Solitary-wave solutions of the Degasperis–Procesi equation by means of the homotopy analysis method

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Abstract

The homotopy analysis method is applied to the Degasperis–Procesi equation in order to find analytic approximations to the known exact solitary-wave solutions for the solitary peakon wave and the family of solitary smooth-hump waves. It is demonstrated that the approximate solutions agree well with the exact solutions. This provides further evidence that the homotopy analysis method is a powerful tool for finding excellent approximations to nonlinear solitary waves.

Key words: Degasperis–Procesi equation; Homotopy analysis method; Solitary-wave solution; Series solution

1 Introduction

The solution of nonlinear problems by analytic techniques is often rather difficult. Recently, the so-called homotopy analysis method (HAM) has been developed by Liao [1]. The HAM has been applied successfully to many nonlinear problems in engineering and science, such as applications in heat transfer [2], solving the generalized Hirota–Satsuma coupled KdV equation [3], in heat radiation [4], finding solitary-wave solutions for the fifth-order KdV equation [5], finding solitary wave solutions for the Kuramoto–Sivashinsky equation [6], finding the root of nonlinear

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equations [7], boundary-layer flows over an impermeable stretched plate [8], unsteady boundary-layer flows over a stretching flat plate [9], exponentially decaying boundary layers [10], a nonlinear model of combined convective and radiative cooling of a spherical body [11], and many other problems (see [12–19], for example).

As discussed in [20], the family of equations

$$u_t - u_{xxt} + (\gamma + 1)uu_x = \gamma u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where $\gamma > 1$ is a constant, contains only two integrable equations, namely the dispersionless Camassa–Holm (dCH) equation for which $\gamma = 2$ and the Degasperis–Procesi (DP) equation for which $\gamma = 3$. Explicit periodic and solitary travelling-wave solutions to the dCH and DP equations were found in [21] and [22], respectively. Weak travelling-wave solutions of the CH and DP equations, including exotic composite solutions, have been classified by Lenells [23,24]; these two papers are a useful source of references regarding properties of the CH and DP equations.

In [25] we found approximate analytic solutions for the family of solitary smooth-hump waves of the dCH equation by using the HAM. The approximate analytic solution for the solitary peakon wave, which has a discontinuity at the crest, was found in [13] by using the HAM.

The aim of this paper is to apply the HAM to the DP equation in order to find analytic approximations to the solitary peakon wave and the family of solitary smooth-hump waves, all given in exact form by Vakhnenko and Parkes [22].

In Section 2 we give the exact solution for the solitary peakon, and present the exact solution for the family of solitary smooth-hump waves in an explicit and more convenient form than in [22]. In Sections 3 and 4 we formulate the HAM for finding approximate analytic solutions for the solitary peakon wave and the family of solitary smooth-hump waves, respectively. A brief conclusion is given in Section 5.

2 Exact solitary-wave solutions

Vakhnenko and Parkes [22] looked for periodic and solitary-wave solutions of the DP equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \tag{2.1}$$

by seeking solutions in the form $u(x, t) = U(\eta)$, where $\eta := x - ct - x_0$, and c (> 0) and x_0 are constants. They found three types of solitary wave characterized by a single parameter A [22, Section 2]. For $A = 1$ there is a corner-wave solution known as a peakon. For $1 < A < 9/8$ there is a family of smooth-hump solitary waves.

In [22] the solitary-wave solutions were presented in terms of a new dependent variable $Z(\eta)$ related to $U(\eta)$ by

$$U(\eta) = c[1 + Z(\eta)]. \tag{2.2}$$

Here we summarize these solutions; the smooth-hump family is presented in an explicit and more convenient form than in [22].

2.1 The peakon wave

In [22, Section 2.3] the peakon is given by

$$Z = e^{-|\eta|} - 1 \tag{2.3}$$

so that

$$U = ce^{-|\eta|}. \tag{2.4}$$

The peakon is shown in [22, Fig. 2(c)].

