Solitary smooth hump solutions of the Camassa–Holm equation by means of the homotopy analysis method

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Abstract

The homotopy analysis method is used to find a family of solitary smooth hump solutions of the Camassa–Holm equation. This approximate solution, which is obtained as a series of exponentials, agrees well with the known exact solution. This paper complements the work of Wu & Liao [Wu W, Liao S. Solving solitary waves with discontinuity by means of the homotopy analysis method. Chaos, Solitons & Fractals 2005;26:177-85] who used the homotopy analysis method to find a different family of solitary wave solutions.

Key words: Camassa–Holm equation; Homotopy analysis method; Soliton solution; Series solution

1 Introduction

The homotopy analysis method (HAM) is a means of finding approximate analytic solutions to nonlinear equations. It was first introduced by Liao in 1992 [1]. The method has been applied successfully to many nonlinear problems in engineering and science, such as boundary-layer flows over an impermeable stretched plate [2], unsteady boundary-layer flows over a stretching flat plate [3], exponentially decaying boundary layers [4], a nonlinear model of combined convective and radiative cooling

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of a spherical body [5], travelling-wave solutions of the Fisher equation [6], solitary-wave solutions of the Camassa–Holm (CH) equation [7], and many other problems (see [8–17], for example).

A related method for finding an analytic approximation to the exact solution to a nonlinear problem is the homotopy perturbation method (HPM) that was introduced by He in 1998 (see [18] and references therein). The HAM and HPM are based on Taylor series with respect to an embedding parameter, and both methods can give a very good approximation by means of a few terms if the initial guess for the solution and for an auxiliary linear operator are good enough [19]. However, Liao [19] pointed out that the HAM has the advantage that it contains an auxiliary parameter $\hbar$ and an auxiliary function $H(\eta)$ that provide a simple way to control the convergence region and rate of convergence of the solution series. Formally, the HPM is just the HAM with $\hbar = -1$ and $H(\eta) = 1$. In the HAM, a range of values for $\hbar$ that gives convergence can be identified from the so-called $\hbar$-curves. Clearly, if $\hbar = -1$ is outside this range, the HPM does not give a convergent series solution. This crucial difference between the HPM and the HAM was clearly illustrated recently in [20] where the flow of a fourth-grade fluid down a vertical cylinder was investigated by both methods. It was found that the HPM results are divergent for strong nonlinearity, whereas $\hbar$ can be adjusted in this case to obtain convergent HAM results.

The CH equation is

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where k is a constant parameter. When $k \neq 0$, the CH equation describes shallow water waves, where u denotes the velocity, x and t denote the spatial and temporal variables, respectively, and k is related to the critical shallow-water wave-speed [21]. When $k = 0$, the CH equation describes dispersive waves in a compressible hyperelastic rod [22]. The CH equation has been investigated at length in the literature (see [23–34], for example).

Camassa et al [24] looked for solitary travelling-wave solutions of (1.1) with $k \neq 0$ by seeking solutions of the form $u(x,t) = U(\eta)$, where $\eta := x - ct - x_0$, and $c(> 0)$ and x_0 are constants. Using the boundary conditions that U and its derivatives tend to zero as $|\eta| \to \infty$, they showed that there is a family of solitary smooth hump waves when $0 < k < c/2$ (so that k is the family-parameter) that have amplitude $c-2k$. Here, 'smooth' means that U' is continuous for all η, where the prime denotes the derivative with respect to η. If the crest of the wave is located at $\eta = 0$ then, for this family, $U(0) = c - 2k$ and $U'(0) = 0$. When $k = 0$, so that (1.1) becomes

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.2}$$

Camassa et al [24] showed that the solitary-wave solution is the solitary peakon

$$u(x,t) = ce^{-|\eta|}. \tag{1.3}$$

In this case $U(0) = c$ but U' is discontinuous at the crest so that $U'(0)$ is undefined. The peakon is a so-called 'corner wave' [23]. The HAM was used in [7] to find analytic approximations to the aforementioned solitary smooth hump solutions and to the solitary peakon.

In [30] Parkes and Vakhnenko found periodic and solitary travelling-wave solutions to the CH equation with $k = 0$, i.e. to Eq. (1.2). In particular they found a family of solitary smooth hump solutions for which the boundary conditions are $U \to U_\infty$ and the derivatives of U tend to zero as $|\eta| \to \infty$, where U_∞ is a constant such that $0 < U_\infty < c/3$ (so that U_∞ is the family-parameter). If the crest of the wave is located at $\eta = 0$ then, for this family, $U(0) = c - 2U_\infty$, so that the amplitude of the wave is $c - 3U_\infty$, and $U'(0) = 0$. In the limit $U_\infty = 0$, the solitary wave is the peakon given by (1.3).

The aim of this paper is to use the HAM to find an analytic approximation to the family of solitary smooth hump waves given in exact form by Parkes and Vakhnenko [30]. In this respect, our work complements the work of Wu and Liao in [7]. In Section 2 we present the exact solution in an explicit and more convenient form than in [30]. In Section 3 we formulate the HAM for finding an approximate analytic solution for the waves given in Section 2. In Section 4 we apply the HAM for particular values of the family-parameter U_∞ and show that the approximate solutions are in excellent agreement with the exact solution. In Section 5 we give the formulation and results for a 'reduced' HAM that is expressed in terms of the new dependent variable z that was used in [30]. A brief conclusion is given in Section 6.

