Multivariate Normal Approximation in Geometric Probability

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Abstract

Consider a measure $\mu_\lambda = \sum_x \xi_x \delta_x$ where the sum is over points x of a Poisson point process of intensity λ on a bounded region in d-space, and ξ_x is a functional determined by the Poisson points near to x, i.e. satisfying an exponential stabilization condition, along with a moments condition (examples include statistics for proximity graphs, germ-grain models and random sequential deposition models). A known general result says the μ_λ-measures (suitably scaled and centred) of disjoint sets in $\mathbb{R}^d$ are asymptotically independent normals as $\lambda \to \infty$; here we give an $O(\lambda^{-1/(2d+\varepsilon)})$ bound on the rate of convergence, and also a new criterion for the limiting normals to be non-degenerate. We illustrate our result with an explicit multivariate central limit theorem for the nearest-neighbour graph on Poisson points on a finite collection of disjoint intervals.

Key words and phrases: Multivariate normal approximation; geometric probability; stabilization; central limit theorem; Stein's method; nearest-neighbour graph.

1 Introduction

There has been considerable recent interest in providing central limit theorems (CLTs) for certain functionals in geometric probability defined on spatial Poisson point processes. Such functionals include those associated with random spatial graphs such as the minimal-length spanning tree or the nearest-neighbour graph, as well as with germ-grain models and random sequential packing models. These functionals are random variables given by sums of contributions from points of a Poisson point process in $\mathbb{R}^d$.

A natural extension to random *measures* may be provided by keeping track of the location of each contribution in $\mathbb{R}^d$. In this way one can obtain a random field indexed by test functions on $\mathbb{R}^d$ or by subsets of $\mathbb{R}^d$. For example, one can consider the measure induced by a Poisson process with a point mass at each Poisson point equal to the distance to its nearest-neighbour; then a typical multivariate statistic induced by this measure is the vector of total edge-lengths of the nearest-neighbour graph on Poisson points over a finite collection of disjoint subsets of $\mathbb{R}^d$.

Under certain conditions, it is known [4, 13, 15] that the measures, appropriately scaled and centred, of disjoint sets (or of test functions with disjoint supports) are asymptotically distributed as independent normals in the large-intensity limit. The main contribution of the present paper is to give bounds on the rate of convergence. We illustrate our result with an application to the nearest-neighbour situation mentioned above.

The unifying concept of *stabilization* on Poisson points has proved a useful notion of local dependence in the context of geometric probability. This says, roughly speaking, that the contribution from a Poisson point is unaffected by changes to the configuration of Poisson points beyond a certain (random) distance.

The methodology of stabilization has been fruitfully employed, in various guises, to produce univariate CLTs and laws of large numbers for random quantities in many problems in geometric probability; see e.g. [4,10,12–16,18–21]. The techniques used in this context include a martingale method (see for instance [10], and [18] where the method is presented for general stabilizing functionals in geometric probability), the method of moments [4], and Stein's method [21], which we employ in the present paper.

The multivariate case, in which several collections of random variables are considered, has also received some attention [4, 13–15]. Applications in geometric probability include, for example, the joint normality of certain random spatial graph functionals defined over a finite collection of disjoint regions in $\mathbb{R}^d$. There are potential applications to multivariate statistics, including nonparametric multi-sample tests (see e.g. [22]).

In the present paper, we employ a form of *Stein's method* (see [23]), which has the advantage that it can provide rates of convergence in the CLT. In this context, Stein's method is a useful tool for establishing normal approximations and CLTs for sums of weakly dependent random variables. In this paper, the weak dependency structure is provided by the concept of stabilization on Poisson points.

In the univariate case, the method yields normal approximation of the sum of a single collection of random variables that are 'mostly independent', i.e. exhibiting a local dependency structure. This structure may be captured using dependency graphs. This method was first used in the context of geometric probability by Avram and Bertsimas in [2] (using the normal approximation error bounds of [3]) to provide CLTs for certain random combinatorial structures that are locally determined in some sense, including the j-th nearest-neighbour graph, and the Delaunay and Voronoi graphs.

Using the sharper normal approximation bounds of [6], more general results for univariate normal approximation based on Stein's method for random point measures were given by Penrose and Yukich in [21]. That paper is the foundation for the present work, which is its multivariate analogue.

Multivariate CLTs for random measures in geometric probability have recently been proved via the method of moments [4], via Stein's method [15] and via the martingale method [13]. In particular, [13] also covers lattice processes (such as percolation), and does not require 'exponential' stabilization, and so admits a larger class of measures. The advantage of the results in the present paper is that information on rates of convergence is provided.

Beyond the context of geometric probability, multivariate central limit theory has been well studied. Related results include multivariate central limit theorems for sums of independent random variables given in [8]. In [9,22], multivariate normal approximation bounds are given for sums of (locally) dependent random variables, often chosen in somewhat special ways, including certain statistics defined on random graphs. The results in the present paper have the advantage of being more generally applicable in geometric probability.

As a further contribution, we provide a general criterion for bounding variances below. It is characteristic of proofs of normality using Stein's method that one needs to prove such lower bounds separately. Our criterion adds to those given previously in [4,18], which are not always directly applicable in the present setting.

The rest of this paper is organized as follows. We state our main general results in Sections 2.1 and 2.2, and prove them in Sections 3 and 4. We briefly discuss in general terms the applicability of the general results in Section 2.3, and illustrate further with a concrete example (the k-nearest neighbour graph) in Section 5.

2 Main results

The basic setting follows that of [21]. Let $d \in \mathbb{N}$. As in [21], we consider marked point processes in $\mathbb{R}^d$ for the sake of generality. Let $(\mathcal{M}, \mathcal{F}_{\mathcal{M}}, \mathbb{P}_{\mathcal{M}})$ be a probability space (the mark space). Let $\xi(x, s; \mathcal{X})$ be a measurable $[0, \infty)$-valued function defined for all triples $(x, s; \mathcal{X})$, where $x \in \mathbb{R}^d$, $s \in \mathcal{M}$ are such that $(x, s) \in \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$ is finite. When $(x, s) \in (\mathbb{R}^d \times \mathcal{M}) \setminus \mathcal{X}$, we abbreviate notation and write $\xi(x, s; \mathcal{X})$ instead of $\xi(x, s; \mathcal{X} \cup \{(x, s)\})$.

Given $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$, $a > 0$ and $y \in \mathbb{R}^d$, set $y + a\mathcal{X} := \{(y + ax, s) : (x, s) \in \mathcal{X}\}$, i.e. translation and scaling act only on the 'spatial' part of $\mathcal{X}$. For all $\lambda > 0$ let

$$\xi_\lambda(x, s; \mathcal{X}) := \xi(x, s; x + \lambda^{1/d}(-x + \mathcal{X})).$$

Thus ξ_λ is a 'scaled-up' version of ξ, defined on a scaled-up version of the (marked) point set $\mathcal{X}$ dilated around x. We say that ξ is *translation-invariant* if $\xi(x + y, s; y + \mathcal{X}) = \xi(x, s; \mathcal{X})$ for all $y \in \mathbb{R}^d$, all $(x, s) \in \mathbb{R}^d \times \mathcal{M}$ and all finite $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$. When ξ is translation-invariant, the functional ξ_λ simplifies to $\xi_\lambda(x, s; \mathcal{X}) = \xi(\lambda^{1/d}x, s; \lambda^{1/d}\mathcal{X})$.

For $q \in [1, \infty]$, let $\| \cdot \|_q$ denote the ℓ_q norm on $\mathbb{R}^d$. In the sequel we will use $q = 2$ (the Euclidean norm) and $q = \infty$. For measurable $B \subset \mathbb{R}^d$, let $|B|$ denote the (d-dimensional) Lebesgue measure of B.

Let κ be a probability density function on $\mathbb{R}^d$ with compact support $A \subset \mathbb{R}^d$, where A is non-null (i.e. $|A| > 0$). We assume throughout that κ is bounded with supremum denoted by $\|\kappa\|_\infty < \infty$. For all $\lambda > 0$ let $\mathcal{P}_\lambda$ denote a Poisson point process in $\mathbb{R}^d \times \mathcal{M}$ with intensity measure $(\lambda\kappa(x)\mathrm{d}x) \times \mathbb{P}_{\mathcal{M}}(\mathrm{d}s)$.

2.1 Multivariate normal approximation

We use the following notion of exponential stabilization, as given in [21] (taking the A_λ there to be A for all λ). For $x \in \mathbb{R}^d$ and $r > 0$, let $B_r(x)$ denote the Euclidean ball centred at x of radius r. Let U denote a random element of $\mathcal{M}$ with distribution $\mathbb{P}_{\mathcal{M}}$, independent of $\mathcal{P}_\lambda$.

Definition 2.1 ξ *is* exponentially stabilizing *with respect to κ and A if for all $\lambda \geq 1$ and all $x \in A$, there exists a random variable $R := R(x,\lambda)$, (a* radius of stabilization *for ξ at x) such that*

$$\xi_\lambda(x, U; [\mathcal{P}_\lambda \cap (B_{\lambda^{-1/d}R}(x) \times \mathcal{M})] \cup \mathcal{X}) = \xi_\lambda(x, U; \mathcal{P}_\lambda \cap (B_{\lambda^{-1/d}R}(x) \times \mathcal{M})),$$

for all finite $\mathcal{X} \subset (A \setminus B_{\lambda^{-1/d}R}(x)) \times \mathcal{M}$, and moreover

$$\limsup_{t\to\infty} t^{-1} \log \left(\sup_{\lambda \geq 1, x \in A} \mathbb{P}[R(x,\lambda) > t] \right) < 0.$$

Roughly speaking, $R(x,\lambda)$ is a radius of stabilization if the value of ξ_λ at x is unaffected by changes to the configuration of Poisson points outside $B_{\lambda^{-1/d}R}(x)$. Exponential stabilization is known to hold for many 'locally determined' functionals defined on spatial point processes, and in particular in several cases of interest in geometric probability; see for example [21]. Following [21], we also make the following definition.

Definition 2.2 *ξ has a moment of order $p > 0$ (with respect to κ and A) if*

$$\sup_{\lambda \geq 1, x \in A} \mathbb{E}\left[|\xi_\lambda(x, U; \mathcal{P}_\lambda)|^p\right] < \infty. \tag{2.1}$$

For $\lambda > 0$, we define the random weighted point measure μ_λ^ξ on $\mathbb{R}^d$, induced by ξ_λ, by

$$\mu_\lambda^\xi := \sum_{(x,s) \in \mathcal{P}_\lambda \cap (A \times \mathcal{M})} \xi_\lambda(x, s; \mathcal{P}_\lambda)\delta_x,$$

where δ_x is the point measure at $x \in \mathbb{R}^d$.

For $\Gamma \subset \mathbb{R}^d$, let $\mathcal{B}(\Gamma)$ denote the set of bounded Borel-measurable functions on Γ. For $f \in \mathcal{B}(\Gamma)$, let $\langle f, \mu_\lambda^\xi \rangle := \int_\Gamma f \mathrm{d}\mu_\lambda^\xi$. Let Φ denote, as usual, the standard normal distribution function on $\mathbb{R}$. We recall the following univariate normal approximation result of Penrose and Yukich (contained in Theorem 2.1 of [21]).

Proposition 2.1 *[21] Let ξ be exponentially stabilizing and satisfy the moment condition (2.1) for some $p > 3$. For Γ a non-null Borel subset of A, let $f \in \mathcal{B}(\Gamma)$ and put $T := \langle f, \mu_\lambda^\xi \rangle$. Then there exists a constant $C \in (0,\infty)$ depending on d, ξ, f, and κ such that for all $\lambda \geq 2$,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left[\frac{T - \mathbb{E}[T]}{(\mathrm{Var}[T])^{1/2}} \leq t \right] - \Phi(t) \right| \leq C(\log \lambda)^{3d} \lambda (\mathrm{Var}[T])^{-3/2}. \tag{2.2}$$

For fixed $m \in \mathbb{N}$, let Γ_i, $i = 1, \ldots, m$ be non-null Borel subsets of $A \subset \mathbb{R}^d$. For notational simplicity, for $i = 1, \ldots, m$ and for $f_i \in \mathcal{B}(\Gamma_i)$ set $T_i := \langle f_i, \mu_\lambda^\xi \rangle = \int_{\Gamma_i} f_i \mathrm{d}\mu_\lambda^\xi$. These are the quantities of interest to us in the present paper. By Proposition 2.1, under appropriate conditions, we have that, individually, each T_i satisfies a normal approximation result of the form of (2.2). For the present paper, we will impose one extra condition to control variances such as $\mathrm{Var}[T_i]$.