2.2 A family of smooth-hump solitary waves

The family of smooth-hump solitary waves is given in [22, Section 2.4]. In parametric form, with τ as the parameter, Z is given as an implicit function of η by

$$Z = \frac{z_3 - z_4 n \tanh^2 \tau}{1 - n \tanh^2 \tau}, \qquad \eta = \frac{\tau z_2}{s} + 2 \tanh^{-1}(\sqrt{n}\, \tanh \tau), \tag{2.5}$$

where

$$n = \frac{z_3 - z_2}{z_4 - z_2}, \qquad s = \frac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)}, \tag{2.6}$$

$$z_1 = z_2 = z_L < z_3 < 0 < z_4, \quad z_L = b^2 - 1, \quad z_3 = -b^2 - b, \quad z_4 = -b^2 + b. \tag{2.7}$$

The family-parameter is b defined by

$$b^2 := (1 - \sqrt{9 - 8A})/4. \tag{2.8}$$

With $1 < A < 9/8$ we have $0 < b < 1/2$. For each member of this family, $z_L \le Z \le z_3$, $Z = z_3$ at the crest where $\tau = 0$ (so that $\eta = 0$), and $Z \to z_L$ as $|\tau| \to \infty$ (so that $|\eta| \to \infty$). The member of this family with $b = 1/4$ is shown in [22, Fig. 2(d)]. As $b \to 0$, the waves tend to the wave of maximum amplitude, namely the peakon (2.3).

Now, from (2.2) and (2.5)–(2.7), we obtain

$$U = aW(\eta) + U_\infty, \tag{2.9}$$

where $a := c(z_3 - z_L) = c(1 - b - 2b^2)$ is the amplitude, $U_\infty := c(1 + z_L) = cb^2$ and W is given as an implicit function of η by

$$W = \frac{(1 + b - 2b^2)(1 - \tanh^2 \tau)}{(1 + b - 2b^2) - (1 - b - 2b^2)\tanh^2 \tau}, \tag{2.10}$$

$$\eta = -2\tau\sqrt{\frac{1 - b^2}{1 - 4b^2}} + 2\tanh^{-1}\left(\sqrt{\frac{1 - b - 2b^2}{1 + b - 2b^2}}\tanh \tau\right). \tag{2.11}$$

W is a solitary smooth-hump wave with unit amplitude. W and its derivatives tend to zero as $|\eta| \to \infty$, $W(0) = 1$ and $W'(0) = 0$.

3 The HAM for the peakon solitary wave

In this section we formulate the HAM in order to find from Eq. (2.1) an analytic approximation to the symmetric solitary peakon wave. It is convenient to introduce a new dependent variable $w(\eta)$ defined by

$$u(x,t) = aw(\eta), \tag{3.1}$$

where a is the amplitude. Substitution of u given by (3.1) into Eq. (2.1) gives (with $x_0 = 0$)

$$c(w''' - w') + aw(4w' - w''') - 3aw'w'' = 0. \tag{3.2}$$

Due to the assumed symmetry of the peakon, in the HAM we consider $w(\eta)$ only for $\eta \geq 0$. The first derivative at the crest of the peakon is not continuous. Hence the appropriate boundary conditions on w for use in the HAM are

$$w(0) = 1, \quad w(+\infty) = 0. \tag{3.3}$$

Our aim is to use the HAM to find analytic approximations to a and $w(\eta)$. For simplicity, in the rest of this section we set $c = 1$.

According to Eq. (3.2) and the boundary conditions (3.3), the solitary-wave solution can be expressed in the form

$$w(\eta) = \sum_{m=1}^{+\infty} d_m e^{-m\eta}, \tag{3.4}$$

where the d_m $(m = 1, 2, \ldots)$ are coefficients to be determined. According to the *rule of solution expression* denoted by (3.4) and the boundary conditions (3.3), it is natural to choose

$$w_0(\eta) = e^{-\eta} - \epsilon[e^{-2\eta} - e^{-3\eta}] \tag{3.5}$$

as the initial approximation to $w(\eta)$, where ϵ is a parameter to be determined later. This choice follows the strategy adopted in [13] in the context of the peakon solution to the CH equation. We define an auxiliary linear operator $\mathcal{L}$ by

$$\mathcal{L}[\phi(\eta;p)] = \left(\frac{\partial^3}{\partial\eta^3} - 3\frac{\partial^2}{\partial\eta^2} + 2\frac{\partial}{\partial\eta}\right)\phi(\eta;p). \tag{3.6}$$

This has the property that

$$\mathcal{L}[C_1 e^{\eta} + C_2 e^{2\eta} + C_3] = 0, \tag{3.7}$$

where C_1, C_2 and C_3 are constants. This choice of $\mathcal{L}$ is motivated by (3.4) and the later requirement that (3.16) should contain only zero constants, i.e. $C_1 = C_2 = C_3 = 0$.