2 A family of solitary smooth hump waves

A family of solitary smooth hump travelling-wave solutions to Eq. (1.2) was obtained in [30, Section 3.2] in terms of a new dependent variable $Z(\eta)$ related to $U(\eta)$ by

$$U(\eta) = c[1 + Z(\eta)], \tag{2.1}$$

where $c > 0$. In parametric form, with ω as the parameter, Z is given as an implicit function of η by

$$Z = \frac{z_3 - z_4 n \tanh^2 \omega}{1 - n \tanh^2 \omega}, \qquad \eta = \frac{\omega z_2}{s} + 2 \tanh^{-1}(\sqrt{n}\, \tanh \omega), \tag{2.2}$$

where

$$n = \frac{z_3 - z_2}{z_4 - z_2}, \qquad s = \frac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)}, \tag{2.3}$$

$$z_4 = 0, \quad z_1 = z_2 < z_3 < 0, \quad z_1 + z_2 + z_3 = -2. \tag{2.4}$$

For each member of this family, $z_2 \le Z \le z_3$, $Z = z_3$ at the crest where $\omega = 0$ (so that $\eta = 0$), and $Z \to z_2$ as $|\omega| \to \infty$ (so that $|\eta| \to \infty$).

For the purpose of this paper it is convenient to take $z_3 = -2Q$ so that, from (2.4), $z_2 = Q - 1$ and $0 < Q < 1/3$. (Here $Q \equiv -q/2$, where q is as in [30].) Now, from (2.1)–(2.3), we obtain

$$Z = \frac{-2Q(1-Q)}{(1-Q) - (1-3Q)\tanh^2\omega}\,, \tag{2.5}$$

$$U = c\left(\frac{(1-Q)(1-2Q) - (1-3Q)\tanh^2\omega}{(1-Q) - (1-3Q)\tanh^2\omega}\right), \tag{2.6}$$

$$\eta = -2\omega\sqrt{\frac{1-Q}{1-3Q}} + 2\tanh^{-1}\left(\sqrt{\frac{1-3Q}{1-Q}}\tanh\omega\right). \tag{2.7}$$

The solitary wave given in parametric form by (2.5) and (2.7) has amplitude $z_3 - z_2 = 1 - 3Q$, and $Z_\infty := z_2 = Q - 1$. The solitary wave given in parametric form by (2.6) and (2.7) has amplitude $(z_3 - z_2)c = (1-3Q)c$, and $U_\infty := (1 + z_2)c = Qc$.

For later use, we note that the transformation

$$U(\eta) = (1-3Q)cW(\eta) + Qc \tag{2.8}$$

gives

$$W = \frac{1}{1-3Q}\left(\frac{(1-Q)(1-2Q) - (1-3Q)\tanh^2\omega}{(1-Q) - (1-3Q)\tanh^2\omega} - Q\right). \tag{2.9}$$

Clearly, W and its derivatives tend to zero as $|\eta| \to \infty$, $W(0) = 1$ and $W'(0) = 0$.

3 The formulation of the HAM in terms of w

In this section we formulate the HAM in order to find from Eq. (1.2) an analytic approximation to the family of symmetric solitary smooth hump waves, with family-parameter U_∞, given exactly in Section 2.

It is convenient to introduce a new dependent variable $w(\eta)$ defined by

$$u(x,t) = aw(\eta) + U_\infty, \quad 0 < U_\infty < c/3, \tag{3.1}$$

where a is the amplitude, $u \to U_\infty$ as $|\eta| \to \infty$, and $U_\infty := Qc$ with $0 < Q < 1/3$. Clearly, $w(0) = 1$, $w'(0) = 0$, $w \to 0$ as $|\eta| \to \infty$, and $w(\eta) = w(-\eta)$. Substitution of u given by (3.1) into Eq. (1.2) gives

$$c(w''' - w') + (aw + Qc)(3w' - w''') - 2aw'w'' = 0. \tag{3.2}$$

Due to the assumed symmetry of the solitary waves, in the HAM we consider $w(\eta)$ only for $\eta \geq 0$. Hence the appropriate boundary conditions on w for use in the HAM are

$$w(0) = 1, \quad w'(0) = 0, \quad w(+\infty) = 0. \tag{3.3}$$

Our aim is to use the HAM to find analytic approximations to a and $w(\eta)$. In Section 4 we will demonstrate that these approximations are in good agreement with the exact expressions $(1-3Q)c$ and $W(\eta)$, respectively, as derived in Section 2. For simplicity, in the rest of this section and in the next section, we set $c = 1$.

