arXiv:0911.2126v1 [math.DS] 11 Nov 2009

A TOPOLOGICAL SEPARATION CONDITION FOR FRACTAL ATTRACTORS

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Abstract. We consider finite systems of contractive homeomorphisms of a complete metric space, which are non-redundant on every level. In general this separation condition is weaker than the strong open set condition and is not equivalent to the weak separation property. We prove that this separation condition is equivalent to the strong Markov property (see definition below). We also show that the set of N-tuples of contractive homeomorphisms, which are non-redundant on every level, is a G_δ set in the topology of pointwise convergence of every component mapping with an additional requirement that the supremum of contraction coefficients of mappings be strictly less than one. We give several sufficient conditions for this separation property. For every fixed N-tuple of $d \times d$ invertible contraction matrices from a certain class, we obtain density results for N-tuples of fixed points which define N-tuples of mappings non-redundant on every level.

Key words: separation condition, Hausdorff dimension, similarity dimension, open set condition, Markov partition property, self-similar sets
AMS subject classification: Primary 28A80, Secondary 37C70

1. Notation and definitions.

Let X be a complete metric space and d be the distance in X. Recall that a mapping $w : X \to X$ is called a *contracting mapping* (or a *contraction*) if

$$\sigma = \sigma(w) = \sup_{x \neq y \in X} \frac{d(w(x), w(y))}{d(x, y)} < 1.$$

The number $\sigma(w)$ will be referred to as the contraction coefficient of the mapping w.

Let $N \in \mathbb{N}$, $w_1, \ldots, w_N : X \to X$ be contracting homeomorphisms of X onto itself and $A = A(w_1, \ldots, w_N) \subset X$ be the unique non-empty compact set such that

$$A = \bigcup_{i=1}^{N} w_i(A).$$

The set A is known as the invariant set or the attractor of the system $\{w_1, \ldots, w_N\}$. This way to define the attractor first appears in the paper by Hutchinson [5]. Denote $\Sigma = \{1, \ldots, N\}$ and for every vector $\mathbf{i} = \{i_1, \ldots, i_n\} \in \Sigma^n$, let

$$w_{\mathbf{i}} = w_{i_1, \ldots, i_n} = w_{i_1} \ldots w_{i_n} = w_{i_1} \circ \ldots \circ w_{i_n}.$$

Denote by $\mathcal{M}(X)$ the space of all contracting homeomorphisms $w : X \to X$ of the space X onto itself.

Definition 1. For every $n \in \mathbb{N}$, denote by $\mathcal{V}_n$ the set of all ordered N-tuples $(w_1, \ldots, w_N) \in (\mathcal{M}(X))^N$ such that for every $\mathbf{i} \in \Sigma^n$, there holds

$$w_{\mathbf{i}}(A) \not\subseteq \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}} w_{\mathbf{j}}(A).$$

[1] This author was partially supported by NSF grant DMS-0500641.

We say that a system $(w_1,\dots,w_N)\in(\mathcal{M}(X))^N$ satisfies the open set condition (OSC), if there is a non-empty open set $\mathcal{O}\subset X$ such that

1. $w_i(\mathcal{O})\cap w_j(\mathcal{O})=\emptyset$, $i\neq j$;
2. $w_i(\mathcal{O})\subset\mathcal{O}$, $i=1,\dots,N$.

We say that the system $(w_1,\dots,w_N)$ satisfies the strong open set condition (SOSC) if it satisfies the OSC with $\mathcal{O}\cap A\neq\emptyset$.

A mapping $w:X\to X$ is called a *contracting similitude* if there is a number $\sigma\in(0,1)$ such that

$$d(w(x),w(y))=\sigma d(x,y),\quad x,y\in X.$$

The attractor of a finite system of contracting similitudes in X is known as *self-similar set.* When $X=\mathbb{R}^d$, $d\in\mathbb{N}$, and $w_1,\dots,w_N:\mathbb{R}^d\to\mathbb{R}^d$ are contracting similitudes, the SOSC and the OSC are equivalent (cf. the result of Schief [8]). In general, the OSC does not imply the SOSC (cf. e.g. [8]). The above definition of self-similarity is different from the definition given for example in the book by Mattila [7], where additional restrictions on the size of the overlaps are required.

We say that a collection $(w_1,\dots,w_N)\in(\mathcal{M}(X))^N$ satisfies the Markov partition property (MPP) if there exists a subset $V\subset A$ open relatively to A such that

1. $\overline{V}=A$;
2. $w_i(V)\cap w_j(V)=\emptyset$, $i\neq j$.

Definition 2. We say that a system of mappings $(w_1,\dots,w_N)\in(\mathcal{M}(X))^N$ satisfies the strong Markov property (SMP) if for every $n\in\mathbb{N}$, there is an open set $\mathcal{O}_n\subset X$ such that

1. $\overline{\mathcal{O}_n\cap A}=A$;
2. $w_{\mathbf{i}}(\mathcal{O}_n)\cap w_{\mathbf{j}}(\mathcal{O}_n)=\emptyset$, for every $\mathbf{i}\neq\mathbf{j}\in\Sigma^n$.

It is not difficult to see that SMP implies MPP if we let $V=\mathcal{O}_1\cap A$, and that SOSC implies the SMP if we set $\mathcal{O}_n=\mathcal{O}$ for every $n\in\mathbb{N}$ (see Proposition 1). The SMP does not in general imply the SOSC (see Remark 2 below). Hence, MPP is also a weaker property than SOSC. We also remark here that SMP does not follow from the weak separation property of Lau and Ngai (see Example 2 on p. 76 in [6]).

Denote by Σ^∞ the set of all infinite sequences $(i_1,i_2,\dots)$, where $i_j\in\Sigma$, $j=1,2,\dots$. A sequence $(i_1,i_2,\dots)\in\Sigma^\infty$ is called an address of a point $x\in A$, if

$$x\in\bigcap_{n=1}^{\infty}w_{i_1,\dots,i_n}(A).$$

This is equivalent to the fact that for some point $a\in X$,

$$x=\lim_{n\to\infty}w_{i_1,\dots,i_n}(a).$$

It is not difficult to see that every point $x\in A$ has at least one address and every sequence from Σ^∞ is an address of some point from A. The set

$$\mathcal{T}=\bigcup_{i\neq j}w_i(A)\cap w_j(A)$$

is non-empty if and only if there are points in A, which have more than one address.

An interesting question is how generic are any of the above separation conditions in $\mathcal{M}(X)$. One of the results we present below is to show that the SMP condition is a countable intersection of open sets, i.e. a G_δ set. This result should be contrasted with that of Falconer [3] where he considered attractors associated with affine maps and obtained a formula for the Hausdorf dimension that was generic in the sense of Lebesgue measure (see also results by Mattila [7, Theorem 9.13] and Solomyak [9]).

In Section 2 we show that SMP holds if and only if $(w_1,\dots,w_N)$ is non-redundant on every level, i.e. $(w_1,\dots,w_N)\in\cap_{n=1}^{\infty}\mathcal{V}_n$. Furthermore we show that the set of all systems of mappings that satisfy SMP is a G_δ set in a suitable topology. The proofs of these results are presented in Sections 3 and 4. In Section 5 we find certain sufficient conditions for the SMP. In Section 6, we discuss the relation between the SMP for a self-similar set in $\mathbb{R}^d$ and the equality of its similarity and Hausdorff dimension. Section 7 deals with density results for the SMP in the case of self-affine sets in $\mathbb{R}^d$.

2. Main results

Theorem 1. *Let X be a complete metric space. The system $(w_1, \ldots, w_N)$ of contracting homeomorphisms of X onto X satisfies the SMP if and only if*

$$(w_1, \ldots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n.$$

Definition 3. We will call a sequence $\{w^m\}_{m\in\mathbb{N}}$ from $\mathcal{M}(X)$ strongly pointwise convergent to a mapping $w \in \mathcal{M}(X)$ and write $w^m \xrightarrow{s.p.} w$, $m \to \infty$, if

1. $\lim_{m\to\infty} w^m(x) = w(x)$ for every $x \in X$;
2. $\sup_{m\in\mathbb{N}} \sigma(w^m) < 1$.

If $\{w^m\}_{m\in\mathbb{N}} \subset \mathcal{M}(X)$ is a sequence of similitudes and $w \in M(X)$ is a similitude, then strong pointwise convergence is equivalent to the "usual" pointwise convergence.

We introduce a topology $\mathcal{B}_N$ on the space $(\mathcal{M}(X))^N$ by defining a subset $C \subset (\mathcal{M}(X))^N$ to be closed if for every sequence $\{(w_1^m, \ldots, w_N^m)\}_{m\in\mathbb{N}} \subset C$, such that $\{w_i^m\} \xrightarrow{s.p.} w_i \in \mathcal{M}(X)$, $i = 1, \ldots, N$, we have $(w_1, \ldots, w_N) \in C$. We agree here that $\emptyset$ is closed. It is not difficult to see, for example, that the space $(\mathcal{M}(X))^N$ with the topology $\mathcal{B}_N$ is a Hausdorff space.

Theorem 2. *Let $N \in \mathbb{N}$ and X be a complete metric space. The set of systems of mappings $(w_1, \ldots, w_N) \in (\mathcal{M}(X))^N$, which satisfy the SMP is a G-delta set in the topology $\mathcal{B}_N$.*

For a $d \times k$ matrix B, let

$$\|B\| = \max_{\mathbf{x}\in\mathbb{R}^k\setminus\{\mathbf{0}\}} \frac{|B\mathbf{x}|}{|\mathbf{x}|} \tag{1}$$

be its norm. We say that B is a *contraction matrix* if $\|B\| < 1$.

Let $X = \mathbb{R}^d$ and $B_1, \ldots, B_N$ be invertible $d \times d$ contraction matrices. Denote by $E_d(B_1, \ldots, B_N)$ the set of all ordered N-tuples $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ of points from $\mathbb{R}^d$ such that the system of mappings $w_i : \mathbb{R}^d \to \mathbb{R}^d$,

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\alpha}_i) + \boldsymbol{\alpha}_i, \quad i = 1, \ldots, N,$$

satisfies the SMP. We will sometimes consider the set $E_d(B_1, \ldots, B_N)$ as a subset of $\mathbb{R}^{dN}$.

Corollary 1. *For any collection $B_1, \ldots, B_N$ of invertible $d \times d$ contraction matrices, the set $E_d(B_1, \ldots, B_N)$ is a G-delta subset of $\mathbb{R}^{dN}$ (in the topology induced by the Euclidean distance).*

3. Proof of Theorem 1

We will start the proof with the following statement.

Lemma 1. *Let X be a complete metric space and $(w_1, \ldots, w_N) \in (\mathcal{M}(X))^N$. If $(w_1, \ldots, w_N) \in \cap_{n=1}^{\infty} \mathcal{V}_n$, then there is an open set $\mathcal{O} \subset X$ such that $\overline{\mathcal{O} \cap A} = A$ and $w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset$, $i \neq j$. In particular, the system $(w_1, \ldots, w_N)$ will satisfy the MPP.*

Proof. In order to prove Lemma 1 denote

$$K_i(A) = w_i(A) \setminus \bigcup_{\substack{j=1\\ j\neq i}}^{N} w_j(A), \quad i = 1, \ldots, N.$$

Let also

$$Z_i = w_i^{-1}(K_i(A)) \quad \text{and} \quad V = \bigcap_{i=1}^{N} Z_i.$$

For example, if $w_1(x) = x/2$ and $w_2(x) = x/2 + 1/2$, then $A = [0,1]$, $Z_1 = [0,1)$, $Z_2 = (0,1]$, and hence, $V = (0,1)$.