From (3.2) we define a nonlinear operator

$$\mathcal{N}[\phi(\eta;p), A(p)] := \left(\frac{\partial^3\phi}{\partial\eta^3} - \frac{\partial\phi}{\partial\eta}\right) + A(p)\phi\left(4\frac{\partial\phi}{\partial\eta} - \frac{\partial^3\phi}{\partial\eta^3}\right) - 3A(p)\frac{\partial\phi}{\partial\eta}\frac{\partial^2\phi}{\partial\eta^2} \tag{3.8}$$

and then construct the homotopy

$$\mathcal{H}[\phi(\eta;p), A(p)] = (1-p)\mathcal{L}[\phi(\eta;p) - w_0(\eta)] - \hbar p\mathcal{N}[\phi(\eta;p), A(p)], \tag{3.9}$$

where $\hbar$ is a nonzero auxiliary parameter. Setting $\mathcal{H}[\phi(\eta;p), A(p)] = 0$, we have the zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\eta;p) - w_0(\eta)] = \hbar p\mathcal{N}[\phi(\eta;p), A(p)], \tag{3.10}$$

subject to the boundary conditions

$$\phi(0;p) = 1, \qquad \phi(+\infty;p) = 0, \tag{3.11}$$

where $p \in [0,1]$ is an embedding parameter. When the parameter p increases from 0 to 1, the solution $\phi(\eta;p)$ varies from $w_0(\eta)$ to $w(\eta)$, and $A(p)$ varies from a_0 to a, where a_0 is the initial value of the wave amplitude. If this continuous variation is smooth enough, the Maclaurin's series with respect to p can be constructed for $\phi(\eta;p)$ and $A(p)$, and further, if these two series are convergent at $p=1$, we have

$$w(\eta) = w_0(\eta) + \sum_{m=1}^{+\infty} w_m(\eta), \qquad a = a_0 + \sum_{m=1}^{+\infty} a_m, \tag{3.12}$$

where

$$w_m(\eta) = \frac{1}{m!}\frac{\partial^m\phi(\eta;p)}{\partial p^m}\bigg|_{p=0}, \qquad a_m = \frac{1}{m!}\frac{\partial^m A(p)}{\partial p^m}\bigg|_{p=0}. \tag{3.13}$$

For brevity, we define the vectors

$$\overrightarrow{w}_k = \{w_0, w_1, \ldots, w_k\}, \qquad \overrightarrow{a}_k = \{a_0, a_1, \ldots, a_k\}.$$

Differentiating Eqs. (3.10) and (3.11) m times with respect to p, then setting $p = 0$, and finally dividing by $m!$, we obtain the mth-order deformation equation

$$\mathcal{L}[w_m(\eta) - \chi_m w_{m-1}(\eta)] = \hbar R_m(\overrightarrow{w}_{m-1}, \overrightarrow{a}_{m-1}), \qquad (m = 1, 2, 3, \ldots) \tag{3.14}$$

subject to the boundary conditions

$$w_m(0) = 0, \quad w_m(\infty) = 0, \tag{3.15}$$

where

$$R_m = w'''_{m-1} - w'_{m-1} + \sum_{n=0}^{m-1}\sum_{i=0}^{n} a_i \left(w_{n-i}[4w'_{m-n-1} - w'''_{m-n-1}] - 3w'_{n-i}w''_{m-n-1} \right),$$

and

$$\chi_m = \begin{cases} 0, \ m \leq 1, \\ 1, \ m > 1. \end{cases}$$

The general solution of Eq. (3.14) is

$$w_m(\eta) = \hat{w}_m(\eta) + C_1 e^{\eta} + C_2 e^{2\eta} + C_3, \tag{3.16}$$

where C_1, C_2 and C_3 are constants and $\hat{w}_m(\eta)$ is the particular solution of Eq. (3.14) that contains the unknown term a_{m-1}. According to the boundary condition (3.15) at infinity and the rule of solution expression (3.4), the constants C_1, C_2 and C_3 must be zero. Due to the boundary condition (3.15) at $\eta = 0$, the unknown a_{m-1} is determined by the linear algebraic equation

$$\hat{w}_m(0) = 0. \tag{3.17}$$

In this way, we derive $w_m(\eta)$ and a_{m-1} for $m = 1, 2, 3, \ldots$, successively. At the Mth-order approximation, we have the analytic solution of Eq. (3.2), namely

$$w(\eta) \approx W_M(\eta) = \sum_{m=0}^{M} w_m(\eta), \qquad a \approx A_M = \sum_{m=0}^{M} a_m. \tag{3.18}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (3.18) in the homotopy analysis solution. By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [1], the appropriate region for $\hbar$ is a horizontal line segment.