Because of the boundary conditions (3.3), we assume that the solitary-wave solution of Eq. (3.2) for $\eta > 0$ can be expressed in the form

$$w(\eta) = \sum_{m=1}^{+\infty} d_m e^{-m\eta}, \tag{3.4}$$

where the d_m $(m = 1, 2, \ldots)$ are coefficients to be determined. The expression in (3.4) is known as the *rule of solution expression* [1, Section 3.4]. According to (3.4) and the boundary conditions (3.3), it is natural to choose

$$w_0(\eta) = 2e^{-\eta} - e^{-2\eta} \tag{3.5}$$

as the initial approximation to $w(\eta)$. We define an auxiliary linear operator $\mathcal{L}$ by

$$\mathcal{L}[\phi(\eta;p)] := \Big(\frac{\partial^3}{\partial\eta^3} - \frac{\partial}{\partial\eta}\Big)\phi(\eta;p). \tag{3.6}$$

This has the property that

$$\mathcal{L}[C_1 e^{-\eta} + C_2 e^{\eta} + C_3] = 0, \tag{3.7}$$

where C_1, C_2 and C_3 are constants. This choice of $\mathcal{L}$ is motivated by (3.4) and the later requirement that (3.17) should contain only one non-zero constant, namely C_1.

From (3.2) we define a nonlinear operator

$$\mathcal{N}[\phi(\eta;p), A(p)] := \left(\frac{\partial^3\phi}{\partial\eta^3} - \frac{\partial\phi}{\partial\eta}\right) + (A(p)\phi + Q)\left(3\frac{\partial\phi}{\partial\eta} - \frac{\partial^3\phi}{\partial\eta^3}\right) - 2A(p)\frac{\partial\phi}{\partial\eta}\frac{\partial^2\phi}{\partial\eta^2}\,, \tag{3.8}$$

and then construct the homotopy

$$\mathcal{H}[\phi(\eta;p), A(p)] = (1-p)\mathcal{L}[\phi(\eta;p) - w_0(\eta)] - \hbar H(\eta) p \mathcal{N}[\phi(\eta;p), A(p)], \tag{3.9}$$

where $\hbar$ is a nonzero auxiliary parameter and $H(\eta) \neq 0$ is an auxiliary function. Setting $\mathcal{H}[\phi(\eta;p), A(p)] = 0$, we have the *zero-order deformation equation*

$$(1-p)\mathcal{L}[\phi(\eta;p) - w_0(\eta)] = \hbar H(\eta) p \mathcal{N}[\phi(\eta;p), A(p)], \tag{3.10}$$

subject to the boundary conditions

$$\phi(0;p) = 1, \qquad \frac{\partial\phi(\eta;p)}{\partial\eta}\bigg|_{\eta=0} = 0, \qquad \phi(+\infty;p) = 0, \tag{3.11}$$

where $p \in [0, 1]$ is an embedding parameter. When the parameter p increases from 0 to 1, the solution $\phi(\eta; p)$ varies from $w_0(\eta)$ to $w(\eta)$, and $A(p)$ varies from a_0 to a, where a_0 is the initial value of the wave amplitude. If this continuous variation is smooth enough, the Maclaurin's series with respect to p can be constructed for $\phi(\eta; p)$ and $A(p)$, and further, if these two series are convergent at $p = 1$, we have

$$w(\eta) = w_0(\eta) + \sum_{m=1}^{+\infty} w_m(\eta), \qquad a = a_0 + \sum_{m=1}^{+\infty} a_m, \tag{3.12}$$

where

$$w_m(\eta) = \frac{1}{m!} \frac{\partial^m \phi(\eta; p)}{\partial p^m}\Big|_{p=0}, \qquad a_m = \frac{1}{m!} \frac{\partial^m A(p)}{\partial p^m}\Big|_{p=0}. \tag{3.13}$$

For brevity, we define the vectors

$$\overrightarrow{w}_k = \{w_0, w_1, \ldots, w_k\}, \qquad \overrightarrow{a}_k = \{a_0, a_1, \ldots, a_k\}.$$

Differentiating Eqs. (3.10) and (3.11) m times with respect to p then setting $p = 0$ and finally dividing by $m!$, we obtain the *mth-order deformation equation*

$$\mathcal{L}[w_m(\eta) - \chi_m w_{m-1}(\eta)] = \hbar R_m(\overrightarrow{w}_{m-1}, \overrightarrow{a}_{m-1}), \qquad (m = 1, 2, 3, \ldots) \tag{3.14}$$

subject to the boundary conditions

$$w_m(0) = 0, \quad w_m'(0) = 0, \quad w_m(\infty) = 0, \tag{3.15}$$

where

$$\begin{aligned} R_m =& (1-Q)w_{m-1}''' - (1-3Q)w_{m-1}' \\ &+ \sum_{n=0}^{m-1} \sum_{i=0}^{n} a_i \left(w_{n-i}[3w_{m-n-1}' - w_{m-n-1}'''] - 2w_{n-i}' w_{m-n-1}'' \right), \end{aligned}$$

and

$$\chi_m = \begin{cases} 0, \ m \le 1, \\ 1, \ m > 1. \end{cases}$$

In order to obey both the *rule of solution expression* and the *rule of the coefficient ergodicity* [1, Section 3.4], the corresponding auxiliary function can be determined uniquely as

$$H(\eta) = e^{-\eta}. \tag{3.16}$$

The general solution of Eq. (3.14) is

$$w_m(\eta) = \hat{w}_m(\eta) + C_1 e^{-\eta} + C_2 e^{\eta} + C_3, \tag{3.17}$$

where C_1, C_2 and C_3 are constants and $\hat{w}_m(\eta)$ is the particular solution of Eq. (3.14) that contains the unknown term a_{m-1}. Using (3.4), we have $C_2 = C_3 = 0$. According to the boundary conditions (3.15), the unknowns a_{m-1} and C_1 are governed by

