde{y}(t)|^p+\kappa\Big(\sup_{-\tau\le s\le t}|y(s)|^p\Big).$$

Hence,

$$\begin{aligned} h(t) &\le \mathbb{E}\|\xi\|^p + \mathbb{E}\Big(\sup_{0\le s\le t}|y(s)|^p\Big) \\ &\le \mathbb{E}\|\xi\|^p + \kappa h(t) + (1-\kappa)^{1-p}\tilde{h}(t), \end{aligned}$$

which implies

$$h(t) \le \frac{\mathbb{E}\|\xi\|^p}{1-\kappa} + \frac{\tilde{h}(t)}{(1-\kappa)^p}. \tag{3.2}$$

Since

$$\tilde{y}(t) = \tilde{y}(0) + \int_0^t f(\bar{y}_s)ds + \int_0^t g(\bar{y}_s)dw(s)$$

with $\tilde{y}(0) = \bar{y}(0) - u(\bar{y}_{-\triangle})$, by the Hölder inequality, we have

$$|\tilde{y}(t)|^p \le 3^{p-1}\Big[|\tilde{y}(0)|^p + t^{p-1}\int_0^t |f(\bar{y}_s)|^p ds + \Big|\int_0^t g(\bar{y}_s)dw(s)\Big|^p\Big].$$

Hence, for any $t_1 \in [0,T]$

$$\begin{aligned} \tilde{h}(t_1) \le 3^{p-1}\Big[&\mathbb{E}|\tilde{y}(0)|^p + T^{p-1}\mathbb{E}\int_0^{t_1}|f(\bar{y}_s)|^p ds \\ &+\mathbb{E}\Big(\sup_{0\le t\le t_1}\Big|\int_0^t g(\bar{y}_s)dw(s)\Big|^p\Big)\Big]. \end{aligned} \tag{3.3}$$

By Assumption 2.3 and the fact $\|\bar{y}_{-\triangle}\| \le \|\bar{y}_0\|$, we compute that

$$\begin{aligned} \mathbb{E}|\tilde{y}(0)|^p &= \mathbb{E}|\bar{y}(0) - u(\bar{y}_{-\triangle})|^p \\ &\le \mathbb{E}(|\bar{y}(0)| + \kappa\|\bar{y}_{-\triangle}\|)^p \\ &\le \mathbb{E}(|\bar{y}(0)| + \kappa\|\bar{y}_0\|)^p \\ &\le \mathbb{E}(|\xi(0)| + \kappa\|\xi\|)^p \\ &\le (1+\kappa)^p\mathbb{E}\|\xi\|^p. \end{aligned} \tag{3.4}$$

Assumption 2.2 and the Hölder inequality give

$$\begin{aligned} \mathbb{E}\int_0^{t_1}|f(\bar{y}_s)|^p ds &\le \mathbb{E}\int_0^{t_1} K^{\frac{p}{2}}(1+\|\bar{y}_s\|^2)^{\frac{p}{2}}ds \\ &\le K^{\frac{p}{2}}2^{\frac{p-2}{2}}\mathbb{E}\int_0^{t_1}(1+\|\bar{y}_s\|^p)ds \\ &\le K^{\frac{p}{2}}2^{\frac{p-2}{2}}\Big[T + \int_0^{t_1}\mathbb{E}\Big(\sup_{-\tau\le t\le s}|y(t)|^p ds\Big)\Big]. \end{aligned} \tag{3.5}$$

Applying the Burkholder–Davis–Gundy inequality, the Hölder inequality and Assumption 2.2 yields

$$\mathbb{E}\Big(\sup_{0\le t\le t_1}\Big|\int_0^t g(\bar{y}_s)dw(s)\Big|^p\Big) \le C_p\mathbb{E}\Big(\int_0^{t_1}|g(\bar{y}_s)|^2 ds\Big)^{\frac{p}{2}}$$

$$\le C_p T^{\frac{p-2}{2}} \mathbb{E}\int_0^{t_1} |g(\bar{y}_s)|^p ds$$
$$\le C_p T^{\frac{p-2}{2}} \mathbb{E}\int_0^{t_1} K^{\frac{p}{2}}(1+\|\bar{y}_s\|^2)^{\frac{p}{2}} ds$$
$$\le C_p T^{\frac{p-2}{2}} K^{\frac{p}{2}} 2^{\frac{p-2}{2}} \mathbb{E}\int_0^{t_1} \Big(1+\|\bar{y}_s\|^p\Big) ds$$
$$\le C_p T^{\frac{p-2}{2}} K^{\frac{p}{2}} 2^{\frac{p-2}{2}} \Big[T + \int_0^{t_1} \mathbb{E}\Big(\sup_{-\tau\le t\le s} |y(t)|^p\Big) ds\Big], \tag{3.6}$$

where C_p is a constant dependent only on p. Substituting (3.4), (3.5) and (3.6) into (3.3) gives

$$\begin{aligned}\tilde{h}(t_1) \le\ & 3^{p-1}\Big[(1+\kappa)^p \mathbb{E}\|\xi\|^p + K^{\frac{p}{2}} 2^{\frac{p-2}{2}} T^p + C_p (2T)^{\frac{p-2}{2}} K^{\frac{p}{2}} T\Big] \\ & + 3^{p-1}\Big[K^{\frac{p}{2}} 2^{\frac{p-2}{2}} T^{p-1} + C_p(2T)^{\frac{p-2}{2}} K^{\frac{p}{2}}\Big] \int_0^{t_1} \mathbb{E}\Big(\sup_{-\tau\le t\le s}|y(t)|^p\Big) ds \\ =:\ & C_1 + C_2 \int_0^{t_1} h(s) ds.\end{aligned}$$

Hence from (3.2), we have

$$\begin{aligned} h(t_1) &\le \frac{\mathbb{E}\|\xi\|^p}{1-\kappa} + \frac{1}{(1-\kappa)^p}\Big[C_1 + C_2\int_0^{t_1} h(s)ds\Big] \\ &\le \frac{\mathbb{E}\|\xi\|^p}{1-\kappa} + \frac{C_1}{(1-\kappa)^p} + \frac{C_2}{(1-\kappa)^p}\int_0^{t_1} h(s)ds.\end{aligned}$$

By the Gronwall inequality we find that

$$h(T) \le \Big[\frac{\mathbb{E}\|\xi\|^p}{1-\kappa} + \frac{C_1}{(1-\kappa)^p}\Big] e^{\frac{C_2 T}{(1-\kappa)^p}}.$$

From the expressions of C_1 and C_2, we know that they are positive constants dependent only on ξ, κ, K, p and T, but independent of $\triangle$. The required assertion must hold. □

LEMMA 3.2. *If Assumptions 2.2–2.4 hold, then for any integer $l > 1$,*

$$\mathbb{E}\Big(\sup_{0\le k\le M-1} \|\bar{y}_{k\triangle} - \bar{y}_{(k-1)\triangle}\|^2\Big) \le c_1\alpha(\triangle) + \bar{c}_1(l)\triangle^{\frac{l-1}{l}} =: \gamma(\triangle), \tag{3.7}$$

where $c_1 = 1/(1-\kappa)$, $\bar{c}_1(l)$ is a constant dependent on l but independent of $\triangle$.