Our solution series contain the auxiliary parameters $\hbar$ and ϵ. We can choose appropriate values of $\hbar$ and ϵ to ensure that the two solution series converge. For a given $\hbar$, we can investigate the influence of ϵ on the convergence of a by plotting the curve of a versus ϵ, as shown in Fig. 1. We find that the appropriate region for ϵ is $-1 < \epsilon < 1$. In the same way, we can plot $\hbar$-curves for a with any given ϵ, as shown in Fig. 2. Clearly, $a \approx 1$ which agrees with the expected value $c = 1$. It now follows that, for convergent solution series, we can choose $\hbar = -3$ and $\epsilon = -0.5$, for example. The corresponding 20th-order approximation for $w(\eta)$ agrees well with the exact solution, i.e. $w = e^{-|\eta|}$, as shown in Fig. 3.

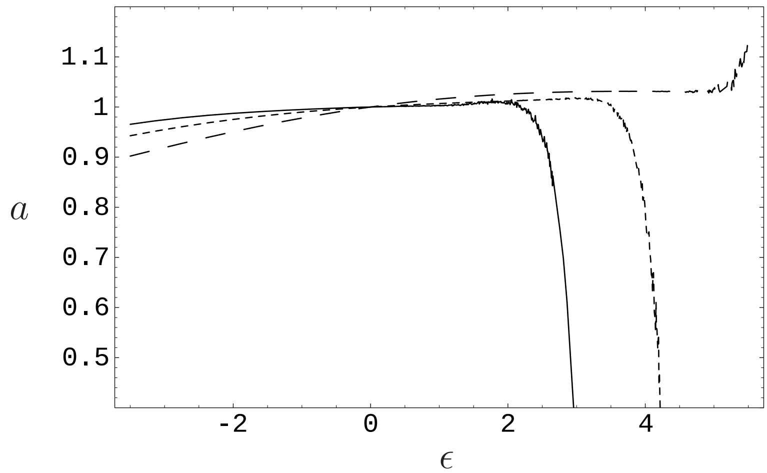

Fig. 1: The curves of the wave amplitude a versus ϵ for the 10th-order approximation. Solid curve: $\hbar = -4$; dotted curve: $\hbar = -3$; dashed curve: $\hbar = -2$.

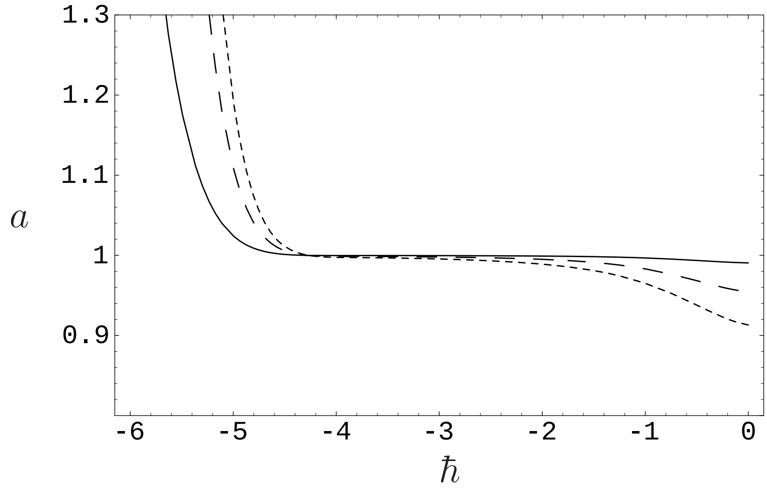

Fig. 2: The curves of the wave amplitude a versus $\hbar$ for the 15th-order approximation. Solid curve: $\epsilon = -0.5$; dotted curve: $\epsilon = -1.5$; dashed curve: $\epsilon = -1$.

4 The HAM for the family of smooth-hump solitary waves

In this section we formulate the HAM in order to find from Eq. (2.1) an analytic approximation to the family of symmetric solitary smooth-hump waves, with family-parameter U_∞, given exactly in Section 2.2.

