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The HELP Inequality on Trees

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Abstract. We establish analogues of Hardy and Littlewood's integro-differential equation for Schrödinger-type operators on metric and discrete trees, based on a generalised strong limit-point property of the graph Laplacian.

1. Introduction

In [**21**] Hardy and Littlewood introduced the inequality

$$\left(\int_0^\infty |f'(x)|^2 \mathrm{d}x\right)^2 \le 4\int_0^\infty |f(x)|^2 \mathrm{d}x \int_0^\infty |f''(x)|^2 \mathrm{d}x, \tag{1.1}$$

which holds for all functions f such that the right-hand side of (1.1) is finite. Equality is attained when, for some $\rho > 0$ and $A \in \mathbb{C}$,

$$f(x) = A\exp\Big(-\frac{\rho x}{2}\Big)\sin\bigg(\frac{\rho\sqrt{3}}{2}x - \frac{\pi}{3}\bigg), \qquad x \in [0,\infty).$$

Their famous book with Pólya [**22**] devotes three different proofs to this inequality. Later, it was extended to what has become known as the HELP inequality, where the second derivative f'' in (1.1) is replaced by a more general Sturm–Liouville operator

$$M[f] := \frac{1}{w}\big(-(pf')' + qf\big).$$

The resulting inequality

$$\left(\int_a^b \Big(p|f'|^2 + q|f|^2\Big)\right)^2 \le K\int_a^b |f|^2 w \int_a^b \big|M[f]\big|^2 w. \tag{1.2}$$

with $b > a > -\infty$, $w > 0$, $p > 0$ and q real-valued, and $1/p$, q and w locally integrable, was proposed by Everitt, who also gave, *inter alia*, a proof of a criterion for the existence of the inequality (finite K) in [**16**]. It is assumed that M is regular at a and singular at b and satisfies the so-called strong limit-point condition at the singular endpoint. The proof in [**16**] is modelled on one of the proofs in [**22**]. In addition, however, a proof which uses only the properties of extensions of symmetric operators is given in [**11, 13**], while [**2**] presents an abstract proof based

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on linear relations. A feature of the results in [**16**] and [**11**] is that the criterion for a finite constant K is given in terms of the Titchmarsh–Weyl m-function associated with the differential expression $M[f]$ while the result [**2**] is applicable in a wider context than (1.2). Many papers have been devoted to finding examples of the HELP inequality for different functions p, q and w, and without any attempt at completeness we mention [**1, 5, 12, 17, 18, 20**]. In addition to these analytic results, we also note the numerical studies of the inequality in [**7**].

The inequality (1.2) has provided the motivation for studying inequalities of this type associated with other differential expressions $M[\cdot]$. For example, when M is an even-order symmetric differential expression of order $2n$, a criterion for the existence of a HELP-type inequality is given in [**3**], while an equivalent result where M is a Hamiltonian system is contained in [**6**]. When M is a second-order symmetric difference expression, an equivalent theory to that in [**11**] is developed [**4**] and it is shown that the so-called Copson inequality may be recovered. Recently, [**24**] developed a very general theory involving only symmetric relations and abstract boundary operators, and this has allowed examples of a HELP type inequality to be found that are associated with block operator matrices. This latter theory will be important in what we wish to report, and we summarise the main results of [**24**] in section 2 below.

All the above examples of inequalities concern operators defined on the half line or an infinite discrete set. Recently, there has been much interest in investigating symmetric problems on trees and graphs and it is this that has provided the impetus for the present work. We shall show that the abstract results reported in [**24**] can be realised on both a metric (continuous) and a discrete (combinatoric) tree. In particular we shall show that the HELP inequality is valid (finite K) for certain operators M and invalid for others.

The paper is structured as follows. In section 2 we review the general framework for HELP inequalities set up in [**24**], giving the result for the abstract inequality which we shall need. In section 3 we introduce the notation for metric trees and show that the limit-point/limit-circle classification for the tree Laplacian is intimately connected with the finiteness or infinity of the volume of the tree. In this section, we focus on symmetric functions on a symmetrically branching tree as an introductory illustration. The following section 4, however, considers general trees of infinite length without any symmetry restriction on the functions, showing that a strong limit-point property holds. This is the essential ingredient for establishing the HELP inequality in section 5; here we also present the explicit example of a regularly branching tree. Finally, sections 6 and 7 concern infinite discrete trees, showing a strong limit-point property and associated HELP inequality.

2. The abstract HELP inequality

In this section we recall an abstract HELP inequality from [**24**]. Since for the HELP inequality on discrete trees we have to deal with linear relations rather than operators, everything is formulated in terms of linear relations and abstract boundary mappings. At the end of the section we specialise the results also to the operator case.

Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$. A *closed linear relation* in $\mathcal{H}$ is a closed subspace of $\mathcal{H}\oplus\mathcal{H}$, where we write $\langle f;g\rangle$ for elements in $\mathcal{H}\oplus\mathcal{H}$; see, e.g. [**9**]. A closed operator T can be identified with its graph and is therefore a closed linear relation with the property that $\langle 0; f\rangle \in T$ implies $f = 0$. The adjoint of a relation T is given by

$$T^* := \{\langle g;\hat{g}\rangle \mid (\hat{f}, g) = (f,\hat{g}) \text{ for all } \langle f;\hat{f}\rangle \in T\},$$

and a relation T is called *symmetric* if $T \subset T^*$ and self-adjoint if $T = T^*$.

For a closed symmetric relation S, the deficiency spaces are given by

$$\widetilde{\mathfrak{N}}_\lambda := \{\langle f;\hat{f}\rangle \in S^* \mid \hat{f} = \lambda f\}, \tag{2.1}$$

$$\mathfrak{N}_\lambda := P_1\widetilde{\mathfrak{N}}_\lambda = \{f \in \mathcal{H} \mid \langle f;\lambda f\rangle \in S^*\} = \ker(S^* - \lambda) \tag{2.2}$$

for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, where P_1 is the projection onto the first component in $\mathcal{H}\oplus\mathcal{H}$. It is well known that $\dim\mathfrak{N}_\lambda$ is constant on the upper and the lower half-plane $\mathbb{C}^\pm$, respectively (see, e.g. [**23**]). The deficiency indices n_+, n_- are defined by $n_\pm := \dim\mathfrak{N}_\lambda$, $\lambda \in \mathbb{C}^\pm$.

Let T be a closed linear relation in a Hilbert space $\mathcal{H}$ whose adjoint is symmetric. The triple $(\mathcal{K},\Gamma_0,\Gamma_1)$ is called a *boundary triple* for T (see, e.g. [**10**]) if $\mathcal{K}$ is a Hilbert space with inner product $(\cdot\,,\cdot)_\mathcal{K}$ and $\Gamma_i : T \to \mathcal{K}$ are linear mappings such that

$$(\hat{f}, g) - (f,\hat{g}) = (\Gamma_1\tilde{f},\Gamma_0\tilde{g})_\mathcal{K} - (\Gamma_0\tilde{f},\Gamma_1\tilde{g})_\mathcal{K} \quad \text{for } \tilde{f} = \langle f;\hat{f}\rangle,\ \tilde{g} = \langle g;\hat{g}\rangle \in T \tag{2.3}$$

and the mapping $\tilde{f} \mapsto \langle\Gamma_0\tilde{f};\Gamma_1\tilde{f}\rangle$ from T into $\mathcal{K}\oplus\mathcal{K}$ is surjective. The Γ_i are called *boundary mappings*. Relation (2.3) can be seen as an abstract Green identity.