It is not difficult to see that $Z_i \subset A$, $i = 1,\ldots,N$. We show that $\overline{Z}_i = A$, $i = 1,\ldots,N$. Let $x \in A$ and let $U \subset X$ be any open set containing x. Denote by $B(a,\rho)$ the open ball in X centered at point a of radius $\rho > 0$. Since $w_i(U)$ is also open, there is $\epsilon > 0$ such that $B(w_i(x),\epsilon) \subset w_i(U)$. Let $r_i = \sigma(w_i) \in (0,1)$ be the contraction coefficient of w_i, $i = 1,\ldots,N$, and define

$$r_{\max} = \max_{i=1,\ldots,N} r_i.$$

Choose a number $m \in \mathbb{N}$ so that $(r_{\max})^m \cdot \mathrm{diam} A < \epsilon$. There exist indices $i_1,\ldots,i_m \in \Sigma$ such that $x \in w_{i_1,\ldots,i_m}(A)$. Then $w_i(x) \in w_{i,i_1,\ldots,i_m}(A)$ and

$$\mathrm{diam}\ w_{i,i_1,\ldots,i_m}(A) \le r_i \cdot r_{i_1} \cdot \ldots \cdot r_{i_m} \cdot \mathrm{diam} A \le (r_{\max})^{m+1} \cdot \mathrm{diam} A < \epsilon.$$

Hence,

$$w_{i,i_1,\ldots,i_m}(A) \subset B(w_i(x),\epsilon) \subset w_i(U). \tag{2}$$

Since $(w_1,\ldots,w_N) \in \mathcal{V}_{m+1}$, we have

$$w_{i,i_1,\ldots,i_m}(A) \not\subseteq \bigcup_{\substack{j_1,\ldots,j_{m+1}\in\Sigma \\ j_1 \ne i}} w_{j_1,\ldots,j_{m+1}}(A) = \bigcup_{\substack{j=1 \\ j\ne i}}^{N} w_j(A).$$

Hence, there is $z \in A$ such that $w_{i,i_1,\ldots,i_m}(z)$ does not belong to $\cup_{j:j\ne i} w_j(A)$. Let $t = w_{i_1,\ldots,i_m}(z)$. Since $w_i(t)$ does not belong to any $w_j(A)$ with $j \ne i$, we must have $w_i(t) \in w_i(A)$, that is, $w_i(t) \in K_i(A)$. Hence, $t \in Z_i$. On the other hand, since $w_i(t) \in w_{i,i_1,\ldots,i_m}(A)$, in view of (2), we have $w_i(t) \in w_i(U)$, that is $t \in U$, which implies that $\overline{Z}_i = A$, $i = 1,\ldots,N$.

We next show that $\overline{V} = A$. Indeed, since each Z_i is open relative to A, there are open sets $W_i \subset X$ such that $Z_i = W_i \cap A$, $i = 1,\ldots,N$. Let y be any element in A and U be any open neighborhood of y. Since $\overline{Z}_1 = A$, there is $z_1 \in Z_1 \cap U = A \cap W_1 \cap U$. Since $\overline{Z}_2 = A$, there is $z_2 \in Z_2$ in the open neighborhood $W_1 \cap U$ of the point $z_1 \in A$, that is $z_2 \in A \cap U \cap W_1 \cap W_2$. Then by induction, there will be an element $z_N \in A \cap U \cap W_1 \cap \ldots \cap W_N = V \cap U$, and the required relation follows.

Note that for every $i \ne j$, there holds

$$w_i(V) \cap w_j(V) \subset w_i(Z_i) \cap w_j(Z_j) = K_i(A) \cap K_j(A) \subset$$
$$\subset (w_i(A) \setminus w_j(A)) \cap w_j(A) = \emptyset.$$

Taking also into account the fact that V is relatively open with respect to A as an intersection of a finite collection of subsets of A, which are open relative to A, we conclude that the system $(w_1,\ldots,w_N)$ possesses the MPP.

For every $x \in V$, denote

$$\rho(x) = \min_{i=1,\ldots,N} \mathrm{dist}\left(w_i(x), \bigcup_{\substack{j=1 \\ j\ne i}}^{N} w_j(A) \right).$$

In view of the relations

$$w_i(V) \subset w_i(Z_i) = K_i(A), \quad i = 1,\ldots,N,$$

point $w_i(x)$, $x \in V$, does not belong to the closed set $\bigcup\limits_{j:j\neq i} w_j(A)$. Hence, $\rho(x) > 0$, $x \in V$, and the set

$$\mathcal{O} = \bigcup_{x\in V} B\left(x, \rho(x)/2\right)$$

is open. Since $\overline{V} = A$ and $V \subset \mathcal{O}\cap A \subset A$, we have $\overline{\mathcal{O}\cap A} = A$. To show that $w_i(\mathcal{O})\cap w_j(\mathcal{O}) = \emptyset$, $i \neq j$, assume to the contrary that there exist indices $i \neq j$ such that $w_i(\mathcal{O}) \cap w_j(\mathcal{O})$ contains some element y. Then $y = w_i(p) = w_j(q)$ for some $p, q \in \mathcal{O}$. There are points $c, b \in V$ such that $d(c, p) < \rho(c)/2$ and $d(b, q) < \rho(b)/2$. Note that

$$d(y, w_i(c)) = d(w_i(p), w_i(c)) \leq r_i \cdot d(p, c) < r_i \cdot \rho(c)/2 \tag{3}$$

and

$$d(y, w_j(b)) = d(w_j(q), w_j(b)) \leq r_j \cdot d(q, b) < r_j \cdot \rho(b)/2. \tag{4}$$

There also hold the following relations

$$\rho(c) \leq \operatorname{dist}\left(w_i(c), \bigcup_{\substack{k=1\\k\neq i}}^{N} w_k(A)\right) \leq \operatorname{dist}(w_i(c), w_j(A)) \leq d(w_i(c), w_j(b)) \tag{5}$$

and

$$\rho(b) \leq \operatorname{dist}\left(w_j(b), \bigcup_{\substack{k=1\\k\neq j}}^{N} w_k(A)\right) \leq \operatorname{dist}(w_j(b), w_i(A)) \leq d(w_j(b), w_i(c)). \tag{6}$$

Then, in view of relations (3)–(6), we obtain

$$\rho(c) + \rho(b) \leq 2d(w_i(c), w_j(b)) \leq 2(d(w_i(c), y) + d(y, w_j(b))) <$$
$$< r_i \cdot \rho(c) + r_j \cdot \rho(b) < \rho(c) + \rho(b),$$

which is impossible. Hence, $w_i(\mathcal{O})$ and $w_j(\mathcal{O})$ are disjoint, which completes the proof of Lemma 1. □

To prove sufficiency in Theorem 1, assume that

$$(w_1, \ldots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n \subset (\mathcal{M}(X))^N.$$

Then for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$, we have $(w_1, \ldots, w_N) \in \mathcal{V}_{nm} \subset (\mathcal{M}(X))^N$, which implies that the system $\{w_{\mathbf{i}}\}_{\mathbf{i}\in\Sigma^m}$ belongs to the set $\mathcal{V}_n \subset (\mathcal{M}(X))^{N^m}$. Hence, $\{w_{\mathbf{i}}\}_{\mathbf{i}\in\Sigma^m} \in \cap_{n=1}^{\infty}\mathcal{V}_n \subset (\mathcal{M}(X))^{N^m}$. By Lemma 1, there is an open set $\mathcal{O}_m \subset X$ such that $\overline{\mathcal{O}_m \cap A} = A$ and $w_{\mathbf{i}}(\mathcal{O}_m) \cap w_{\mathbf{j}}(\mathcal{O}_m) = \emptyset$ for every $\mathbf{i} \neq \mathbf{j} \in \Sigma^m$, $m \in \mathbb{N}$. Hence, the system $(w_1, \ldots, w_N)$ satisfies the SMP.

The proof of the necessity in Theorem 1 is preceded by the following proposition.

Lemma 2. *Let mappings $w_1, \ldots, w_N \in \mathcal{M}(X)$ be such that there is a non-empty open set $\mathcal{O} \subset X$ with the property*

$$w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset, \quad i \neq j.$$

Then for every $i = 1, \ldots, N$,

$$w_i(\mathcal{O}) \cap \bigcup_{j:j\neq i} w_j(\overline{\mathcal{O}}) = \emptyset.$$

Proof. Assume the contrary. Then for some $j_0 \neq i$, there $x \in w_i(\mathcal{O}) \cap w_{j_0}(\overline{\mathcal{O}})$. Let $z \in \overline{\mathcal{O}}$ be such that $x = w_{j_0}(z)$. There is a sequence $\{z_m\}_{m\in\mathbb{N}} \subset \mathcal{O}$ such that $z = \lim\limits_{m\to\infty} z_m$ and hence, $x = \lim\limits_{m\to\infty} w_{j_0}(z_m)$. Since $w_i(\mathcal{O})$ is an open neighborhood of x, we have $w_{j_0}(z_m) \in w_i(\mathcal{O})$ for every m sufficiently large, and hence, $w_i(\mathcal{O}) \cap w_{j_0}(\mathcal{O}) \neq \emptyset$, which contradicts the assumption, thus Lemma 2 is proved. □

Completion of the proof of Theorem 1. Assume that system $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfies the SMP. Let $k \in \mathbb{N}$ be arbitrary. Then there is an open set $\mathcal{O}_k \subset X$ such that $\overline{\mathcal{O}_k \cap A} = A$ and $w_{\mathbf{i}}(\mathcal{O}_k) \cap w_{\mathbf{j}}(\mathcal{O}_k) = \emptyset$ for every $\mathbf{i} \neq \mathbf{j} \in \Sigma^k$. We show that for every $\mathbf{i} \in \Sigma^k$,

$$w_{\mathbf{i}}(A) = \overline{w_{\mathbf{i}}(\mathcal{O}_k) \cap A}. \tag{7}$$

Taking into account Lemma 2 and the fact that $A = \overline{\mathcal{O}_k \cap A} \subset \overline{\mathcal{O}_k}$, we obtain

$$w_{\mathbf{i}}(\mathcal{O}_k) \cap A = (w_{\mathbf{i}}(\mathcal{O}_k) \cap w_{\mathbf{i}}(A)) \cup \left(w_{\mathbf{i}}(\mathcal{O}_k) \cap \bigcup_{\mathbf{j} \in \Sigma^k,\ \mathbf{j} \neq \mathbf{i}} w_{\mathbf{j}}(A) \right) \subset$$

$$\subset w_{\mathbf{i}}(\mathcal{O}_k \cap A) \cup \left(w_{\mathbf{i}}(\mathcal{O}_k) \cap \bigcup_{\mathbf{j} \in \Sigma^k,\ \mathbf{j} \neq \mathbf{i}} w_{\mathbf{j}}(\overline{\mathcal{O}_k}) \right) = w_{\mathbf{i}}(\mathcal{O}_k \cap A).$$

Then

$$\overline{w_{\mathbf{i}}(\mathcal{O}_k) \cap A} \subset \overline{w_{\mathbf{i}}(\mathcal{O}_k \cap A)} = w_{\mathbf{i}}(\overline{\mathcal{O}_k \cap A}) = w_{\mathbf{i}}(A).$$

On the other hand,

$$w_{\mathbf{i}}(A) = w_{\mathbf{i}}(\overline{\mathcal{O}_k \cap A}) = \overline{w_{\mathbf{i}}(\mathcal{O}_k \cap A)} = \overline{w_{\mathbf{i}}(\mathcal{O}_k) \cap w_{\mathbf{i}}(A)} \subset \overline{w_{\mathbf{i}}(\mathcal{O}_k) \cap A},$$

and (7) follows.