(A1) There exist constants $C_i \in (0,\infty)$ such that for each i, for all λ sufficiently large, $\mathrm{Var}[T_i] \geq C_i\lambda$.

Under assumption (A1), the bound on the rate of convergence on the right of (2.2) (in the case $T = T_i$) becomes $O(\lambda^{-1/2}(\log\lambda)^{3d})$ (compare Corollary 2.1 of [21]), and in particular (2.2) yields the central limit theorems

$$\frac{T_i - \mathbb{E}[T_i]}{(\mathrm{Var}[T_i])^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

as $\lambda \to \infty$, where $\mathcal{N}(0,1)$ is the standard normal distribution on $\mathbb{R}$ and '$\xrightarrow{\mathcal{D}}$' denotes convergence in distribution. As discussed in Section 2.2, condition (A1) is true in many cases.

Our main result, Theorem 2.1 below, extends Proposition 2.1 to give a multivariate central limit theorem for $(T_i : i = 1,\dots,m)$, centred and scaled, with a bound on the rate of convergence. We impose the additional assumptions that (A1) holds and that the sub-regions Γ_i are pairwise disjoint and satisfy the natural regularity condition (A2) below. The central difficulty in extending Proposition 2.1 to a multivariate version is that the T_i are not, in general, independent. However, with the aid of stabilization we will show that they are 'asymptotically independent' in an appropriate sense.

To state (A2), we introduce some notation. Let ∂B denote the boundary of $B \subset \mathbb{R}^d$. For $B \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ let $d_q(x,B) := \inf_{y\in B}\|x-y\|_q$. Also, for $B, B' \subset \mathbb{R}^d$ with $B \cap B' = \emptyset$, let $d_q(B,B') := \inf_{x\in B, y\in B'}\|x-y\|_q$, i.e. the shortest distance (in the ℓ_q sense) between B and B'. For $r > 0$, let $\partial_r(B)$ denote the r-neighbourhood of the boundary of $B \subset \mathbb{R}^d$ in the ℓ_∞ norm, that is the set $\{x \in \mathbb{R}^d : d_\infty(x,\partial B) \leq r\}$.

(A2) For each i, $|\partial_r(\Gamma_i)| = O(r)$ as $r \downarrow 0$.

Sufficient conditions for (A2) include that each of the Γ_i is convex, or each is the finite union of convex regions (e.g. polyhedral). We can now state our main result.

Theorem 2.1 *Let ξ be exponentially stabilizing and satisfy the moment condition (2.1) for all $p \geq 1$. Let $m \in \mathbb{N}$. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ be fixed disjoint non-null Borel subsets of A satisfying (A2). For $i = 1,\dots,m$, let $f_i \in \mathcal{B}(\Gamma_i)$ and set $T_i := \langle f_i, \mu_\lambda^\xi\rangle$. Suppose that (A1) holds. Let $\varepsilon > 0$. Then there exists a constant $C \in (0,\infty)$ depending on d, m, ξ, κ, ε, $\{f_i\}$ and $\{\Gamma_i\}$, such that, for all $\lambda \geq 1$,*

$$\sup_{t_1,\dots,t_m\in\mathbb{R}}\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{\frac{T_i - \mathbb{E}[T_i]}{(\mathrm{Var}[T_i])^{1/2}} \leq t_i\right\}\right] - \prod_{i=1}^m \Phi(t_i)\right| \leq C\lambda^{-1/(2d+\varepsilon)}. \tag{2.3}$$

In particular, from (2.3) we obtain the multivariate central limit theorem that says

$$\left(\frac{T_i - \mathbb{E}[T_i]}{(\mathrm{Var}[T_i])^{1/2}} : i = 1,\dots,m\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, I_m), \tag{2.4}$$

as $\lambda \to \infty$, where $\mathcal{N}(0,I_m)$ is the m-dimensional normal distribution with mean 0 and covariance matrix given by the identity matrix I_m. It was already known [4, 13, 15] that under similar conditions to those of Theorem 2.1 we have (2.4), at least when $\lambda^{-1}\mathrm{Var}[T_i] \to \sigma_i^2$ for some $\sigma_i^2 \in (0,\infty)$. Theorem 2.1 adds to this by providing a bound on the rate of convergence.

As an example of the application of Theorem 2.1, one can take $f_i = \mathbf{1}_{\Gamma_i}$ for $i = 1,2,\dots,m$, where $\mathbf{1}_\Gamma$ is the indicator function of $\Gamma \subset \mathbb{R}^d$. We indicate some particular applications of Theorem 2.1 in Sections 2.3 and 5. Under additional technical conditions, one can say more about the asymptotic behaviour of the variance terms in (2.3); see Section 2.2 below.

Remark. The relatively slow rate of convergence in higher dimensions arises primarily due to the possibility of strongly dependent points in the neighbourhood of the interface of adjacent regions. If all of the Γ_i are separated by a strictly positive distance, then our methods can be adapted to yield a rate of convergence of the same order as in the univariate result (Proposition 2.1), that is $O(\lambda^{-1/2}(\log \lambda)^{3d})$.

For ease of presentation, we prove Theorem 2.1 in Section 3 under the conditions that ξ is translation-invariant and that the mark space is degenerate (i.e. $\mathcal{M} = \{1\}$), and so from now on we suppress any mention of $\mathcal{M}$. In particular, point sets such as $\mathcal{X}$ and $\mathcal{P}_\lambda$ will be treated as (their corresponding) subsets of $\mathbb{R}^d$, and we will write $\xi(x; \mathcal{X})$ rather than $\xi(x, 1; \mathcal{X})$ (and similarly with ξ_λ). The proof can be adapted for the general marked case, as in [21].

2.2 Control of variances

Recall that Theorem 2.1 is stated under assumption (A1). In this section we discuss conditions under which one can say something about the variances $\mathrm{Var}[T_i]$, and in particular give criteria for verifying (A1).

It is known [4,15] that if ξ stabilizes exponentially, has a moment of order $p > 2$, and satisfies certain extra conditions in the same spirit (see e.g. Theorem 2.1 of [15] for a clear statement), then

$$\lambda^{-1}\mathrm{Var}[T_i] \to \sigma_i^2, \tag{2.5}$$

for some $\sigma_i^2 \in [0, \infty)$, given explicitly as an integral in [4, 15]. In cases where (2.5) holds, then clearly (A1) is equivalent to the condition that $\sigma_i^2 > 0$.

One approach to the verification of (A1) is to compute σ_i^2 by evaluating the integral formula explicitly or at least to verify that it is strictly positive. In some cases, this is possible; see Section 5 for an example.

However, such cases seem to be the exception rather than the rule, so it is useful to have some other criterion for verifying (A1). No attempt is made in [15] to establish general conditions for σ_i^2 to be strictly positive. While [4] does contain such conditions, these are given only for $f_i \equiv 1$, and since here we are interested in multivariate CLTs with at least two different f_is, it is desirable to improve on the criteria in [4].

We now give a sufficient condition for (A1) to hold, similar in spirit to those used to bound limiting variances away from zero in [18] and [4]. As in those papers, we use a form of *external* stabilization, which roughly speaking says that not only do Poisson points beyond the radius of stabilization for x not influence x, but also x does not influence these points.

For notational convenience we here consider only the unmarked case (equivalent to $\mathcal{M} = \{1\}$). For simplicity, *we also assume translation-invariance.* Given translation-invariant $\xi(x; \mathcal{X})$ defined for $x \in \mathcal{X}$ and finite $\mathcal{X} \subset \mathbb{R}^d$, set

$$H^\xi(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x; \mathcal{X}); \qquad H_\lambda^{\xi,f}(\mathcal{X}) := \sum_{x \in \mathcal{X}} f(x)\xi_\lambda(x; \mathcal{X}).$$

Let $\mathbf{0}$ be the origin in $\mathbb{R}^d$ and write B_r for the ball $B_r(\mathbf{0})$. For $K > 0$, a locally finite set $\mathcal{X}$ in $\mathbb{R}^d$ is said to be K-*externally stable* if for all finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_K$, $x \in \mathcal{A}$ and $y \in \mathcal{X} \cap B_K$, we have the three conditions:

$$\begin{aligned}
\xi(\mathbf{0}; (\mathcal{X} \cap B_K) \cup \mathcal{A}) &= \xi(\mathbf{0}; \mathcal{X} \cap B_K); \\
\xi(x; (\mathcal{X} \cap B_K) \cup \mathcal{A} \cup \{\mathbf{0}\}) &= \xi(x; (\mathcal{X} \cap B_K) \cup \mathcal{A}); \\
\xi(y; (\mathcal{X} \cap B_K) \cup \mathcal{A} \cup \{\mathbf{0}\}) - \xi(y; (\mathcal{X} \cap B_K) \cup \mathcal{A}) &= \xi(y; (\mathcal{X} \cap B_K) \cup \{\mathbf{0}\}) - \xi(y; \mathcal{X} \cap B_K).
\end{aligned}$$

Suppose $\mathcal{X}$ is a point process (i.e. a random locally finite set in $\mathbb{R}^d$). A random variable R, taking values in $(0, \infty]$, is said to be a *radius of external stabilization* for $\mathcal{X}$ if, whenever R is finite, the point process $\mathcal{X}$ is R-externally stable. Note that by definition $R \equiv \infty$ is a radius of external stabilization but we are interested in the cases where a radius of external stabilization R exists with $\mathbb{P}[R < \infty] > 0$. When $R < \infty$, we say that *external stabilization* holds for $\mathcal{X}$.

For a point process $\mathcal{X}$ in $\mathbb{R}^d$ with radius of external stabilization R, there is a random variable $\Delta(\mathcal{X}; R)$ such that whenever $R < \infty$, we have

$$H^\xi((\mathcal{X} \cap B_R) \cup \mathcal{A} \cup \{\mathbf{0}\}) - H^\xi((\mathcal{X} \cap B_R) \cup \mathcal{A}) = \Delta(\mathcal{X}; R)$$

for all finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_R$. As a matter of convention we put $\Delta(\mathcal{X}; R) = 0$ whenever $R = \infty$. We refer to $\Delta(\mathcal{X}; R)$ as the *add one cost* associated with $\mathcal{X}$ and R. In principle, the external stabilization condition $R < \infty$ is a stronger condition than the existence of this $\Delta(\mathcal{X}; R)$, which is referred to as 'strong stabilization' in [4]. In all the examples that we are aware of, however, if strong stabilization holds then so does external stabilization. Also, the external stabilization condition that we shall require is only that $\mathbb{P}[R < \infty] > 0$, whereas the corresponding strong stabilization in [4] is required to hold with probability 1.

Let $\tilde{\mathcal{B}}(\Gamma)$ be the class of functions in $\mathcal{B}(\Gamma)$ which are almost everywhere continuous. Let X denote a random d-vector with density κ, independent of $\mathcal{P}_\lambda$, and for $\lambda > 0$ let $\mathcal{H}_\lambda$ denote a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity λ. We now state our second result, which provides general criteria for (A1) to hold based on external stabilization. It is proved in Section 4.

Theorem 2.2 *Suppose that ξ is translation-invariant. Suppose that there is a radius of external stabilization R for $\mathcal{H}_1$ with associated add one cost $\Delta(\mathcal{H}_1; R)$, such that*

$$\mathbb{P}[\{R < \infty\} \cap \{\Delta(\mathcal{H}_1; R) \neq 0\}] > 0. \tag{2.6}$$

Let $f \in \tilde{\mathcal{B}}(\Gamma)$, with $f(x)\kappa(x)$ not almost everywhere zero, and suppose for some $\lambda_0 > 0$ and $p > 2$ that

$$\sup_{\lambda \geq \lambda_0} \mathbb{E}\left[\left|H_\lambda^{\xi,f}(\mathcal{P}_\lambda \cup \{X\}) - H_\lambda^{\xi,f}(\mathcal{P}_\lambda)\right|^p\right] < \infty. \tag{2.7}$$

Suppose, also, either that $f \equiv 1$ or that for all $K > 0$ and Lebesgue-almost all $x \in \mathbb{R}^d$ with $\kappa(x) > 0$ we have

$$\int_{B_K} \mathbb{E}[|\xi(y; \mathcal{H}_{\kappa(x)} \cup \{\mathbf{0}\}) - \xi(y; \mathcal{H}_{\kappa(x)})|]\mathrm{d}y < \infty. \tag{2.8}$$

Then $\liminf_{\lambda \to \infty}(\lambda^{-1}\mathrm{Var}[\langle f, \mu_\lambda^\xi \rangle]) > 0$.