One can easily show that for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, the restriction of Γ_0 to $\widetilde{\mathfrak{N}}_\lambda$ is bijective onto $\mathcal{K}$ (see, e.g. [**10**]). Hence the following definition of the abstract *Titchmarsh–Weyl function* makes sense:

$$m(\lambda) := \Gamma_1(\Gamma_0 \restriction \widetilde{\mathfrak{N}}_\lambda)^{-1}, \tag{2.4}$$

which, for every λ, is an operator in $\mathcal{K}$. Moreover, we set

$$\tilde{\gamma}(\lambda) := (\Gamma_0 \restriction \widetilde{\mathfrak{N}}_\lambda)^{-1} : \mathcal{K} \to \widetilde{\mathfrak{N}}_\lambda \tag{2.5}$$

and define the *Dirichlet form* by

$$D[\tilde{f},\tilde{g}] := (\hat{f}, g) + (\Gamma_0\tilde{f},\Gamma_1\tilde{g})_\mathcal{K} \qquad \text{for } \tilde{f} = \langle f;\hat{f}\rangle,\ \tilde{g} = \langle g;\hat{g}\rangle \in T. \tag{2.6}$$

It follows from (2.3) that D is a symmetric form on T. Moreover, we set $D[\tilde{f}] := D[\tilde{f},\tilde{f}]$ for $\tilde{f} \in T$.

Remark 2.1. Assume that T is a closed linear relation in a Hilbert space $\mathcal{H}$ whose adjoint is symmetric. Let $\Gamma_0,\Gamma_1 : T \to \mathcal{K}$, where $\mathcal{K}$ is another Hilbert space, be linear mappings such that $\tilde{f} \mapsto \langle\Gamma_0\tilde{f};\Gamma_1\tilde{f}\rangle$ is surjective, and let D be a symmetric form on T such that

$$(\hat{f}, g) = D[\tilde{f},\tilde{g}] - (\Gamma_0\tilde{f},\Gamma_1\tilde{g})_\mathcal{K}, \qquad \tilde{f} = \langle f;\hat{f}\rangle,\ \tilde{g} = \langle g;\hat{g}\rangle \in T;$$

then $(\mathcal{K},\Gamma_0,\Gamma_1)$ is a boundary triple and D the corresponding Dirichlet form. This follows from

$$\begin{aligned}(\hat{f}, g) - (f,\hat{g}) &= (\hat{f}, g) - \overline{(\hat{g}, f)} = D[\tilde{f},\tilde{g}] - (\Gamma_0\tilde{f},\Gamma_1\tilde{g})_\mathcal{K} - \overline{D[\tilde{g},\tilde{f}]} + \overline{(\Gamma_0\tilde{g},\Gamma_1\tilde{f})_\mathcal{K}}\\ &= (\Gamma_1\tilde{f},\Gamma_0\tilde{g})_\mathcal{K} - (\Gamma_0\tilde{f},\Gamma_1\tilde{g})_\mathcal{K},\end{aligned}$$

which gives (2.3).

The following theorem from [**24**] gives a characterisation of an abstract HELP inequality in terms of the Titchmarsh–Weyl function.

Theorem 2.2. [**24**] *Let T be a closed symmetric relation in $\mathcal{H}$ whose adjoint is symmetric and let $(\mathcal{K}, \Gamma_0, \Gamma_1)$ be a corresponding boundary triple. Moreover, let D be the Dirichlet form defined in* (2.6) *and m be the Titchmarsh–Weyl function defined in* (2.4)*. Then the following assertions are equivalent:*

(i) *there exists a positive constant C such that*

$$\big|D[\tilde{f}]\big| \le C\|f\|\,\|\hat{f}\| \tag{2.7}$$

for all $\tilde{f} = \langle f; \hat{f}\rangle \in T$;

(ii) *there exist $\theta_+, \theta_- \in [0, \pi/2)$ such that*

$$\operatorname{Im}(-\lambda^2 m(\lambda)) \ge 0 \tag{2.8}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ with $\arg\lambda \in [\theta_+, \pi - \theta_-]$.

Let θ_+, θ_- be minimal in (ii) *and put $\theta_0 := \max\{\theta_+, \theta_-\}$. If the Dirichlet form does not vanish identically on T (it can vanish identically only if $\theta_0 = 0$), then the best possible constant in* (2.7) *is $C = 1/\cos\theta_0$.*

Equality holds in (2.7) *if and only if $f = 0$ or $\hat{f} = 0$ or*

$$\tilde{f} = \lambda\tilde{\gamma}(\lambda)u - \overline{\lambda}\tilde{\gamma}(\overline{\lambda})u$$

with $\arg\lambda = \theta_+$ (if $\theta_0 = \theta_+$) or $\arg\lambda = \pi - \theta_-$ (if $\theta_0 = \theta_-$) and

$$u \in \ker(\operatorname{Im}(\lambda^2 m(\lambda))).$$

Let us now consider the case that T is an operator, i.e. T^* is a densely defined operator. In this case the boundary mappings are determined by the first component of elements in T and can hence be defined just for elements in $\mathcal{D}(T)$. The abstract Green identity reduces to

$$(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{K}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{K}}, \qquad f, g \in \mathcal{D}(T). \tag{2.9}$$

The relations (2.4) and (2.5) have to be replaced by

$$m(\lambda) = \Gamma_1(\Gamma_0 \restriction \mathfrak{N}_\lambda)^{-1} \quad \text{and} \quad \gamma(\lambda) = (\Gamma_0 \restriction \mathfrak{N}_\lambda)^{-1},$$

respectively, and the Dirichlet form is defined by

$$D[f, g] = (Tf, g) + (\Gamma_0 f, \Gamma_1 g)_{\mathcal{K}}, \qquad f, g \in \mathcal{D}(T), \tag{2.10}$$

and $D[f] = D[f, f]$. The HELP inequality (2.7) reduces to the following inequality:

$$|D[f]| \le C\|f\|\,\|Tf\|, \qquad f \in \mathcal{D}(T), \tag{2.11}$$

and the cases of equality are $f = 0$, $Tf = 0$ and $f = \lambda\gamma(\lambda)u - \overline{\lambda}\gamma(\overline{\lambda})u$ with u as in the theorem.

Let us recall some criteria for the validity of a HELP inequality in terms of the behaviour of m at 0 and at infinity; see [**24**, section 4].

Proposition 2.3. *Assume that the Titchmarsh–Weyl function m is scalar, i.e. $\dim\mathcal{K} = 1$. The function m satisfies condition* (2.8) *if and only if it satisfies this inequality locally both at* 0 *and at infinity. Moreover, the following is true.*

(i) *If $m(\lambda) \sim c\lambda^\alpha$ for $\lambda \to \infty$ in a sector around the positive imaginary axis with $c \ne 0$ and $\alpha \in [-1, 1] \setminus \{0\}$, then* (2.8) *is satisfied at infinity.*

(ii) *Assume that the non-tangential limit $\lim_{z\to 0} m(z) =: m_0$ from the upper half-plane exists. If $m_0 \in \mathbb{C}\setminus\mathbb{R}$, then* (2.8) *is satisfied at* 0*; if $m_0 \in \mathbb{R}\setminus\{0\}$, then* (2.8) *is not satisfied at* 0*. If, in addition, m is analytic at* 0 *and $m_0 = 0$, then* (2.8) *is satisfied at* 0.

3. Symmetric functions on a binary radial tree

In this section, we consider a *binary radial tree*, i.e. a tree which, starting from a first interval, splits into two branches at each vertex, with the jth generation intervals (of which there are 2^j) having equal length l_j ($j \in \mathbb{N}_0$). As usual, we require functions in the domain of the Laplacian on the tree to be absolutely continuous with absolutely continuous derivative on each tree edge (interval), with the requirement of continuity and the *Kirchhoff condition* (sum of directed derivatives vanishes) at each vertex.