Proof. For $\theta \in [i\triangle, (i+1)\triangle]$, where $i = -N, -(N+1), \cdots, -1$, from (2.9),

$$\begin{aligned}|\bar{y}_{k\triangle} - \bar{y}_{(k-1)\triangle}| \le\ & \frac{(i+1)\triangle - \theta}{\triangle}|\bar{y}((k+i)\triangle) - \bar{y}((k-1+i)\triangle)| \\ & + \frac{\theta - i\triangle}{\triangle}|\bar{y}((k+i+1)\triangle) - \bar{y}((k+i)\triangle)| \\ \le\ & |\bar{y}((k+i)\triangle) - \bar{y}((k-1+i)\triangle)| \vee |\bar{y}((k+i+1)\triangle) - \bar{y}((k+i)\triangle)|,\end{aligned}$$

so

$$\|\bar{y}_{k\triangle} - \bar{y}_{(k-1)\triangle}\| \le \sup_{-N\le i\le 0} |\bar{y}((k+i)\triangle) - \bar{y}((k-1+i)\triangle)|.$$

We therefore have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le k\le M-1}\|\bar{y}_{k\triangle}-\bar{y}_{(k-1)\triangle}\|^2\Big)\\
&\le \mathbb{E}\Big[\sup_{0\le k\le M-1}\Big(\sup_{-N\le i\le 0}|\bar{y}((k+i)\triangle)-\bar{y}((k-1+i)\triangle)|^2\Big)\Big]\\
&\le \mathbb{E}\Big(\sup_{-N\le k\le M-1}|\bar{y}(k\triangle)-\bar{y}((k-1)\triangle)|^2\Big). \qquad (3.8)
\end{aligned}
$$

When $-N\le k\le 0$, by Assumption 2.4 and $\bar{y}(-(N+1)\triangle)=\xi(-N\triangle)$,

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{-N\le k\le 0}|\bar{y}(k\triangle)-\bar{y}((k-1)\triangle)|^2\Big) &\le \mathbb{E}\Big(\sup_{-N\le k\le 0}|\xi(k\triangle)-\xi((k-1)\triangle)|^2\Big)\\
&\le \alpha(\triangle). \qquad (3.9)
\end{aligned}
$$

When $1\le k\le M-1$, from (2.7), we have

$$
\bar{y}(k\triangle)-\bar{y}((k-1)\triangle)=u(\bar{y}_{(k-1)\triangle})-u(\bar{y}_{(k-2)\triangle})+f(\bar{y}_{(k-1)\triangle})\triangle+g(\bar{y}_{(k-1)\triangle})\triangle w_k.
$$

Recall the elementary inequality, for any $x,y>0$ and $\varepsilon\in(0,1)$,

$$
(x+y)^2\le \frac{x^2}{\varepsilon}+\frac{y^2}{1-\varepsilon}.
$$

Then we have,

$$
\begin{aligned}
&|\bar{y}(k\triangle)-\bar{y}((k-1)\triangle)|^2\\
&\le \frac{1}{\varepsilon}|u(\bar{y}_{(k-1)\triangle})-u(\bar{y}_{(k-2)\triangle})|^2+\frac{1}{1-\varepsilon}|f(\bar{y}_{(k-1)\triangle})\triangle+g(\bar{y}_{(k-1)\triangle})\triangle w_k|^2\\
&\le \frac{\kappa^2}{\varepsilon}\|\bar{y}_{(k-1)\triangle}-\bar{y}_{(k-2)\triangle}\|^2+\frac{2}{1-\varepsilon}|f(\bar{y}_{(k-1)\triangle})|^2\triangle^2+\frac{2}{1-\varepsilon}|g(\bar{y}_{(k-1)\triangle})\triangle w_k|^2,
\end{aligned}
$$

consequently,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{1\le k\le M-1}|\bar{y}(k\triangle)-\bar{y}((k-1)\triangle)|^2\Big)\\
&\le \frac{\kappa^2}{\varepsilon}\mathbb{E}\Big(\sup_{1\le k\le M-1}\|\bar{y}_{(k-1)\triangle}-\bar{y}_{(k-2)\triangle}\|^2\Big)\\
&\quad+\frac{2\triangle^2}{1-\varepsilon}\mathbb{E}\Big(\sup_{1\le k\le M-1}|f(\bar{y}_{(k-1)\triangle})|^2\Big)\\
&\quad+\frac{2}{1-\varepsilon}\mathbb{E}\Big(\sup_{1\le k\le M-1}|g(\bar{y}_{(k-1)\triangle})\triangle w_k|^2\Big). \qquad (3.10)
\end{aligned}
$$

We deal with these three terms, separately. By (3.9),

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{1\le k\le M-1}\|\bar{y}_{(k-1)\triangle}-\bar{y}_{(k-2)\triangle}\|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{1\le k\le M-1}\sup_{-N\le i\le 0}|\bar{y}((k+i-1)\triangle)-\bar{y}((k-2+i)\triangle)|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{-N\le k\le M-1}|\bar{y}(k\triangle)-\bar{y}((k-1)\triangle)|^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le \mathbb{E}\Big(\sup_{-N\le k\le 0}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\big)\\
&\quad+\mathbb{E}\Big(\sup_{1\le k\le M-1}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\Big)\\
&\le \alpha(\triangle)+\mathbb{E}\Big(\sup_{1\le k\le M-1}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\Big). \qquad (3.11)
\end{aligned}
$$

Noting that $\mathbb{E}[\sup_{-\tau\le t\le T}|\bar y(t)|^2]\le \mathbb{E}[\sup_{-\tau\le t\le T}|y(t)|^2]\le H(2)$ (where $H(p)$ has been defined in Lemma 3.1), by Assumption 2.2 and (2.10),

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{1\le k\le M-1}|f(\bar y_{(k-1)\triangle})|^2\Big) &\le \mathbb{E}\Big(\sup_{1\le k\le M-1}K(1+\|\bar y_{(k-1)\triangle}\|^2)\Big)\\
&\le K+K\mathbb{E}\Big(\sup_{1\le k\le M-1}\sup_{-N\le i\le 0}|\bar y((k-1+i)\triangle)|^2\Big)\\
&\le K+K\mathbb{E}\Big(\sup_{-N\le k\le M-1}|\bar y(k\triangle)|^2\Big)\\
&\le K+K\mathbb{E}\Big(\sup_{-\tau\le t\le T}|\bar y(t)|^2\Big)\\
&\le K(1+H(2)). \qquad (3.12)
\end{aligned}
$$