Assume that $(w_1, \dots, w_N)$ does not belong to $\cap_{n=1}^{\infty} \mathcal{V}_n$. Then there is $n \in \mathbb{N}$ and $\mathbf{i}_n \in \Sigma^n$ such that

$$w_{\mathbf{i}_n}(A) \subset \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}_n} w_{\mathbf{j}}(A).$$

Then, taking into account (7) we obtain

$$w_{\mathbf{i}_n}(\mathcal{O}_n) \cap A \subset \overline{w_{\mathbf{i}_n}(\mathcal{O}_n) \cap A} = w_{\mathbf{i}_n}(A) \subset \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}_n} w_{\mathbf{j}}(A) =$$

$$= \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}_n} \overline{w_{\mathbf{j}}(\mathcal{O}_n) \cap A} \subset \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}_n} w_{\mathbf{j}}(\overline{\mathcal{O}_n}).$$

Since $\overline{w_{\mathbf{i}_n}(\mathcal{O}_n) \cap A} = w_{\mathbf{i}_n}(A) \neq \emptyset$, there is a point $x \in w_{\mathbf{i}_n}(\mathcal{O}_n) \cap A \subset w_{\mathbf{i}_n}(\mathcal{O}_n)$. Then $x \in \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}_n} w_{\mathbf{j}}(\overline{\mathcal{O}_n})$, Hence,

$$w_{\mathbf{i}_n}(\mathcal{O}_n) \cap \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}_n} w_{\mathbf{j}}(\overline{\mathcal{O}_n}) \neq \emptyset,$$

which contradicts to Lemma 2. Theorem 1 is proved. □

4. Proof of Theorem 2.

The proof of some statements in this section is standard, but we include it for the convenience of the reader.

Lemma 3. *If a sequence $\{w^m\}_{m \in \mathbb{N}} \subset \mathcal{M}(X)$ converges strongly pointwise to a mapping $w \in \mathcal{M}(X)$, then the sequence of fixed points of mappings w^m converges to the fixed point of w.*

Proof. Let $x^m \in X$ be the fixed point of w^m, $m \in \mathbb{N}$, and $x \in X$ be the fixed point of w. Denote also

$$\sigma = \sup_{m \in \mathbb{N}} \sigma(w^m).$$

Then

$$d(x^m, x) \leq d(x^m, w^m(x)) + d(w^m(x), x) =$$
$$= d(w^m(x^m), w^m(x)) + d(w^m(x), w(x)) \leq \sigma d(x^m, x) + d(w^m(x), w(x)).$$

Hence,

$$d(x^m, x) \le \frac{1}{1-\sigma} d(w^m(x), w(x)),$$

and we have

$$\lim_{m\to\infty} d(x^m, x) = 0.$$

Lemma 3 is proved. $\square$

Lemma 4. *Let A be the attractor of a system of mappings $w_1, \dots, w_N \in \mathcal{M}(X)$ with contraction coefficients not exceeding a given number $\sigma \in (0,1)$. Let also $B[a,r]$ be a closed ball containing the fixed point of every mapping $w_1, \dots, w_N$. Then $A \subset B[a,R]$, where $R = \frac{1+\sigma}{1-\sigma} r$.*

Proof. Assume the contrary. Denote by $y_1, \dots, y_N$ the fixed points of mappings $w_1, \dots, w_N$ respectively. Let z be a point in A furthest from a. Then we must have $d(z,a) > R$. Let $1 \le i \le N$ be such index that $z = w_i(z_1)$ for some $z_1 \in A$. Then

$$d(z_1, a) \ge d(z_1, y_i) - d(y_i, a) \ge \frac{1}{\sigma} d(w_i(z_1), w_i(y_i)) - r =$$
$$= \frac{1}{\sigma} d(z, y_i) - r \ge \frac{1}{\sigma} d(z, a) - \frac{1}{\sigma} d(y_i, a) - r \ge \frac{1}{\sigma} d(z,a) - \frac{r}{\sigma} - r.$$

Hence,

$$\frac{d(z_1,a)}{d(z,a)} \ge \frac{1}{\sigma} - \frac{(1+\sigma)r}{\sigma d(z,a)} > \frac{1}{\sigma} - \frac{(1+\sigma)r}{\sigma R} = 1,$$

which contradicts to the fact that z is a point in A furthest from a. $\square$

Lemma 5. *Let $\{w_1^m\}_{m\in\mathbb{N}}, \dots, \{w_n^m\}_{m\in\mathbb{N}}$ be sequences of mappings from $\mathcal{M}(X)$ such that $w_i^m \xrightarrow{s.p.} w_i \in \mathcal{M}(X)$, $i = 1, \dots, n$. Then $w_1^m \circ \dots \circ w_n^m \xrightarrow{s.p.} w_1 \circ \dots \circ w_n$, $m \to \infty$.*

Proof. We will use induction. For $n = 1$, the assertion of the lemma is trivial. Assume that the assertion is true for a given value of $n \ge 1$ and show that it holds for any $n+1$ sequences satisfying the assumptions of the lemma. For every $x \in X$, we will have

$$\begin{aligned} d(w_1^m w_2^m \dots w_{n+1}^m(x), w_1 w_2 \dots w_{n+1}(x)) &\le d(w_1^m(w_2^m \dots w_{n+1}^m(x)), w_1^m(w_2 \dots w_{n+1}(x))) \\ &\quad + d(w_1^m(w_2 \dots w_{n+1}(x)), w_1(w_2 \dots w_{n+1}(x))) \\ &\le d(w_2^m \dots w_{n+1}^m(x), w_2 \dots w_{n+1}(x)) \\ &\quad + d(w_1^m(w_2 \dots w_{n+1}(x)), w_1(w_2 \dots w_{n+1}(x))). \end{aligned}$$

By the assumption of the induction, both distances in the last line vanish as $m \to \infty$ and we have

$$\lim_{m\to\infty} w_1^m w_2^m \dots w_{n+1}^m(x) = w_1 w_2 \dots w_{n+1}(x), \quad x \in X.$$

Since

$$\sigma = \max_{i=1,\dots,n+1} \sup_{m\in\mathbb{N}} \sigma(w_i^m) < 1,$$

we have

$$\sigma(w_1^m w_2^m \dots w_{n+1}^m) \le \sigma^{n+1} < 1, \quad m \in \mathbb{N},$$

which implies strong pointwise convergence. Lemma 5 is proved. $\square$

Given a system $W = (w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ and an address $\mathbf{i} \in \Sigma^\infty$, let $\Pi_{\mathbf{i}}(W)$ be the point in the attractor of W with address $\mathbf{i}$.

Lemma 6. *Let $W_m = (w_1^m, \dots, w_N^m)$, $m \in \mathbb{N}$, be a sequence from $(\mathcal{M}(X))^N$ such that for every $i = 1, \dots, N$, the sequence $\{w_i^m\}_{m\in\mathbb{N}}$ converges strongly pointwise to some mapping $w_i \in \mathcal{M}(X)$. Then for every address $\mathbf{i} \in \Sigma^\infty$,*

$$\lim_{m\to\infty} \Pi_{\mathbf{i}}(W_m) = \Pi_{\mathbf{i}}(W),$$

where $W = (w_1, \dots, w_N)$.

Proof. Given an arbitrary address $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma^\infty$, denote by $x_{i_1 \dots i_n}$ the fixed point of the mapping $w_{i_1 \dots i_n}$. Let also

$$\delta = \max_{i=1,\dots,N} \sup_{m\in\mathbb{N}} \sigma(w_i^m).$$

Let $B(a,r)$ be a ball containing the attractor A of the system W and $R = \frac{1+\delta}{1-\delta} r$.

Choose an arbitrary $\epsilon > 0$ and let $n \in \mathbb{N}$ be large enough so that

$$d(\Pi_{\mathbf{i}}(W), x_{i_1 \dots i_n}) < \epsilon \quad \text{and} \quad R\delta^n < \epsilon. \tag{8}$$

Denote by $x^m_{\alpha_1\dots\alpha_n}$ the fixed point of the mapping $w^m_{\alpha_1} \circ \dots \circ w^m_{\alpha_n}$, $\alpha_1, \dots, \alpha_n \in \Sigma$. By Lemma 5, we have $w^m_{\alpha_1} \circ \dots \circ w^m_{\alpha_n} \xrightarrow{s.p.} w_{\alpha_1\dots\alpha_n}$, $m \to \infty$. Then by Lemma 3, we have $\lim_{m\to\infty} x^m_{\alpha_1\dots\alpha_n} = x_{\alpha_1\dots\alpha_n}$, for every $\alpha_1, \dots, \alpha_n \in \Sigma$. Since $x_{\alpha_1\dots\alpha_n} \in A \subset B(a,r)$, there is a number $m_n \in \mathbb{N}$ such that for every $m > m_n$ and $\alpha_1, \dots, \alpha_n \in \Sigma$, we have $x^m_{\alpha_1\dots\alpha_n} \in B(a,r)$. For every $m > m_n$, we obtain

$$\begin{aligned} d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) &\le d(\Pi_{\mathbf{i}}(W), x_{i_1\dots i_n}) + d(w_{i_1\dots i_n}(x_{i_1\dots i_n}), w^m_{i_1} \dots w^m_{i_n}(x_{i_1\dots i_n})) \\ &\quad + d(w^m_{i_1} \dots w^m_{i_n}(x_{i_1\dots i_n}), \Pi_{\mathbf{i}}(W_m)) \\ &\le \epsilon + d(w_{i_1\dots i_n}(x_{i_1\dots i_n}), w^m_{i_1} \dots w^m_{i_n}(x_{i_1\dots i_n})) \\ &\quad + d(w^m_{i_1} \dots w^m_{i_n}(x_{i_1\dots i_n}), w^m_{i_1} \dots w^m_{i_n}(z_{\mathbf{i},m})), \end{aligned}$$

where $z_{\mathbf{i},m}$ is some point in the attractor A_m of the system W_m. Taking into account Lemma 5, we will have

$$d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) \le \epsilon + o(1) + \delta^n d(x_{i_1\dots i_n}, z_{\mathbf{i},m}).$$

For every $i = 1, \dots, N$, the fixed point x_i^m of w_i^m is also the fixed point of the n-th power of w_i^m, and as it was noted above, $x_i^m \in B(a,r)$, $m > m_n$. By Lemma 4, we have $z_{\mathbf{i},m} \in A_m \subset B[a,R]$. Since $x_{i_1\dots i_n} \in A \subset B(a,r) \subset B[a,R]$, in view of (8), we obtain

$$d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) \le \epsilon + o(1) + 2R\delta^n \le 3\epsilon + o(1).$$

Hence,

$$\limsup_{m\to\infty} d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) \le 3\epsilon.$$

In view of arbitrariness of ϵ, we have

$$\lim_{m\to\infty} d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) = 0,$$

and the assertion of Lemma 6 follows. □

Let

$$\mathcal{F} = \bigcup_{n\in\mathbb{N}} \Sigma^n.$$

Lemma 7. *Let $N, n \in \mathbb{N}$. If a sequence $\{(w_1^m, \dots, w_N^m)\}_{m\in\mathbb{N}} \subset (\mathcal{M}(X))^N \setminus \mathcal{V}_n$ converges strongly pointwise in every component to a system $W = (w_1, \dots, w_N) \in (\mathcal{M}(X))^N$, then we have $W \in (\mathcal{M}(X))^N \setminus \mathcal{V}_n$.*

From Lemma 7 we obtain the following statement, which in view of Theorem 1, implies the assertion of Theorem 2.