A *symmetric* function u on the tree is a function which depends only on the distance from the root vertex. If we define

$$x_j := \sum_{k=0}^{j-1} l_k, \quad j \in \mathbb{N}_0,$$

and $L := \lim_{j\to\infty} x_j$ (the *length* of the tree), then a symmetric function can be identified with a function on the interval $[0, L)$ which is continuous throughout; the Kirchhoff condition translates into the jump condition for the derivative

$$u'(x_j-) = 2u'(x_j+).$$

To remove these discontinuities of the derivative, we define

$$m_k := 2^{-k} l_k, \quad k \in \mathbb{N}_0, \qquad y_j := \sum_{k=0}^{j-1} m_k, \quad j \in \mathbb{N}_0,$$

and $v(y_j + \eta) = u(x_j + 2^j \eta)$ for $\eta \in [0, m_j]$: then clearly v is a continuous function on $[0, B)$, where $B := \lim_{j\to\infty} y_j$, and on the interval (y_j, y_{j+1}) we have

$$v(y_j + \eta) = u(x_j + 2^j\eta), \qquad v'(y_j + \eta) = 2^j u'(x_j + 2^j\eta),$$
$$v''(y_j + \eta) = 4^j u''(x_j + 2^j\eta).$$

In particular,

$$\begin{aligned} v'(y_j-) &= 2^{j-1}u'(x_{j-1} + 2^{j-1}m_{j-1}-) = 2^{j-1}u'(x_j-) = 2^j u'(x_j+) \\ &= 2^j u'(x_j + 2^j 0+) = v'(y_j+), \end{aligned}$$

so v' is continuous as well.

The space $L^2(\Gamma)_{\text{symm}}$ of square-integrable symmetric functions on the tree clearly corresponds to $L^2([0, L), w)$ with the weight

$$w(x) := 2^j \qquad (x \in [x_j, x_{j+1})),$$

which is the total branching number at distance x from the root. Now, with $\tilde{w}(y) := 4^j$ $(y \in [y_j, y_{j+1}))$, we have that $u \in L^2([0, L), w) \Leftrightarrow v \in L^2([0, B), \tilde{w})$; indeed,

$$\begin{aligned} \int_0^L |u|^2(x)\, w(x)\, dx &= \sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} |u(x)|^2\, 2^j\, dx = \sum_{j=0}^{\infty} 2^j \int_0^{l_j} |u(x_j + \xi)|^2\, d\xi \\ &= \sum_{j=0}^{\infty} 2^j \int_0^{m_j} |u(x_j + 2^j\eta)|^2\, 2^j\, d\eta = \sum_{j=0}^{\infty} 4^j \int_0^{m_j} |v(y_j + \eta)|^2\, d\eta \\ &= \sum_{j=0}^{\infty} 4^j \int_{y_j}^{y_{j+1}} |v(y)|^2\, dy = \int_0^B |v|^2(y)\, \tilde{w}(y)\, dy. \end{aligned}$$

The Laplacian on the tree gives rise to the differential equation

$$-u'' = \lambda u$$

on $[0, L)$ — without weight, but with jumps in the derivative at x_j. The corresponding function v satisfies the differential equation (without further conditions)

$$-v'' = \lambda \tilde{w} v$$

on the interval $[0, B)$. This is a differential equation of Sturm–Liouville type, and we have Weyl's Alternative for the singular end-point B: this point is in the limit-circle case if for all complex λ, every solution is in $L^2([0, B), \tilde{w})$; otherwise it is in the limit-point case.

Taking $\lambda = 0$, every solution of the above differential equation is of the form $v(y) = ay + b$ with constants a, b. If $\int_0^B \tilde{w} = \infty$, then the constant solution ($a = 0$) is not in $L^2([0, B), \tilde{w})$, so we have the limit-point case. If, on the other hand $\int_0^B \tilde{w} < \infty$, then $B < \infty$ (as $\tilde{w} \geq 1$), so all solutions are bounded and hence in $L^2([0, B), \tilde{w})$, and we have the limit-circle case. We observe that this classification is intimately connected with the *volume* of the tree, defined as the sum of the lengths of all vertices: indeed, the volume is

$$\operatorname{vol}(\Gamma) = \sum_{j=0}^{\infty} 2^j l_j = \sum_{j=0}^{\infty} 4^j m_j = \int_0^B \tilde{w}.$$

One can show that the limit-point case at B is equivalent to the property that

$$\lim_{y \to B} \big(f'(y) \bar{g}(y) - f(y) \bar{g}'(y) \big) = 0$$

for all f, g in the maximal domain of the Sturm–Liouville operator [**14**]. If the two terms in this limit tend to zero separately, one speaks of a *strong limit-point case.*

We now show that if the tree has infinite volume, the end-point B is in the strong limit-point case. Our proof is heavily inspired by [**15**] (see also [**19**]).

THEOREM 3.1. *Let* $\mathcal{D} := \{ f : [0, B) \to \mathbb{C} \mid f, f' \in AC_{\mathrm{loc}}[0, B), f, f''/\tilde{w} \in L^2([0, B), \tilde{w}) \}$. *Then* $\lim_{y \to B} f'(y) \bar{g}(y) = 0$ $(f, g \in \mathcal{D})$.

PROOF. Let $f \in \mathcal{D}$; then by integration by parts

$$\int_0^y |f'|^2 = f'(y) \bar{f}(y) - f'(0) \bar{f}(0) - \int_0^y \frac{f''}{\tilde{w}} \, \bar{f} \, \tilde{w}.$$

As the l.h.s. is increasing in y, it either converges to a finite limit or tends to ∞ as $y \to B$. In the latter case, we conclude that $\lim_{y \to B} \operatorname{Re} f'(y) \bar{f}(y) = \infty$ (the other terms on the r.h.s. are bounded), so that $(|f|^2)' \to \infty$: then $|f|^2$ is eventually bounded below by a positive constant and thus cannot be in $L^2([0, B), \tilde{w})$ — a contradiction.

Hence $\int\limits_0^B |f'|^2 < \infty$ and $\lim_{y \to B} f'(y) \bar{f}(y)$ exists for each $f \in \mathcal{D}$. Consequently, $\int_0^B f' \bar{g}'$ exists for all $f, g \in \mathcal{D}$, and from

$$\int_0^y f' \bar{g}' = f'(y) \bar{g}(y) - f'(0) \bar{g}(0) - \int_0^y \frac{f''}{\tilde{w}} \, \bar{g} \, \tilde{w}$$

we conclude that the limit $\lim_{y \to B} f'(y) \bar{g}(y)$ exists as well.

It remains to show that the limit is 0. Define $W(y) := \int\limits_0^y \tilde{w}$ $(y \in [0, B))$; then $\lim\limits_{y\to B} W(y) = \infty$. Let $f, g \in \mathcal{D}$; then

$$\begin{aligned}\frac{f'(y)}{\sqrt{W(y)}} &= \frac{f'(0)}{\sqrt{W(y)}} + \frac{1}{\sqrt{W(y)}} \int_0^y \frac{f''}{\tilde{w}} \sqrt{\tilde{w}}\sqrt{\tilde{w}} \\ &\le o(1) + \sqrt{\int_0^y \left|\frac{f''}{\tilde{w}}\right|^2 \tilde{w}}\; \frac{\sqrt{\int_0^y \tilde{w}}}{\sqrt{W(y)}} = O(1)\end{aligned}$$

as $y \to B$. Now if $\liminf\limits_{y\to B} |g(y)|^2 W(y) > 0$, then there exist $y_0 \in (0, B)$ and $\delta > 0$ such that $|g(y)|^2 W(y) \ge \delta$ for $y \in [y_0, B)$. Then

$$\infty > \int_0^B |g|^2 \tilde{w} \ge \int_{y_0}^y |g|^2 W \frac{\tilde{w}}{W} \ge \delta \int_{y_0}^y \frac{\tilde{w}}{W} = \delta\big(\log W(y) - \log W(y_0)\big) \to \infty$$

as $y \to B$ — a contradiction.

Hence there is a sequence $z_n \to B$ such that $|g(z_n)|^2 W(z_n) \to 0$ $(n \to \infty)$, so

$$f'(z_n)\bar{g}(z_n) = \frac{f'(z_n)}{\sqrt{W(z_n)}} \sqrt{W(z_n)}\bar{g}(z_n) \to 0 \qquad (n \to \infty).$$

As we had already shown that the limit of $f'\bar{g}$ exists, it must be 0. □

We remark that the methods and conclusions of this section are very similar to some results in [**8**], but it seems that the proof given there for the limit-point property if the tree volume is infinite is rather more complicated than ours.