By the Hölder inequality, for any integer $l>1$,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{1\le k\le M-1}|g(\bar y_{k\triangle})\triangle w_k|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{1\le k\le M-1}|g(\bar y_{(k-1)\triangle})|^2\sup_{1\le k\le M-1}|\triangle w_k|^2\Big)\\
&\le \Big[\mathbb{E}\Big(\sup_{1\le k\le M-1}|g(\bar y_{(k-1)\triangle})|^{\frac{2l}{l-1}}\Big)\Big]^{\frac{l-1}{l}}\Big[\mathbb{E}\Big(\sup_{0\le k\le M-1}|\triangle w_k|^{2l}\Big)\Big]^{\frac{1}{l}}\\
&\le \Big[\mathbb{E}\Big(\sup_{0\le k\le M-1}(K(1+\|\bar y_{k\triangle}\|^2))^{\frac{l}{l-1}}\Big)\Big]^{\frac{l-1}{l}}\Big[\mathbb{E}\Big(\sum_{k=0}^{M-1}|\triangle w_k|^{2l}\Big)\Big]^{\frac{1}{l}}\\
&\le \Big[K^{\frac{l}{1-l}}\mathbb{E}\Big(1+\Big(\sup_{0\le k\le M-1}\|\bar y_{k\triangle}\|^2\Big)\Big)^{\frac{l}{l-1}}\Big]^{\frac{l-1}{l}}\Big[\Big(\sum_{k=0}^{M-1}\mathbb{E}|\triangle w_k|^{2l}\Big)\Big]^{\frac{1}{l}}\\
&\le \Big[2^{\frac{1}{l-1}}K^{\frac{l}{1-l}}\Big(1+\mathbb{E}\Big(\sup_{0\le k\le M-1}\|\bar y_{k\triangle}\|^{\frac{2l}{l-1}}\Big)\Big)\Big]^{\frac{l-1}{l}}\Big[\Big(\sum_{k=0}^{M-1}(2l-1)!!\triangle^l\Big)\Big]^{\frac{1}{l}}\\
&\le \Big[2^{\frac{1}{l-1}}K^{\frac{l}{1-l}}\Big(1+H\Big(\frac{2l}{l-1}\Big)\Big)\Big]^{\frac{l-1}{l}}[(2l-1)!!T\triangle^{l-1}]^{\frac{1}{l}}\\
&\le D(l)\triangle^{\frac{l-1}{l}}, \qquad (3.13)
\end{aligned}
$$

where $(2l-1)!!=1\cdot 3\cdots(2l-1)$, $D(l)$ is a constant dependent on l.

Substituting (3.11), (3.12) and (3.13) into (3.10), choosing $\varepsilon=\kappa$ and noting $\triangle\in(0,1)$, we have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{1\le k\le M-1}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\Big)\\
&\le \frac{\kappa}{1-\kappa}\alpha(\triangle)+\frac{2K(1+H(2))+2D(l)}{(1-\kappa)^2}\triangle^{\frac{l-1}{l}}. \qquad (3.14)
\end{aligned}
$$

Combining (3.9) with (3.14), from (3.8) we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le k\le M-1}\|\bar y_{k\triangle}-\bar y_{(k-1)\triangle}\|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{-N\le k\le M-1}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{-N\le k\le 0}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\Big)\\
&\quad+\mathbb{E}\Big(\sup_{1\le k\le M-1}|\bar y(k\triangle)-\bar y((k-1)\triangle)|^2\Big)\\
&\le \frac{1}{1-\kappa}\alpha(\triangle)+\frac{2K(1+H(2))+2D(l)}{(1-\kappa)^2}\triangle^{\frac{l-1}{l}}, \qquad (3.15)
\end{aligned}$$

as required. □

LEMMA 3.3. *If Assumptions 2.2–2.4 hold,*

$$\mathbb{E}\Big(\sup_{0\le s\le T}\|y_s-\bar y_s\|^2\Big)\le c_2\alpha(2\triangle)+\bar c_2(l)\triangle^{\frac{l-1}{l}}=:\beta(\triangle), \qquad (3.16)$$

where c_2 is a constant independent of l and $\triangle$, $\bar c_2(l)$ is a constant dependent on l but independent of $\triangle$.

Proof. Fix any $s\in[0,T]$ and $\theta\in[-\tau,0]$. Let $k_s\in\{0,1,2,\cdots,M-1\}$, $k_\theta\in\{-N,-N+1,\cdots,-1\}$ and $k_{s\theta}\in\{-N,-N+1,\cdots,M-1\}$ be the integers for which $s\in[k_s\triangle,(k_s+1)\triangle]$, $\theta\in[k_\theta\triangle,(k_\theta+1)\triangle]$ and $s+\theta\in[k_{s\theta}\triangle,(k_{s\theta}+1)\triangle]$, respectively. Clearly,

$$0\le s+\theta-(k_s+k_\theta)\le 2\triangle \qquad (3.17)$$

$$k_{s\theta}-(k_s+k_\theta)\in\{0,1,2\}. \qquad (3.18)$$

From (2.8),

$$\begin{aligned}
\bar y_s&=\bar y_{k_s\triangle}(\theta)\\
&=\bar y((k_s+k_\theta)\triangle)+\frac{\theta-k_\theta\triangle}{\triangle}(\bar y((k_s+k_\theta+1)\triangle)-\bar y(k_s+k_\theta)\triangle),
\end{aligned}$$

which yields

$$\begin{aligned}
|y_s-\bar y_s|&=|y(s+\theta)-\bar y_{k_s\triangle}(\theta)|\\
&\le|y(s+\theta)-\bar y((k_s+k_\theta)\triangle)|+\frac{\theta-k_\theta\triangle}{\triangle}|\bar y((k_s+k_\theta+1)\triangle)-\bar y((k_s+k_\theta)\triangle)|\\
&\le|y(s+\theta)-\bar y((k_s+k_\theta)\triangle)|+|\bar y((k_s+k_\theta+1)\triangle)-\bar y((k_s+k_\theta)\triangle)|,
\end{aligned}$$

so by (3.8) and Lemma 3.2, noting $\bar y(M\triangle)=\bar y((M-1)\triangle)$ from (2.11),

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le T}\|y_s-\bar y_s\|^2\Big)\\
&\le 2\mathbb{E}\Big[\sup_{0\le s\le T}\Big(\sup_{-\tau\le\theta\le 0}|y(s+\theta)-\bar y((k_s+k_\theta)\triangle)|^2\Big)\Big]\\
&\quad+2\mathbb{E}\Big[\sup_{0\le k_s\le M-1}\Big(\sup_{-N\le k_\theta\le 0}|\bar y((k_s+k_\theta+1)\triangle)-\bar y((k_s+k_\theta)\triangle)|^2\Big)\Big]\\
&\le 2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T}|y(s+\theta)-\bar y((k_s+k_\theta)\triangle)|^2\Big)+2\gamma(\triangle). \qquad (3.19)
\end{aligned}$$