Corollary 2. *For every positive integers n and N, the set $\mathcal{V}_n$ is open in the topology $\mathcal{B}_N$, and hence, $\cap_{n=1}^\infty \mathcal{V}_n$ is a G-delta set.*

Proof of Lemma 7. Let $W_m = (w_1^m, \dots, w_N^m) \in (\mathcal{M}(X))^N \setminus \mathcal{V}_n$ be a sequence, where every component is convergent strongly pointwise to the corresponding component of the system $W = (w_1, \dots, w_N) \in (\mathcal{M}(X))^N$. Denote $w^m_{\mathbf{k}} = w^m_{k_1} \circ \dots \circ w^m_{k_p}$, $\mathbf{k} = (k_1, \dots, k_p) \in \mathcal{F}$. For every $m \in \mathbb{N}$, there is a vector $\mathbf{i}_m \in \Sigma^n$ such that

$$w^m_{\mathbf{i}_m}(A_m) \subset \bigcup_{\mathbf{k}\in\Sigma^n,\ \mathbf{k}\ne\mathbf{i}_m} w^m_{\mathbf{k}}(A_m),$$

where A_m is the attractor of the system W_m. There is an index $\mathbf{i} \in \Sigma^n$ and infinite subsequence $\mathcal{N} \subset \mathbb{N}$ such that

$$w_{\mathbf{i}}^m(A_m) \subset \bigcup_{\mathbf{k}\in\Sigma^n,\ \mathbf{k}\neq\mathbf{i}} w_{\mathbf{k}}^m(A_m), \quad m \in \mathcal{N}. \tag{9}$$

Let A be the attractor of the system W and $x \in w_{\mathbf{i}}(A)$ be an arbitrary point. Then $x = \Pi_{\mathbf{i}\beta}(W)$ for some $\beta \in \Sigma^\infty$. In view of (9), for every $m \in \mathcal{N}$, there holds

$$\Pi_{\mathbf{i}\beta}(W_m) \in w_{\mathbf{i}}^m(A_m) \cap w_{\mathbf{j}_m}^m(A_m)$$

for some $\mathbf{j}_m \in \Sigma^n$ distinct from $\mathbf{i}$. There are index $\mathbf{j} \in \Sigma^n$, $\mathbf{j} \neq \mathbf{i}$, and infinite subsequence $\mathcal{N}' \subset \mathcal{N}$ such that

$$\Pi_{\mathbf{i}\beta}(W_m) \in w_{\mathbf{i}}^m(A_m) \cap w_{\mathbf{j}}^m(A_m), \quad m \in \mathcal{N}'.$$

Hence, there is a sequence $\gamma_m = (\gamma_1^m, \gamma_2^m, \ldots) \in \Sigma^\infty$ such that

$$\Pi_{\mathbf{i}\beta}(W_m) = \Pi_{\mathbf{j}\gamma_m}(W_m), \quad m \in \mathcal{N}'. \tag{10}$$

One can find an infinite subsequence $\mathcal{N}_1 \subset \mathcal{N}'$ and an index $\gamma_1 \in \Sigma$ such that $\gamma_1^m = \gamma_1$, $m \in \mathcal{N}_1$. One can find an infinite subsequence $\mathcal{N}_2 \subset \mathcal{N}_1$ and an index $\gamma_2 \in \Sigma$ such that $\gamma_1^m = \gamma_1$ and $\gamma_2^m = \gamma_2$, $m \in \mathcal{N}_2$. Continuing this process indefinitely, we obtain an address $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Sigma^\infty$ and a sequence of embedded infinite sets $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \ldots \supset \mathcal{N}_k \supset \ldots$ such that $\gamma_k^m = \gamma_k$, $m \in \mathcal{N}_k$, $k \in \mathbb{N}$.

Let as above, $B(a, r)$ be an open ball containing A. Since A contains the fixed points of mappings $w_1, \ldots, w_N$, by Lemma 3, there is $m_0 \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $m > m_0$, the fixed points of mappings $w_1^m, \ldots, w_N^m$ will be in $B(a, r)$. Then, by Lemma 4, $A_m \subset B[a, R]$, where $R = \frac{1+\delta}{1-\delta} r$ and $\delta = \max\limits_{i=1,N} \sup\limits_{m\in\mathbb{N}} \sigma(w_i^m) \in (0, 1)$. For every $k \in \mathbb{N}$ and $m \in \mathcal{N}_k$, $m > m_0$, there are points b and c in A_m such that

$$\begin{aligned} d(\Pi_{\mathbf{j}\gamma_m}(W_m), \Pi_{\mathbf{j}\gamma}(W_m)) &= d(w_{\mathbf{j}}^m w_{\gamma_1^m\ldots\gamma_k^m}^m(b), w_{\mathbf{j}}^m w_{\gamma_1\ldots\gamma_k}^m(c)) \\ &= d(w_{\mathbf{j}}^m w_{\gamma_1\ldots\gamma_k}^m(b), w_{\mathbf{j}}^m w_{\gamma_1\ldots\gamma_k}^m(c)) \leq \sigma(w_{\mathbf{j}}^m) \cdot \sigma(w_{\gamma_1}^m) \cdot \ldots \cdot \sigma(w_{\gamma_k}^m) d(b, c) \\ &\leq \delta^{n+k} \mathrm{diam} A_m \leq 2R\delta^{n+k}. \end{aligned}$$

By Lemma 6 and relation (10), for every $m \in \mathcal{N}_k$, $m > m_0$, we obtain

$$\begin{aligned} d(x, \Pi_{\mathbf{j}\gamma}(W)) &\leq d(\Pi_{\mathbf{i}\beta}(W), \Pi_{\mathbf{i}\beta}(W_m)) + d(\Pi_{\mathbf{j}\gamma_m}(W_m), \Pi_{\mathbf{j}\gamma}(W_m)) \\ &\quad + d(\Pi_{\mathbf{j}\gamma}(W_m), \Pi_{\mathbf{j}\gamma}(W)) \leq 2R\delta^{n+k} + o(1). \end{aligned}$$

Hence, letting $m \to \infty$ along the sequence $\mathcal{N}_k$, we will have

$$d(x, \Pi_{\mathbf{j}\gamma}(W)) \leq 2R\delta^{n+k}, \quad k \in \mathbb{N}.$$

Letting now $k \to \infty$ we get that $d(x, \Pi_{\mathbf{j}\gamma}(W)) = 0$, which implies that

$$x = \Pi_{\mathbf{j}\gamma}(W) \in w_{\mathbf{j}}(A) \subset \bigcup_{\mathbf{k}\in\Sigma^n,\ \mathbf{k}\neq\mathbf{i}} w_{\mathbf{k}}(A),$$

where vector $\mathbf{j}$ was chosen to be distinct from $\mathbf{i}$. Since $x \in w_{\mathbf{i}}(A)$ was chosen arbitrarily, we obtain that

$$w_{\mathbf{i}}(A) \subset \bigcup_{\mathbf{k}\in\Sigma^n,\ \mathbf{k}\neq\mathbf{i}} w_{\mathbf{k}}(A),$$

and hence, $W \in (\mathcal{M}(X))^N \setminus \mathcal{V}_n$. Lemma 7 is proved, which completes the proof of Theorem 2.□

Proof of Corollary 1. Let U_n, $n \in \mathbb{N}$, be the set of ordered N-tuples $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N) \in (\mathbb{R}^d)^N$ such that the system of mappings

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\alpha}_i) + \boldsymbol{\alpha}_i, \quad i = 1, \ldots, N, \tag{11}$$

belongs to $\mathcal{V}_n$. By Theorem 1, we have

$$E_d(B_1,\dots,B_N)=\bigcap_{n=1}^{\infty}U_n.$$

It remains to show that for every $n\in\mathbb{N}$, the set U_n is open. Assume the contrary and let $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_N)\in U_n$ be not an interior point of U_n. Then there is a sequence $\{\boldsymbol{\beta}_m\}_{m=1}^{\infty}\subset(\mathbb{R}^d)^N\setminus U_n$ such that $\boldsymbol{\alpha}=\lim_{m\to\infty}\boldsymbol{\beta}_m$. Let $\boldsymbol{\beta}_m=(\boldsymbol{\beta}_1^m,\dots,\boldsymbol{\beta}_N^m)$, $m\in\mathbb{N}$, where $\boldsymbol{\beta}_i^m\in\mathbb{R}^d$, $i=1,\dots,N$. Then for every $m\in\mathbb{N}$, the system of contracting mappings

$$w_i^m(\mathbf{x})=B_i(\mathbf{x}-\boldsymbol{\beta}_i^m)+\boldsymbol{\beta}_i^m,\quad i=1,\dots,N,$$

does not belong to $\mathcal{V}_n$. Since for every $i=1,\dots,N$ and $\mathbf{x}\in\mathbb{R}^d$,

$$\lim_{m\to\infty}w_i^m(\mathbf{x})=w_i(\mathbf{x}),$$

where w_i is defined as in (11), and

$$\max_{i=1,\dots,N}\|B_i\|<1,$$

we have a strong pointwise convergence of the sequence $\{w_i^m\}_{m=1}^{\infty}$ to w_i, $i=1,\dots,N$. By Lemma 7, we have that $(w_1,\dots,w_N)$ does not belong to $\mathcal{V}_n$, i.e. $\boldsymbol{\alpha}\notin U_n$. This contradiction shows that U_n is an open set for every n and the assertion of Corollary 1 follows. □

5. Sufficient conditions for the SMP

Proposition 1. *Let X be a complete metric space and $(w_1,\dots,w_N)$ be a collection of contracting homeomorphisms of X onto X. If $(w_1,\dots,w_N)$ satisfies the SOSC, then*

$$(w_1,\dots,w_N)\in\bigcap_{n=1}^{\infty}\mathcal{V}_n,$$

or, equivalently, $(w_1,\dots,w_N)$ satisfies the SMP.

The converse is not true (see Remark 2 below).

Proof. Let $\mathcal{O}\subset X$ be the open set from the definition of the SOSC. Show that $\overline{A\cap\mathcal{O}}=A$. Indeed, if $x\in A$ and $\epsilon>0$ are arbitrary, for some $m\in\mathbb{N}$ sufficiently large and $\mathbf{i}\in\Sigma^m$ we have $x\in w_{\mathbf{i}}(A)\subset B(x,\epsilon)$. Since $w_{\mathbf{i}}(\mathcal{O}\cap A)\neq\emptyset$, $w_{\mathbf{i}}(\mathcal{O}\cap A)\subset B(x,\epsilon)$, and

$$w_{\mathbf{i}}(\mathcal{O}\cap A)=w_{\mathbf{i}}(\mathcal{O})\cap w_{\mathbf{i}}(A)\subset\mathcal{O}\cap A,$$

we have $(\mathcal{O}\cap A)\cap B(x,\epsilon)\neq\emptyset$. Hence, $A\subset\overline{\mathcal{O}\cap A}$. Since the opposite inclusion is trivial, we have $\overline{\mathcal{O}\cap A}=A$.

If for every $n\in\mathbb{N}$, we let $\mathcal{O}_n=\mathcal{O}$, then condition 1 in the definition of the SMP holds. For every $\mathbf{i}\neq\mathbf{j}\in\Sigma^n$, if $1\le k\le n$ is the smallest index such that $i_k\neq j_k$, then

$$\begin{aligned}w_{\mathbf{i}}(\mathcal{O}_n)\cap w_{\mathbf{j}}(\mathcal{O}_n)&=w_{\mathbf{i}}(\mathcal{O})\cap w_{\mathbf{j}}(\mathcal{O})\subset w_{i_1\dots i_{k-1}}(w_{i_k}(\mathcal{O}))\cap w_{i_1\dots i_{k-1}}(w_{j_k}(\mathcal{O}))\\&=w_{i_1\dots i_{k-1}}(w_{i_k}(\mathcal{O})\cap w_{j_k}(\mathcal{O}))=\emptyset.\end{aligned}$$

Thus, the system $(w_1,\dots,w_N)$ satisfies the SMP and, by Theorem 1, we have $(w_1,\dots,w_N)\in\cap_{n=1}^{\infty}\mathcal{V}_n$. Proposition 1 is proved. □

Recall that

$$\mathcal{T}=\bigcup_{i\neq j}w_i(A)\cap w_j(A)$$

and denote

$$\mathcal{D}=\mathcal{D}(w_1,\dots,w_N)=A\setminus\bigcup_{\mathbf{i}\in\mathcal{F}}w_{\mathbf{i}}^{-1}(\mathcal{T}).$$

The following result holds.