Remarks. The additional moments condition (2.7) is weaker in principle than the corresponding moments conditions required for similar lower bounds on variances in [18] and [4] because in those references the moments condition is for a point inserted into a binomial point process but here the moments conditions (2.7) and (2.8) are for a point inserted into a Poisson point process. Note that the argument in [4] requires the 'bounded moments condition' (eqn (2.5) of [4]) as a condition for parts (ii) and (iii) of Theorem 2.2 of [4] as well as for part (i); we thank Joseph Yukich for confirming this in a personal communication.

As in [4, 18], Theorem 2.2 requires external stabilization. Our condition (2.6) says that the add one cost $\Delta(\mathcal{X}; R)$ is not identically zero. In principle this is weaker than the corresponding condition in [4, 18] where it is assumed that $\Delta(\mathcal{X}; R)$ has a non-degenerate distribution. These slightly weaker conditions are possible because in the current paper we are considering only Poisson point processes, not binomial point processes as in [4, 18].

The conclusion in Theorem 2.2 provides a lower bound for the variance of $\lambda^{-1/2}\langle f, \mu_\lambda^\xi \rangle$, as required in condition (A1). The lower bounds for the limiting variances given in [4, 18] (see e.g. Theorem 2.2(iii) and the subsequent Remark (iii) of [4]) are given only for the case with $f \equiv 1$. Here, in Theorem 2.2 we allow for other choices of f but restrict attention to almost everywhere continuous f and also (except for the case $f \equiv 1$) require the extra moments condition (2.8). We note that a sufficient condition for (2.8) is that we have the two conditions

$$\int_{B_K} \mathbb{E}[|\xi(y; \mathcal{H}_{\kappa(x)} \cup \{\mathbf{0}\})|]\mathrm{d}y < \infty; \tag{2.9}$$

$$\int_{B_K} \mathbb{E}[|\xi(y; \mathcal{H}_{\kappa(x)})|]\mathrm{d}y < \infty. \tag{2.10}$$

Condition (2.10) is similar to the moments condition (2.1) used earlier. Condition (2.9) is slightly different but usually true in examples satisfying (2.1). For example, if $\xi(x; \mathcal{X})$ is the logarithm of the distance from x to its nearest neighbour in $\mathcal{X}$ (examples of this type appear in [5]), then the integrand in (2.9) blows up as y approaches $\mathbf{0}$, but only slowly, and (2.9) holds.

2.3 Indication of applications

In applying Theorem 2.1, one needs to check that the stabilization and moments conditions given in Definitions 2.1 and 2.2 hold. These conditions, or related versions thereof, are known to hold for many problems of interest in geometric probability; see [4] and [21] for an indication of problems for which exponential stabilization and moment bounds are satisfied.

One also needs to verify the variance bound (A1), as discussed in Section 2.2. For the special case with f_i constant, condition (A1) has been demonstrated for many examples, see for example [2, 4, 18].

For the general case under consideration here, we can often verify (A1) via Theorem 2.2. As mentioned before, the moments conditions in Theorem 2.2 usually hold for examples satisfying the moments condition of Definition 2.2. Likewise, external stabilization holds for many of the examples satisfying the exponential stabilization condition of Definition 2.1. In fact, external stabilization is demonstrated for numerous examples in [18]; see also [4, 19]. In many cases, external stabilization can be shown by constructing a configuration, having positive probability, of many points in an 'annulus' around the origin and an empty 'moat' in a smaller annulus, that ensures sufficient independence; see [2] for such a construction (in a similar context but not explicitly mentioning stabilization) for the total length of the j-th nearest-neighbour, Voronoi, and Delaunay graphs. These examples are also considered in [18], along with other examples such as the sphere of influence graph and Gabriel graph. External stabilization for random sequential adsorption and related deposition processes (with Poisson input) is demonstrated in [19].

To sum up; for many examples the conditions of both Theorem 2.1 and Theorem 2.2 hold, with Theorem 2.2 providing the means to verify the condition (A1) for Theorem 2.1, so that one can conclude (2.3) in these examples.

In Section 5 we give an example of our result as applied to the k-nearest neighbour graph. In particular, we give a multivariate CLT with explicit variance scalings in the case of the nearest-neighbour (directed) graph on disjoint subsets of the real line (Theorem 5.1 below).

3 Towards a proof of Theorem 2.1

Throughout this section, we assume that $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are (arbitrary) non-null Borel subsets of the bounded region $A \subset \mathbb{R}^d$, such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and condition (A2) holds. Also, for each i we have a function $f_i \in \mathcal{B}(\Gamma_i)$.

For fixed $\alpha > 0$, let $s_\lambda := \alpha\lambda^{-1/d}\log\lambda$, and let Γ_i^{bd} denote the s_λ 'boundary region' of $\Gamma_i \subseteq A$, in the sense

$$\Gamma_i^{\mathrm{bd}} := \left\{x \in \Gamma_i : d_\infty(x, \partial\Gamma_i) \leq \alpha\lambda^{-1/d}\log\lambda\right\} = \Gamma_i \cap \partial_{s_\lambda}(\Gamma_i). \tag{3.1}$$

The remainder of the set Γ_i we simply call the 'interior' and denote by Γ_i^{in}, where

$$\Gamma_i^{\mathrm{in}} := \left\{x \in \Gamma_i : d_\infty(x, \partial\Gamma_i) > \alpha\lambda^{-1/d}\log\lambda\right\} = \Gamma_i \setminus \partial_{s_\lambda}(\Gamma_i).$$

As previously mentioned, we assume that ξ is translation-invariant, and that $\mathcal{M} = \{1\}$.

Define

$$T_i^{\mathrm{bd}} := \int_{\Gamma_i^{\mathrm{bd}}} f_i \mathrm{d}\mu_\lambda^\xi; \quad \text{and} \quad T_i^{\mathrm{in}} := \int_{\Gamma_i^{\mathrm{in}}} f_i \mathrm{d}\mu_\lambda^\xi,$$

so that $T_i = T_i^{\mathrm{in}} + T_i^{\mathrm{bd}}$. To prepare for the proof of Theorem 2.1 we need some auxiliary lemmas. For the subsequent results, we will need the following covering of scaled-up Borel regions $\lambda^{1/d}B \subset \mathbb{R}^d$ by cubes of side 1.

First we need some more notation. Let $\mathrm{card}(\mathcal{X})$ denote the cardinality of set $\mathcal{X}$. For $x \in \mathbb{R}^d$, let Q_x denote the unit-volume ℓ_∞ ball in $\mathbb{R}^d$ with centre x (i.e., the unit d-cube at x). For a Borel set $B \subseteq A \subset \mathbb{R}^d$, let

$$\mathcal{Z}_\lambda(B) := \left\{x \in \mathbb{Z}^d : Q_x \cap \lambda^{1/d}B \neq \emptyset\right\}, \tag{3.2}$$

and set $n_\lambda(B) := \mathrm{card}(\mathcal{Z}_\lambda(B))$. Then the covering of $\lambda^{1/d}B$ is

$$\mathcal{Q}_\lambda(B) := \{Q_z : z \in \mathcal{Z}_\lambda(B)\}. \tag{3.3}$$

The next result gives error bounds for approximating the volume of $\lambda^{1/d}\Gamma_i$ or of $\lambda^{1/d}\Gamma_i^{\mathrm{bd}}$ (as defined at (3.1)) by the number of unit cubes in $\mathbb{Z}^d$ in its covering (as defined at (3.2) and (3.3)).

Lemma 3.1 *Let Γ_i be a non-null Borel subset of $A \subset \mathbb{R}^d$ such that $|\partial_r(\Gamma_i)| = O(r)$ as $r \downarrow 0$. Then, as $\lambda \to \infty$,*

$$n_\lambda(\Gamma_i) - |\lambda^{1/d}\Gamma_i| = O\left(\lambda^{(d-1)/d}\right). \tag{3.4}$$

Define Γ_i^{bd} as at (3.1). Then, as $\lambda \to \infty$,

$$n_\lambda(\Gamma_i^{\mathrm{bd}}) - |\lambda^{1/d}\Gamma_i^{\mathrm{bd}}| = O\left(\lambda^{(d-1)/d}\log\lambda\right). \tag{3.5}$$

Proof. There exists a constant $c \in (0, \infty)$ (depending only on d) such that, for any $\lambda > 0$, and any non-null Borel subset B of A,

$$\lambda^{1/d} B \subseteq \bigcup_{z \in \mathcal{Z}_\lambda(B)} Q_z \subseteq \lambda^{1/d} B \cup \partial_c(\lambda^{1/d} B),$$

and hence

$$|\lambda^{1/d} B| \le n_\lambda(B) \le |\lambda^{1/d} B| + |\partial_c(\lambda^{1/d} B)| = |\lambda^{1/d} B| + \lambda |\partial_{c\lambda^{-1/d}}(B)|. \tag{3.6}$$

In the case $B = \Gamma_i$, the regularity assumption that $|\partial_r(\Gamma_i)| = O(r)$ as $r \downarrow 0$ implies that $|\partial_{c\lambda^{-1/d}}(\Gamma_i)| = O(\lambda^{-1/d})$. Thus (3.4) follows from (3.6).

In the case $B = \Gamma_i^{\mathrm{bd}}$, we have that

$$|\partial_{c\lambda^{-1/d}}(\Gamma_i^{\mathrm{bd}})| \le |\partial_{c\lambda^{-1/d} + s_\lambda}(\Gamma_i)| = O(s_\lambda),$$

as $\lambda \to \infty$, again by the regularity assumption on Γ_i. Thus (3.6) yields (3.5) in this case. $\square$

Once more consider a Borel subset B of $A \subset \mathbb{R}^d$ and the covering $\mathcal{Q}_\lambda(B)$ of $\lambda^{1/d} B$. For all $z \in \mathcal{Z}_\lambda(B)$, the number of points of $\mathcal{P}_\lambda \cap \lambda^{-1/d} Q_z$ is a Poisson random variable N_z with parameter $\nu_z := \lambda \int_{\lambda^{-1/d} Q_z} \kappa(x) \mathrm{d}x$. Assuming $\nu_z > 0$, choose an ordering on the points of $\mathcal{P}_\lambda \cap \lambda^{-1/d} Q_z$ uniformly at random from all $N_z!$ possibilities. List the points as $X_{z,1}, \ldots, X_{z,N_z}$, where conditional on the value of N_z, the random variables $X_{z,k}$, $k = 1, 2, \ldots, N_z$ are i.i.d. on $\lambda^{-1/d} Q_z$ with a density $\kappa_z(\cdot) := \kappa(\cdot) / \int_{\lambda^{-1/d} Q_z} \kappa(x) \mathrm{d}x$. Thus we have the representation

$$\mathcal{P}_\lambda \cap B = \bigcup_{z \in \mathcal{Z}_\lambda(B)} \bigcup_{k=1}^{N_z} (\{X_{z,k}\} \cap B).$$

Then for f in $\mathcal{B}(B)$, we can express $\langle f, \mu_\lambda^\xi \rangle$ as follows:

$$\langle f, \mu_\lambda^\xi \rangle = \sum_{z \in \mathcal{Z}_\lambda(B)} \sum_{k=1}^{N_z} \xi_\lambda(X_{z,k}; \mathcal{P}_\lambda) \cdot f(X_{z,k}) \cdot \mathbf{1}_B(X_{z,k}). \tag{3.7}$$

For all $z \in \mathcal{Z}_\lambda(B)$ and for all $k \in \mathbb{N}$, let $R_{z,k}$ denote the radius of stabilization of ξ at $X_{z,k}$ if $1 \le k \le N_z$ and let $R_{z,k} = 0$ otherwise. Define the event $E_{z,k} := \{R_{z,k} \le \alpha \log \lambda\}$. We define here the function $\tilde{T}(B; f)$ as follows, the idea being that $\tilde{T}(B; f)$ is, with high probability, the same as $\langle f, \mu_\lambda^\xi \rangle$, but exhibits a much more localized dependency structure. Set

$$\tilde{T}(B; f) := \sum_{z \in \mathcal{Z}_\lambda(B)} \sum_{k=1}^{N_z} \xi_\lambda(X_{z,k}; \mathcal{P}_\lambda) \cdot \mathbf{1}_{E_{z,k}} \cdot f(X_{z,k}) \cdot \mathbf{1}_B(X_{z,k}), \tag{3.8}$$

where we use $\mathbf{1}_E$ to denote the indicator random variable of the event E.