It is convenient to introduce a new dependent variable $w(\eta)$ defined by

$$u(x,t) = U(\eta) := aw(\eta) + U_\infty, \quad 0 < U_\infty < c/4, \tag{4.1}$$

where a is the amplitude, $u \to U_\infty$ as $|\eta| \to \infty$, $U_\infty := cb^2$ with $0 < b < 1/2$, and $w(\eta)$ is a solitary smooth-hump wave of unit amplitude such that $w(0) = 1$, $w'(0) = 0$, $w \to 0$ as $|\eta| \to \infty$, and $w(\eta) = w(-\eta)$. Substitution of u given by (4.1) into Eq. (2.1) gives

$$c(w''' - w') + (aw + b^2c)(4w' - w''') - 3aw'w'' = 0. \tag{4.2}$$

Due to the assumed symmetry of the solitary waves, in the HAM we consider $w(\eta)$ only for $\eta \geq 0$. Hence the appropriate boundary conditions on w for use in the HAM are

$$w(0) = 1, \quad w'(0) = 0, \quad w(+\infty) = 0. \tag{4.3}$$

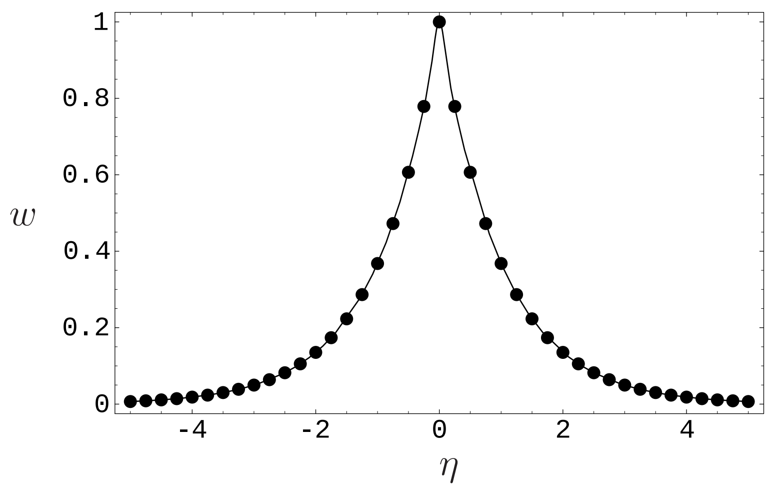

Fig. 3: The analytic approximation for $w(\eta)$ when $\hbar = -3$ with $\epsilon = -0.5$ and the exact solution $w(\eta) = -e^{|\eta|}$. Solid curve: 20th-order approximation; symbols: exact solution.

Our aim is to use the HAM to find analytic approximations to a and $w(\eta)$. We will demonstrate that these approximations are in good agreement with the exact expressions $c(1 - b - 2b^2)$ and $W(\eta)$, respectively, as derived in Section 2.2.

First we write

$$w(\eta) \approx B \exp(-\mu\eta), \quad \text{as } \eta \to \infty, \tag{4.4}$$

where $\mu > 0$ and B are constants. Substituting (4.4) into (4.2) and balancing the main terms, we have

$$\mu^2 = \frac{1 - 4b^2}{1 - b^2}.$$

Now we put $\xi = \mu\eta$ so that Eq. (4.2) becomes

$$c(1 - 4b^2)(w''' - w') + aw(4w' - \mu^2 w''') - 3\mu^2 aw'w'' = 0, \tag{4.5}$$

where the prime denotes the derivative with respect to ξ. For simplicity, in the rest of this section we set $c = 1$.

According to Eq. (4.5) and the boundary conditions (4.3), the solitary-wave solution can be expressed in the form

$$w(\xi) = \sum_{m=1}^{+\infty} d_m e^{-m\xi}, \tag{4.6}$$

where the d_m $(m = 1, 2, \ldots)$ are coefficients to be determined.

We define an auxiliary linear operator $\mathcal{L}$ by

$$\mathcal{L}[\phi(\xi;p)] = (1 - 4b^2)\left(\frac{\partial^3}{\partial\xi^3} - \frac{\partial}{\partial\xi}\right)\phi(\xi;p), \tag{4.7}$$

with the property

$$\mathcal{L}[C_1 e^{-\xi} + C_2 e^{\xi} + C_3] = 0, \tag{4.8}$$

where C_1, C_2 and C_3 are constants. This choice of $\mathcal{L}$ is motivated by (4.6) and the later requirement that (4.14) should contain only one non-zero constant, namely C_1.