Proposition 2. *Let X be a complete metric space and $w_1, \dots, w_N$ be contracting homeomorphisms of the space X onto X. If $\mathcal{D} = \mathcal{D}(w_1, \dots, w_N) \neq \emptyset$, then*

1. $w_i(\mathcal{D}) \subset \mathcal{D}$, $i = 1, \dots, N$;
2. $w_i(\mathcal{D}) \cap w_j(\mathcal{D}) = \emptyset$, $i \neq j$;
3. $\overline{\mathcal{D}} = A$;
4. $(w_1, \dots, w_N) \in \bigcap\limits_{n=1}^{\infty} \mathcal{V}_n$.

Proof. To prove the first statement assume the contrary, i.e. for some $1 \le k \le N$, there is $y \in w_k(\mathcal{D}) \setminus \mathcal{D}$. Then there is a point $x \in \mathcal{D}$ such that $y = w_k(x)$. On the other hand, since y is not in $\mathcal{D}$, there is a vector $\mathbf{p} = (p_1, \dots, p_s) \in \mathcal{F}$ such that $w_{\mathbf{p}}(y) \in \mathcal{T}$. Hence, $w_{p_1,\dots,p_s,k}(x) \in \mathcal{T}$, which contradicts to the fact that $x \in \mathcal{D}$.

To prove the second statement, assume again the contrary, i.e. for some indexes $1 \le i \neq j \le N$, there is a point $x \in w_i(\mathcal{D}) \cap w_j(\mathcal{D})$. Then $x = w_i(t)$, $t \in \mathcal{D}$. Since

$$w_i(t) \in w_i(\mathcal{D}) \cap w_j(\mathcal{D}) \subset w_i(A) \cap w_j(A) \subset \mathcal{T},$$

we have a contradiction with the fact that $t \in \mathcal{D}$.

To show the third statement, choose any point $z \in A$ and a ball $B(z, \epsilon)$, $\epsilon > 0$. Denote $r_{\max} = \max\limits_{i=1,\dots,N} \sigma(w_i)$. Let $m \in \mathbb{N}$ be such number that $r_{\max}^m \cdot \mathrm{diam} A < \epsilon$ and $\mathbf{i} = (i_1, \dots, i_m) \in \Sigma^m$ be such that $z \in w_{\mathbf{i}}(A)$. Let point $q \in A$ be such that $z = w_{\mathbf{i}}(q)$ and x be some point in $\mathcal{D}$. Then, by the first statement, $w_{\mathbf{i}}(x) \in \mathcal{D}$. Since

$$d(z, w_{\mathbf{i}}(x)) = d(w_{\mathbf{i}}(q), w_{\mathbf{i}}(x)) \le \sigma(w_{i_1}) \cdot \ldots \cdot \sigma(w_{i_m}) \cdot d(q, x) \le r_{\max}^m \cdot \mathrm{diam} A < \epsilon,$$

we have $\mathcal{D} \cap B(z, \epsilon) \neq \emptyset$ for every $z \in A$ and $\epsilon > 0$. Taking into account that $\mathcal{D} \subset A$, we have $\overline{\mathcal{D}} = A$.

Statement 4 is also proved by contradiction. Assume that $\mathcal{D} \neq \emptyset$, but $(w_1, \dots, w_N)$ does not belong to $\mathcal{V}_n$ for some $n \in \mathbb{N}$. Then there is a vector $\mathbf{i} = (i_1, \dots, i_n) \in \Sigma^n$ such that

$$w_{\mathbf{i}}(A) \subset \bigcup_{\substack{\mathbf{j} \in \Sigma^n \\ \mathbf{j} \neq \mathbf{i}}} w_{\mathbf{j}}(A).$$

Let x be any point in $\mathcal{D}$. There is a vector $\mathbf{k} \in \Sigma^n$, $\mathbf{k} = (k_1, \dots, k_n) \neq \mathbf{i}$, such that $w_{\mathbf{i}}(x) \in w_{\mathbf{i}}(A) \cap w_{\mathbf{k}}(A)$. If $i_1 \neq k_1$, then $w_{\mathbf{i}}(x) \in w_{i_1}(A) \cap w_{k_1}(A) \subset \mathcal{T}$, which contradicts to the fact that $x \in \mathcal{D}$.

If $i_1 = k_1$, let $1 \le s < n$ be an index that $i_1 = k_1$, ..., $i_s = k_s$, but $i_{s+1} \neq k_{s+1}$. Then

$$\begin{aligned} w_{\mathbf{i}}(x) &\in w_{i_1,\dots,i_s}\left(w_{i_{s+1},\dots,i_n}(A)\right) \cap w_{i_1,\dots,i_s}\left(w_{k_{s+1},\dots,k_n}(A)\right) \\ &= w_{i_1,\dots,i_s}\left(w_{i_{s+1},\dots,i_n}(A) \cap w_{k_{s+1},\dots,k_n}(A)\right) \\ &\subset w_{i_1,\dots,i_s}\left(w_{i_{s+1}}(A) \cap w_{k_{s+1}}(A)\right) \subset w_{i_1,\dots,i_s}\left(\mathcal{T}\right). \end{aligned}$$

Hence, $w_{i_{s+1},\dots,i_n}(x) \in \mathcal{T}$, which again implies that x does not belong to $\mathcal{D}$. Thus, our assumption is wrong and the fourth statement holds. Proposition 2 is proved. □

The following statement shows the relation between the cardinality of the overlaps of sets $w_i(A)$ and the SMP.

Proposition 3. *Let $w_1, \dots, w_N \in \mathcal{M}(X)$ be such that the corresponding attractor A is uncountable and every set $w_i(A) \cap w_j(A)$, $i \neq j$, is at most countable. Then the system $(w_1, \dots, w_N)$ satisfies the SMP.*

Proof. By assumption, the set $\mathcal{T}$ is at most countable. Then the set $\cup_{\mathbf{i} \in \mathcal{F}} w_{\mathbf{i}}^{-1}(\mathcal{T})$ is also at most countable. Since A is uncountable, we have $D(w_1, \dots, w_N) \neq \emptyset$. By Proposition 2, we have $(w_1, \dots, w_N) \in \cap_{n=1}^{\infty} \mathcal{V}_n$, which in view of Theorem 1, implies the SMP. Proposition 3 is proved. □

Proposition 4. *Let X be a complete metric space and $w_1, \dots, w_N : X \to X$ be contracting homeomorphisms of X onto X. Assume that every point in the attractor A of this system has a finite number of addresses. Then*

$$(w_1, \dots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n,$$

or equivalently, the system $(w_1, \dots, w_N)$ satisfies the SMP.

Proof. Assume the contrary, i.e. there exist $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^n$ such that

$$w_{\mathbf{i}}(A) \subset \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \neq \mathbf{i}} w_{\mathbf{j}}(A). \tag{12}$$

Denote by x the fixed point of $w_{\mathbf{i}}$. In view of (12), $x \in w_{\mathbf{j}}(A)$ for some $\mathbf{j} \in \Sigma^n$, $\mathbf{j} \neq \mathbf{i}$. Let $p \in A$ be such point that $x = w_{\mathbf{j}}(p)$. Then for every $m \in \mathbb{N}$, we have $x = (w_{\mathbf{i}})^m(x) = (w_{\mathbf{i}})^m w_{\mathbf{j}}(p)$. Hence, for every $m \in \mathbb{N}$, the point x will have an address starting with m vectors $\mathbf{i}$ followed by vector $\mathbf{j}$ different from $\mathbf{i}$, thus having infinitely many addresses, which contradicts our assumption. Proposition 4 is proved. □

We say that two vectors $\mathbf{i}, \mathbf{j} \in \mathcal{F}$ are *incomparable* if neither $\mathbf{i}$ is an initial word of $\mathbf{j}$ nor $\mathbf{j}$ is an initial word of $\mathbf{i}$. Denote

$$\mathcal{E} = \{w_{\mathbf{j}}^{-1} w_{\mathbf{i}} : \mathbf{i}, \mathbf{j} \in \mathcal{F},\ \mathbf{i}, \mathbf{j} \text{ incomparable}\}.$$

Denote by I the identity mapping from X to X. In the case when $X = \mathbb{R}^d$ and w_i's are contractive similitudes, the results of papers by Hutchinson [5], Bandt and Graf [1], and Schief [8] imply that SOSC is equivalent to the condition that $I \notin \overline{\mathcal{E}}$ in the topology of pointwise convergence of similitudes. The weak separation property (WSP) introduced by Lau and Ngai in [6] was shown to be equivalent to the condition that $I \notin \overline{\mathcal{E} \setminus \{I\}}$ for a wide class of self-similar sets (cf. the work by Zerner [10]). We see that SOSC does not allow $I \in \mathcal{E}$. The WSP allows I to be in $\mathcal{E}$ as an isolated point. The following proposition shows the relation between the condition that $I \notin \mathcal{E}$ and the SMP.

Proposition 5. *Let X be a complete metric space and let the system $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfy the SMP. Then $I \notin \mathcal{E}$. The converse is not true.*

Remark 1. This proposition together with above mentioned results implies that for a wide class of systems of contracting similitudes in $\mathbb{R}^d$, SMP together with WSP is equivalent to SOSC.

Proof. Assume that $I \in \mathcal{E}$. Then $I = w_{\mathbf{j}}^{-1} w_{\mathbf{i}}$ for some incomparable $\mathbf{i}, \mathbf{j} \in \mathcal{F}$. Hence, $w_{\mathbf{j}} = w_{\mathbf{i}}$. Without loss of generality we can assume that vector-index $\mathbf{i}$ is of the same or of a shorter length than $\mathbf{j}$. Since $\mathbf{i}$ is not a prefix of $\mathbf{j}$, we have

$$w_{\mathbf{i}}(A) = w_{\mathbf{j}}(A) \subset \bigcup_{\substack{\mathbf{k} \neq \mathbf{i} \\ |\mathbf{k}| = |\mathbf{i}|}} w_{\mathbf{k}}(A),$$

which implies that the SMP does not hold. Hence, SMP implies that $I \notin \mathcal{E}$.

The following counterexample shows that the converse is not true. Let $w_1(x) = x/2$, $w_2(x) = (x+1)/2$, and $w_3(x) = (x+a)/2$, where a is an irrational number from $(0,1)$. It is not difficult to see that interval $[0,1]$ is the attractor of the system (w_1, w_2, w_3). Since $w_3([0,1]) \subset w_1([0,1]) \cup w_2([0,1])$, the system (w_1, w_2, w_3) does not satisfy the SMP. If we assumed that $I \in \mathcal{E}$, there would be incomparable indexes $\mathbf{i} = (i_1, \dots, i_n), \mathbf{j} = (j_1, \dots, j_m) \in \mathcal{F}$ such that $w_{\mathbf{i}} = w_{\mathbf{j}}$. Hence,

$$w_{\mathbf{i}}(x) = \frac{1}{2^n} x + \sum_{\substack{k=1 \\ i_k = 2}}^{n} \frac{1}{2^k} + \sum_{\substack{k=1 \\ i_k = 3}}^{n} \frac{1}{2^k} a = w_{\mathbf{j}}(x) = \frac{1}{2^m} x + \sum_{\substack{k=1 \\ j_k = 2}}^{m} \frac{1}{2^k} + \sum_{\substack{k=1 \\ j_k = 3}}^{m} \frac{1}{2^k} a.$$

Then $n = m$ and

$$a\left(\sum_{k\,:\,j_k=3}\frac{1}{2^k}-\sum_{k\,:\,i_k=3}\frac{1}{2^k}\right)=\sum_{k\,:\,i_k=2}\frac{1}{2^k}-\sum_{k\,:\,j_k=2}\frac{1}{2^k}.$$

Since a is irrational, we must have

$$\sum_{k\,:\,j_k=3}\frac{1}{2^k}=\sum_{k\,:\,i_k=3}\frac{1}{2^k}.$$

Hence, $\{k : i_k = 3\} = \{k : j_k = 3\}$. But then

$$\sum_{k\,:\,i_k=2}\frac{1}{2^k}=\sum_{k\,:\,j_k=2}\frac{1}{2^k}.$$

Hence, $\{k : i_k = 2\} = \{k : j_k = 2\}$. This implies that $\{k : i_k = 1\} = \{k : j_k = 1\}$ and $\mathbf{i} = \mathbf{j}$, which contradicts to the incomparability of $\mathbf{i}$ and $\mathbf{j}$. This contradiction shows that for the system (w_1, w_2, w_3) we have $I \notin \mathcal{E}$ but SMP does not hold. □

6. Some results for self-similar sets in $\mathbb{R}^d$.

An address $(i_1, i_2, \ldots) \in \Sigma^\infty$ is called *universal*, if for any vector $\mathbf{j} = (j_1, \ldots, j_s) \in \mathcal{F}$, there is $k \geq 0$ such that $i_{k+1} = j_1$, ..., $i_{k+s} = j_s$. An address $(i_1, i_2, \ldots) \in \Sigma^\infty$ is called *recurrent*, if for every $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $i_{k+1} = i_1$, ..., $i_{k+n} = i_n$. In other words, a universal address is an address, which contains every finite sequence of numbers from Σ, and a recurrent address is an address where any finite prefix occurs further in that address.