$$\hat{w}_m(0) + C_1 = 0, \qquad \hat{w}'_m(0) - C_1 = 0. \tag{3.18}$$

Thus, the unknown a_{m-1} is obtained by solving the linear algebraic equation

$$\hat{w}_m(0) + \hat{w}'_m(0) = 0, \tag{3.19}$$

and thereafter C_1 is given by

$$C_1 = -\hat{w}_m(0). \tag{3.20}$$

In this way, we derive $w_m(\eta)$ and a_{m-1} for $m = 1, 2, 3, \ldots$, successively. At the Mth-order approximation, we have the analytic solution of Eq. (3.2), namely

$$w(\eta) \approx W_M(\eta) = \sum_{m=0}^{M} w_m(\eta), \qquad a \approx A_M = \sum_{m=0}^{M} a_m. \tag{3.21}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (3.21) in the homotopy analysis solution. By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [1], the appropriate region for $\hbar$ is a horizontal line segment.

4 Results

First, we investigate the influence of $\hbar$ on the series solution for a, the wave amplitude. Fig. 1 shows the $\hbar$-curve for the amplitude corresponding to $Q = 1/10$ from which we can clearly identify an appropriate region for $\hbar$. Note that this gives $a \approx 0.7$ which agrees with the expected value, namely $(1 - 3Q)c$.

Generally, it is found that as long as the series solution for the amplitude a is convergent to the expected value, the corresponding series solution for $w(\eta)$ is also convergent. Also, to investigate the influence of $\hbar$ on the series solution (3.21), we can consider the convergence of some related series such $w'(0)$, $w''(0)$, $w'''(0)$, and so on. But here, $w'(0) = 0$ holds for all results at any order of approximation and so it cannot provide us with any useful information about the choice of $\hbar$. However, $w''(0)$ and $w'''(0)$ are dependent on $\hbar$. The $\hbar$-curves for $w''(0)$ and $w'''(0)$ with $Q = 1/10$ are shown in Fig. 2. It is clear that the series for $w''(0)$ and $w'''(0)$ are convergent when $-9 < \hbar < -2$.

For example, for $\hbar = -5$ and $Q = 1/10$, our approximate analytic solution for w converges rapidly to the exact solution given by (2.7) and (2.9). This is shown in Fig. 3 and Fig. 4. Fig. 5 shows the *residual error* for different orders of approximation with $Q = 1/10$, and clearly indicates that the HAM gives rapid convergence.

The initial approximation (3.5) is independent of Q. For $Q = 1/10$, the initial approximation is already quite a good approximation to the exact solution and so it is not surprising that the convergence of the approximate solution is rapid. However, as Q increases, the profile of the exact solution gets wider and so the HAM does not converge quite so rapidly. This is illustrated in Fig. 6 where our approximate solution with $\hbar = -8$ and $Q = 2/10$ is compared with the exact solution.

5 The formulation of the HAM in terms of z

In this section we formulate the HAM in terms of the dependent variable z defined by

$$u(x,t) = c[1 + z(\eta)], \tag{5.1}$$

where $c > 0$. The motivation for this is that z is the dependent variable used in [30], and it is of interest to see how the formulation in terms of z leads to a different choice of $\mathcal{L}$, $\mathcal{N}$ and H than in the formulation in terms of w given in Section 3.

Substitution of u given by (5.1) into Eq. (1.2) gives

$$zz''' + 2z'z'' - 3zz' - 2z' = 0. \tag{5.2}$$

The appropriate boundary conditions on z for use in the HAM are

$$z(0) = b - 1 + Q, \quad z'(0) = 0, \quad z(+\infty) = -1 + Q, \tag{5.3}$$

where b is the amplitude of the wave profile in terms of z. From the exact solution given in Section 2 we know that exact value of b is $1 - 3Q$. However, if we treat b as an unknown constant to be determined by the HAM, we observe that it is in the boundary conditions (5.3) and not in the ODE (5.2). This is in contrast to the formulation in Section 3 where the unknown constant a is in the ODE (3.2). In order to move b from the boundary conditions into an ODE we could write

$$z = bw(\eta) + Z_\infty \tag{5.4}$$

and recover the HAM in terms of w as in Section 3. Here, however, we wish to formulate a HAM in terms of z. Hence we present a 'reduced' HAM in which we assume that $b = 1 - 3Q$ so that the boundary conditions (5.3) become

$$z(0) = -2Q, \quad z'(0) = 0, \quad z(+\infty) = -1 + Q, \tag{5.5}$$

where $0 < Q < 1/3$.