Therefore, it is a key to compute $\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big)$. We discuss the following four possible cases.
Case 1: $k_s+k_\theta\ge 0$. We again divide this case into three possible subcases according to $k_{s\theta}-(k_s+k_\theta)\in\{0,1,2\}$.
Subcase 1: $k_{s\theta}-(k_s+k_\theta)=0$. From (2.13)

$$\begin{aligned}&y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)\\ =\;& u\Big(\bar{y}_{(k_{s\theta}-1)\triangle}+\frac{s+\theta-k_{s\theta}\triangle}{\triangle}(\bar{y}_{k_{s\theta}\triangle}-\bar{y}_{(k_{s\theta}-1)\triangle})\Big)-u(\bar{y}_{(k_{s\theta}-1)\triangle})\\ &+\int_{k_{s\theta}\triangle}^{s+\theta}f(\bar{y}_r)dr+\int_{k_{s\theta}\triangle}^{s+\theta}g(\bar{y}_r)dw(r),\end{aligned}$$

which yields

$$\begin{aligned}&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big)\\ \le\;& 3\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\Big|u\Big(\bar{y}_{(k_{s\theta}-1)\triangle}+\frac{s+\theta-k_{s\theta}\triangle}{\triangle}(\bar{y}_{k_{s\theta}\triangle}-\bar{y}_{(k_{s\theta}-1)\triangle})\Big)\\ &-u(\bar{y}_{(k_{s\theta}-1)\triangle})\Big)\Big|^2+3\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\Big|\int_{k_{s\theta}\triangle}^{s+\theta}f(\bar{y}_r)dr\Big|^2\Big)\\ &+3\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\Big|\int_{k_{s\theta}\triangle}^{s+\theta}g(\bar{y}_r)dw(r)\Big|^2\Big).\end{aligned}\tag{3.20}$$

From Assumption 2.3 and Lemma 3.2, we have

$$\begin{aligned}&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\Big|u\Big(\bar{y}_{(k_{s\theta}-1)\triangle}\\ &+\frac{s+\theta-k_{s\theta}\triangle}{\triangle}(\bar{y}_{k_{s\theta}\triangle}-\bar{y}_{(k_{s\theta}-1)\triangle})\Big)-u(\bar{y}_{(k_{s\theta}-1)\triangle})\Big)\Big|^2\\ \le\;&\kappa^2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\Big\|\frac{s+\theta-k_{s\theta}\triangle}{\triangle}(\bar{y}_{k_{s\theta}\triangle}-\bar{y}_{(k_{s\theta}-1)\triangle})\Big\|^2\Big)\\ \le\;&\kappa^2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\|(\bar{y}_{k_{s\theta}\triangle}-\bar{y}_{(k_{s\theta}-1)\triangle})\|^2\Big)\\ \le\;&\kappa^2\mathbb{E}\Big(\sup_{0\le k_{s\theta}\le M-1}\|(\bar{y}_{k_{s\theta}\triangle}-\bar{y}_{(k_{s\theta}-1)\triangle})\|^2\Big)\\ \le\;&\kappa^2\gamma(\triangle).\end{aligned}\tag{3.21}$$

By the Hölder inequality, Assumption 2.2 and Lemma 3.1,

$$\begin{aligned}&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\Big|\int_{k_{s\theta}\triangle}^{s+\theta}f(\bar{y}_r)dr\Big|^2\Big)\\ \le\;&\triangle\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\int_{k_{s\theta}\triangle}^{s+\theta}|f(\bar{y}_r)|^2dr\Big)\\ \le\;&K\triangle\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\int_{k_{s\theta}\triangle}^{s+\theta}(1+\|\bar{y}_r\|^2)dr\Big)\\ \le\;&K\triangle\mathbb{E}\Big[\int_0^T\Big(1+\sup_{0\le r\le T}\|\bar{y}_r\|^2\Big)dr\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq K\triangle \int_0^T \Big[1+\mathbb{E}\Big(\sup_{-\tau\leq t\leq T}|y(t)|^2\Big)\Big]dr \\
&\leq K\triangle \int_0^T [1+H(2)]dr \\
&\leq KT[1+H(2)]\triangle. \qquad (3.22)
\end{aligned}
$$

Setting $v = s+\theta$ and $k_v = k_{s\theta}$ and applying the Hölder inequality yield

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\leq\theta\leq 0, 0\leq s\leq T, k_{s\theta}\geq 0}\Big|\int_{k_{s\theta}\triangle}^{s+\theta} g(\bar{y}_r)dw(r)\Big|^2\Big) \\
&= \mathbb{E}\Big(\sup_{0\leq v\leq T, 0\leq k_v\leq M-1}|g(\bar{y}_{k_v\triangle})(w(v)-w(k_v\triangle))|^2\Big) \\
&\leq \Big[\mathbb{E}\Big(\sup_{0\leq v\leq T, 0\leq k_v\leq M-1}|g(\bar{y}_{k_v\triangle})|^{\frac{2l}{l-1}}\Big)\Big]^{\frac{l-1}{l}}\Big[\mathbb{E}\Big(\sup_{0\leq v\leq T, 0\leq k_v\leq M-1}|w(v)-w(k_v\triangle)|^{2l}\Big)\Big]^{\frac{1}{l}}.
\end{aligned}
$$

The Doob martingale inequality gives

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq v\leq T, 0\leq k_v\leq M-1}|w(v)-w(k_v\triangle)|^{2l}\Big) \\
&= \mathbb{E}\Big(\sup_{0\leq k_v\leq M-1}\Big(\sup_{k_v\triangle\leq v\leq (k_v+1)\triangle}|w(v)-w(k_v\triangle)|^{2l}\Big)\Big) \\
&\leq \mathbb{E}\Big(\sum_{k_v=0}^{M-1}\Big(\sup_{k_v\triangle\leq v\leq (k_v+1)\triangle}|w(v)-w(k_v\triangle)|^{2l}\Big)\Big) \\
&= \sum_{k_v=0}^{M-1}\mathbb{E}\Big(\sup_{k_v\triangle\leq v\leq (k_v+1)\triangle}|w(v)-w(k_v\triangle)|^{2l}\Big) \\
&\leq \Big(\frac{2l}{2l-1}\Big)^{2l}\sum_{k_v=0}^{M-1}\mathbb{E}|w((k_v+1)\triangle)-w(k_v\triangle)|^{2l}.
\end{aligned}
$$