Recall that a mapping $w : \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{N}$, is a contracting similitude, if there is a number $r \in (0, 1)$ such that $d(w(x), w(y)) = r \cdot d(x, y)$, $x, y \in \mathbb{R}^d$. Here d will denote the Euclidian distance in $\mathbb{R}^d$. The attractor of a system of finite contracting similitudes is called a self-similar set.

We will need the following result.

Theorem 3. *(Bandt and Rao* [2]*). Let $w_1, \ldots, w_N : \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{N}$, be a system of contracting similitudes and A be the attractor of this system. If one point $a \in w_{s_1}(A)$ with a recurrent address $(s_1, s_2, \ldots)$ belongs to the set $w_{t_1}(A)$ with $t_1 \neq s_1$, then the OSC cannot hold.*

We obtain the following statement in the case of self-similar attractors in $\mathbb{R}^d$.

Proposition 6. *Let $X = \mathbb{R}^d$, $d \in \mathbb{N}$, and mappings $w_1, \ldots, w_N$ be contracting similitudes in $\mathbb{R}^d$. If $D(w_1, \ldots, w_N) = \emptyset$, then the system $(w_1, \ldots, w_N)$ does not satisfy the OSC.*

Proof. Let $x \in A$ be a point with a universal address $\mathbf{j} = (j_1, j_2, \ldots) \in \Sigma^\infty$. Since $\mathcal{D} = \emptyset$, we have

$$x \in A \subset \bigcup_{\mathbf{i}\in\mathcal{F}} w_{\mathbf{i}}^{-1}(\mathcal{T}).$$

Hence, there exists a vector $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{F}$ such that $w_{\mathbf{i}}(x) \in \mathcal{T}$. This imlies that there is an index $l \in \Sigma$, $l \neq i_1$, such that $w_{\mathbf{i}}(x) \in w_{i_1}(A) \cap w_l(A)$. The address $\mathbf{ij} = (i_1, \ldots, i_k, j_1, j_2, \ldots) \in \Sigma^\infty$ is also universal. Since

$$w_{\mathbf{i}}(x) = w_{\mathbf{i}}\left(\lim_{n\to\infty} w_{j_1,\ldots,j_n}(\mathbf{0})\right) = \lim_{n\to\infty} w_{i_1,\ldots,i_k,j_1,\ldots,j_n}(\mathbf{0}),$$

sequence $\mathbf{ij}$ is an address of the point $w_{\mathbf{i}}(x)$. Since every universal address is also a recurrent address, by Theorem 3, the OSC does not hold for the system $(w_1, \ldots, w_N)$. □

Let $W = (w_1, \ldots, w_N)$, where $w_1, \ldots, w_N : \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{N}$, are similitudes with similarity coefficients $r_1, \ldots, r_N \in (0, 1)$ respectively. Denote by $\alpha = \alpha(W)$ the unique positive number such that

$$r_1^\alpha + \ldots + r_N^\alpha = 1.$$

This number is known as the similarity dimension of the attractor A associated with the system W. Denote by $\dim A$ the Hausdorff dimension of the set A and by $\mathcal{H}_\lambda$, $\lambda > 0$, the λ-dimensional Hausdorff measure in $\mathbb{R}^d$. The standard covering argument shows that

$$\dim A(W) \le \alpha(W). \tag{13}$$

Proposition 7. *Let $W = (w_1, \dots, w_N)$ be a system of contracting similitudes in $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\dim A(W) = \alpha(W)$. Then A satisfies the SMP.*

Remark 2. The results by Hutchinson [5, Theorem 1, Section 5.3] combined with the results by Schief [8, Theorem 2.1] imply that for the attractor $A(W)$ of a finite system W of contracting similitudes in $\mathbb{R}^d$, we have $\mathcal{H}_{\alpha(W)}(A(W)) > 0$ if and only if W satisfies the OSC. Proposition 7 implies that any finite system of contracting similitudes W such that $\dim A(W) = \alpha(W)$ and $\mathcal{H}_{\alpha(W)}(A(W)) = 0$ (existence of such systems is proved by Solomyak [9]), will still belong to $\cap_{n=1}^{\infty} \mathcal{V}_n$ and in view of Theorem 1, will have the SMP. But such system will not satisfy the OSC. This disproves the conjecture about the equivalence of these two properties. Since SMP implies MPP as asserted by Lemma 1, we conclude that MPP is also weaker than OSC.

Proof of Proposition 7. Assume the contrary. Then in view of Theorem 1, there is $n \in \mathbb{N}$ such that $(w_1, \dots, w_N)$ does not belong to $\mathcal{V}_n$. Then there is a vector $\mathbf{i} \in \Sigma^n$ such that

$$w_{\mathbf{i}}(A) \subset \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \ne \mathbf{i}} w_{\mathbf{j}}(A).$$

Hence,

$$A = \bigcup_{\mathbf{j} \in \Sigma^n} w_{\mathbf{j}}(A) = \bigcup_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \ne \mathbf{i}} w_{\mathbf{j}}(A)$$

and A will be also the attractor for the system of mappings $S = \{w_{\mathbf{j}}\}_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \ne \mathbf{i}}$. In this case the similarity dimension of A associated with system S satisfies

$$\sum_{\mathbf{j} \in \Sigma^n,\ \mathbf{j} \ne \mathbf{i}} r_{\mathbf{j}}^{\alpha(S)} = 1,$$

where $r_{\mathbf{j}}$ is the contraction coefficient of the mapping $w_{\mathbf{j}}$, $\mathbf{j} \in \Sigma^n$. Since

$$\sum_{\mathbf{j} \in \Sigma^n} r_{\mathbf{j}}^{\alpha(W)} = 1,$$

we have $\alpha(S) < \alpha(W)$. Then, by (13), we obtain $\dim A \le \alpha(S) < \alpha(W)$, which contradicts to the assumptions of the proposition. Proposition 7 is proved. $\square$

7. Density of the SMP on certain classes of self-similar sets

Let $B_1, \dots, B_N$ be invertible $d \times d$ contraction matrices, $d \in \mathbb{N}$. Recall that $E_d(B_1, \dots, B_N)$ is the set of ordered point collections $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_N) \in \left(\mathbb{R}^d\right)^N$ such that the system of mappings

$$u_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\alpha}_i) + \boldsymbol{\alpha}_i, \quad i = 1, \dots, N,$$

has the SMP. We will consider the set $E_d(B_1, \dots, B_N)$ as a subset of $\mathbb{R}^{dN}$.

Remark 3. When matrices $B_1, \dots, B_N$ are orthogonal and
1) $\|B_1\| < \frac{1}{2}$, ..., $\|B_N\| < \frac{1}{2}$,
2) $\sum_{i=1}^{N} \|B_i\|^d < 1$,

the set $E_d(B_1, \dots, B_N)$ is a subset of $\mathbb{R}^{dN}$ of full measure. This follows from results of Falconer [3, Theorem 5.3], Solomyak [9, Proposition 3.1]), and Proposition 7. (Recent results of Falconer and Miao [4] imply an upper estimate for the Hausdorff dimension of the complement of $E_d(B_1, \dots, B_N)$.) Hence, $E_d(B_1, \dots, B_N)$ will be dense in $\mathbb{R}^{dN}$. In this paper we can show that $E_d(B_1, \dots, B_N)$ is dense when assumption 1) is replaced with certain other assumptions.

Theorem 4. *Let $B_1, \ldots, B_N$ be invertible $d \times d$ contraction matrices such that $\sum\limits_{i=1}^{N} \|B_i\| < 1$. Then the set $E_d(B_1, \ldots, B_N)$ is a dense G_δ-subset of $\mathbb{R}^{dN}$.*

When $d = 1$ the density result of Theorem 4 immediately follows from the result of Mattila [7, Theorem 9.13]. He also mentions without proof that his result can be extended to certain cases of contractive multidimensional similitudes.

Theorem 5. *Let $B_i = \sigma_i U_i$, where $\sigma_i \in (0, 1)$, U_i is a 2×2 rotation matrix, $i = 1, \ldots, N$, and $\sum_{i=1}^{N} \sigma_i^2 < 1$. Then the set $E_2(B_1, \ldots, B_N)$ is either empty or is a dense G_δ-subset of $\mathbb{R}^{2N}$.*

Remark 4. The set $E_2(B_1, \ldots, B_N)$ can be empty under assumptions of Theorem 5 as the following example shows. Let $\sigma_1, \sigma_2 > 0$ be such that $\sigma_1 + \sigma_2 > 1$ and $\sigma_1^2 + \sigma_2^2 < 1$, and $B_1 = \sigma_1 I_2$, $B_2 = \sigma_2 I_2$ (here and below I_d denotes the $d \times d$ identity matrix). For any ordered pair $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ of points in $\mathbb{R}^2$, the attractor A of the system of mappings

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\alpha}_i) + \boldsymbol{\alpha}_i = \sigma_i \mathbf{x} + (1 - \sigma_i)\boldsymbol{\alpha}_i, \quad i = 1, 2,$$

is the closed segment with endpoints $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. The set $w_1(A) \cap w_2(A)$ is a segment of positive length. For $n \in \mathbb{N}$ sufficiently large and some index $\mathbf{i} \in \Sigma^n$, there holds $w_{\mathbf{i}}(A) \subset w_1(A) \cap w_2(A)$. If $\mathbf{i}$ starts with 1, we have

$$w_{\mathbf{i}}(A) \subset w_2(A) = \bigcup_{\mathbf{j} \in \Sigma^{n-1}} w_2 w_{\mathbf{j}}(A) \subset \bigcup_{\substack{\mathbf{j} \in \Sigma^n \\ \mathbf{j} \neq \mathbf{i}}} w_{\mathbf{j}}(A).$$

If $\mathbf{i}$ starts with 2 we use analogous argument. Thus, the system (w_1, w_2) does not posses the SMP for any collection of fixed points $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ and hence, $E_2(B_1, B_2) = \emptyset$.

The proof of Theorems 4 and 5 will follow from the statement presented below. For an ordered collection of points $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N) \in (\mathbb{R}^d)^N$, denote by $\Pi_{\mathbf{k}}(\boldsymbol{\beta})$ the element with the address $\mathbf{k} \in \Sigma^\infty$ in the attractor of the system of mappings

$$u_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\beta}_i) + \boldsymbol{\beta}_i, \quad i = 1, \ldots, N.$$

Proposition 8. *Let $1 \le k \le d$ be integers and $B_1, \ldots, B_N$ be invertible $d \times d$ contraction matrices such that $\sum\limits_{i=1}^{N} \|B_i\|^k < 1$. Assume that there is an ordered collection $\boldsymbol{\gamma}_1 = \left(\boldsymbol{\gamma}_1^1, \ldots, \boldsymbol{\gamma}_N^1\right) \in (\mathbb{R}^d)^N$ such that the system $W = (w_1, \ldots, w_N)$, where*

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\gamma}_i^1) + \boldsymbol{\gamma}_i^1, \quad i = 1, \ldots, N,$$

has the SMP. In the case $k \ge 2$ assume also that there are collections $\boldsymbol{\gamma}_j = \left(\boldsymbol{\gamma}_1^j, \ldots, \boldsymbol{\gamma}_N^j\right) \in (\mathbb{R}^d)^N$, $j = 2, \ldots, k$, such that for every pair of addresses $\mathbf{i} \neq \mathbf{j} \in \Sigma^\infty$ such that $\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) \neq \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1)$, the system of vectors $\{\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_i) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_i) : i = 1, \ldots, k\}$ is linearly independent.