In this case the *rule of solution expression* is

$$z(\eta) = \sum_{m=0}^{+\infty} e_m e^{-m\eta}, \tag{5.6}$$

where the e_m $(m = 0, 1, 2, \ldots)$ are coefficients to be determined. According to (5.6) and (5.5), it is natural to choose

$$z_0(\eta) = (-1 + Q) + 2(1 - 3Q)e^{-\eta} - (1 - 3Q)e^{-2\eta} \tag{5.7}$$

as the initial approximation to $z(\eta)$. We define an auxiliary linear operator $\mathcal{L}$ by

$$\mathcal{L}[\phi(\eta; p)] = \left(\frac{\partial^3}{\partial\eta^3} + 3\frac{\partial^2}{\partial\eta^2} + 2\frac{\partial}{\partial\eta}\right)\phi(\eta; p). \tag{5.8}$$

This has the property that

$$\mathcal{L}[C_1 + C_2 e^{-\eta} + C_3 e^{-2\eta}] = 0, \tag{5.9}$$

where C_1, C_2 and C_3 are constants. This choice of $\mathcal{L}$ is motivated by (5.6) and the later requirement that (5.19) should contain only two non-zero constants, namely C_2 and C_3.

From (5.2) we define a nonlinear operator

$$\mathcal{N}[\phi(\eta; p)] = \phi\frac{\partial^3\phi}{\partial\eta^3} + 2\frac{\partial\phi}{\partial\eta}\frac{\partial^2\phi}{\partial\eta^2} - 3\phi\frac{\partial\phi}{\partial\eta} - 2\frac{\partial\phi}{\partial\eta}, \tag{5.10}$$

and then construct the homotopy

$$\mathcal{H}[\phi(\eta; p)] = (1 - p)\mathcal{L}[\phi(\eta; p) - z_0] - \hbar H(\eta) p \mathcal{N}[\phi(\eta; p)], \tag{5.11}$$

where $\hbar$ is a nonzero auxiliary parameter and $H(\eta) \neq 0$ is an auxiliary function. Setting $\mathcal{H}[\phi(\eta; p)] = 0$, we have the *zero-order deformation equation*

$$(1 - p)\mathcal{L}[\phi(\eta; p) - z_0] = \hbar H(\eta) p \mathcal{N}[\phi(\eta; p)], \tag{5.12}$$

subject to the boundary conditions

$$\phi(0; p) = -2Q, \qquad \left.\frac{\partial\phi(\eta; p)}{\partial\eta}\right|_{\eta=0} = 0, \qquad \phi(+\infty; p) = -1 + Q, \tag{5.13}$$

where $p \in [0, 1]$ is an embedding parameter. When the parameter p increases from 0 to 1, the solution $\phi(\eta; p)$ varies from $z_0(\eta)$ to $z(\eta)$. If this continuous variation is smooth enough, the Maclaurin's series with respect to p can be constructed for $\phi(\eta; p)$ and if this series is convergent at $p = 1$, we have

$$z(\eta) = z_0(\eta) + \sum_{m=1}^{+\infty} z_m(\eta), \tag{5.14}$$

where

$$z_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\phi(\eta; p)}{\partial p^m}\right|_{p=0}. \tag{5.15}$$

For brevity, define the vector

$$\overrightarrow{z}_k = \{z_0, z_1, \ldots, z_k\}.$$

Differentiating Eqs. (5.12) and (5.13) m times with respect to p then setting $p = 0$ and finally dividing by $m!$, we obtain the *mth-order deformation equation*

$$\mathcal{L}[z_m(\eta) - \chi_m z_{m-1}(\eta)] = \hbar R_m(\overrightarrow{z}_{m-1}), \qquad (m = 1, 2, 3, \ldots) \tag{5.16}$$

subject to the boundary conditions

$$z_m(0) = 0, \quad z'_m(0) = 0, \quad z_m(\infty) = 0, \tag{5.17}$$

where

$$R_m = \sum_{n=0}^{m-1} \left(z_n(z'''_{m-n-1} - 3z'_{m-n-1}) + 2z'_n z''_{m-n-1} \right) - 2z'_{m-1}.$$

In order to obey both the *rule of solution expression* and the *rule of the coefficient ergodicity* [1, Section 3.4], the corresponding auxiliary function can be determined uniquely as

$$H(\eta) = e^{-2\eta}. \tag{5.18}$$

The general solution of Eq. (5.16) is

$$z_m(\eta) = \hat{z}_m(\eta) + C_1 + C_2 e^{-\eta} + C_3 e^{-2\eta}, \tag{5.19}$$

where C_1, C_2 and C_3 are constants and $\hat{z}_m(\eta)$ is a particular solution of Eq. (5.16). Using (5.6), we have $C_1 = 0$. The unknowns C_2 and C_3 are determined by using the boundary conditions (5.17). As in Section 4, we can find an appropriate range of $\hbar$ for convergence. The $\hbar$-curves for $z''(0)$ and $z'''(0)$ with $Q = 1/10$ are shown in Fig. 7; it is clear that we require $1 < \hbar < 9$ for convergence. Fig. 8 illustrates the rapid convergence of our approximate analytic solution and Fig. 9 shows that our approximate solution is in excellent agreement with the exact solution given by (2.5) and (2.7). Fig. 10 shows the *residual error* for different orders of approximation and clearly indicates that the HAM gives rapid convergence.