Recall that Γ_i, $i = 1, 2, \ldots, m$ are disjoint non-null Borel regions in $A \subset \mathbb{R}^d$ and $f_i \in \mathcal{B}(\Gamma_i)$ for $i = 1, 2, \ldots, m$. Then for each i, $\tilde{T}(\Gamma_i; f_i)$ is defined by (3.8). In the same way as we use the abbreviations T_i, T_i^{bd} and T_i^{in}, we let $\tilde{T}_i := \tilde{T}(\Gamma_i; f_i)$, $\tilde{T}_i^{\mathrm{bd}} := \tilde{T}(\Gamma_i^{\mathrm{bd}}; f_i)$, and $\tilde{T}_i^{\mathrm{in}} := \tilde{T}(\Gamma_i^{\mathrm{in}}; f_i)$. Thus $\tilde{T}_i = \tilde{T}_i^{\mathrm{bd}} + \tilde{T}_i^{\mathrm{in}}$.

For $z \in \mathcal{Z}_\lambda(B)$ let $Y_z(B; f)$ be the contribution to $\tilde{T}(B; f)$ from the points in $\lambda^{-1/d} Q_z$, i.e.

$$Y_z(B; f) := \sum_{k=1}^{N_z} \xi_\lambda(X_{z,k}; \mathcal{P}_\lambda) \cdot \mathbf{1}_{E_{z,k}} \cdot f(X_{z,k}) \cdot \mathbf{1}_B(X_{z,k}), \tag{3.9}$$

so that $\tilde{T}(B;f) = \sum_{z\in\mathcal{Z}_\lambda(B)} Y_z(B;f)$.

Let A_λ, $\lambda \geq 1$ be a family of Borel subsets of $A \subset \mathbb{R}^d$. The next two results show that the moments condition (2.1) implies bounds on the moments of $Y_z(A_\lambda;f)$ for $f \in \mathcal{B}(A)$. When we come to apply the two lemmas below, we will be taking $A_\lambda = \Gamma_i$ or $A_\lambda = \Gamma_i^{\mathrm{bd}}$.

Lemma 3.2 *Let A_λ, $\lambda \geq 1$ be a family of Borel subsets of $A \subset \mathbb{R}^d$. If (2.1) holds for some $p > 0$, then there is a constant $C \in (0,\infty)$ such that for all $\lambda \geq 1$, all $k \geq 1$ and $z \in \mathcal{Z}_\lambda(A_\lambda)$*

$$\mathbb{E}[|\xi_\lambda(X_{z,k};\mathcal{P}_\lambda)\cdot \mathbf{1}_{A_\lambda}(X_{z,k})\cdot \mathbf{1}_{\{k\leq N_z\}}|^p] \leq C. \tag{3.10}$$

Proof. It suffices to consider the case with $A_\lambda = A$ for all λ. The proof of the lemma closely follows that of Lemma 4.2 in [21], although our covering is somewhat different. In the notation of the proof of Lemma 4.2 of [21], we have $\rho_\lambda = 1$ and $\nu_i = \nu_{z_i} \equiv \lambda\int_{\lambda^{-1/d}Q_{z_i}} \kappa(x)\mathrm{d}x \leq \|\kappa\|_\infty$, where we have written $\mathcal{Z}_\lambda(B) = \{z_1,\ldots,z_{n_\lambda(B)}\}$. Then, following the argument in [21], we obtain (3.10). $\square$

Lemma 3.3 *Let A_λ, $\lambda \geq 1$, be a sequence of Borel subsets of $A \subset \mathbb{R}^d$, and suppose $f \in \mathcal{B}(A)$. If (2.1) holds for some $p > 1$, then for any $q \in (1,p)$ there is a constant $C \in (0,\infty)$ such that for all $\lambda \geq 1$ and all $z \in \mathcal{Z}_\lambda(A_\lambda)$*

$$\|Y_z(A_\lambda;f)\|_q^q \leq C. \tag{3.11}$$

Proof. The proof closely follows that of Lemma 4.3 in [21], again with ρ_λ there equal to 1 (and $\nu_i \leq \|\kappa\|_\infty$). Thus, with the use of Lemma 3.2 (and the boundedness of f), we obtain (3.11). $\square$

Lemma 3.4 *Suppose that ξ is exponentially stabilizing and satisfies the moments condition (2.1) for some $p > 3$. Then there exists a constant $C \in (0,\infty)$ such that for all $\lambda \geq 2$*

$$\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left[\frac{\tilde{T}_i - \mathbb{E}[\tilde{T}_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} \leq t\right] - \Phi(t)\right| \leq C\lambda(\mathrm{Var}[\tilde{T}_i])^{-3/2}(\log\lambda)^{3d}. \tag{3.12}$$

Moreover, (3.12) holds with $\tilde{T}_i$ replaced by $\tilde{T}_i^{\mathrm{in}}$ everywhere.

Proof. The statement for $\tilde{T}_i$ follows from equation (4.18) in [21] with $\rho_\lambda = O(\log\lambda)$, $q = 3$, and taking the A_λ of [21] to be Γ_i. In equation (4.18) of [21], T'_λ is the equivalent of our $\tilde{T}_i$, T_λ is our T_i, and S is our $(\tilde{T}_i - \mathbb{E}[\tilde{T}_i])(\mathrm{Var}[\tilde{T}_i])^{-1/2}$. The statement for $\tilde{T}_i^{\mathrm{in}}$ follows in the same way, this time taking the A_λ of [21] to be Γ_i^{in}. $\square$

Lemma 3.5 *Suppose that (2.1) holds for some $p > 2$. Then there exist constants $C_1, C_2, C_3 \in (0,\infty)$ such that, for all $\lambda \geq 2$,*

$$\mathrm{Var}[\tilde{T}_i^{\mathrm{bd}}] \leq C_1\lambda^{(d-1)/d}(\log\lambda)^{d+1}, \tag{3.13}$$

$$\mathrm{Var}[\tilde{T}_i] \leq C_2\lambda(\log\lambda)^d, \quad \textit{and} \tag{3.14}$$

$$\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}] \leq C_3\lambda(\log\lambda)^d. \tag{3.15}$$

Proof. First we prove (3.13). Consider the covering $\mathcal{Q}_\lambda(\Gamma_i^{\rm bd})$ of $\lambda^{1/d}\Gamma_i^{\rm bd}$ by unit d-cubes, as defined at (3.3). For $z \in \mathcal{Z}_\lambda(\Gamma_i^{\rm bd})$ let $Y_z(\Gamma_i^{\rm bd}; f_i)$ be the contribution to $\tilde{T}_i^{\rm bd}$ from the points in $\lambda^{-1/d}Q_z$, as defined at (3.9), that is

$$Y_z(\Gamma_i^{\rm bd}; f_i) := \sum_{k=1}^{N_z} \xi_\lambda(X_{z,k}; \mathcal{P}_\lambda) \cdot \mathbf{1}_{E_{z,k}} \cdot f_i(X_{z,k}) \cdot \mathbf{1}_{\Gamma_i^{\rm bd}}(X_{z,k}). \tag{3.16}$$

Now, using the representation $\tilde{T}_i^{\rm bd} = \sum_{z\in\mathcal{Z}_\lambda(\Gamma_i^{\rm bd})} Y_z(\Gamma_i^{\rm bd}; f_i)$, we have

$$\mathrm{Var}[\tilde{T}_i^{\rm bd}] = \sum_z \mathrm{Var}[Y_z(\Gamma_i^{\rm bd}; f_i)] + \sum_{z\neq w} \mathrm{Cov}[Y_z(\Gamma_i^{\rm bd}; f_i), Y_w(\Gamma_i^{\rm bd}; f_i)]. \tag{3.17}$$

By the assumption that (2.1) holds for some $p > 2$, by taking $q = 2$ and $A_\lambda = \Gamma_i^{\rm bd}$ in Lemma 3.3 we have that $\mathrm{Var}[Y_z(\Gamma_i^{\rm bd}; f_i)] \leq V$, for some constant $V < \infty$, for all $z \in \mathcal{Z}_\lambda(\Gamma_i^{\rm bd})$. So by the Cauchy-Schwarz inequality we have $\mathrm{Cov}[Y_z(\Gamma_i^{\rm bd}; f_i), Y_w(\Gamma_i^{\rm bd}; f_i)] \leq V$. Also, $Y_z(\Gamma_i^{\rm bd}; f_i)$ and $Y_w(\Gamma_i^{\rm bd}; f_i)$ are independent if $d_2(Q_z, Q_w) > 2\alpha \log\lambda$ (by the definition of $E_{z,k}$). Further, given z, the number of w for which $d_2(Q_z, Q_w) \leq 2\alpha\log\lambda$ is $O((\log\lambda)^d)$. Hence (3.17) implies that

$$\mathrm{Var}[\tilde{T}_i^{\rm bd}] \quad\leq\quad n_\lambda(\Gamma_i^{\rm bd})(V + O((\log\lambda)^d)). \tag{3.18}$$

Then by (3.5) we have that

$$n_\lambda(\Gamma_i^{\rm bd}) = \lambda|\Gamma_i^{\rm bd}| + O\left(\lambda^{(d-1)/d}\log\lambda\right) = O(\lambda^{(d-1)/d}\log\lambda), \tag{3.19}$$

using (3.1) and (A2). So from (3.18) and (3.19) we obtain (3.13).

The proof of (3.14) follows similarly, using $A_\lambda = \Gamma_i$ for all λ in Lemma 3.3 and (3.4) in place of (3.5). Finally, (3.15) follows from (3.14), (3.13) and the Cauchy-Schwarz inequality, since $\tilde{T}_i^{\rm in} = \tilde{T}_i - \tilde{T}_i^{\rm bd}$. $\square$

Lemma 3.6 *Suppose that ξ is exponentially stabilizing and satisfies the moments condition (2.1) for some $p > 3$. Then there exists a constant $C \in (0,\infty)$ such that for any $\delta > 0$, all $\lambda \geq 2$, and any $t \in \mathbb{R}$*

$$\mathbb{P}\left[\left|\frac{\tilde{T}_i - \mathbb{E}[\tilde{T}_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} - t\right| \leq \delta\right] \leq \sqrt{\frac{2}{\pi}}\delta + C(\log\lambda)^{3d}\lambda(\mathrm{Var}[\tilde{T}_i])^{-3/2}, \tag{3.20}$$

and also

$$\mathbb{P}\left[\left|\frac{\tilde{T}_i^{\rm in} - \mathbb{E}[\tilde{T}_i^{\rm in}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} - t\right| \leq \delta\right] \leq 2\sqrt{\frac{2}{\pi}}\delta + C(\log\lambda)^{3d}\lambda(\mathrm{Var}[\tilde{T}_i])^{-3/2} + \mathbb{P}\left[\left|\frac{\tilde{T}_i^{\rm bd} - \mathbb{E}[\tilde{T}_i^{\rm bd}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}}\right| > \delta\right]. \tag{3.21}$$

Proof. First we prove (3.20). For the duration of this proof, write

$$F(t) = \mathbb{P}\left[\frac{\tilde{T}_i - \mathbb{E}[\tilde{T}_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} \leq t\right].$$

Then we have that for $t \in \mathbb{R}$ and $\delta > 0$

$$
\begin{aligned}
& \mathbb{P}\left[\left|\frac{\tilde{T}_i - \mathbb{E}[\tilde{T}_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} - t\right| \leq \delta\right] = F(t+\delta) - F(t-\delta) \\
= \;& \Phi(t+\delta) - \Phi(t-\delta) + [F(t+\delta) - \Phi(t+\delta)] - [F(t-\delta) - \Phi(t-\delta)] \\
\leq \;& |\Phi(t+\delta) - \Phi(t-\delta)| + |F(t+\delta) - \Phi(t+\delta)| + |F(t-\delta) - \Phi(t-\delta)|\,.
\end{aligned}
$$