Then the set $E_d(B_1, \ldots, B_N)$ is a dense G_δ-subset of $\mathbb{R}^{dN}$.

Proof. Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N) \in (\mathbb{R}^d)^N$ be arbitrary. For every $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$, denote by $W_{\mathbf{t}} = (w_1^{\mathbf{t}}, \ldots, w_N^{\mathbf{t}})$ the system of mappings

$$w_i^{\mathbf{t}}(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\alpha}_i - t_1\boldsymbol{\gamma}_i^1 - \ldots - t_k\boldsymbol{\gamma}_i^k) + \boldsymbol{\alpha}_i + t_1\boldsymbol{\gamma}_i^1 + \ldots + t_k\boldsymbol{\gamma}_i^k, \; i = 1, \ldots, N.$$

Let $A_{\mathbf{t}} = A(W_{\mathbf{t}})$ be the attractor of the system $W_{\mathbf{t}}$ and $A = A(W)$ be the attractor of the system W. Denote

$$P(\boldsymbol{\alpha}) = \{\mathbf{t} \in \mathbb{R}^k : W_{\mathbf{t}} \text{ has no SMP}\}$$

and for an index $\mathbf{i} = (i_1, \ldots, i_n) \in \Sigma^n$, let

$$w_{\mathbf{i}}^{\mathbf{t}} = w_{i_1}^{\mathbf{t}} \circ \ldots \circ w_{i_n}^{\mathbf{t}}.$$

Then

$$P(\boldsymbol{\alpha}) = \bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i}\in\Sigma^n} \left\{ \mathbf{t} \in \mathbb{R}^k : w^{\mathbf{t}}_{\mathbf{i}}(A_{\mathbf{t}}) \subset \bigcup_{\mathbf{j}\in\Sigma^n,\ \mathbf{j}\neq\mathbf{i}} w^{\mathbf{t}}_{\mathbf{j}}(A_{\mathbf{t}}) \right\}.$$

Denote by $\Pi_{\mathbf{k}}$, $\mathbf{k} \in \Sigma^\infty$, the element $\mathbf{x}$ in A with address $\mathbf{k}$. Let also $\Pi^{\mathbf{t}}_{\mathbf{k}}$, $\mathbf{k} \in \Sigma^\infty$, be the element in $A_{\mathbf{t}}$ with address $\mathbf{k}$. For every $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^n$, let $\mathbf{k}(\mathbf{i}) \in \Sigma^\infty$ be such sequence that

$$\Pi_{\mathbf{ik(i)}} \notin \bigcup_{\substack{\mathbf{j}\in\Sigma^n \\ \mathbf{j}\neq\mathbf{i}}} w_{\mathbf{j}}(A)$$

(such $\mathbf{k}(\mathbf{i})$ exists since W satisfies the SMP). For every $\mathbf{i} \neq \mathbf{j} \in \Sigma^n$, let

$$Q_{\mathbf{i},\mathbf{j}} = \{\mathbf{t} \in \mathbb{R}^k : \Pi^{\mathbf{t}}_{\mathbf{ik(i)}} \in w^{\mathbf{t}}_{\mathbf{j}}(A_{\mathbf{t}})\}.$$

Then

$$P(\boldsymbol{\alpha}) \subset \bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i}\in\Sigma^n} \bigcup_{\substack{\mathbf{j}\in\Sigma^n \\ \mathbf{j}\neq\mathbf{i}}} Q_{\mathbf{i},\mathbf{j}}.$$

We now fix a number $n \in \mathbb{N}$ and indices $\mathbf{i} \neq \mathbf{j} \in \Sigma^n$. For every $m \in \mathbb{N}$ and $\mathbf{k} \in \Sigma^m$, denote

$$Q^{\mathbf{k}}_{\mathbf{i},\mathbf{j}} = \{\mathbf{t} \in \mathbb{R}^k : \Pi^{\mathbf{t}}_{\mathbf{ik(i)}} \in w^{\mathbf{t}}_{\mathbf{jk}}(A_{\mathbf{t}})\} = \{\mathbf{t} \in \mathbb{R}^k : \Pi^{\mathbf{t}}_{\mathbf{ik(i)}} = \Pi^{\mathbf{t}}_{\mathbf{jkp}} \text{ for some } \mathbf{p} \in \Sigma^\infty\}.$$

Then

$$Q_{\mathbf{i},\mathbf{j}} = \bigcup_{\mathbf{k}\in\Sigma^m} Q^{\mathbf{k}}_{\mathbf{i},\mathbf{j}}.$$

It is a straightforward argument to verify that for every address $\mathbf{q} = (q_1, q_2, \ldots) \in \Sigma^\infty$, we have

$$\begin{aligned}
\Pi^{\mathbf{t}}_{\mathbf{q}} &= \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}} (I_d - B_{q_i})(\boldsymbol{\alpha}_{q_i} + t_1 \boldsymbol{\gamma}^1_{q_i} + \ldots + t_k \boldsymbol{\gamma}^k_{q_i}) \\
&= \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}} (I_d - B_{q_i}) \boldsymbol{\alpha}_{q_i} + t_1 \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}} (I_d - B_{q_i}) \boldsymbol{\gamma}^1_{q_i} \\
&\quad + \ldots + t_k \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}} (I_d - B_{q_i}) \boldsymbol{\gamma}^k_{q_i} = \Pi_{\mathbf{q}}(\boldsymbol{\alpha}) + t_1 \Pi_{\mathbf{q}}(\boldsymbol{\gamma}_1) + \ldots + t_k \Pi_{\mathbf{q}}(\boldsymbol{\gamma}_k).
\end{aligned}$$

Then

$$\begin{aligned}
Q^{\mathbf{k}}_{\mathbf{i},\mathbf{j}} &= \left\{ \mathbf{t} \in \mathbb{R}^k : \Pi_{\mathbf{ik(i)}}(\boldsymbol{\alpha}) + \sum_{i=1}^{k} t_i \Pi_{\mathbf{ik(i)}}(\boldsymbol{\gamma}_i) = \Pi_{\mathbf{jkp}}(\boldsymbol{\alpha}) + \sum_{i=1}^{k} t_i \Pi_{\mathbf{jkp}}(\boldsymbol{\gamma}_i) \text{ for some } p \in \Sigma^\infty \right\} \\
&= \left\{ \mathbf{t} \in \mathbb{R}^k : \sum_{i=1}^{k} t_i (\Pi_{\mathbf{ik(i)}}(\boldsymbol{\gamma}_i) - \Pi_{\mathbf{jkp}}(\boldsymbol{\gamma}_i)) = \Pi_{\mathbf{jkp}}(\boldsymbol{\alpha}) - \Pi_{\mathbf{ik(i)}}(\boldsymbol{\alpha}) \text{ for some } p \in \Sigma^\infty \right\}.
\end{aligned}$$

Given an address $q \in \Sigma^\infty$, let

$$B(\mathbf{q}) = \left[\Pi_{\mathbf{ik(i)}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{jq}}(\boldsymbol{\gamma}_1), \ldots, \Pi_{\mathbf{ik(i)}}(\boldsymbol{\gamma}_k) - \Pi_{\mathbf{jq}}(\boldsymbol{\gamma}_k)\right]$$

be the $d \times k$ matrix with columns $\Pi_{\mathbf{ik(i)}}(\boldsymbol{\gamma}_i) - \Pi_{\mathbf{jq}}(\boldsymbol{\gamma}_i)$, $i = 1, \ldots, k$. Let also $\mathbf{b}(\mathbf{q}) = \Pi_{\mathbf{jq}}(\boldsymbol{\alpha}) - \Pi_{\mathbf{ik(i)}}(\boldsymbol{\alpha})$, $\sigma = \max_{i=1,\ldots,N} \|B_i\|$,

$$a = \max\{\text{diam } A(\boldsymbol{\alpha}), \text{diam} A(\boldsymbol{\gamma}_1), \ldots, \text{diam} A(\boldsymbol{\gamma}_k)\},$$

where $A(\mathbf{c})$, $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_N)$, denotes the attractor of the system $u_i(\mathbf{x}) = B_i(\mathbf{x} - \mathbf{c}_i) + \mathbf{c}_i$, $i = 1, \ldots, N$, and for an index $\mathbf{j} = (j_1, \ldots, j_n) \in \Sigma^n$, denote $\sigma_{\mathbf{j}} = \|B_{j_1}\| \cdot \ldots \cdot \|B_{j_n}\|$. Then

$$Q^{\mathbf{k}}_{\mathbf{i},\mathbf{j}} = \{\mathbf{t} \in \mathbb{R}^k : B(\mathbf{kp}) \cdot \mathbf{t} = \mathbf{b}(\mathbf{kp}) \text{ for some } \mathbf{p} \in \Sigma^\infty\}.$$

We will need the following auxiliary statement.

Lemma 8. *Let $1 \le k \le d$ be integers, $\mathcal{C}$ be a set of $d \times k$ matrices of rank k, which has diameter δ with respect to the matrix norm (1), and $\mathcal{P}$ be a set of vectors from $\mathbb{R}^d$, which has diameter ϵ with respect to the Euclidean distance. Assume that there exists a finite and positive number $M > 0$ such that for every matrix $B \in \mathcal{C}$,*

$$\left\| \left(B^T B\right)^{-1} \right\| \le M.$$

Denote also by L and K positive numbers such that $\|B\| \le L$ for every matrix $B \in \mathcal{C}$, and $|\mathbf{b}| \le K$ for every vector $\mathbf{b} \in \mathcal{P}$. Let Q be the set of all vectors $\mathbf{t} \in \mathbb{R}^k$, which are solutions to the equation

$$B\mathbf{t} = \mathbf{b}$$

for some matrix $B \in \mathcal{C}$ and vector $\mathbf{b} \in \mathcal{P}$. Then

$$\text{diam } Q \le \epsilon M L + \delta M K + 2\delta M^2 L^2 K. \tag{14}$$

Proof. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be arbitrary points from Q. There exist matrices $B_1, B_2 \in \mathcal{C}$ and vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{P}$ such that

$$B_i \mathbf{t}_i = \mathbf{b}_i, \quad i = 1, 2. \tag{15}$$

Since matrices B_1 and B_2 have rank k, each equation (15) has a unique solution $\mathbf{t}_i = \left(B_i^T B_i\right)^{-1} B_i^T \mathbf{b}_i$, $i = 1, 2$. Then

$$\begin{aligned}
|\mathbf{t}_1 - \mathbf{t}_2| &= \left| \left(B_1^T B_1\right)^{-1} B_1^T \mathbf{b}_1 - \left(B_2^T B_2\right)^{-1} B_2^T \mathbf{b}_2 \right| \\
&\le \left| \left(B_1^T B_1\right)^{-1} B_1^T \mathbf{b}_1 - \left(B_2^T B_2\right)^{-1} B_1^T \mathbf{b}_1 \right| + \left| \left(B_2^T B_2\right)^{-1} B_1^T \mathbf{b}_1 - \left(B_2^T B_2\right)^{-1} B_2^T \mathbf{b}_1 \right| \\
&\quad + \left| \left(B_2^T B_2\right)^{-1} B_2^T \mathbf{b}_1 - \left(B_2^T B_2\right)^{-1} B_2^T \mathbf{b}_2 \right| \\
&\le \| \left(B_1^T B_1\right)^{-1} - \left(B_2^T B_2\right)^{-1} \| \cdot \|B_1^T\| \cdot |\mathbf{b}_1| \\
&\quad + \| \left(B_2^T B_2\right)^{-1} \| \cdot \|B_1^T - B_2^T\| \cdot |\mathbf{b}_1| + \| \left(B_2^T B_2\right)^{-1} \| \cdot \|B_2^T\| \cdot |\mathbf{b}_1 - \mathbf{b}_2| \,.
\end{aligned}$$