6 Conclusions

We have applied the homotopy analysis method (HAM) to the Camassa–Holm equation (1.1) with $k = 0$ to obtain an excellent analytic approximation to the family of solitary smooth hump waves given in exact form in [30]. In the formulation of the HAM in Section 3, the amplitude of the solitary waves was treated as an unknown to be determined by the HAM; in Section 5 we formulated a 'reduced' HAM in which the amplitude of the waves was assumed. The former formulation, and the corresponding results, complement the work of Wu and Liao [7] who performed a similar investigation for the family of solitary smooth hump waves for which $0 < k < c/2$ and which was given in exact implicit form in [24].

The HAM provides us with a convenient way to control the convergence of approximation series; this is a fundamental qualitative difference in analysis between the HAM and other methods. The example in this paper is further confirmation of the flexibility and potential of the HAM for complicated nonlinear problems in science and engineering.

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List of Figures

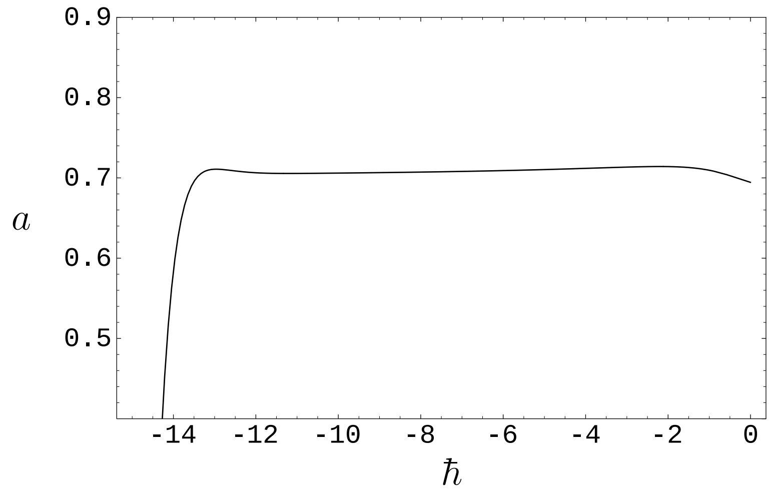

Fig. 1. The $\hbar$-curve for the wave amplitude a at the 20th-order approximation with $Q = 1/10$.

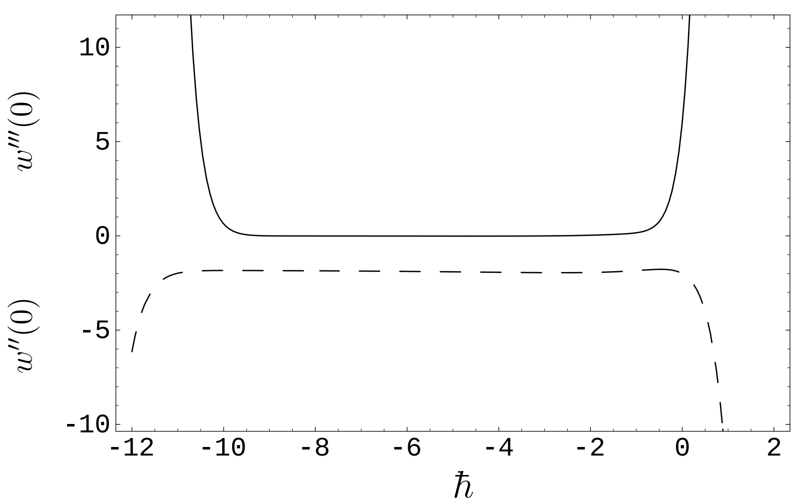

Fig. 2. The $\hbar$-curves for $w''(0)$ and $w'''(0)$ at the 20th-order approximation with $Q = 1/10$. Dashed curve: $w''(0)$; solid curve: $w'''(0)$.

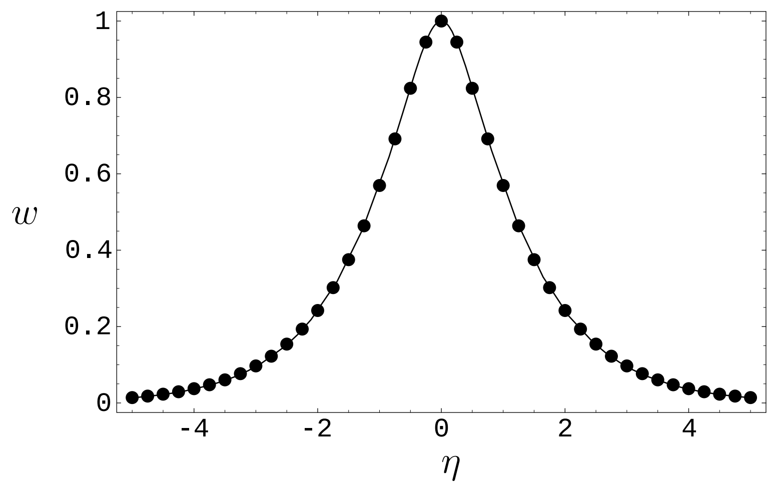

Fig. 3. The analytic approximation for $w(\eta)$ when $\hbar = -5$ with $Q = 1/10$. Solid curve: 5th-order approximation; symbols: 10th-order approximation.