4. The strong limit-point property on metric trees of infinite length

In this section, we consider a general tree Γ of infinite length, i.e. such that any regular path in the tree can be extended to a regular path of infinite length. However, we impose no restrictions or symmetry conditions on the branching numbers and edge lengths. Taking the proof of Theorem 3.1 as a guide, we show that the Laplacian on such a tree has a generalised strong limit-point property where the singular end-point is replaced by the limit set of infinite ends of the tree.

Let $\mathcal{D} := \{f : \Gamma \to \mathbb{C} \mid f, f' \text{ a.c. on edges}, f \text{ continuous, Kirchhoff condition at each vertex}, f, f'' \in L^2(\Gamma)\}$.

For $x \in \Gamma$, denote by $|x|$ the distance of x from the root, and let $\Gamma_r := \big\{x \in \Gamma \bigm| |x| \le r\big\}$ $(r > 0)$. Then we have the following theorem (note that here f, g are general functions in $\mathcal{D}$ with no assumptions of symmetry).

Theorem 4.1. *For all $f, g \in \mathcal{D}$,*

$$\lim_{r\to\infty} \sum_{|x|=r} f(x)\bar{g}'(x) = 0.$$

Proof. For $f \in \mathcal{D}$, we have

$$\int_{\Gamma_r} |f'|^2 = -\int_{\Gamma_r} f''\bar{f} + \sum_{|x|=r} f'(x)\bar{f}(x) - f'(0)\bar{f}(0)$$

by integration by parts; the contributions of the inner vertices vanish because of the Kirchhoff condition. The integral on the l.h.s. is increasing in r, so it either converges to a finite limit or tends to infinity as $r \to \infty$. In the latter case, this implies that

$$\sum_{|x|=r} (|f|^2)'(x) = 2\,\mathrm{Re} \sum_{|x|=r} f'(x)\bar{f}(x) \to \infty \qquad (r \to \infty).$$

Hence, if we set $F(r) := \sum_{|x|=r} |f|^2(x)$, we have $F'(r) \to \infty$ $(r \to \infty)$, so F is eventually bounded below by a positive constant, and

$$\int_\Gamma |f|^2 = \int_0^\infty F(r)\,dr = \infty,$$

a contradiction.

Thus we conclude that the integral on the l.h.s. has a finite limit as r tends to infinity. But then

$$\lim_{r\to\infty} \sum_{|x|=r} (|f|^2)'(x)$$

exists as well. If this limit is non-zero, then it must be positive since $F(r) \geq 0$; then $\sum_{|x|=r} |f|^2$ will be eventually growing at least linearly, contradicting $f \in L^2(\Gamma)$.

Thus we find that the limit of $\sum_{|x|=r} f'(x)\bar{f}(x)$ as $r \to \infty$ exists and that

$$2\,\mathrm{Re} \lim_{r\to\infty} \sum_{|x|=r} f'(x)\bar{f}(x) = \lim_{r\to\infty} \sum_{|x|=r} (|f|^2)'(x) = 0.$$

In order to show that the imaginary part of the above limit vanishes as well, we proceed as follows. Let S be the minimal operator associated with the (negative) Laplacian on the tree; $\mathcal{D}(S) := \{f \in C_0(\Gamma) \mid f \text{ is } C^\infty \text{ on each edge, Kirchhoff condition at each vertex}\}$. Then clearly S is symmetric, and as for a single interval (cf. [**26**, Theorem 3.6 b]) one can show that $\mathcal{D}(S^*) = \mathcal{D}$. Furthermore, denote by A the operator S^* restricted by a Dirichlet boundary condition at the tree root, a one-dimensional restriction.

To estimate the deficiency indices of the operator S^*, consider a solution $f \in \mathcal{D}(A)$ of $-f'' = if$. Then, integrating by parts as above and using the boundary condition, we find

$$\int_\Gamma |f'|^2 = i \int_\Gamma |f|^2 + \lim_{r\to\infty} \sum_{|x|=r} f'(x)\bar{f}(x).$$

Here the l.h.s. is real, the r.h.s. purely imaginary! Hence $f' = 0$ on the graph edges, and so f is constant. From the differential equation, it must be identically zero.

An analogous argument applies to $-f'' = -if$. Therefore the kernel of $A \pm i$ is trivial, and we conclude that S^* has deficiency indices at most $(1, 1)$.

Consequently, the Dirichlet operator A is self-adjoint. For each $f \in \mathcal{D}(A)$, we have

$$\mathrm{Im} \lim_{r\to\infty} \sum_{|x|=r} f'(x)\bar{f}(x) = \mathrm{Im} \left(\int_\Gamma |f'|^2 - \int_\Gamma (Af)\bar{f} \right) = 0.$$

As the elements of $\mathcal{D}$ differ from those of $\mathcal{D}(A)$ only at the tree root, we conclude that

$$\lim_{r\to\infty} \sum_{|x|=r} f'(x)\bar{f}(x) = 0 \qquad (f \in \mathcal{D}).$$

Now if $f, g \in \mathcal{D}$, then

$$0 = \lim_{r\to\infty} \sum_{|x|=r} (f+g)'(x)\overline{(f+g)(x)} = \lim_{r\to\infty} \sum_{|x|=r} (g'(x)\bar{f}(x) + f'(x)\bar{g}(x)),$$

$$0 = \lim_{r\to\infty} \sum_{|x|=r} (f+ig)'(x)\overline{(f+ig)(x)} = i\lim_{r\to\infty} \sum_{|x|=r} (g'(x)\bar{f}(x) - f'(x)\bar{g}(x)),$$

whence the assertion follows. □

5. The HELP inequality on metric trees

Let Γ be a tree with infinite length as in Section 4 and q be a bounded function on Γ. The maximal operator T is defined by

$$Tf = -f'' + qf$$

with domain

$$\mathcal{D}(T) = \big\{ f \in L^2(\Gamma) \;\big|\; f, f' \text{ a.c. on edges}, f \text{ continuous and satisfies Kirchhoff condition at each vertex}, f'' \in L^2(\Gamma)\big\},$$

which is equal to $\mathcal{D}$ in the previous section.

Lemma 5.1. *Let Γ and T be as above. The triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ with*

$$\Gamma_0 f := f'(0), \quad \Gamma_1 f := -f(0), \qquad f \in \mathcal{D}(T),$$

is a boundary triple for T. The corresponding Dirichlet form is given by

$$D[f,g] = \int_\Gamma \big(f'\overline{g'} + qf\overline{g}\big), \qquad f, g \in \mathcal{D}(T).$$

Proof. It is clear that the mapping $f \mapsto \langle \Gamma_0 f; \Gamma_1 f\rangle$ maps $\mathcal{D}(T)$ onto $\mathbb{C}^2$. Let $f, g \in \mathcal{D}(T)$ and $\Gamma_r := \big\{x \in \Gamma \;\big|\; |x| \le r\big\}$; then

$$\int_{\Gamma_r} (-f'' + qf)\overline{g} = \int_{\Gamma_r} (f'\overline{g'} + qf\overline{g}) + f'(0)\overline{g(0)} - \sum_{|x|=r} f'(x)\overline{g(x)}.$$

Letting $r \to \infty$ and using Theorem 4.1 we obtain

$$(Tf, g) = \int_\Gamma (-f'' + qf)\overline{g} = \int_\Gamma (f'\overline{g'} + qf\overline{g}) + f'(0)\overline{g(0)} = D[f,g] - \Gamma_0 f \cdot \overline{\Gamma_1 g}.$$

This together with Remark 2.1 shows all assertions of the lemma. □

The deficiency subspaces $\mathfrak{N}_\lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are one-dimensional (see the proof of Theorem 4.1). Let ψ_λ be a solution of $T\psi_\lambda = \lambda\psi_\lambda$, normalised such that $\Gamma_0\psi_\lambda = \psi_\lambda'(0) = 1$. Then ψ_λ spans $\mathfrak{N}_\lambda$, and the Titchmarsh–Weyl function is given by

$$m(\lambda) = \Gamma_1\psi_\lambda = -\psi_\lambda(0). \tag{5.1}$$

With this we can formulate a criterion for the validity of a HELP inequality on a metric tree.