Then (3.20) follows from the Mean Value Theorem (applied to the first term on the right of the above inequality) and Lemma 3.4 (applied to the other two terms). Finally, we have that for $\delta > 0$

$$
\mathbb{P}\left[\left|\frac{\tilde{T}_i^{\mathrm{in}} - \mathbb{E}[\tilde{T}_i^{\mathrm{in}}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} - t\right| \leq \delta\right] \leq \mathbb{P}\left[\left|\frac{\tilde{T}_i - \mathbb{E}[\tilde{T}_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} - t\right| \leq 2\delta\right] + \mathbb{P}\left[\left|\frac{\tilde{T}_i^{\mathrm{bd}} - \mathbb{E}[\tilde{T}_i^{\mathrm{bd}}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}}\right| > \delta\right].
$$

Then using (3.20) yields (3.21). $\square$

Lemma 3.7 *Suppose that the moments condition (2.1) holds for all $p \geq 1$, and condition (A2) holds. Let k be an even positive integer. Then there exists a constant $C \in (0, \infty)$ (depending on k) such that for all $\lambda \geq 2$,*

$$
\mathbb{E}\left[\left|\tilde{T}_i^{\mathrm{bd}} - \mathbb{E}[\tilde{T}_i^{\mathrm{bd}}]\right|^k\right] \leq C\lambda^{k(d-1)/(2d)}(\log \lambda)^{k(1+d)/2}. \tag{3.22}
$$

Proof. Again consider the covering $\mathcal{Q}_\lambda(\Gamma_i^{\mathrm{bd}})$ of $\lambda^{1/d}\Gamma_i^{\mathrm{bd}}$ as defined at (3.3). For $z \in \mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})$, let $\bar{Y}_z$ be the contribution to $\tilde{T}_i^{\mathrm{bd}} - \mathbb{E}[\tilde{T}_i^{\mathrm{bd}}]$ from cube Q_z, that is $\bar{Y}_z := Y_z(\Gamma_i^{\mathrm{bd}}; f_i) - \mathbb{E}[Y_z(\Gamma_i^{\mathrm{bd}}; f_i)]$ where $Y_z(\Gamma_i^{\mathrm{bd}}; f_i)$ is given by (3.16). Thus, for all $z \in \mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})$, $\mathbb{E}[\bar{Y}_z] = 0$ and $\mathrm{Var}[\bar{Y}_z] \leq V$ for constant V, by Lemma 3.3.

Let k be an even positive integer. Then

$$
\mathbb{E}\left[\left|\tilde{T}_i^{\mathrm{bd}} - \mathbb{E}\tilde{T}_i^{\mathrm{bd}}\right|^k\right] = \sum_{z_1 \in \mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})} \sum_{z_2 \in \mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})} \cdots \sum_{z_k \in \mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})} \mathbb{E}\left[\bar{Y}_{z_1}\bar{Y}_{z_2}\cdots\bar{Y}_{z_k}\right].
$$

The term $\mathbb{E}\left[\bar{Y}_{z_1}\bar{Y}_{z_2}\cdots\bar{Y}_{z_k}\right]$ will vanish if any of the cubes corresponding to the $\bar{Y}_{z_j}$ is farther than $2\alpha\log\lambda$ from all the other cubes (since then it will be independent of the other $\bar{Y}_{z_j}$ and has expectation zero). Thus many of the terms in the last sum are zero. We proceed to count the possible non-zero contributions. Consider constructing the geometric graph (in the sense of [12]) on vertices $z_1, z_2, \ldots, z_k$: that is, connect by an edge any two vertices that lie within distance $2 + 2\alpha\log\lambda$ of each other. In particular, the absence of an edge between z_1 and z_2 implies that the cubes Q_{z_1}, Q_{z_2} are at distance more than $2\alpha\log\lambda$. For a non-zero contribution to the sum from a particular vertex set $z_1, z_2, \ldots, z_k$, a necessary condition is that the geometric graph just described has no isolated vertices. It follows that the graph must have no more than $k/2$ connected components. Let i be denoted a 'free' index if z_i is lexicographically the first vertex of a component of the graph; each 'free' index determines the location of a connected component. There are at most $k/2$ 'free' indices of $(z_1, \ldots, z_k)$. Each index that is not 'free' (and so lies in the same connected component as a 'free' index) has $O((\log\lambda)^d)$ possible values.

Further, $\mathbb{E}\left[\bar{Y}_{z_1}\bar{Y}_{z_2}\cdots\bar{Y}_{z_k}\right] \leq C$ for some constant C, by Lemma 3.3 (given the moments condition (2.1) for all $p \geq 1$) and Hölder's inequality. Thus for some other constant also denoted C,

$$\begin{aligned}\sum_{z_1\in\mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})}\sum_{z_2\in\mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})}\cdots\sum_{z_k\in\mathcal{Z}_\lambda(\Gamma_i^{\mathrm{bd}})}\mathbb{E}\left[\bar{Y}_{z_1}\bar{Y}_{z_2}\cdots\bar{Y}_{z_k}\right] &\leq C(n_\lambda(\Gamma_i^{\mathrm{bd}}))^{k/2}(\log\lambda)^{kd/2}\\ &\leq C\lambda^{k(d-1)/(2d)}(\log\lambda)^{k/2}(\log\lambda)^{kd/2},\end{aligned}$$

the final inequality by (3.5), (3.1) and (A2). Hence we have (3.22). □

The next lemma says that given condition (A1), we can obtain lower bounds on the variances of $\tilde{T}_i^{\mathrm{in}}$ and $\tilde{T}_i$. We will need the following result from [21] (see (4.17) therein), which says that if ξ is exponentially stabilizing and satisfies the moments condition (2.1) for some $p>2$, then

$$\left|\mathrm{Var}[\tilde{T}_i]-\mathrm{Var}[T_i]\right| \leq C\lambda^{-2}. \tag{3.23}$$

Lemma 3.8 *Suppose that (A1) and (A2) are satisfied, and that the moments condition (2.1) holds for all $p\geq 1$. Then there exist constants $C\in(0,\infty)$ and $\lambda_0\in[1,\infty)$ such that for all $\lambda\geq\lambda_0$*

$$\mathrm{Var}[\tilde{T}_i] \geq C\lambda, \quad \textit{and} \tag{3.24}$$

$$\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}] \geq C\lambda. \tag{3.25}$$

Proof. These follow in a straightforward manner from (3.23), (A1), (3.13), (3.14) and the Cauchy-Schwarz inequality. □

Lemma 3.9 *Suppose that ξ is exponentially stabilizing and satisfies the moments condition (2.1) for all $p\geq 1$. Suppose conditions (A1) and (A2) hold. Then for $m\in\mathbb{N}$ and any $\varepsilon>0$, there exists $C=C(m,\varepsilon)\in(0,\infty)$ such that for all $\lambda\geq 1$ and all $t_i\in\mathbb{R}$*

$$\begin{aligned}&\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{\frac{\tilde{T}_i-\mathbb{E}[\tilde{T}_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}}\leq t_i\right\}\right]-\prod_{i=1}^m\Phi(t_i)\right|\\ &\leq C\lambda^{-1/(2d+\varepsilon)}+\left|\prod_{i=1}^m\mathbb{P}\left[\frac{\tilde{T}_i^{\mathrm{in}}-\mathbb{E}[\tilde{T}_i^{\mathrm{in}}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}}\leq t_i\right]-\prod_{i=1}^m\Phi(t_i)\right|.\end{aligned} \tag{3.26}$$

Proof. We abbreviate our notation for the duration of the current proof by setting $\sigma_i := (\mathrm{Var}[\tilde{T}_i])^{1/2}$. Then we have

$$\begin{aligned}&\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{(\tilde{T}_i-\mathbb{E}[\tilde{T}_i])\sigma_i^{-1}\leq t_i\right\}\right]-\prod_{i=1}^m\Phi(t_i)\right|\\ \leq\ &\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{(\tilde{T}_i^{\mathrm{in}}-\mathbb{E}[\tilde{T}_i^{\mathrm{in}}])\sigma_i^{-1}\leq t_i\right\}\right]-\prod_{i=1}^m\Phi(t_i)\right|\\ &+\sum_{i=1}^m\mathbb{P}\left[(\tilde{T}_i^{\mathrm{in}}-\mathbb{E}[\tilde{T}_i^{\mathrm{in}}])\sigma_i^{-1}\leq t_i,\ (\tilde{T}_i-\mathbb{E}[\tilde{T}_i])\sigma_i^{-1}>t_i\right]\\ &+\sum_{i=1}^m\mathbb{P}\left[(\tilde{T}_i^{\mathrm{in}}-\mathbb{E}[\tilde{T}_i^{\mathrm{in}}])\sigma_i^{-1}>t_i,\ (\tilde{T}_i-\mathbb{E}[\tilde{T}_i])\sigma^{-1}\leq t_i\right].\end{aligned} \tag{3.27}$$

For random variables X, X_1, X_2 with $X = X_1 + X_2$ it holds that for any $t \in \mathbb{R}$ and any $z > 0$

$$\mathbb{P}[X_1 \le t, X > t] \le \mathbb{P}[|X_2| > z] + \mathbb{P}[|X_1 - t| \le z],$$

and the same upper bound holds for $\mathbb{P}[X_1 > t, X \le t]$. Hence for any $\lambda \ge 1$ and $\beta > 0$ we have

$$\begin{aligned} &\max\Big(\mathbb{P}\left[(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} \le t,\ (\tilde{T}_i - \mathbb{E}[\tilde{T}_i])\sigma_i^{-1} > t\right], \\ &\qquad \mathbb{P}\left[(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} > t,\ (\tilde{T}_i - \mathbb{E}[\tilde{T}_i])\sigma_i^{-1} \le t\right]\Big) \\ \le\ & \mathbb{P}\left[\left|(\tilde{T}_i^{\text{bd}} - \mathbb{E}[\tilde{T}_i^{\text{bd}}])\sigma_i^{-1}\right| > \lambda^{-\beta}\right] + \mathbb{P}\left[\left|(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} - t\right| \le \lambda^{-\beta}\right]. \end{aligned} \tag{3.28}$$

Then, from (3.27) and (3.28)

$$\begin{aligned} &\left|\mathbb{P}\left[\bigcap_{i=1}^m \left\{(\tilde{T}_i - \mathbb{E}[\tilde{T}_i])\sigma_i^{-1} \le t_i\right\}\right] - \prod_{i=1}^m \Phi(t_i)\right| \\ &\le \left|\mathbb{P}\left[\bigcap_{i=1}^m \left\{(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} \le t_i\right\}\right] - \prod_{i=1}^m \Phi(t_i)\right| \\ &+2\sum_{i=1}^m \mathbb{P}\left[\left|(\tilde{T}_i^{\text{bd}} - \mathbb{E}[\tilde{T}_i^{\text{bd}}])\sigma_i^{-1}\right| > \lambda^{-\beta}\right] + 2\sum_{i=1}^m \mathbb{P}\left[\left|(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} - t_i\right| \le \lambda^{-\beta}\right]. \end{aligned} \tag{3.29}$$

Since $d_2(\lambda^{1/d}\Gamma_i^{\text{in}}, \lambda^{1/d}\Gamma_j^{\text{in}})$ is at least $2\alpha \log \lambda$ for $i \ne j$, $\tilde{T}_i^{\text{in}}$, $1 \le i \le m$ is a sequence of mutually independent random variables, so that

$$\mathbb{P}\left[\bigcap_{i=1}^m \left\{(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} \le t_i\right\}\right] = \prod_{i=1}^m \mathbb{P}\left[(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} \le t_i\right]. \tag{3.30}$$

Also, from Markov's inequality, we have that, for $k \in 2\mathbb{N}$,

$$\mathbb{P}\left[\left|(\tilde{T}_i^{\text{bd}} - \mathbb{E}[\tilde{T}_i^{\text{bd}}])\sigma_i^{-1}\right| > \lambda^{-\beta}\right] \le \mathbb{E}\left[\left|\tilde{T}_i^{\text{bd}} - \mathbb{E}[\tilde{T}_i^{\text{bd}}]\right|^k\right]\left(\text{Var}[\tilde{T}_i]\right)^{-k/2}\lambda^{k\beta}. \tag{3.31}$$