Due to equality $\|B^T\| = \|B\|$ and definition of numbers M, L and K, we have

$$|\mathbf{t}_1 - \mathbf{t}_2| \le LK \cdot \| \left(B_1^T B_1\right)^{-1} - \left(B_2^T B_2\right)^{-1} \| + \delta M K + \epsilon M L.$$

Using the following estimate

$$\begin{aligned}
\| \left(B_1^T B_1\right)^{-1} - \left(B_2^T B_2\right)^{-1} \| &= \| \left(B_2^T B_2\right)^{-1} B_2^T B_2 \left(B_1^T B_1\right)^{-1} - \left(B_2^T B_2\right)^{-1} B_1^T B_1 \left(B_1^T B_1\right)^{-1} \| \\
&= \| \left(B_2^T B_2\right)^{-1} \left(B_2^T B_2 - B_1^T B_1\right) \left(B_1^T B_1\right)^{-1} \| \le M^2 \|B_2^T B_2 - B_1^T B_1\| \\
&\le M^2 \left(\|B_2^T B_2 - B_2^T B_1\| + \|B_2^T B_1 - B_1^T B_1\| \right) \\
&\le M^2 \left(\|B_2^T\| \cdot \|B_2 - B_1\| + \|B_2^T - B_1^T\| \cdot \|B_1\| \right) \le 2\delta M^2 L, \qquad (16)
\end{aligned}$$

for every $\mathbf{t}_1, \mathbf{t}_2 \in Q$, we obtain

$$|\mathbf{t}_1 - \mathbf{t}_2| \le 2\delta M^2 L^2 K + \delta M K + \epsilon M L,$$

and estimate (14) follows. Lemma 8 is proved. □

Completion of the proof of Proposition 8. We apply Lemma 8 with $\mathcal{C} = \{B(\mathbf{kp}) : \mathbf{p} \in \Sigma^\infty\}$ and $\mathcal{P} = \{\mathbf{b}(\mathbf{kp}) : \mathbf{p} \in \Sigma^\infty\}$. For a matrix $B = [\mathbf{b}_1, \dots, \mathbf{b}_k]$, denote

$$\|B\|_{2,\infty} = \max_{i=1,\dots,k} |\mathbf{b}_i|.$$

It is not difficult to see that for any $d \times k$ matrix B,

$$\|B\|_{2,\infty} \le \|B\| \le \sqrt{k} \|B\|_{2,\infty}. \tag{17}$$

Let

$$M_{\mathbf{i},\mathbf{j}} = \sup_{\mathbf{q}\in\Sigma^\infty} \|(B(\mathbf{q})^T B(\mathbf{q}))^{-1}\|.$$

Denote $\mathcal{Y} = \{B(\mathbf{q}) : \mathbf{q} \in \Sigma^\infty\}$. By assumption, the columns of matrix $B(\mathbf{q})$ are linearly independent for every $\mathbf{q} \in \Sigma^\infty$. In view of the fact that $\det B^T B \neq 0$, $B \in \mathcal{Y}$, and continuity of $\det B^T B$ and of the algebraic complement to every element of $B^T B$, we have that $\|(B^T B)^{-1}\|$ is also continuous with respect to matrix $B \in \mathcal{Y}$. Since $\mathcal{Y}$ is compact with respect to the matrix norm (1), we obtain that $M_{\mathbf{i},\mathbf{j}}$ is finite.

It is not difficult to see that diam $\mathcal{C} \leq a\sqrt{k}\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}$ and diam $\mathcal{P} \leq a\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}$. Denote

$$L_{\mathbf{i},\mathbf{j}} = \sup_{\mathbf{q}\in\Sigma^\infty} \|B(\mathbf{q})\|,$$

and let

$$K_{\mathbf{i},\mathbf{j}} = \sup_{\mathbf{q}\in\Sigma^\infty} |\mathbf{b}(\mathbf{q})|.$$

Then by Lemma 8,

$$\text{diam } Q^{\mathbf{k}}_{\mathbf{i},\mathbf{j}} \leq \sigma_{\mathbf{jk}} a M_{\mathbf{i},\mathbf{j}} L_{\mathbf{i},\mathbf{j}} + \sigma_{\mathbf{jk}}\sqrt{k} a M_{\mathbf{i},\mathbf{j}} K_{\mathbf{i},\mathbf{j}} + 2\sigma_{\mathbf{jk}}\sqrt{k} a M^2_{\mathbf{i},\mathbf{j}} L^2_{\mathbf{i},\mathbf{j}} K_{\mathbf{i},\mathbf{j}} =: \sigma_{\mathbf{j}}\sigma_{\mathbf{k}} U_{\mathbf{i},\mathbf{j}}.$$

Denote by λ such number that

$$\sum_{i=1}^{N} \|B_i\|^\lambda = 1.$$

Then

$$\mathcal{H}_\lambda(Q_{\mathbf{i},\mathbf{j}}) \leq \limsup_{m\to\infty} \sum_{\mathbf{k}\in\Sigma^m} \left(\text{diam } Q^{\mathbf{k}}_{\mathbf{i},\mathbf{j}}\right)^\lambda \leq \lim_{m\to\infty} \sum_{\mathbf{k}\in\Sigma^m} \sigma^\lambda_{\mathbf{j}} \sigma^\lambda_{\mathbf{k}} U^\lambda_{\mathbf{i},\mathbf{j}} = \sigma^\lambda_{\mathbf{j}} U^\lambda_{\mathbf{i},\mathbf{j}} < \infty.$$

Since $P(\boldsymbol{\alpha})$ is covered by a countable collection of sets of Hausdorff dimension at most λ, we have $\dim P(\boldsymbol{\alpha}) \leq \lambda < k$. Hence, the complement of $P(\boldsymbol{\alpha})$ is dense in $\mathbb{R}^k$ and we can find vector $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$ such that point $\boldsymbol{\alpha}_i + \sum\limits_{j=1}^{k} t_j \gamma^j_i$ will be arbitrarily close to α_i, $i = 1, \ldots, N$, and the system $W_{\mathbf{t}}$ will have the SMP. This implies that $E_d(B_1, \ldots, B_N)$ is dense in $\mathbb{R}^{dN}$. Corollary 1 implies that $E_d(B_1, \ldots, B_N)$ is a G_δ-set. Proposition 8 is proved. □

Proof of Theorem 4. Let $\mathbf{u} \in \mathbb{R}^d$ be a unit vector. Since $\sum_{i=1}^N \|B_i\| < 1$, there are numbers $c_1, \ldots, c_N \in (-1, 1)$ such that balls $B[c_i\mathbf{u}, \|B_i\|]$, $i = 1, \ldots, N$, are pairwise disjoint and are contained in $B[\mathbf{0}, 1]$. Let $\boldsymbol{\gamma}^1_i = c_i(I_d - B_i)^{-1}\mathbf{u}$, and

$$w_i(\mathbf{x}) = B_i\mathbf{x} + c_i\mathbf{u} = B_i(\mathbf{x} - \boldsymbol{\gamma}^1_i) + \boldsymbol{\gamma}^1_i, \quad i = 1, \ldots, N,$$

and $A = A(w_1, \ldots, w_N)$ be the attractor of the system $(w_1, \ldots, w_N)$. It is not difficult to see that

$$w_i(B[\mathbf{0}, 1]) \subset B[c_i\mathbf{u}, \|B_i\|] \subset B[\mathbf{0}, 1], \quad i = 1, \ldots, N.$$

This implies that $A \subset B[\mathbf{0}, 1]$. Indeed, for every element $\mathbf{x} \in A$, there is a sequence $(i_1, i_2, \ldots) \in \Sigma^\infty$ such that $\mathbf{x} = \lim\limits_{n\to\infty} w_{i_1\ldots i_n}(\mathbf{0})$. Since $w_{i_1\ldots i_n}(\mathbf{0}) \in B[\mathbf{0}, 1]$ for every $n \in \mathbb{N}$, we have $\mathbf{x} \in B[\mathbf{0}, 1]$. Then we also have

$$w_i(A) \cap w_j(A) \subset w_i(B[\mathbf{0}, 1]) \cap w_j(B[\mathbf{0}, 1]) \subset B[c_i\mathbf{u}, \|B_i\|] \cap B[c_j\mathbf{u}, \|B_j\|] = \emptyset, \quad i \neq j,$$

which implies $w_{\mathbf{i}}(A) \cap w_{\mathbf{j}}(A) = \emptyset$, $\mathbf{i}, \mathbf{j} \in \Sigma^n$, $\mathbf{i} \neq \mathbf{j}$, $n \in \mathbb{N}$. Hence, system of mappings $(w_1, \ldots, w_N)$ has the SMP and we have $\boldsymbol{\gamma}_1 = (\boldsymbol{\gamma}^1_1, \ldots, \boldsymbol{\gamma}^1_N) \in E_d(B_1, \ldots, B_N)$. Since $k = 1$, the other assumption of Proposition 8 does not apply and the density of $E_d(B_1, \ldots, B_N)$ as well as the fact that it is a G_δ-set follows. Theorem 4 is proved. □

Proof of Theorem 5. Assume that $E_2(B_1, \ldots, B_N) \neq \emptyset$ and let $\boldsymbol{\gamma}_1 = (\boldsymbol{\gamma}^1_1, \ldots, \boldsymbol{\gamma}^1_N) \in (\mathbb{R}^2)^N$ be such collection of points that the system of mappings

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\gamma}^1_i) + \boldsymbol{\gamma}^1_i, \quad i = 1, \ldots, N,$$

satisfies the SMP. Denote

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(V is a rotation matrix) and let $\boldsymbol{\gamma}_2 = \left(V\boldsymbol{\gamma}_1^1, \ldots, V\boldsymbol{\gamma}_N^1\right)$. Note that for any non-zero vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we have

$$\det[\mathbf{x}, V\mathbf{x}] = \left(\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix}\right) = x_1^2 + x_2^2 \neq 0.$$

Since rotation matrices commute, for every address $\mathbf{q} = (q_1, q_2, \ldots)\Sigma^\infty$, we obtain,

$$\begin{aligned}\Pi_{\mathbf{q}}(\boldsymbol{\gamma}_2) &= \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}}(I_2 - B_{q_i})V\boldsymbol{\gamma}_i^1 = \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}}(V - B_{q_i}V)\boldsymbol{\gamma}_i^1 \\ &= \sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}}(V - VB_{q_i})\boldsymbol{\gamma}_i^1 = V\sum_{i=1}^{\infty} B_{q_1} \cdot \ldots \cdot B_{q_{i-1}}(I_2 - B_{q_i})\boldsymbol{\gamma}_i^1 = V\Pi_{\mathbf{q}}(\boldsymbol{\gamma}_1).\end{aligned}$$

Then for every pair of addresses $\mathbf{i} \neq \mathbf{j} \in \Sigma^\infty$ such that $\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) \neq \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1)$, we have

$$\begin{aligned}&\det\ [\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1), \Pi_{\mathbf{i}}(\boldsymbol{\gamma}_2) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_2)] \\ &\quad = \det\ [\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1), V(\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1))] \neq 0.\end{aligned}$$

Then vectors $\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_i) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_i)$, $i = 1, 2$, are linearly independent and by Proposition 8 we obtain that $E_2(B_1, \ldots, B_N)$ is a dense G_δ-subset of $\mathbb{R}^{2N}$. □

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