Theorem 5.2. *Let the metric tree Γ and the operator T be as above and m the Titchmarsh–Weyl function defined in* (5.1). *Then the following are equivalent:*

(i) *there exists a positive constant K such that*

$$\left(\int_\Gamma \left(|f'|^2 + q|f|^2\right)\right)^2 \leq K \int_\Gamma |f|^2 \int_\Gamma \left|-f'' + qf\right|^2 \tag{5.2}$$

for all $f \in \mathcal{D}(T)$;

(ii) *there exist $\theta_+, \theta_- \in [0, \pi/2)$ such that*

$$\operatorname{Im}(-\lambda^2 m(\lambda)) \geq 0 \tag{5.3}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ with $\arg\lambda \in [\theta_+, \pi - \theta_-]$.

Let θ_+, θ_- be minimal in (ii) *and put $\theta_0 := \max\{\theta_+, \theta_-\}$. Then the best possible constant in* (5.2) *is $K = 1/(\cos\theta_0)^2$.*

Let A_D and A_N be the Dirichlet and Neumann operators, i.e. the restrictions of T to functions that satisfy $f(0) = 0$ or $f'(0) = 0$. Equality holds in (5.2) *if and only if 0 is an eigenvalue of A_D or A_N and f is a corresponding eigenfunction, or $f = \alpha \operatorname{Im}(\lambda\psi_\lambda)$ with $\alpha \in \mathbb{C}$ and λ such that $\operatorname{Im}(\lambda^2 m(\lambda)) = 0$ and $\arg\lambda = \theta_+$ if $\theta_0 = \theta_+$ or $\arg\lambda = \pi - \theta_-$ if $\theta_0 = \theta_-$.*

Proof. The theorem follows almost immediately from Theorem 2.2 since (5.2) is the square of (2.7). Concerning the cases of equality: if $Tf = 0$, then (5.2) implies $D[f] = 0$, and hence by (2.10) also $\Gamma_0 f = 0$ or $\Gamma_1 f = 0$. The third case of equality in Theorem 2.2 yields (with $c \in \mathbb{C}$)

$$f = \lambda\gamma(\lambda)c - \overline{\lambda}\gamma(\overline{\lambda})c = \lambda c\psi_\lambda - \overline{\lambda} c\overline{\psi_\lambda} = 2ic \operatorname{Im}(\lambda\psi_\lambda).$$

□

Let us now consider a regularly branching tree with branching number b and assume that $q(x)$ is a symmetric function, i.e. depends only on $|x|$. Then also $\psi_\lambda(x)$ depends only on $|x|$. For assume that $\psi_\lambda(x_1) \neq \psi_\lambda(x_2)$ with $|x_1| = |x_2|$. Let $\pi_\Gamma : \Gamma \to \Gamma$ be a bijection that swaps two branches that contain x_1 and x_2, respectively. Then $\psi_\lambda \circ \pi_\Gamma$ is also in $\mathfrak{N}_\lambda$, and ψ_λ and $\psi_\lambda \circ \pi_\Gamma$ are linearly independent, which is a contradiction to the fact that $\dim \mathfrak{N}_\lambda = 1$.

To calculate the Titchmarsh–Weyl function more explicitly, consider a binary tree (i.e. $b = 2$) where all edges are of equal length l and q is the same on every edge. Let ψ_λ be the defect element with $\Gamma_0\psi_\lambda = \psi_\lambda'(0) = 1$. Since ψ_λ is symmetric, we can identify it with a function on $[0, \infty)$. Let $\theta(\,\cdot\,;\lambda)$, $\phi(\,\cdot\,;\lambda)$ be the solutions of $-y'' + qy = \lambda y$ on $[0, l]$ with

$$\theta(0;\lambda) = 1, \quad \theta'(0;\lambda) = 0, \qquad \phi(0;\lambda) = 0, \quad \phi'(0;\lambda) = 1.$$

Then for $n \in \mathbb{N}$,

$$\begin{pmatrix} \psi_\lambda\big((n+1)l+\big) \\ \psi_\lambda'\big((n+1)l+\big) \end{pmatrix} = \begin{pmatrix} \psi_\lambda\big((n+1)l-\big) \\ \frac{1}{2}\psi_\lambda'\big((n+1)l-\big) \end{pmatrix} = \underbrace{\begin{pmatrix} \theta(l-;\lambda) & \phi(l-;\lambda) \\ \frac{1}{2}\theta'(l-;\lambda) & \frac{1}{2}\phi'(l-;\lambda) \end{pmatrix}}_{=:A(\lambda)} \begin{pmatrix} \psi_\lambda(nl+) \\ \psi_\lambda'(nl+) \end{pmatrix}.$$

Since $\dim \mathfrak{N}_\lambda = 1$, we have $\psi_\lambda(x + l) = \mu(\lambda)\psi_\lambda(x)$ with some $\mu(\lambda) \in \mathbb{C}$ and hence $m(\lambda) = -\psi_\lambda(0) = -\frac{\psi_\lambda(nl+)}{\psi_\lambda'(nl+)}$, which implies that $\mu(\lambda)$ is an eigenvalue of $A(\lambda)$ with eigenvector $\binom{m(\lambda)}{-1}$. Since $\det A(\lambda) = \frac{1}{2}$, we have (we write $\theta(l;\lambda)$ for $\theta(l-;\lambda)$)

$$\mu(\lambda)^2 - \Big(\theta(l;\lambda) + \frac{1}{2}\phi'(l;\lambda)\Big)\mu(\lambda) + \frac{1}{2} = 0. \tag{5.4}$$

As $\psi_\lambda \in L^2(\Gamma)$, $\mu(\lambda)$ must be the solution of (5.4) with $|\mu(\lambda)| < \frac{1}{2}$. The eigenvalue equation for $A(\lambda)$, the form of the eigenvector and (5.4) show that

$$m(\lambda) = \frac{\phi(l;\lambda)}{\theta(l;\lambda) - \mu(\lambda)} = \frac{4\phi(l;\lambda)}{2\theta(l;\lambda) - \phi'(l;\lambda) \pm \sqrt{\big(2\theta(l;\lambda) + \phi'(l;\lambda)\big)^2 - 8}} \,.$$

If $l = 1$ and $q \equiv -\tau$ with $\tau \in \mathbb{R}$, we obtain

$$m(\lambda) = \frac{\sin\sqrt{\lambda+\tau}}{\sqrt{\lambda+\tau}\big(\cos\sqrt{\lambda+\tau} - \mu(\lambda)\big)},$$

where

$$\mu(\lambda) = \frac{1}{4}\Big(3\cos\sqrt{\lambda+\tau} \pm \sqrt{9\cos^2\sqrt{\lambda+\tau} - 8}\Big)$$

is chosen such that $|\mu(\lambda)| < \frac{1}{2}$.

To determine whether there is a HELP inequality, we use Proposition 2.3. In a sector around the positive imaginary axis we have $m(\lambda) \sim i/\sqrt{\lambda}$ for $\lambda \to \infty$. Hence (2.8) is satisfied at infinity. The non-tangential limit from the upper half-plane $m_0 = \lim_{z\to 0} m(z)$ exists for every τ. It is non-real if and only if $9\cos^2\sqrt{\lambda+\tau} < 8$; moreover, $m_0 = 0$ exactly for $\tau = n\pi$, $n = 1, 2, \ldots$. So according to Proposition 2.3 there is a HELP inequality if and only if

$$\tau \in \left(\Big((n-1)\pi + \arccos\tfrac{\sqrt{8}}{3}\Big)^2, \Big(n\pi - \arccos\tfrac{\sqrt{8}}{3}\Big)^2\right) \quad \text{for some } n \in \mathbb{N} \tag{5.5}$$

$$\text{or} \quad \tau = n^2\pi^2 \qquad \text{for some } n \in \mathbb{N}.$$

These intervals are exactly the interiors of the continuous spectrum of the operator with $\tau = 0$, cf. [**25**]. For the values in the first two intervals we calculated the best constants numerically by directly testing the condition (2.8); see figure 1. Note that the first two intervals in (5.5) are (0.115, 7.85) and (12.12, 35.32). If τ is in the first interval, then there are cases of equality only in the interval (0.115, 0.49). There are no cases of equality if τ is in the second interval.