Then we obtain, from (3.31), with (3.22) and (3.24),

$$\mathbb{P}\left[\left|(\tilde{T}_i^{\text{bd}} - \mathbb{E}[\tilde{T}_i^{\text{bd}}])\sigma_i^{-1}\right| > \lambda^{-\beta}\right] \le C\lambda^{k(\beta - 1/(2d))}(\log \lambda)^{k(1+d)/2}; \tag{3.32}$$

this then gives a bound for the penultimate sum in (3.29). To bound the final sum in (3.29), taking $\delta = \lambda^{-\beta}$ we have from (3.21), (3.32) and (3.24) that

$$\begin{aligned} &\mathbb{P}\left[\left|(\tilde{T}_i^{\text{in}} - \mathbb{E}[\tilde{T}_i^{\text{in}}])\sigma_i^{-1} - t_i\right| \le \lambda^{-\beta}\right] \\ &\le 2\sqrt{\frac{2}{\pi}}\lambda^{-\beta} + C(\log \lambda)^{3d}\lambda^{-1/2} + C\lambda^{k(\beta - 1/(2d))}(\log \lambda)^{k(1+d)/2}. \end{aligned} \tag{3.33}$$

To obtain the best rates of convergence via this method, we want to maximize the lowest power of λ^{-1} on the right-hand sides of (3.32) and (3.33). So we choose β such that $-\beta = k(\beta - 1/(2d))$, that is, take

$$\beta = \frac{k}{2d(k+1)}. \tag{3.34}$$

For any $\varepsilon > 0$ we can choose k large enough in (3.34) to give $1/(2d) > \beta \geq 1/(2d+\varepsilon/2)$. Then, for λ sufficiently large, $\lambda^{-1/(2d+\varepsilon)} \geq \lambda^{-1/(2d+\varepsilon/2)}(\log\lambda)^{k(1+d)/2}$. Now from (3.29) and (3.30), with the bounds (3.32) and (3.33) we obtain (3.26). This completes the proof of the lemma. □

Proof of Theorem 2.1. To complete the proof we proceed in a similar manner to [21]. Let

$$E_\lambda := \bigcap_{i=1}^{m} \bigcap_{z \in \mathcal{Z}_\lambda(\Gamma_i)} \bigcap_{k=1}^{N_z} E_{z,k},$$

recalling the definition of the event $E_{z,k}$ just below (3.7). By standard Palm theory (e.g. Theorem 1.6 in [12]) and exponential stabilization (see (4.11) in [21]), we have that $\mathbb{P}[E_\lambda^c] \leq C\lambda^{-3}$ for α sufficiently large and some $C \in (0,\infty)$. Then $|\tilde{T}_i - T_i| = 0$ except possibly on the set E_λ^c, which has probability less than $C\lambda^{-3}$.

For $i = 1,\dots,m$, let $K_i := (\mathrm{Var}[\tilde{T}_i])^{-1/2}(\tilde{T}_i - \mathbb{E}[\tilde{T}_i])$ and $Z_i := (\mathrm{Var}[\tilde{T}_i])^{-1/2}(T_i - \mathbb{E}[T_i])$. Then for $\delta > 0$ we have that for any $t_i \in \mathbb{R}$

$$\{(Z_i \leq t_i)\Delta(K_i \leq t_i)\} \subseteq \{|K_i - t_i| \leq \delta\} \cup \{|Z_i - K_i| \geq \delta\},$$

so that

$$\begin{aligned} \left|\mathbb{P}\left[\bigcap_{i=1}^{m}\{Z_i \leq t_i\}\right] - \prod_{i=1}^{m}\Phi(t_i)\right| &\leq \left|\mathbb{P}\left[\bigcap_{i=1}^{m}\{K_i \leq t_i\}\right] - \prod_{i=1}^{m}\Phi(t_i)\right| \\ &\quad + \sum_{i=1}^{m}\mathbb{P}\left[|K_i - t_i| \leq \delta\right] + \sum_{i=1}^{m}\mathbb{P}\left[|Z_i - K_i| \geq \delta\right]. \end{aligned} \tag{3.35}$$

Then, using (3.26) for the first term on the right-hand side of the inequality in (3.35), and (3.20) with (3.24) for the second, we obtain

$$\begin{aligned} \left|\mathbb{P}\left[\bigcap_{i=1}^{m}\{Z_i \leq t_i\}\right] - \prod_{i=1}^{m}\Phi(t_i)\right| &\leq C\lambda^{-1/(2d+\varepsilon)} + \left|\prod_{i=1}^{m}\mathbb{P}\left[\frac{\tilde{T}_i^{\mathrm{in}} - \mathbb{E}[\tilde{T}_i^{\mathrm{in}}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} \leq t_i\right] - \prod_{i=1}^{m}\Phi(t_i)\right| \\ &\quad + C\delta + C(\log\lambda)^{3d}\lambda^{-1/2} + \sum_{i=1}^{m}\mathbb{P}\left[|Z_i - K_i| \geq \delta\right]. \end{aligned} \tag{3.36}$$

We now consider the second term on the right-hand side of (3.36). For ease of notation, write

$$G_i(t) := \mathbb{P}\left[\frac{\tilde{T}_i^{\mathrm{in}} - \mathbb{E}[\tilde{T}_i^{\mathrm{in}}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} \leq t\right],$$

for $i = 1,\dots,m$. For complex $x_1,\dots,x_n$, $y_1,\dots,y_n$ with modulus at most 1 we have $|\prod_{i=1}^n x_i - \prod_{i=1}^n y_i| \leq \sum_{i=1}^n |x_i - y_i|$ (see e.g. p. 110 of [7]). Hence

$$\left|\prod_{i=1}^{m} G_i(t_i) - \prod_{i=1}^{m}\Phi(t_i)\right| \leq \sum_{i=1}^{m}|G_i(t_i) - \Phi(t_i)|. \tag{3.37}$$

Writing

$$H_i(t) := \mathbb{P}\left[\frac{\tilde{T}_i^{\mathrm{in}} - \mathbb{E}[\tilde{T}_i^{\mathrm{in}}]}{(\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}])^{1/2}} \leq t\right],$$

we have that, for $i = 1, \dots, m$

$$|G_i(t_i) - \Phi(t_i)| \leq |H_i(t_i(1+\gamma_i)) - \Phi(t_i(1+\gamma_i))| + |\Phi(t_i(1+\gamma_i)) - \Phi(t_i)|, \tag{3.38}$$

where $1 + \gamma_i := \left(\frac{\mathrm{Var}[\tilde{T}_i]}{\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}]}\right)^{1/2}$. Then, using Lemma 3.4 we have that the first term on the right-hand side of (3.38) satisfies

$$|H_i(t_i(1+\gamma_i)) - \Phi(t_i(1+\gamma_i))| \leq C(\log\lambda)^{3d}\lambda(\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}])^{-3/2} \leq C\lambda^{-1/2}(\log\lambda)^{3d}, \tag{3.39}$$

by (3.25). In order to deal with the second term on the right-hand side of (3.38), we need to estimate γ_i. We note that

$$\frac{\mathrm{Var}[\tilde{T}_i]}{\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}]} = 1 + \frac{\mathrm{Var}[\tilde{T}_i^{\mathrm{bd}}]}{\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}]} + \frac{2\mathrm{Cov}[\tilde{T}_i^{\mathrm{in}}, \tilde{T}_i^{\mathrm{bd}}]}{\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}]}.$$

Then using the upper and lower variance bounds (3.13), (3.15), (3.25), and the Cauchy-Schwarz inequality, yields

$$\frac{\mathrm{Var}[\tilde{T}_i]}{\mathrm{Var}[\tilde{T}_i^{\mathrm{in}}]} = 1 + O(\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2}),$$

so that

$$\gamma_i = O(\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2}). \tag{3.40}$$

Since for all $s \leq t$ we have $|\Phi(s) - \Phi(t)| \leq (t-s)\sup_{s\leq u\leq t}\varphi(u)$ (where φ is the standard normal density function), we have

$$\begin{aligned} &\sup_{t_i} |\Phi(t_i(1+\gamma_i)) - \Phi(t_i)| \\ \leq\ & C\sup_{t_i}\left(|t_i|\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2} \sup_{|u-t_i|\leq t_i C\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2}} \varphi(u)\right) \\ \leq\ & C\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2}. \end{aligned} \tag{3.41}$$

So, for the second term on the right-hand side in (3.36), we obtain from (3.37), (3.38), (3.39) and (3.41)

$$\begin{aligned} &\sup_{t_1,\dots,t_m}\left|\prod_{i=1}^m \mathbb{P}\left[\frac{\tilde{T}_i^{\mathrm{in}} - \mathbb{E}[\tilde{T}_i^{\mathrm{in}}]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}} \leq t_i\right] - \prod_{i=1}^m \Phi(t_i)\right| \\ \leq\ & C(\log\lambda)^{3d}\lambda^{-1/2} + C\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2}. \end{aligned} \tag{3.42}$$

We now move on to the fifth term on the right-hand side of (3.36). We have

$$|Z_i - K_i| = (\mathrm{Var}[\tilde{T}_i])^{-1/2}|(T_i - \mathbb{E}[T_i]) - (\tilde{T}_i - \mathbb{E}[\tilde{T}_i])| \leq (\mathrm{Var}[\tilde{T}_i])^{-1/2}\left(|T_i - \tilde{T}_i| + \mathbb{E}[|T_i - \tilde{T}_i|]\right),$$

and from just below (4.19) in [21], we have that this is bounded by $C\lambda^{-3}$ except possibly on the set E_λ^c which has probability less than $C\lambda^{-3}$. Thus by (3.36) with $\delta = C\lambda^{-3}$, and using (3.42) for the second term on the right-hand side of (3.36), we obtain

$$\begin{aligned} \sup_{t_1,\dots,t_m}\left|\mathbb{P}\left[\bigcap_{i=1}^m \{Z_i \leq t_i\}\right] - \prod_{i=1}^m \Phi(t_i)\right| &\leq C\lambda^{-1/(2d+\varepsilon)} + C(\log\lambda)^{3d}\lambda^{-1/2} \\ &\quad + C\lambda^{-1/(2d)}(\log\lambda)^{(2d+1)/2} + C\lambda^{-3} = O(\lambda^{-1/(2d+\varepsilon)}). \end{aligned} \tag{3.43}$$

By the triangle inequality we have

$$
\begin{aligned}
&\sup_{t_1,\ldots,t_m}\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{\frac{T_i-\mathbb{E}[T_i]}{(\mathrm{Var}[T_i])^{1/2}}\le t_i\right\}\right]-\prod_{i=1}^m\Phi(t_i)\right|\\
&\le \sup_{t_1,\ldots,t_m}\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{\frac{T_i-\mathbb{E}[T_i]}{(\mathrm{Var}[\tilde{T}_i])^{1/2}}\le t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}\right\}\right]-\prod_{i=1}^m\Phi\left(t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}\right)\right|\\
&\quad+\sup_{t_1,\ldots,t_m}\left|\prod_{i=1}^m\Phi\left(t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}\right)-\prod_{i=1}^m\Phi(t_i)\right|. \qquad (3.44)
\end{aligned}
$$

Now from (3.23) and (3.24), there is a constant $C\in(0,\infty)$ such that for all $\lambda\ge 1$ and all $t_i\in\mathbb{R}$

$$
\begin{aligned}
\left|t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}-t_i\right| &= |t_i|\left|\left(1+\frac{\mathrm{Var}[T_i]-\mathrm{Var}[\tilde{T}_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}-1\right|\\
&=|t_i|\left|\big(1+O(\lambda^{-3})\big)^{1/2}-1\right|\le C|t_i|\lambda^{-3};
\end{aligned}
$$

then since for all $s\le t$ we have $|\Phi(s)-\Phi(t)|\le (t-s)\max_{s\le u\le t}\varphi(u)$, we get

$$
\sup_{t_i}\left|\Phi\left(t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}\right)-\Phi(t_i)\right|\le C\sup_{t_i}\left(|t_i|\lambda^{-3}\sup_{u:|u-t_i|\le t_iC\lambda^{-3}}\varphi(u)\right)\le C\lambda^{-3}. \qquad (3.45)
$$

Then, considering the second term on the right-hand side of (3.44), arguing as at (3.37) we have

$$
\begin{aligned}
\sup_{t_1,\ldots,t_m}\left|\prod_{i=1}^m\Phi\left(t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}\right)-\prod_{i=1}^m\Phi(t_i)\right| &\le \sup_{t_1,\ldots,t_m}\sum_{i=1}^m\left|\Phi\left(t_i\cdot\left(\frac{\mathrm{Var}[T_i]}{\mathrm{Var}[\tilde{T}_i]}\right)^{1/2}\right)-\Phi(t_i)\right|\\
&\le C\lambda^{-3}, \qquad (3.46)
\end{aligned}
$$

by (3.45). Thus for any $\varepsilon>0$, from (3.44) and (3.43) with (3.46),

$$
\sup_{t_1,\ldots,t_m}\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{\frac{T_i-\mathbb{E}[T_i]}{(\mathrm{Var}[T_i])^{1/2}}\le t_i\right\}\right]-\prod_{i=1}^m\Phi(t_i)\right|\le C\lambda^{-1/(2d+\varepsilon)}+C\lambda^{-3}=O(\lambda^{-1/(2d+\varepsilon)}).
$$

This completes the proof of Theorem 2.1. □

4 Proof of Theorem 2.2

Recall from Section 2.2 that we here assume $\xi(x,\mathcal{X})$ is translation invariant and defined for $x\in\mathcal{X}$ and finite $\mathcal{X}\subset\mathbb{R}^d$ (i.e. for unmarked point sets), and that X denotes a random d-vector with density κ. Let $\mathcal{H}_{\kappa(X)}$ be a point process in $\mathbb{R}^d$ whose distribution, given X, is that of a homogeneous Poisson point process of intensity $\kappa(X)$ (i.e., a certain Cox point process). We will employ an ensemble version the *pivoted coupling* of $(\mathcal{P}_t : t>0)$ and $\mathcal{H}_{\kappa(X)}$ as in [20] (see also [16]), which will in particular enable us to approximate the nonhomogeneous Poisson point process $\mathcal{P}_\lambda$ locally by a Poisson process which is homogeneous (given X).