By (3.13), we therefore have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\leq\theta\leq 0, 0\leq s\leq T, k_{s\theta}\geq 0}\Big|\int_{k_{s\theta}\triangle}^{s+\theta} g(\bar{y}_r)dw(r)\Big|^2\Big) \\
&\leq \Big(\frac{2l}{2l-1}\Big)^2\Big[\mathbb{E}\Big(\sup_{0\leq k_v\leq M-1}|g(\bar{y}_{k_v\triangle})|^{\frac{2l}{l-1}}\Big)\Big]^{\frac{l-1}{l}} \\
&\quad\times\Big[\sum_{k_v=0}^{M-1}\mathbb{E}|w((k_v+1)\triangle)-w(k_v\triangle)|^{2l}\Big]^{\frac{1}{l}} \\
&\leq \Big(\frac{2l}{2l-1}\Big)^2 D(l)\triangle^{\frac{l-1}{l}}. \qquad (3.23)
\end{aligned}
$$

Substituting (3.21), (3.22) and (3.23) into (3.20) and noting $\triangle\in(0,1)$ give

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\leq\theta\leq 0, 0\leq s\leq T, k_s+k_\theta\geq 0}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big) \\
&\leq 3\kappa^2\gamma(\triangle)+3\Big(KT(1+H(2))+\Big(\frac{2l}{2l-1}\Big)^2 D(l)\Big)\triangle^{\frac{l-1}{l}} \\
&=: 3\kappa^2\gamma(\triangle)+c_1(l)\triangle^{\frac{l-1}{l}}. \qquad (3.24)
\end{aligned}
$$

Subcase 2: $k_{s\theta} - (k_s + k_\theta) = 1$. From (2.13) and (2.7),

$$\begin{aligned}
&y(s+\theta) - \bar{y}((k_s + k_\theta)\triangle) \\
&= y(k_{s\theta}\triangle) - \bar{y}((k_s + k_\theta)\triangle) + y(s+\theta) - \bar{y}(k_{s\theta}\triangle) \\
&\le u(\bar{y}_{(k_s+k_\theta)\triangle}) - u(\bar{y}_{(k_s+k_\theta-1)\triangle}) + f(\bar{y}_{(k_s+k_\theta)\triangle})\triangle + g(\bar{y}_{(k_s+k_\theta)\triangle})\triangle w_{k_s+k_\theta} \\
&\quad + y(s+\theta) - \bar{y}(k_{s\theta}\triangle),
\end{aligned}$$

so we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |y(s+\theta) - \bar{y}((k_s+k_\theta)\triangle)|^2\Big) \\
&\le 4\Big\{\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |u(\bar{y}_{(k_s+k_\theta)\triangle}) - u(\bar{y}_{(k_s+k_\theta-1)\triangle})|^2\Big) \\
&\quad + \mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |f(\bar{y}_{(k_s+k_\theta)\triangle})\triangle|^2\Big) \\
&\quad + \mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |g(\bar{y}_{(k_s+k_\theta)\triangle})\triangle w_{k_s+k_\theta}|^2\Big) \\
&\quad + \mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |y(s+\theta) - \bar{y}(k_{s\theta}\triangle)|^2\Big)\Big\}.
\end{aligned} \tag{3.25}$$

Since

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |u(\bar{y}_{(k_s+k_\theta)\triangle}) - u(\bar{y}_{(k_s+k_\theta-1)\triangle})|^2\Big) \\
&\le \kappa^2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} \|\bar{y}_{(k_s+k_\theta)\triangle} - \bar{y}_{(k_s+k_\theta-1)\triangle}\|^2\Big) \\
&\le \kappa^2\gamma(\triangle),
\end{aligned} \tag{3.26}$$

from (3.12), (3.13), and the subcase 1, noting $\triangle \in (0,1)$, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |y(s+\theta) - \bar{y}((k_s+k_\theta)\triangle)|^2\Big) \\
&\le 4\Big\{\kappa^2\gamma(\triangle) + K[1+H(2)]\triangle^2 + D(l)\triangle^{\frac{l-1}{l}} + 3\kappa^2\gamma(\triangle) + c_1(l)\triangle^{\frac{l-1}{l}}\Big\} \\
&=: 16\kappa^2\gamma(\triangle) + c_2(l)\triangle^{\frac{l-1}{l}}.
\end{aligned} \tag{3.27}$$

Subcase 3: $k_{s\theta} - (k_s + k_\theta) = 2$. From (2.13) and (2.7), we have

$$\begin{aligned}
&y(s+\theta) - \bar{y}((k_s + k_\theta)\triangle) \\
&= y(s+\theta) - y((k_{s\theta}-1)\triangle) + y((k_{s\theta}-1)\triangle) - \bar{y}((k_s+k_\theta)\triangle),
\end{aligned}$$

so from the subcase 2, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |y(s+\theta) - \bar{y}((k_s+k_\theta)\triangle)|^2\Big) \\
&\le 2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |y(s+\theta) - \bar{y}((k_{s\theta}-1)\triangle)|^2\Big) \\
&\quad + 2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0} |y((k_{s\theta}-1)\triangle) - \bar{y}((k_s+k_\theta)\triangle)|^2\Big) \\
&\le 2[16\kappa^2\gamma(\triangle) + c_2(l)\triangle^{\frac{l-1}{l}}] + 2[\kappa^2\gamma(\triangle) + K[1+H(2)]\triangle^2 + D(l)\triangle^{\frac{l-1}{l}}] \\
&=: 34\kappa^2\gamma(\triangle) + c_3(l)\triangle^{\frac{l-1}{l}}.
\end{aligned} \tag{3.28}$$

From these three subcases, we have

$$\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big)$$
$$\le 53\kappa^2\gamma(\triangle)+[c_1(l)+c_2(l)+c_3(l)]\triangle^{\frac{l-1}{l}}. \tag{3.29}$$

Case 2: $k_s+k_\theta=-1$ and $0\le s+\theta\le\triangle$. In this case, applying Assumption 2.4 and case 1, we have

$$\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big)$$
$$\le 2\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T}|y(s+\theta)-\bar{y}(0)|^2\Big)+2E|y(0)-y(-\triangle)|^2$$
$$\le 106\kappa^2\gamma(\triangle)+2[c_1(l)+c_2(l)+c_3(l)]\triangle^{\frac{l-1}{l}}+2\alpha(\triangle). \tag{3.30}$$

Case 3: $k_s+k_\theta=-1$ and $-\triangle\le s+\theta\le 0$. In this case, using Assumption 2.4

$$\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big)\le\alpha(\triangle). \tag{3.31}$$

Case 4: $k_s+k_\theta\le -2$. In this case, $s+\theta\le 0$, so using Assumption 2.4

$$\mathbb{E}\Big(\sup_{-\tau\le\theta\le 0,0\le s\le T}|y(s+\theta)-\bar{y}((k_s+k_\theta)\triangle)|^2\Big)\le\alpha(2\triangle). \tag{3.32}$$