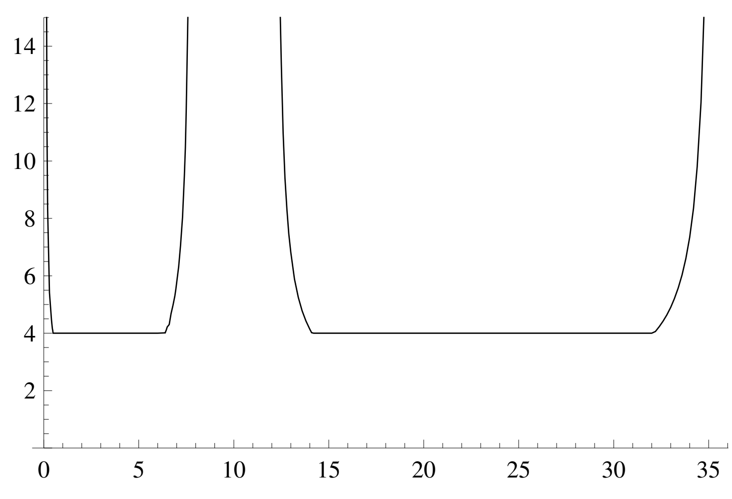

FIGURE 1. The first two bands of the continuous problem

6. The strong limit-point property on discrete trees

In this section, we consider a discrete tree. This means that the edges of the tree do not have a length, and the functions under consideration are defined only on the discrete set of vertices V. For each vertex $x \in V$, its distance $|x|$ from the root vertex (which we denote by 0) is just the integer number of edges connecting this vertex to the root. The set $A(x)$ of vertices adjacent to x consists of one vertex, which we denote by $x-$, such that $|x-| = |x| - 1$, and a set $A_+(x)$ of vertices y such that $|y| = |x|+1$ and $y- = x$. We denote by $N_+(x)$ the number of elements in $A_+(x)$. We generally assume that the tree has infinite length, i.e. that $N_+(x) > 0$ for all $x \in V$.

For a function $f : V \to \mathbb{C}$ we then consider the (negative) discrete Laplacian $\mathcal{L}_0 f$, defined as

$$\mathcal{L}_0 f(x) = -\sum_{y \in A(x)} (f(y) - f(x)) \qquad (x \in V).$$

We also define the derivative (backward difference) of f as $f'(0) = 0$ and

$$f'(x) := f(x) - f(x-)$$

for $x \in V \setminus \{0\}$.

As usual, we denote by $\ell^2(V)$ the space of complex-valued functions which are square-summable on the discrete tree.

THEOREM 6.1. *Let $f \in \ell^2(V)$ such that $\mathcal{L}_0 f \in \ell^2(V)$. Then $f' \in \ell^2(V)$ as well.*

We remark that if the branching number N_+ is bounded throughout the tree, i.e. if there is $N \in \mathbb{N}$ such that $N_+(x) \le N$ $(x \in V)$, then the statement of Theorem 6.1 is trivial, as the derivative is then a bounded linear operator in $\ell^2(V)$. However, this is not the case if N_+ is unbounded.

PROOF. Let $f, g : V \to \mathbb{C}$ and $r \in \mathbb{N}$. Then we have

$$\begin{aligned}
\sum_{0<|x|\le r} f'(x)\overline{g'(x)} &= \sum_{0<|x|\le r} \big(f(x) - f(x-)\big)\overline{\big(g(x) - g(x-)\big)} \\
&= \sum_{0<|x|\le r} \big(f(x) - f(x-)\big)\overline{g(x)} - \sum_{0\le|x|<r} \sum_{y\in A_+(x)} \big(f(y) - f(x)\big)\overline{g(x)} \\
&= -\sum_{0<|x|<r} \Bigg(f(x-) - f(x) + \sum_{y\in A_+(x)} \big(f(y) - f(x)\big)\Bigg)\overline{g(x)} \\
&\quad + \sum_{|x|=r} \big(f(x) - f(x-)\big)\overline{g(x)} - \sum_{y\in A_+(0)} \big(f(y) - f(0)\big)\overline{g(0)} \\
&= \sum_{0<|x|<r} (\mathcal{L}_0 f)(x)\overline{g(x)} + \sum_{|x|=r} f'(x)\overline{g(x)} - \sum_{y\in A_+(0)} f'(y)\overline{g(0)}. \qquad (6.1)
\end{aligned}$$

Now if we assume that $f = g \in \ell^2(V)$ and $\mathcal{L}_0 f \in \ell^2(V)$, we find, by applying the Cauchy–Schwarz inequality to the first and the second term on the right-hand side of the above identity, that for any $r \in \mathbb{N}$,

$$\sum_{|x|=r} |f'(x)|^2 \le \sum_{0<|x|\le r} |f'(x)|^2$$

$$\le \|\mathcal{L}_0 f\|\,\|f\| + \sqrt{\sum_{|x|=r} |f'(x)|^2}\sqrt{\sum_{|x|=r} |f(x)|^2} + \left|\sum_{y\in A_+(0)} f'(y)\bar{f}(0)\right|. \tag{6.2}$$

Hence, setting

$$a_r := \sqrt{\sum_{|x|=r} |f'(x)|^2}, \qquad b_r := \sqrt{\sum_{|x|=r} |f(x)|^2},$$

and observing that $\lim_{r\to\infty} b_r = 0$, we conclude that $a_r^2 \le \text{const} + b_r a_r$, or equivalently

$$\left(a_r - \frac{b_r}{2}\right)^2 \le \text{const} + \frac{b_r^2}{4} \qquad (r \in \mathbb{N}).$$

Therefore the sequence $(a_r)_{r\in\mathbb{N}}$ is bounded, and again by (6.2), $\sum\limits_{0<|x|\le r} |f'(x)|^2$ remains bounded as $r \to \infty$, and the assertion of the theorem follows. □

As an immediate consequence, we obtain the following strong limit-point property for the discrete tree.

Corollary 6.2. *Let $\mathcal{D} := \{f \in \ell^2(V) \mid \mathcal{L}_0 f \in \ell^2(V)\}$. Then*

$$\lim_{r\to\infty} \sum_{|x|=r} f'(x)\,\bar{g}(x) = 0.$$

Proof. We have

$$\sum_{|x|=r} f'(x)\bar{g}(x) \le \sqrt{\sum_{|x|=r} |f'(x)|^2}\sqrt{\sum_{|x|=r} |g(x)|^2},$$

and the assertion follows from

$$\|f'\|^2 = \sum_{r=0}^{\infty}\sum_{|x|=r} |f'(x)|^2 < \infty, \quad \|g\|^2 = \sum_{r=0}^{\infty}\sum_{|x|=r} |g(x)|^2 < \infty.$$

□

7. The HELP inequality on discrete trees

Consider a discrete tree as in the previous section such that $N_+(x) > 0$ for every $x \in V$; $N_+(x)$, $A(x)$ and $A_+(x)$ are defined as at the beginning of section 6. Moreover, let q be a bounded function on V and

$$(\mathcal{L}f)(x) := -\sum_{y\in A(x)} \big(f(y) - f(x)\big) + q(x)f(x).$$

Define the minimal operator $T_{\min}$ in $\ell^2(V)$ by

$$\mathcal{D}(T_{\min}) = \big\{f \in \ell^2(V) \bigm| f(x) = 0 \text{ for almost all } x;\ f(0) = 0\big\},$$

$$(T_{\min} f)(x) = (\mathcal{L}f)(x), \qquad x \in V,\ f \in \mathcal{D}(T_{\min}).$$

Lemma 7.1. *The adjoint of $T_{\min}$ is given by the relation*

$$T := \big\{\langle f; \hat{f}\rangle \in \ell^2(V) \oplus \ell^2(V) \bigm| \hat{f}(x) = (\mathcal{L}f)(x) \text{ for all } x \in V \setminus \{0\}\big\}.$$

Note that $\hat{f}(0)$ is arbitrary; so T is a proper relation.