On a suitable probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $\mathcal{H}^+$ denote a homogeneous Poisson process of unit intensity in $\mathbb{R}^d\times[0,\infty)$ that is independent of X. For $t\ge 0$, let $\mathcal{P}'_t$ denote the image of the restriction of $\mathcal{H}^+$ to the set $\{(x,s)\in\mathbb{R}^d\times[0,\infty): s\le t\kappa(x)\}$ under the mapping $(x,s)\mapsto x$. For $\lambda>0$, let $\mathcal{H}'_\lambda$ denote the image of the restriction of $\mathcal{H}^+$ to the set $\{(x,s)\in\mathbb{R}^d\times[0,\infty): s\le\lambda\kappa(X)\}$ under the mapping $(x,s)\mapsto\lambda^{1/d}(x-X)$.

Lemma 4.1 *Let $\lambda > 0$. For each $t > 0$ we have that $\mathcal{P}'_t \stackrel{\mathcal{D}}{=} \mathcal{P}_t$, and for each $\lambda > 0$ the conditional distribution of $\mathcal{H}'_\lambda$, given X, is the same as that of $\mathcal{H}_{\kappa(X)}$.*

Moreover, for all $K > 0$ the event $A_\lambda(K)$ defined by

$$A_\lambda(K) := \{\lambda^{1/d}(-X + \mathcal{P}'_\lambda) \cap B_K = \mathcal{H}'_\lambda \cap B_K\} \tag{4.1}$$

satisfies $\mathbb{P}[A_\lambda(K)] \to 1$ as $\lambda \to \infty$.

Proof. The proof is based on that of Theorem 2.1 of [20] (see Section 3 of [20]). By the Mapping Theorem [11], $\mathcal{P}'_t$ has the distribution of $\mathcal{P}_t$, and the conditional distribution of $\mathcal{H}'_\lambda$, given X, is the same as that of $\mathcal{H}_{\kappa(X)}$.

The final statement in the lemma follows as in the proof of Lemma 3.1 in [20] (our λ, $\mathcal{P}'_\lambda$, and $\mathcal{H}'_\lambda$ are there called n, $\mathcal{P}(n)$, and $\mathcal{H}_n$ respectively). □

In proving Theorem 2.2, it is useful to use a radius of stabilization which is determined entirely by the set of points within a fixed distance of the origin, having non-zero associated Δ with positive probability. The next lemma enables us to do this.

Lemma 4.2 *Suppose R is a radius of external stabilization for $\mathcal{H}_1$ satisfying* (2.6)*. Then there exist a constant $K \in (0, \infty)$, a nonnegative integer m, and a measurable subset E of $(B_K)^m$ with strictly positive (dm)-dimensional Lebesgue measure, such that if $(x_1, \ldots, x_m) \in E$, then the point set $\{x_1, \ldots, x_m\}$ is K-externally stable with $H^\xi(\{\mathbf{0}, x_1, \ldots, x_m\}) - H^\xi(\{x_1, \ldots, x_m\}) \neq 0$.*

Proof. By (2.6) and the continuity of measure, we can (and do) choose $K < \infty$ such that $\mathbb{P}[R \leq K, \Delta(\mathcal{H}_1; R) \neq 0] > 0$, and then we can (and do) choose a nonnegative integer m such that $\mathbb{P}[R \leq K, \Delta(\mathcal{H}_1; R) \neq 0, \mathrm{card}(\mathcal{H}_1 \cap B_K) = m] > 0$. Let $\mathcal{H}_1^{(1)}$ be the restriction of $\mathcal{H}_1$ to B_K and let $\mathcal{H}_1^{(2)}$ to be the restriction of $\mathcal{H}_1$ to $\mathbb{R}^d \setminus B_K$. Then $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(2)}$ are independent Poisson point processes on complementary regions of $\mathbb{R}^d$.

In general, the event $\{R \leq K\}$ might not be determined by the restriction of $\mathcal{H}_1$ to B_K. To get around this, observe that there exists a version of the conditional probability $\mathbb{P}[R \leq K, \Delta(\mathcal{H}_1; R) \neq 0 | \mathcal{H}_1^{(1)}]$ that is a (measurable) function of the configuration $\mathcal{H}_1^{(1)}$ alone; let $\psi(\mathcal{H}_1^{(1)})$ denote such a version of this conditional probability. Let ω_d be the volume of the unit-radius ball in $\mathbb{R}^d$. Then

$$\begin{aligned} 0 &< \mathbb{P}[R \leq K, \Delta(\mathcal{H}_1; R) \neq 0 | \mathrm{card}(\mathcal{H}_1 \cap B_K) = m] \\ &= (K^d \omega_d)^{-m} \int_{B_K} \mathrm{d}x_1 \int_{B_K} \mathrm{d}x_2 \cdots \int_{B_K} \mathrm{d}x_m \psi(\{x_1, \ldots, x_m\}). \end{aligned}$$

Hence, denoting by E the set of $(x_1, \ldots, x_m) \in (B_K)^m$ such that $\psi(\{x_1, \ldots, x_m\}) > 0$, we see that E must have strictly positive Lebesgue measure in $\mathbb{R}^{dm}$.

For almost every configuration $(x_1, \ldots, x_m) \in E$, there is a non-zero probability that $\{x_1, \ldots, x_m\} \cup \mathcal{H}_1^{(2)}$ is K-externally stable, and hence there exists a configuration $\mathcal{X}$ of points outside B_K such that $\{x_1, \ldots, x_m\} \cup \mathcal{X}$ is K-externally stable with $\Delta \neq 0$; but in this case, by the definition of K-external stability, $\{x_1, \ldots, x_m\}$ itself must be K-externally stable with $\Delta \neq 0$, and therefore the set E (possibly amended by a set of measure zero) has the properties claimed. □

Throughout the rest of this section, we assume that there exists a radius of external stabilization R for $\mathcal{H}_1$ satisfying (2.6). Choose K, m and E as in the preceding lemma, and fix these for the rest of this section. Below we shall write simply A_λ for $A_\lambda(K)$. Let E' be the set of m-point configurations $\{x_1,\ldots,x_m\}$ in B_K such that $(x_1,\ldots,x_m)\in E$, and for any point process $\mathcal{X}$ in $\mathbb{R}^d$ define

$$R_K(\mathcal{X}) := \begin{cases} K & \text{if } \mathcal{X}\cap B_K \in E' \\ +\infty & \text{otherwise} \end{cases} . \tag{4.2}$$

By the properties of E in Lemma 4.2, $R_K(\mathcal{X})$ is a radius of external stabilization for $\mathcal{X}$ and $\Delta(\mathcal{X}; R_K(\mathcal{X})) \neq 0$ whenever $R_K(\mathcal{X}) < \infty$ (i.e., when $R_K(\mathcal{X}) = K$). Moreover, $R_K(\mathcal{X})$ is measurable with respect to the σ-algebra generated by $\mathcal{X}\cap B_K$, and $\mathbb{P}[R_K(\mathcal{H}_1) = K] > 0$. We shall be interested in $R_K(\mathcal{X})$ for various choices of $\mathcal{X}$ including $\mathcal{H}_{\kappa(X)}$ and $\lambda^{1/d}(-X+\mathcal{P}_\lambda)$.

Recall that $\tilde{\mathcal{B}}(\Gamma)$ denotes the class of functions in $B(\Gamma)$ that are almost everywhere continuous. As a further step towards proving Theorem 2.2, we give the following lemma.

Lemma 4.3 *Suppose that $f\in\tilde{\mathcal{B}}(\Gamma)$, and that either (i) f is constant, or (ii)* (2.8) *holds. Then as $\lambda\to\infty$ we have*

$$(H_\lambda^{\xi,f}(\mathcal{P}'_\lambda\cup\{X\}) - H_\lambda^{\xi,f}(\mathcal{P}'_\lambda))\mathbf{1}_{\{R_K(\lambda^{1/d}(-X+\mathcal{P}'_\lambda))=K\}} \xrightarrow{\mathcal{D}} f(X)\Delta(\mathcal{H}_{\kappa(X)}; R_K(\mathcal{H}_{\kappa(X)})).$$

Proof. If $R_K(\lambda^{1/d}(-X+\mathcal{P}'_\lambda)) = K$ and A_λ occurs, then by (4.1), $R_K(\mathcal{H}'_\lambda) = K$ and

$$\begin{aligned} & H_\lambda^{\xi,f}(\mathcal{P}'_\lambda\cup\{X\}) - H_\lambda^{\xi,f}(\mathcal{P}'_\lambda) - f(X)\Delta(\mathcal{H}'_\lambda;K) \\ = & \sum_{y\in\mathcal{P}'_\lambda\cap B_{\lambda^{-1/d}K}(X)} (f(y)-f(X))(\xi_\lambda(y;(\mathcal{P}'_\lambda\cap B_{\lambda^{-1/d}K}(X))\cup\{X\}) - \xi_\lambda(y;\mathcal{P}'_\lambda\cap B_{\lambda^{-1/d}K}(X))) \\ = & \sum_{y\in\mathcal{H}'_\lambda\cap B_K} (f(X+\lambda^{-1/d}y)-f(X))(\xi(y;(\mathcal{H}'_\lambda\cap B_K)\cup\{\mathbf{0}\}) - \xi(y;\mathcal{H}'_\lambda\cap B_K)), \end{aligned} \tag{4.3}$$

using translation-invariance in the final equality. If f is constant then the last expression in (4.3) is zero, and for this case the result follows from the two facts (given by Lemma 4.1) that $\mathbb{P}[A_\lambda]\to 1$ as $\lambda\to\infty$ and $\mathcal{H}'_\lambda$ has the same distribution, given X, as $\mathcal{H}_{\kappa(X)}$. So it remains to consider the case where $f\in\tilde{\mathcal{B}}(\Gamma)$ and (2.8) holds. Put $\phi_\varepsilon(x) := \sup_{y\in B_\varepsilon(x)}\{|f(y)-f(x)|\}$. Then by (4.3) and the fact that $\mathcal{H}'_\lambda$ has the same distribution, given X, as $\mathcal{H}_{\kappa(X)}$, we have