In this case, the nonlinear operator $\mathcal{N}[\phi(\xi;p)]$ is defined as

$$\begin{aligned}\mathcal{N}[\phi(\xi;p), A(p)] := (1 - 4b^2)\left(\frac{\partial^3\phi}{\partial\xi^3} - \frac{\partial\phi}{\partial\xi}\right) + A(p)\phi\left(4\frac{\partial\phi}{\partial\xi} - \mu^2\frac{\partial^3\phi}{\partial\xi^3}\right)\\ -3\mu^2 A(p)\frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial\xi^2},\end{aligned} \tag{4.9}$$

and the homotopy $\mathcal{H}$ is defined as in (3.9). Also, the zero-order deformation equation is defined as

$$(1 - p)\mathcal{L}[\phi(\xi;p) - w_0(\xi)] = \hbar p\mathcal{N}[\phi(\xi;p), A(p)], \tag{4.10}$$

subject to the boundary conditions

$$\phi(0;p) = 1, \qquad \left.\frac{\partial\phi(\xi;p)}{\partial\xi}\right|_{\xi=0} = 0, \qquad \phi(+\infty;p) = 0, \tag{4.11}$$

where $p \in [0, 1]$ is an embedding parameter and $w_0(\xi) = 2e^{-\xi} - e^{-2\xi}$. Differentiating Eqs. (4.10) and (4.11) m times with respect to p, then setting $p = 0$, and finally dividing by $m!$, we obtain the mth-order deformation equation

$$\mathcal{L}[w_m(\xi) - \chi_m w_{m-1}(\xi)] = \hbar R_m(\overrightarrow{w}_{m-1}), \qquad (m = 1, 2, 3, \ldots) \tag{4.12}$$

subject to the boundary conditions

$$w_m(0) = 0, \quad w'_m(0) = 0, \quad w_m(\infty) = 0, \tag{4.13}$$

where R_m is defined as

$$R_m = (1-4b^2)(w'''_{m-1} - w'_{m-1})$$
$$+\sum_{n=0}^{m-1}\sum_{i=0}^{n} a_i\left(w_{n-i}[4w'_{m-n-1} - \mu^2 w'''_{m-n-1}] - 3\mu^2 w'_{n-i}w''_{m-n-1}\right).$$

The general solution of Eq. (4.12) is

$$w_m(\xi) = \hat{w}_m(\xi) + C_1 e^{-\xi} + C_2 e^{\xi} + C_3, \tag{4.14}$$

where C_1, C_2 and C_3 are constants and $\hat{w}_m(\xi)$ is a particular solution of Eq. (4.12). Using (4.6), we have $C_2 = C_3 = 0$. The unknowns C_1 and a_{m-1} are governed by

$$\hat{w}_m(0) + C_1 = 0, \qquad \hat{w}'_m(0) - C_1 = 0. \tag{4.15}$$

Thus, the unknown a_{m-1} is obtained by solving the linear algebraic equation

$$\hat{w}_m(0) + \hat{w}'_m(0) = 0, \tag{4.16}$$

and thereafter C_1 is given by

$$C_1 = -\hat{w}_m(0). \tag{4.17}$$

To ensure of convergence of the HAM, we first focus on how to choose an appropriate value of $\hbar$. We can investigate the influence of $\hbar$ on the series of a by means of the $\hbar$-curve. The appropriate region for $\hbar$ in this case is $-3.1 < \hbar < -1.2$, as shown in Fig. 4 for $b = \frac{1}{4}$. In this case the exact value of $a = c(1-b-2b^2)$ is 0.625. In general, as long as the series of amplitude is convergent, the corresponding series for $w(\xi)$ is also convergent. For example, when $\hbar = -2$, our analytic solution converges. This is demonstrated in Fig. 5 where it can be seen that the 10th-order approximation for w as a function of η $(\equiv \xi/\mu)$ agrees well with the exact solution given by Eqs. (2.10) and (2.11).