PROOF. Let $x_0 \in V \setminus \{0\}$ and define

$$g_{x_0}(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0, \end{cases}$$

which is in $\mathcal{D}(T_{\min})$; then we have

$$(T_{\min} g_{x_0})(x) = \begin{cases} |A(x_0)| + q(x_0), & x = x_0, \\ -1, & x \in A(x_0), \\ 0, & \text{otherwise}. \end{cases}$$

If $\tilde{f} = \langle f; \hat{f} \rangle \in T_{\min}^*$, then for $x_0 \in V \setminus \{0\}$:

$$\begin{aligned} \hat{f}(x_0) &= (\hat{f}, g_{x_0}) = (f, T_{\min} g_{x_0}) \\ &= \big(|A(x_0)| + q(x_0)\big) f(x_0) - \sum_{y \in A(x_0)} f(y) = (\mathcal{L} f)(x_0), \end{aligned}$$

which implies that $\tilde{f} \in T$.

On the other hand, if $\tilde{f} = \langle f; \hat{f} \rangle \in T$, then $(\hat{f}, g) = (f, T_{\min} g)$ for all $g \in \mathcal{D}(T_{\min})$ since all these g are linear combinations of g_{x_0}; hence $\tilde{f} \in T_{\min}^*$. □

Since $T_{\min} \subset T$, the operator $T_{\min}$ is symmetric. In the next lemma a boundary triple is constructed.

LEMMA 7.2. *The triple* $(\mathbb{C}, \Gamma_0, \Gamma_1)$ *is a boundary triple for* T*, where the boundary mappings* $\Gamma_0, \Gamma_1 : T \to \mathbb{C}$ *are defined by*

$$\begin{aligned} \Gamma_0 \tilde{f} &:= -\hat{f}(0) - \sum_{y \in A_+(0)} f'(y), \\ \Gamma_1 \tilde{f} &:= f(0), \end{aligned} \qquad \tilde{f} = \langle f; \hat{f} \rangle \in T.$$

The corresponding Dirichlet form is given by

$$D[\tilde{f}, \tilde{g}] := \sum_{x \neq 0} f'(x) \overline{g'(x)} + \sum_{x \in V} q(x) f(x) \overline{g(x)}, \qquad \tilde{f} = \langle f; \hat{f} \rangle,\ \tilde{g} = \langle g; \hat{g} \rangle \in T.$$

PROOF. Let $r \in \mathbb{N}$ and $\tilde{f} = \langle f; \hat{f} \rangle$, $\tilde{g} = \langle g; \hat{g} \rangle \in T$; then according to Corollary 6.2 the second term on the r.h.s. of (6.1) converges to 0 for $r \to \infty$, and hence

$$\sum_{x \neq 0} f'(x) \overline{g'(x)} + \sum_{x \in V} q(x) f(x) \overline{g(x)} = \sum_{x \neq 0} (\mathcal{L} f)(x) \overline{g(x)} - \sum_{y \in A_+(0)} f'(y) \overline{g(0)},$$

which implies

$$\begin{aligned} (\hat{f}, g) &= \hat{f}(0) \overline{g(0)} + \sum_{x \neq 0} (\mathcal{L} f)(x) \overline{g(x)} \\ &= \sum_{x \neq 0} f'(x) \overline{g'(x)} + \sum_{x \in V} q(x) f(x) \overline{g(x)} + \Big(\hat{f}(0) + \sum_{y \in A_+(0)} f'(y) \Big) \overline{g(0)} \\ &= D[\tilde{f}, \tilde{g}] - \Gamma_0 \tilde{f} \cdot \overline{\Gamma_1 \tilde{g}}. \end{aligned}$$

Since D is a symmetric form on T and $\tilde f \mapsto \langle \Gamma_0 \tilde f; \Gamma_1 \tilde f\rangle$ is surjective, this shows that Γ_i are boundary mappings for T and D is the corresponding Dirichlet form; see Remark 2.1. □

The previous lemma implies that $\ker\Gamma_0 \cap \ker\Gamma_1$ is a symmetric relation (actually it is an operator since f determines $\hat f(0)$ by $\Gamma_0 \tilde f = 0$) whose adjoint is T; cf., e.g. [**24**, Lemma 2.1]. The deficiency subspace $\widetilde{\mathfrak{N}}_\lambda$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, defined in (2.1), is one-dimensional and spanned by a non-trivial element $\tilde\psi_\lambda = \langle \psi_\lambda; \hat\psi_\lambda\rangle = \langle \psi_\lambda; \lambda\psi_\lambda\rangle \in T$, i.e.

$$\mathcal{L}\psi_\lambda(x) = \lambda\psi_\lambda(x), \qquad x \in V \setminus \{0\},$$

which we normalise such that $\Gamma_0\tilde\psi_\lambda = 1$, i.e.

$$-\hat\psi_\lambda(0) - \sum_{y\in A_+(0)} \psi'_\lambda(y) = (N_+(0) - \lambda)\psi_\lambda(0) - \sum_{y\in A_+(0)} \psi_\lambda(y) = 1.$$

The function $\tilde\gamma(\lambda)$ defined in (2.5) is a mapping from $\mathbb{C}$ onto $\widetilde{\mathfrak{N}}_\lambda$ that maps $\alpha \in \mathbb{C}$ onto $\alpha\tilde\psi_\lambda$; the Titchmarsh–Weyl function is a multiplication operator in $\mathbb{C}$, which we can identify with a scalar function:

$$m(\lambda) = \Gamma_1\tilde\psi_\lambda = \psi_\lambda(0). \tag{7.1}$$

Now we can give a criterion for a HELP inequality on a discrete tree.

THEOREM 7.3. *Let the discrete tree* Γ *and* $\mathcal{L}$ *be as above and* m *the Titchmarsh–Weyl function defined in* (7.1)*. Then the following are equivalent:*

(i) *there exists a positive constant* K *such that*

$$\left(\sum_{x\in V\setminus\{0\}} \left|f'(x)\right|^2 + \sum_{x\in V} q(x)\left|f(x)\right|^2\right)^2 \le K \sum_{x\in V}\left|f(x)\right|^2 \sum_{x\in V\setminus\{0\}} \left|(\mathcal{L}f)(x)\right|^2 \tag{7.2}$$

for all $f \in \ell^2(V)$ *such that* $\mathcal{L}f \in \ell^2(V)$*;*

(ii) *there exist* $\theta_+, \theta_- \in [0, \pi/2)$ *such that*

$$\operatorname{Im}(-\lambda^2 m(\lambda)) \ge 0 \tag{7.3}$$

for all $\lambda \in \mathbb{C}\setminus\{0\}$ *with* $\arg\lambda \in [\theta_+, \pi - \theta_-]$.

Let θ_+, θ_- *be minimal in* (ii) *and put* $\theta_0 := \max\{\theta_+, \theta_-\}$*. Then the best possible constant in* (7.2) *is* $K = 1/(\cos\theta_0)^2$.

Equality holds in (7.2) *if and only if* 0 *is an eigenvalue of* $\ker\Gamma_0$ *or* $\ker\Gamma_1$ *and* f *is a corresponding eigenvector, or* $f = \alpha\operatorname{Im}(\lambda\psi_\lambda)$ *with* $\alpha \in \mathbb{C}$ *and* λ *such that* $\operatorname{Im}(\lambda^2 m(\lambda)) = 0$ *and* $\arg\lambda = \theta_+$ *if* $\theta_0 = \theta_+$ *or* $\arg\lambda = \pi - \theta_-$ *if* $\theta_0 = \theta_-$.