Substituting these four cases into (3.19) and noting the expression of $\gamma(\triangle)$, there exist c_2 and $\bar{c}_2(l)$ such that

$$\mathbb{E}\Big(\sup_{0\le s\le T}\|y_s-\bar{y}_s\|^2\Big)\le c_2\alpha(2\triangle)+\bar{c}_2(l)\triangle^{\frac{l-1}{l}}, \tag{3.33}$$

namely, the required assertion holds. □

4. Proof of Theorem 2.2. From Lemma 3.1 and Theorem 2.1, there exists a positive constant $\widetilde{H}$ such that

$$\mathbb{E}\Big(\sup_{-\tau\le t\le T}|x(t)|^p\Big)\vee\mathbb{E}\Big(\sup_{-\tau\le t\le T}|y(t)|^p\Big)\le\widetilde{H}. \tag{4.1}$$

Let j be a sufficient large integer. Define the stopping times

$$u_j:=\inf\{t\ge 0:\|x_t\|\ge j\},\quad v_j:=\inf\{t\ge 0:\|y_t\|\ge j\},\quad \rho_j:=u_j\wedge v_j,$$

where we set $\inf\emptyset=\infty$ as usual. Let

$$e(t):=x(t)-y(t).$$

Obviously,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big)=\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2 1_{\{u_j>T,\ v_j>T\}}\Big)+\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2 1_{\{u_j\le T\text{ or }v_j\le T\}}\Big).$$

Recall the following elementary inequality:

$$a^\gamma b^{1-\gamma}\le\gamma a+(1-\gamma)b,\quad\forall a,b>0,\ \gamma\in[0,1]. \tag{4.2}$$

We thus have, for any $\delta > 0$,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2 1_{\{u_j\le T \text{ or } v_j\le T\}}\Big) \le \mathbb{E}\Big[\Big(\delta \sup_{\{0\le t\le T\}}|e(t)|^p\Big)^{\frac{2}{p}}\Big(\delta^{-\frac{2}{p-2}}1_{\{u_j\le T \text{ or } v_j\le T\}}\Big)^{\frac{p-2}{p}}\Big]$$
$$\le \frac{2\delta}{p}\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^p\Big) + \frac{p-2}{p\delta^{2/(p-2)}}\mathbb{P}(u_j\le T \text{ or } v_j\le T).$$

Hence

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2 1_{\{\rho_j>T\}}\Big) + \frac{2\delta}{p}\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^p\Big) + \frac{p-2}{p\delta^{2/(p-2)}}\mathbb{P}(u_j\le T \text{ or } v_j\le T). \tag{4.3}$$

Now,

$$\mathbb{P}(u_j\le T) \le \mathbb{E}\Big(1_{\{u_j\le T\}}\frac{\|x_t\|^p}{j^p}\Big) \le \frac{1}{j^p}\mathbb{E}\Big(\sup_{-\tau\le t\le T}|x(t)|^p\Big) \le \frac{\widetilde{H}}{j^p}.$$

Similarly,

$$\mathbb{P}(v_j\le T) \le \frac{\widetilde{H}}{j^p}.$$

Thus

$$\mathbb{P}(v_j\le T \text{ or } u_j\le T) \le \mathbb{P}(v_j\le T) + \mathbb{P}(u_j\le T) \le \frac{2\widetilde{H}}{j^p}. \tag{4.4}$$

We also have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^p\Big) \le 2^{p-1}\mathbb{E}\Big(\sup_{0\le t\le T}(|x_t|^p+|y_t|^p)\Big) \le 2^p\widetilde{H}. \tag{4.5}$$

Moreover,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2 1_{\{\rho_j>T\}}\Big) = \mathbb{E}\Big(\sup_{0\le t\le T}|e(t\wedge\rho_j)|^2 1_{\{\rho_j>T\}}\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T}|e(t\wedge\rho_j)|^2\Big).$$

Using these bounds in (4.3) yields

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T}|e(t\wedge\rho_j)|^2\Big) + \frac{2^{p+1}\delta\widetilde{H}}{p} + \frac{2(p-2)\widetilde{H}}{p\delta^{2/(p-2)}j^p}. \tag{4.6}$$

Setting $r := t\wedge\rho_j$ and for any $\varepsilon\in(0,1)$, by the Hölder inequality, when $r\in[k\triangle,(k+1)\triangle]$, for $k=0,1,2,\cdots,M-1$,

$$\begin{aligned}|e(r)|^2 &= |x(r)-y(r)|^2\\ &\le \Big|u(x_r)-u(\bar y_{(k-1)\triangle}+(r-k\triangle)(\bar y_{k\triangle}-\bar y_{(k-1)\triangle})/\triangle)\\ &\quad+\int_0^r[f(x_s)-f(\bar y_s)]ds+\int_0^r[g(x_s)-g(\bar y_s)]dw(s)\Big|^2\\ &\le \frac{1}{\varepsilon}|u(x_r)-u(\bar y_{(k-1)\triangle}+(r-k\triangle)(\bar y_{k\triangle}-\bar y_{(k-1)\triangle})/\triangle)|^2\\ &\quad+\frac{2}{1-\varepsilon}\Big[T\int_0^r[f(x_s)-f(\bar y_s)]^2ds+\Big|\int_0^r[g(x_s)-g(\bar y_s)]dw(s)\Big|^2\Big].\end{aligned}$$

Since

$$\begin{aligned}&\Big|u(x_r)-u\Big(\bar y_{(k-1)\triangle}+\frac{r-k\triangle}{\triangle}(\bar y_{k\triangle}-\bar y_{(k-1)\triangle})\Big)\Big|^2\\ &\le \kappa^2\Big\|x_r-\bar y_{(k-1)\triangle}-\frac{r-k\triangle}{\triangle}(\bar y_{k\triangle}-\bar y_{(k-1)\triangle})\Big\|^2\\ &\le \kappa^2\||x_r-y_r|+|y_r-\bar y_r|+|\bar y_{k\triangle}-\bar y_{(k-1)\triangle}|-\frac{r-k\triangle}{\triangle}|\bar y_{k\triangle}-\bar y_{(k-1)\triangle}|\|^2\\ &\le \frac{\kappa^2}{\varepsilon}\|x_r-y_r\|^2+\frac{2\kappa^2}{1-\varepsilon}\Big(\|y_r-\bar y_r\|^2+\|\bar y_{k\triangle}-\bar y_{(k-1)\triangle}\|^2\Big),\end{aligned}$$

we have

$$\begin{aligned}|e(r)|^2 &\le \frac{\kappa^2}{\varepsilon^2}\|x_r-y_r\|^2+\frac{2\kappa^2}{\varepsilon(1-\varepsilon)}(\|y_r-\bar y_r\|^2+\|\bar y_{k\triangle}-\bar y_{(k-1)\triangle}\|^2)\\ &\quad+\frac{2}{1-\varepsilon}\Big[T\int_0^r[f(x_s)-f(\bar y_s)]^2ds+\Big|\int_0^r[g(x_s)-g(\bar y_s)]dw(s)\Big|^2\Big].\end{aligned}$$