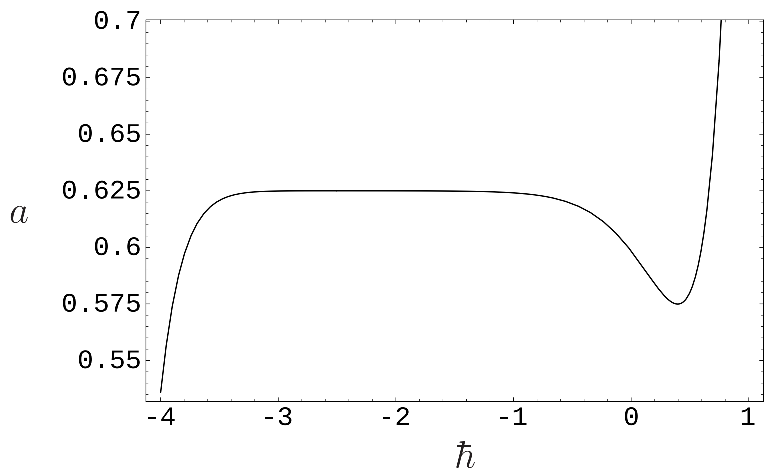

Fig. 4: The $\hbar$-curve for the wave amplitude a at the 15th-order approximation with $b = \frac{1}{4}$.

The value of the amplitude is shown in Table 1. The so-called homotopy-Padé technique (see [1]) is employed, which greatly accelerates the convergence. Clearly, the amplitude converges to the exact value 0.625.

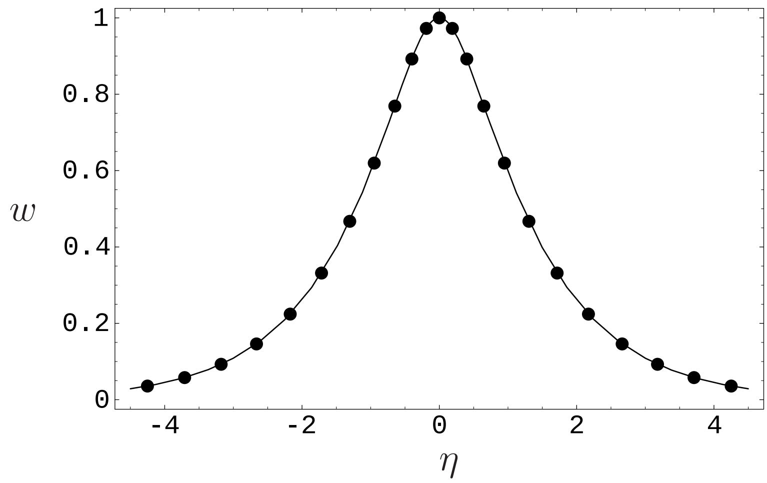

Fig. 5: The analytic approximation for w when $\hbar = -2$ with $b = \frac{1}{4}$ and the exact solution given by Eqs. (2.10) and (2.11). Solid curve: the 10th-order approximation; symbols: exact solution.

Table 1: Results for $[m, m]$ Homotopy-Padé approach

Order of approximation	$[m, m]$	a
2	[1,1]	0.644857
4	[2,2]	0.624729
6	[3,3]	0.625302
8	[4,4]	0.624991
10	[5,5]	0.625132
12	[6,6]	0.625
14	[7,7]	0.625

5 Conclusions

We have applied the homotopy analysis method (HAM) to the Degasperis–Procesi equation (2.1) to obtain analytic approximations to known solitary-wave solutions as given in [22]. In Sections 3 and 4, the methods used are very similar to the ones used for the dCH equation in [13] and [25], respectively. For the peakon solution in Section 3, and the family of smooth-hump waves in Section 4, the amplitude of the solitary waves was treated as an unknown to be determined by the HAM. In all the cases considered, the HAM gave excellent agreement with the known solutions.

In [22] it was shown that the DP equation has a solitary loop-like solution. In [26] we attempted to formulate the HAM in order to find an analytic approximation to this exact solution. The formulation involved the introduction of a new independent variable as was done in [27] for the short-pulse equation; the resulting equation, corresponding to (4.2), involved higher-order nonlinearities. We found that the approximate solution did not agree well with the exact solution. Resolution of this problem is ongoing.

The HAM provides us with a convenient way to control the convergence of approximation series; this is a fundamental qualitative difference between the HAM and other methods for finding approximate solutions. The examples in this paper give further confirmation of the power of the HAM to solve complicated nonlinear problems.

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