PROOF. We can apply Theorem 2.2, which yields the equivalence of (ii) and the existence of a positive constant C such that

$$\begin{aligned}&\left(\sum_{x\in V\setminus\{0\}} \left|f'(x)\right|^2 + \sum_{x\in V} q(x)\left|f(x)\right|^2\right)^2 \\ &\qquad\le C^2 \sum_{x\in V} |f(x)|^2 \left(\left|\hat f(0)\right|^2 + \sum_{x\in V\setminus\{0\}} \left|(\mathcal{L}f)(x)\right|^2\right)\end{aligned} \tag{7.4}$$

for every $\tilde f = \langle f; \hat f\rangle \in T$. If $f \in \ell^2(V)$ such that $\mathcal{L}f \in \ell^2(V)$, then $\langle f; \hat f\rangle \in T$ where $\hat f(0) = 0$ and $\hat f(x) = (\mathcal{L}f)(x)$ for $x \ne 0$. Hence (7.4) implies (7.2) with $K = C^2$. The converse implication is trivial. Since the Dirichlet form does not vanish identically, the best constant in (7.2) is always given by $K = C^2 = 1/(\cos\theta_0)^2$.

Now consider the cases of equality. If $\hat{f} = 0$, then also $D[\tilde{f}] = 0$, which implies that $\Gamma_0\tilde{f} \cdot \overline{\Gamma_1\tilde{f}} = 0$. Hence $\tilde{f} = \langle f; 0\rangle \in \ker\Gamma_0$ or $\tilde{f} = \langle f; 0\rangle \in \ker\Gamma_1$. The third case in Theorem 2.2 leads to $f = \alpha\,\mathrm{Im}(\lambda\psi_\lambda)$ because of the form of $\tilde{\gamma}(\lambda)$. □

Let us now consider a regularly branching tree for which $N_+(x) = b$, $x \neq 0$, and $N_+(0) = 1$, where $b \in \mathbb{N}$, $b > 1$. So V consists of vertices of the form $0 = (0; 0)$ and $(n; k)$, $n = 1, 2, \ldots$; $k = 1, \ldots, b^{n-1}$. Here $|(n; k)| = n$ and $A_+((n; k)) = \{(n+1; (k-1)b+l) \mid l \in \{1, \ldots, b\}\}$ for $n > 0$ and $A_+((0; 0)) = \{(1; 1)\}$. Moreover, let us assume that $q(x) \equiv -\tau$ is constant. In this case the deficiency subspaces consist of symmetric functions only. For assume that $\widetilde{\mathfrak{N}}_\lambda$ contains a non-symmetric element $\tilde{f} = \langle f; \hat{f}\rangle$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then f differs on two vertices x_1, x_2 in the same generation, i.e. for the same n. Let g be the function on V that is obtained from f by swapping the two branches that contain x_1 and x_2, respectively. Obviously, also $\tilde{g} = \langle g; \lambda g\rangle \in \widetilde{\mathfrak{N}}_\lambda$, and $\tilde{f}$ and $\tilde{g}$ are linearly independent, which is a contradiction to the fact that $\dim\widetilde{\mathfrak{N}}_\lambda = 1$. Now it is easy to calculate the Titchmarsh–Weyl function.

Proposition 7.4. *The Titchmarsh–Weyl function $m(\lambda)$ for a discrete regularly branching tree with branching number b and $q(x) \equiv -\tau$ is given by*

$$m(\lambda) = \frac{1}{1 - \lambda - \alpha(\lambda)}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{7.5}$$

where $\alpha(\lambda)$ is the unique number

$$\frac{b + 1 - \tau - \lambda \pm \sqrt{(b + 1 - \tau - \lambda)^2 - 4b}}{2b} \tag{7.6}$$

for which $|\alpha(\lambda)| < 1/\sqrt{b}$.

Proof. First we find the element $\tilde{\psi}_\lambda = \langle\psi_\lambda; \lambda\psi_\lambda\rangle$ that spans the deficiency space $\widetilde{\mathfrak{N}}_\lambda$ and satisfies $\Gamma_0\tilde{\psi}_\lambda = 1$. Let $h(n) := \psi_\lambda(n; k)$ for $n = 0, 1, \ldots$; this is well defined since ψ_λ is symmetric. The equality $\mathcal{L}\psi_\lambda = \lambda\psi_\lambda$ implies

$$-bh(n + 1) - h(n - 1) + (b + 1)h(n) - \tau h(n) = \lambda h(n), \qquad n = 1, 2, \ldots,$$

which is solved by $h(n) = A\alpha_1^n + B\alpha_2^n$, $n = 0, 1, \ldots$, where α_1, α_2 are the two solutions of the equation $b\alpha^2 - (b + 1 - \tau - \lambda)\alpha + 1 = 0$, i.e. the numbers

$$\frac{b + 1 - \tau - \lambda \pm \sqrt{(b + 1 - \tau - \lambda)^2 - 4b}}{2b}$$

such that $|\alpha_1| < 1/\sqrt{b}$ and $|\alpha_2| > 1/\sqrt{b}$. The choice of α_i is possible since $\alpha_1\alpha_2 = 1/b$ and $\alpha_1 + \alpha_2 = (b+1-\tau-\lambda)/b \notin \mathbb{R}$ for $\lambda \notin \mathbb{R}$, and hence $|\alpha_1| \neq |\alpha_2|$. We have to choose $B = 0$ in order that $\psi_\lambda \in \ell^2(V)$ as an easy calculation shows. Since

$$\Gamma_0\tilde{\psi}_\lambda = -\lambda\psi_\lambda(0; 0) - \psi_\lambda(1; 1) + \psi_\lambda(0; 0) = A(-\lambda - \alpha_1 + 1),$$

we have to take $A = 1/(-\lambda - \alpha_1 + 1)$. Hence $m(\lambda) = \Gamma_1\tilde{\psi}_\lambda = A$, which is (7.5) if we observe that $\alpha(\lambda) = \alpha_1$. □

Now let us consider for which τ there is a HELP inequality. We use again Proposition 2.3. Since $m(\lambda) \sim -1/\lambda$, the condition (2.8) is satisfied at infinity. The non-tangential limit from the upper half-plane $m_0 = \lim_{z\to 0} m(z)$ exists for every τ. It is non-real if and only if the expression under the square root in (7.6) is negative, which is exactly the case if $\tau \in \big(b+1-2\sqrt{b},\, b+1+2\sqrt{b}\big)$. Otherwise, $m_0 \neq 0$; hence there is a HELP inequality if and only if

$$\tau \in \big(b+1-2\sqrt{b},\, b+1+2\sqrt{b}\,\big).$$

For $b = 2$ and every $\tau \in (3-2\sqrt{2},\, 3+2\sqrt{2}\,) \approx (0.17157,\, 5.8284)$ we calculated the best constant K in the HELP inequality numerically using (7.3); see figure 2. There are no cases of equality.

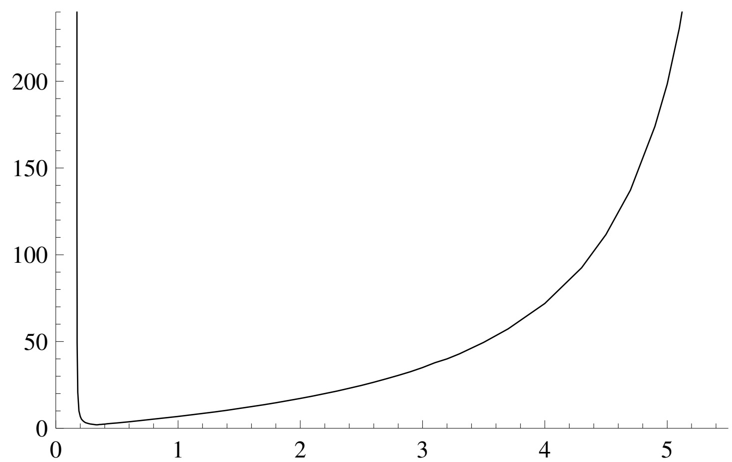

FIGURE 2. The best constants for the discrete problem in dependence of τ

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