Hence, for any $t_1\in[0,T]$, by Lemma 3.2 and 3.3,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le t_1}|e(t\wedge\rho_j)|^2\Big]\\ &\le \frac{\kappa^2}{\varepsilon^2}\mathbb{E}\Big(\sup_{0\le t\le t_1}\|x_{t\wedge\rho_j}-y_{t\wedge\rho_j}\|^2\Big)+\frac{2\kappa^2}{\varepsilon(1-\varepsilon)}\Big[\mathbb{E}\Big(\sup_{0\le t\le t_1}\|y_{t\wedge\rho_j}-\bar y_{t\wedge\rho_j}\|^2\Big)\\ &\quad+\mathbb{E}\Big(\sup_{0\le k\le M-1}\|\bar y_{k\triangle\wedge\rho_j}-\bar y_{(k-1)\triangle\wedge\rho_j}\|^2\Big)\Big]\\ &\quad+\frac{2T}{1-\varepsilon}\mathbb{E}\int_0^{t_1\wedge\rho_j}[f(x_s)-f(\bar y_s)]^2ds\\ &\quad+\frac{2}{1-\varepsilon}\mathbb{E}\Big[\sup_{0\le t\le t_1}\Big|\int_0^{t\wedge\rho_j}[g(x_s)-g(\bar y_s)]dw(s)\Big|^2\Big]\\ &\le \frac{\kappa^2}{\varepsilon^2}\mathbb{E}\Big(\sup_{0\le t\le t_1}\|x_{t\wedge\rho_j}-y_{t\wedge\rho_j}\|^2\Big)+\frac{2\kappa^2}{\varepsilon(1-\varepsilon)}(\gamma(\triangle)+\beta(\triangle))\\ &\quad+\frac{2T}{1-\varepsilon}\mathbb{E}\int_0^{t_1\wedge\rho_j}[f(x_s)-f(\bar y_s)]^2ds\\ &\quad+\frac{2}{1-\varepsilon}\mathbb{E}\Big[\sup_{0\le t\le t_1}\Big|\int_0^{t\wedge\rho_j}[g(x_s)-g(\bar y_s)]dw(s)\Big|^2\Big].\end{aligned}\tag{4.7}$$

Since $x(t) = y(t) = \xi(t)$ when $t \in [-\tau, 0]$, we have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le t_1} \|x_{t\wedge\rho_j} - y_{t\wedge\rho_j}\|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{-\tau\le\theta\le 0}\sup_{0\le t\le t_1} |x(t\wedge\rho_j+\theta) - y(t\wedge\rho_j+\theta)|^2\Big)\\
&\le \mathbb{E}\Big(\sup_{-\tau\le t\le t_1} |x(t\wedge\rho_j) - y(t\wedge\rho_j)|^2\Big)\\
&= \mathbb{E}\Big(\sup_{0\le t\le t_1} |x(t\wedge\rho_j) - y(t\wedge\rho_j)|^2\Big). \qquad (4.8)
\end{aligned}
$$

By Assumption 2.1, and Lemma 3.3, we may compute

$$
\begin{aligned}
&\mathbb{E}\int_0^{t_1\wedge\rho_j} [f(x_s) - f(\bar{y}_s)]^2 ds\\
&\le C_j\mathbb{E}\int_0^{t_1\wedge\rho_j} \|x_s - \bar{y}_s\|^2 ds\\
&\le 2C_j\mathbb{E}\int_0^{t_1\wedge\rho_j} \|x_s - y_s\|^2 ds + 2C_j\mathbb{E}\int_0^{t_1\wedge\rho_j} \|y_s - \bar{y}_s\|^2 ds\\
&\le 2C_j\mathbb{E}\int_0^{t_1\wedge\rho_j} \sup_{-\tau\le\theta\le 0} |x(s+\theta) - y(s+\theta)|^2 ds + 2C_j\int_0^{t_1} \mathbb{E}\|y_s - \bar{y}_s\|^2 ds\\
&\le 2C_j\mathbb{E}\int_0^{t_1} \sup_{-\tau\le\theta\le 0} |x(s\wedge\rho_j+\theta) - y(s\wedge\rho_j+\theta)|^2 ds + 2C_jT\beta(\triangle)\\
&\le 2C_j\mathbb{E}\int_0^{t_1} \sup_{-\tau\le r\le s} |x(r\wedge\rho_j) - y(r\wedge\rho_j)|^2 ds + 2C_jT\beta(\triangle)\\
&= 2C_j\mathbb{E}\int_0^{t_1} \sup_{0\le r\le s} |x(r\wedge\rho_j) - y(r\wedge\rho_j)|^2 ds + 2C_jT\beta(\triangle). \qquad (4.9)
\end{aligned}
$$

By the Doob martingale inequality and Assumption 2.1,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le t_1}\Big|\int_0^{t\wedge\rho_j} [g(x_s) - g(\bar{y}_s)]dw(s)\Big|^2\Big]\\
&\le 4\mathbb{E}\int_0^{t_1\wedge\rho_j} |g(x_s) - g(\bar{y}_s)|^2 ds\\
&\le 4C_j\mathbb{E}\int_0^{t_1\wedge\rho_j} \|x_s - \bar{y}_s\|^2 ds\\
&\le 8C_j\int_0^{t_1} \mathbb{E}\Big(\sup_{0\le r\le s} |x(r\wedge\rho_j) - y(r\wedge\rho_j)|^2\Big) ds + 8C_jT\beta(\triangle). \qquad (4.10)
\end{aligned}
$$

Substituting (4.8), (4.9), (4.10) into (4.7) yields

$$
\begin{aligned}
\Big(1 - \frac{\kappa^2}{\varepsilon^2}\Big)\mathbb{E}\Big[\sup_{0\le t\le t_1} |e(t\wedge\rho_j)|^2\Big] \le{}& \frac{2\kappa^2}{\varepsilon(1-\varepsilon)}[\beta(\triangle) + \gamma(\triangle)] + \frac{4C_jT(T+4)}{1-\varepsilon}\beta(\triangle)\\
&+ \frac{4C_j(T+4)}{1-\varepsilon}\int_0^{t_1} \mathbb{E}\Big(\sup_{0\le r\le s} |e(r\wedge\rho_j)|^2\Big) ds \qquad (4.11)
\end{aligned}
$$