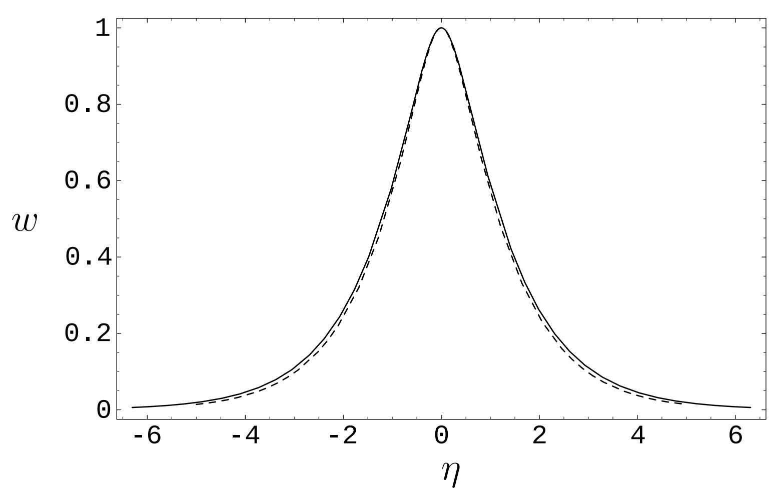

Fig. 4. The analytic approximation for $w(\eta)$ when $\hbar = -5$ and the exact solution given by (2.7) and (2.9) with $Q = 1/10$. Solid curve: exact solution; dashed curve: 10th-order approximation.

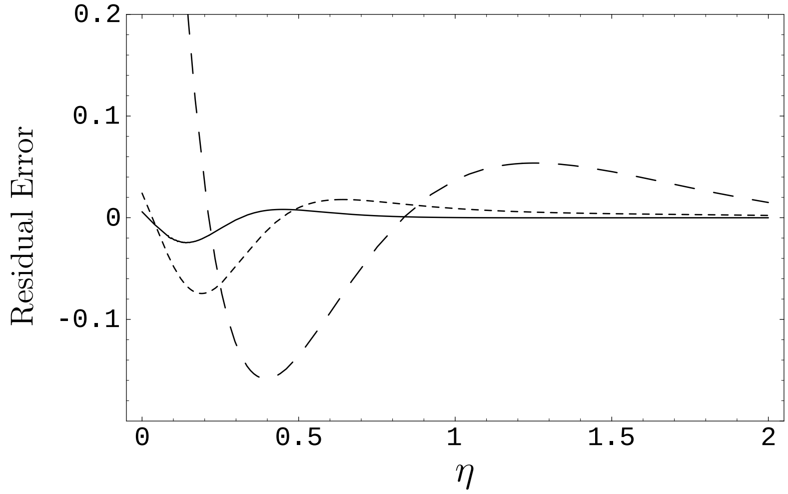

Fig. 5. The residual error for Eq. (3.2) when $\hbar = -4$ with $Q = 1/10$. Solid curve: 10th-order approximation; dotted curve: 5th-order approximation; dashed curve: initial guess.

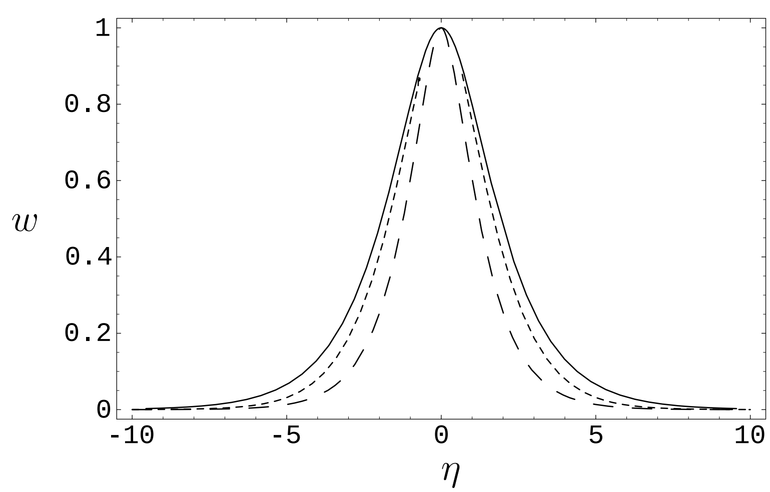

Fig. 6. The analytic approximation for $w(\eta)$ when $\hbar = -8$ and the exact solution given by (2.7) and (2.9) with $Q = 2/10$. Solid curve: exact solution; dashed curve: initial guess; dotted curve: 20th-order approximation.

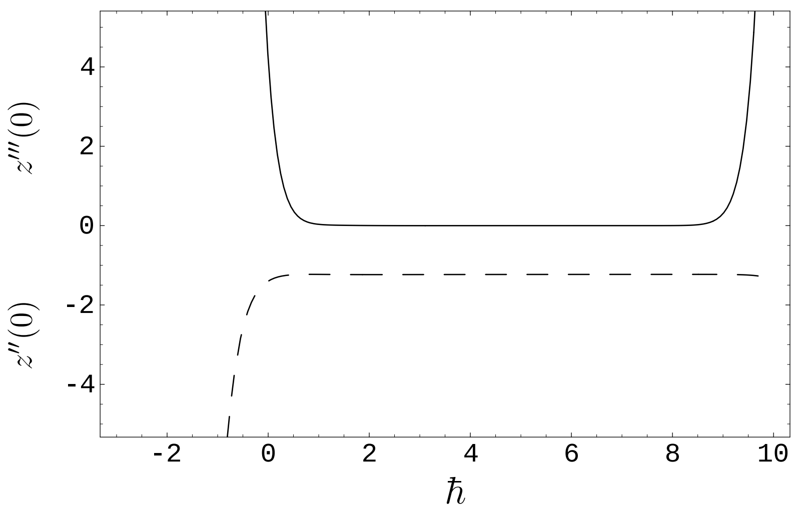

Fig. 7. The $\hbar$-curves for $z''(0)$ and $z'''(0)$ at the 20th-order approximation with $Q = 1/10$. Dashed curve: $z''(0)$; solid curve: $z'''(0)$.