Choosing $\varepsilon = (1+\kappa)/2$ and noting $\kappa \in (0,1)$, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le t_1} |e(t\wedge\rho_j)|^2\Big)\\
&\le \frac{8\kappa^2(1+\kappa)}{(1-\kappa)^2(1+3\kappa)}[\beta(\triangle)+\gamma(\triangle)] + \frac{8C_jT(1+\kappa)^2(T+4)}{(1-\kappa)^2(1+3\kappa)}\beta(\triangle)\\
&\quad + \frac{8C_j(1+\kappa)^2(T+4)}{(1-\kappa)^2(1+3\kappa)} \int_0^{t_1} \mathbb{E}\Big(\sup_{0\le s\le r} |e(s\wedge\rho_j)|^2\Big) dr\\
&\le \frac{8}{(1-\kappa)^2}[\beta(\triangle)+\gamma(\triangle)] + \frac{16}{(1-\kappa)^2} C_jT(T+4)\beta(\triangle)\\
&\quad + \frac{16}{(1-\kappa)^2} C_j(T+4) \int_0^{t_1} \mathbb{E}\Big(\sup_{0\le s\le r} |e(s\wedge\rho_j)|^2\Big) dr.
\end{aligned}$$

The Gronwall inequality yields

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le t_1} |e(t\wedge\rho_j)|^2\Big] \le{}& \frac{8}{(1-\kappa)^2}[\beta(\triangle)+\gamma(\triangle)+2C_jT(T+4)\beta(\triangle)]\\
&\times e^{\frac{16}{(1-\kappa)^2}C_jT(T+4)}.
\end{aligned} \tag{4.12}$$

By Eq. (4.6),

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T} |e(t)|^2\Big) \le{}& \frac{8}{(1-\kappa)^2}[\beta(\triangle)+\gamma(\triangle)+2C_jT(T+4)\beta(\triangle)]e^{\frac{16}{(1-\kappa)^2}C_jT(T+4)}\\
&+ \frac{2^{p+1}\delta\widetilde{H}}{p} + \frac{2(p-2)\widetilde{H}}{p\delta^{2/(p-2)}j^p}.
\end{aligned} \tag{4.13}$$

Given any $\epsilon > 0$, we can now choose δ sufficient small such that $2^{p+1}\delta\widetilde{H}/p \le \epsilon/3$, then choose j sufficient large such that

$$\frac{2(p-2)\widetilde{H}}{p\delta^{2/(p-2)}j^p} < \frac{\epsilon}{3}$$

and finally choose $\triangle$ so small that

$$\frac{8}{(1-\kappa)^2}[\beta(\triangle)+\gamma(\triangle)+2C_jT(T+4)\beta(\triangle)]e^{\frac{16}{(1-\kappa)^2}C_jT(T+4)} < \frac{\epsilon}{3}$$

and thus, $\mathbb{E}(\sup_{0\le t\le T} |e(t)|^2) \le \epsilon$ as required. □

Remark It should be pointed out that much simpler proofs of Lemmas 3.2 and 3.3 can be obtained by chosing $l = 2$ if we only want to prove Theorem 2.2. The reason why we want to control the stochastic term by $c_1(l)\Delta^{\frac{l-1}{l}}$ for any $l > 1$ in section 3 is for the use in the next section where we will show the order of the strong convergence.

5. Order of convergence under the global Lipschitz condition. Theorem 2.2 shows that under Assumptions 2.1–2.4 the Euler–Maruyama approximate solutions strongly converge to the true solution. However, this theorem does not give the order of the convergence. In this section we reveal the order of the convergence, but here we need to replace the local Lipschitz condition by the global Lipschitz condition. To be more precise, we state the assumption as follows.

ASSUMPTION 5.1. *(Global Lipschitz condition) There exists a constant C such that for all $\varphi, \psi \in C([-\tau, 0]; \mathbb{R}^n)$ and $t \in [0, T]$*

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \leq C\|\varphi - \psi\|^2. \tag{5.1}$$

It is easy to see from the global Lipschitz condition that for any $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$,

$$|f(\varphi)|^2 \vee |f(\psi)|^2 \leq 2(|f(0)|^2 \vee |f(0)|^2) + C\|\varphi\|^2. \tag{5.2}$$

In other words, the global Lipschitz condition implies linear growth condition with the growth coefficient

$$K = 2[|f(0)|^2 \vee |g(0)|^2 \vee C].$$

In addition, we need slightly strength Assumption 2.4 on the initial data.

ASSUMPTION 5.2. *$\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ for some $p \geq 2$ and there exists a positive constant λ such that*

$$\mathbb{E}\Big(\sup_{-\tau \leq s \leq t \leq 0} |\xi(t) - \xi(s)|^2\Big) \leq \lambda(t - s). \tag{5.3}$$

We can state another theorem, which reveals the order of the convergence.

THEOREM 5.1. *Under Assumptions 5.1, 5.2 and 2.3, for any positive constant $\varepsilon \in (0, 1)$,*

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} |x(t) - y(t)|^2\Big) \leq O(\triangle^{1-\varepsilon}). \tag{5.4}$$

Proof. Since $\alpha(\triangle)$ may be replaced by $\lambda\triangle$, from Lemmas 3.2 and 3.3, there exists constants $\tilde{c}_1(l)$ and $\tilde{c}_2(l)$ such that $\beta(\triangle) \leq \tilde{c}_1(l)\triangle^{\frac{l-1}{l}}$ and $\gamma(\triangle) \leq \tilde{c}_2(l)\triangle^{\frac{l-1}{l}}$. Here we do not need to define the stopping times u_j and v_j, we may repeat the proof in section 4 and directly compute

$$\begin{aligned}\mathbb{E}\Big(\sup_{0 \leq t \leq T} |e(t)|^2\Big) \leq\ & \frac{8}{(1-\kappa)^2}[\tilde{c}_1(l) + \tilde{c}_2(l) + 2\tilde{c}_1(l)CT(T+4)] \\ & \times e^{\frac{16CT(T+4)}{(1-\kappa)^2}} \triangle^{\frac{l-1}{l}}.\end{aligned} \tag{5.5}$$

Choosing $l \geq 1/\varepsilon$ yields the required assertion. □

Remark Let us give some comments to close our paper. Theorem 5.1 reveals that the order of convergence in mean square is closed to 1, although the order of the strong convergence in mean square for the Euler–Maruyama scheme applied to both SDEs and SFDEs is one (see [8, 16]). Noting that $\tilde{c}_i(l)(i = 1, 2)$ appeared in (5.5) will tend to infinity as $l \to \infty$, we cannot simply let $l \to \infty$ to show that the order of the convergence is 1. It remains open whether the order of the convergence is 1 in the case of NSFDEs.

We would also like to point out that although we concentrate on the convergence in mean square (namely in L^2), our methods can easily be extended to establish the strong convergence in L^p for any $p \geq 2$. In particular, under the global Lipschitz condition, we can show that the order of the strong convergence in L^p is $p/2 - \varepsilon$ for any $\varepsilon > 0$.

Acknowledgements. The authors would like to thank the financial support from Chinese Scholarship Council. They also wish to thank the referees for their detailed comments and helpful suggestions.

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