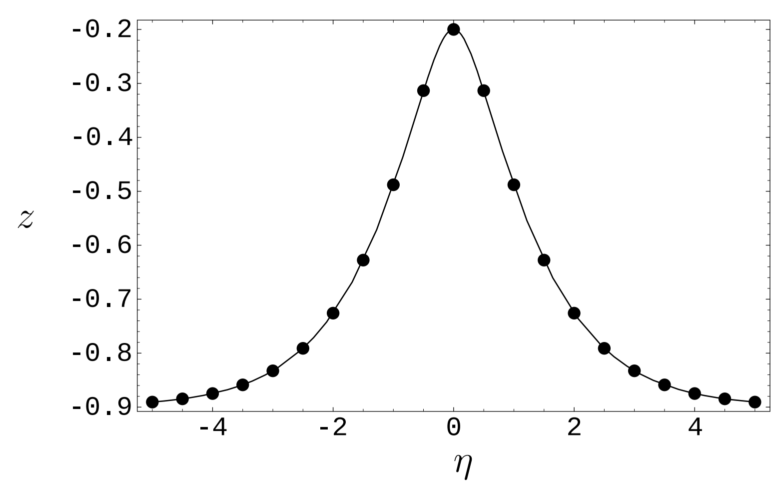

Fig. 8. The analytic approximation for $z(\eta)$ when $\hbar = 1$ with $Q = 1/10$. Solid curve: 20th-order approximation; symbols: 15th-order approximation.

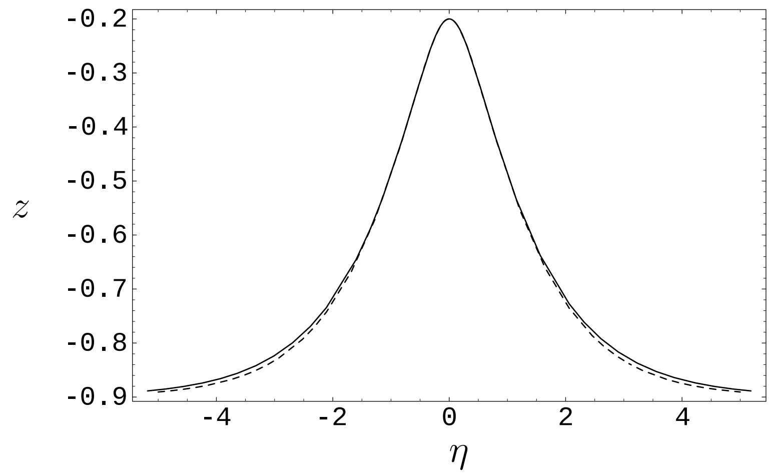

Fig. 9. The analytic approximation of $z(\eta)$ when $\hbar = 1$ and the exact solution given by (2.5) and (2.7) with $Q = 1/10$. Solid curve: exact solution; dashed curve: 20th-order approximation.

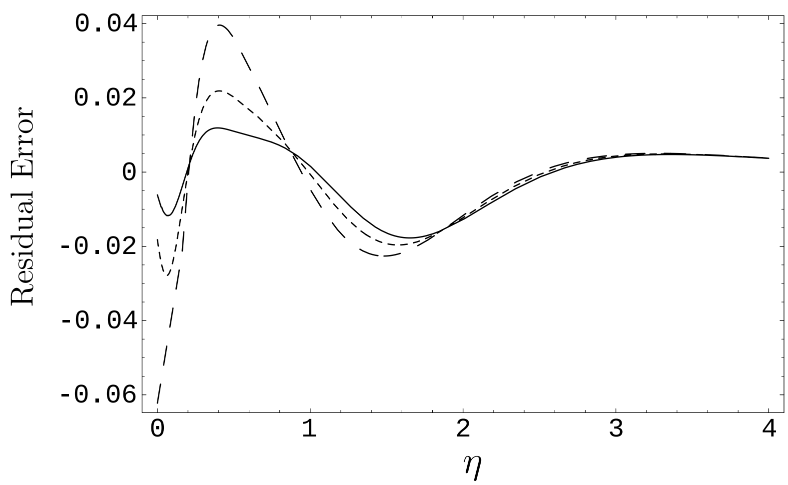

Fig. 10. The residual error for Eq. (5.2) when $\hbar = 1$ with $Q = 1/10$. Solid curve: 20th-order approximation; dotted curve: 15th-order approximation; dashed curve: 10th-order approximation.