$$\begin{aligned} & \mathbb{E}[(|H_\lambda^{\xi,f}(\mathcal{P}'_\lambda\cup\{X\}) - H_\lambda^{\xi,f}(\mathcal{P}'_\lambda) - f(X)\Delta(\mathcal{H}'_\lambda;R_K(\mathcal{H}'_\lambda))|\mathbf{1}_{A_\lambda\cap\{R_K(\mathcal{H}'_\lambda)=K\}})|X] \\ \le & \mathbb{E}\Big[\Big(\sum_{y\in\mathcal{H}_{\kappa(X)}\cap B_K} |f(X+\lambda^{-1/d}y)-f(X)|\cdot|\xi(y;(\mathcal{H}_{\kappa(X)}\cap B_K)\cup\{\mathbf{0}\}) - \xi(y;\mathcal{H}_{\kappa(X)}\cap B_K)|\Big)\Big|X\Big] \\ = & \int_{B_K} |f(X+\lambda^{-1/d}y)-f(X)|\mathbb{E}[(|\xi(y;(\mathcal{H}_{\kappa(X)}\cap B_K)\cup\{\mathbf{0}\}) - \xi(y;\mathcal{H}_{\kappa(X)}\cap B_K)|)|X]\kappa(X)\mathrm{d}y \\ \le & \kappa(X)\phi_{\lambda^{-1/d}K}(X)\int_{B_K}\mathbb{E}[(|\xi(y;(\mathcal{H}_{\kappa(X)}\cap B_K)\cup\{\mathbf{0}\}) - \xi(y;\mathcal{H}_{\kappa(X)}\cap B_K)|)|X]\mathrm{d}y, \end{aligned}$$

where for the equality we have used standard Palm theory (Theorem 1.6 of [12]). Since we assume $f\in\tilde{\mathcal{B}}(\Gamma)$ and (2.8) holds, almost surely with respect to X the above expression is finite and tends to zero as $\lambda\to\infty$. Since by (4.1) $R_K(\lambda^{1/d}(-X+\mathcal{P}'_\lambda)) = R_K(\mathcal{H}'_\lambda)$ on A_λ and by Lemma 4.1 $\mathbb{P}[A_\lambda]\to 1$, this shows that for any $\varepsilon>0$,

$$\mathbb{P}[|(H_\lambda^{\xi,f}(\mathcal{P}'_\lambda\cup\{X\}) - H_\lambda^{\xi,f}(\mathcal{P}'_\lambda))\mathbf{1}_{\{R_K(\lambda^{1/d}(-X+\mathcal{P}'_\lambda))=K\}} - f(X)\Delta(\mathcal{H}'_\lambda;R_K(\mathcal{H}'_\lambda))\mathbf{1}_{\{R_K(\mathcal{H}'_\lambda)=K\}}| > \varepsilon]$$

tends to 0 as $\lambda \to \infty$. Since (by Lemma 4.1) $\mathcal{H}'_\lambda$ has the same distribution, given X, as $\mathcal{H}_{\kappa(X)}$, this demonstrates the result. $\square$

Recall that X and $\mathcal{H}^+$, along with $\mathcal{P}'_t$, are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0$ be the trivial σ-algebra, and for $t > 0$ let $\mathcal{F}_t$ be the sub-σ-algebra of $\mathcal{F}$ generated by $\{X\} \cup \{(x,s) \in \mathcal{H}^+ : s \leq t\kappa(x)\}$; that is, the smallest σ-algebra with respect to which the random variable X is measurable and the number of points in $\mathcal{H}^+$ in F is measurable for each Borel $F \subset \{(x,s) : s \leq t\kappa(x)\}$. In particular, $\mathcal{P}_t$ in the pivoted coupling (i.e., $\mathcal{P}'_t$) is $\mathcal{F}_t$-measurable.

A key step in the proof of Theorem 2.2 is the following lemma which could also be given in more general terms as a formula for the variance of a function of a Markov process of pure jump type with bounded jump rates.

Lemma 4.4 *Let $\lambda > 1$. Suppose that g is a measurable function defined on all finite subsets of $\mathbb{R}^d$ with the property that*

$$\mathbb{E}[|g(\mathcal{P}'_\lambda \cup \{X\}) - g(\mathcal{P}'_\lambda)|^p] < \infty \tag{4.4}$$

for some $p > 2$. Then

$$\mathrm{Var}[g(\mathcal{P}_\lambda)] = \mathbb{E}\int_0^\lambda (\mathbb{E}[g(\mathcal{P}'_\lambda \cup \{X\}) - g(\mathcal{P}'_\lambda)|\mathcal{F}_t])^2 \mathrm{d}t. \tag{4.5}$$

We shall prove Lemma 4.4 by discrete-time approximation, but it it seems likely that it can also be proved by an argument based on generators (S.C. Harris, personal communication). It may be that with such a proof, the condition $p > 2$ can be relaxed to $p = 2$.

Before proving Lemma 4.4, we give a preliminary result.

Lemma 4.5 *Let $\lambda > 1$. Suppose that g is a measurable function defined on all finite subsets of $\mathbb{R}^d$, and that (4.4) holds for some $p > 2$. Then for any $q \in [2, p)$, we have that as $h \downarrow 0$,*

$$\mathbb{E}[|g(\mathcal{P}'_\lambda) - g(\mathcal{P}'_{\lambda-h})|^q] = O(h). \tag{4.6}$$

Proof. In this proof we write simply $\mathcal{P}_t$ for $\mathcal{P}'_t$. Let $\mathcal{Y}_k$ denote the point process consisting of k independent random κ-distributed points on $\mathbb{R}^d$. Note first that setting $\alpha_k := \mathbb{E}[|g(\mathcal{Y}_k \cup \{X\}) - g(\mathcal{Y}_k)|^p]$, we have for $0 < t \leq \lambda$ that

$$\begin{aligned}\mathbb{E}[|g(\mathcal{P}_t \cup \{X\}) - g(\mathcal{P}_t)|^p] &= \sum_{k=0}^\infty \alpha_k e^{-t}\frac{t^k}{k!} \leq e^\lambda \sum_{k=0}^\infty \alpha_k e^{-\lambda}\frac{\lambda^k}{k!} \\ &= e^\lambda \mathbb{E}[|g(\mathcal{P}_\lambda \cup \{X\}) - g(\mathcal{P}_\lambda)|^p] < \infty,\end{aligned} \tag{4.7}$$

by (4.4), a fact that we shall use later.

For $t > 0$ let N_t denote the number of points of $\mathcal{P}_t$ (a Poisson variable with mean t). Setting $M := N_\lambda - N_{\lambda-h}$, by conditioning on M and $N_{\lambda-h}$ we can write, for $q \geq 2$,

$$\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})|^q] = \sum_{j=1}^\infty \sum_{k=0}^\infty \mathbb{E}[|g(\mathcal{Y}_{k+j}) - g(\mathcal{Y}_k)|^q]\mathbb{P}[M = j]\mathbb{P}[N_{\lambda-h} = k]. \tag{4.8}$$

It follows from Minkowski's inequality that, for $q \geq 2$,

$$\mathbb{E}[|g(\mathcal{Y}_{k+j}) - g(\mathcal{Y}_k)|^q] \leq j^q \sum_{\ell=0}^{j-1} \mathbb{E}[|g(\mathcal{Y}_{k+\ell+1}) - g(\mathcal{Y}_{k+\ell})|^q]. \tag{4.9}$$

Moreover, setting $r = k + \ell$,

$$\sum_{k=0}^{\infty} \mathbb{E}[|g(\mathcal{Y}_{k+\ell+1}) - g(\mathcal{Y}_{k+\ell})|^q]\mathbb{P}[N_{\lambda-h} = k] \leq \sum_{r=0}^{\infty} \mathbb{E}[|g(\mathcal{Y}_{r+1}) - g(\mathcal{Y}_r)|^q]\mathbb{P}[N_{\lambda-h} = r](\lambda - h)^{-\ell} r^{\ell},$$

using the fact that $N_{\lambda-h}$ is Poisson with mean $\lambda - h$. Hence

$$\begin{aligned} &\sum_{k=0}^{\infty} \mathbb{E}[|g(\mathcal{Y}_{k+\ell+1}) - g(\mathcal{Y}_{k+\ell})|^q]\mathbb{P}[N_{\lambda-h} = k] \\ &\leq \mathbb{E}[|g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|^q N_{\lambda-h}^{\ell}](\lambda - h)^{-\ell}. \end{aligned} \tag{4.10}$$

Thus we obtain from (4.8) with (4.9) and (4.10) that

$$\begin{aligned} \mathbb{E}[|g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})|^q] &\leq \sum_{j=1}^{\infty} j^q \sum_{\ell=0}^{j-1} \mathbb{E}[|g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|^q N_{\lambda-h}^{\ell}](\lambda - h)^{-\ell}\mathbb{P}[M = j] \\ &\leq \sum_{j=1}^{\infty} j^q \mathbb{E}\left[|g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|^q \sum_{\ell=0}^{j-1} N_{\lambda-h}^{\ell}\right]\mathbb{P}[M = j], \end{aligned}$$

taking h small enough such that $\lambda - h > 1$. Since $N_{\lambda-h}$ takes nonnegative integer values, we have $|\sum_{\ell=0}^{j-1} N_{\lambda-h}^{\ell}| \leq j(N_{\lambda-h}^{j} + 1)$. Also $\mathcal{P}_{\lambda-h}$ and $N_{\lambda-h}$ are independent of M, so

$$\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})|^q] \leq \mathbb{E}[|g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|^q M^{q+1}(N_{\lambda-h}^{M} + 1)\mathbf{1}_{\{M\geq 1\}}].$$

Suppose that (4.4) holds for some $p > q$, and set $r = p/q > 1$. Then Hölder's inequality implies that

$$\begin{aligned} &\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})|^q] \\ &\leq \mathbb{E}[|g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|^p \mathbf{1}_{\{M\geq 1\}}]^{1/r}\mathbb{E}[((N_{\lambda-h}^{M} + 1)M^{q+1})^{r/(r-1)}\mathbf{1}_{\{M\geq 1\}}]^{1-(1/r)}. \end{aligned} \tag{4.11}$$

Since M is independent of $\mathcal{P}_{\lambda-h}$, we have from (4.4) and (4.7) that

$$\mathbb{E}[|g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|^p \mathbf{1}_{\{M\geq 1\}}] = O(\mathbb{P}[M \geq 1]) = O(h). \tag{4.12}$$

Also,

$$\mathbb{E}[(N_{\lambda-h}^{M} M^{q+1})^{r/(r-1)}\mathbf{1}_{\{M\geq 1\}}] = \sum_{i=1}^{\infty} e^{-h}\frac{h^i}{i!}\mathbb{E}[N_{\lambda-h}^{ir/(r-1)}]i^{(q+1)r/(r-1)},$$

which for $\lambda - h > 1$, using standard Poisson moment bounds, is bounded above by (setting $j = i - 1$)

$$h\sum_{j=0}^{\infty} e^{-h}\frac{h^j(\lambda - h)^{(j+1)r/(r-1)}}{(j+1)!}j^{(q+1)r/(r-1)}.$$

Now for fixed $\lambda > 1$ and $r > 1$, we can choose h small enough such that $h(\lambda - h)^{r/(r-1)} \leq 1$, and hence

$$\mathbb{E}[(N^M_{\lambda-h} M^{q+1})^{r/(r-1)} \mathbf{1}_{\{M \geq 1\}}] \leq O(h) \cdot \sum_{j=0}^{\infty} \frac{j^{(q+1)r/(r-1)}}{(j+1)!} = O(h), \tag{4.13}$$

as $h \downarrow 0$. Then (4.11) with (4.12) and (4.13) gives the result. $\square$

Proof of Lemma 4.4. Again we write $\mathcal{P}_t$ for $\mathcal{P}'_t$ throughout this proof. For $0 \leq s < t$ set

$$D_{s,t} := \mathbb{E}[g(\mathcal{P}_\lambda)|\mathcal{F}_t] - \mathbb{E}[g(\mathcal{P}_\lambda)|\mathcal{F}_s] = \mathbb{E}[g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,s,t})|\mathcal{F}_t], \tag{4.14}$$

where $\tilde{\mathcal{P}}_{\lambda,s,t}$ is obtained by resampling that part of the Poisson process which arrives between times s and t. More formally, letting $\mathcal{P}''_r$ be an independent copy of $\mathcal{P}_r$, we can obtain $\tilde{\mathcal{P}}_{\lambda,s,t}$ (for $0 \leq s < t \leq \lambda$) by setting

$$\tilde{\mathcal{P}}_{\lambda,s,t} := \mathcal{P}_s \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_t) \cup \mathcal{P}''_{t-s}.$$

Let $n > 0$ be an integer. Then $g(\mathcal{P}_\lambda) - \mathbb{E}[g(\mathcal{P}_\lambda)] = \sum_{i=1}^n D_{(i-1)\lambda/n, i\lambda/n}$ and by the orthogonality of martingale differences,

$$\mathrm{Var}[g(\mathcal{P}_\lambda)] = \sum_{i=1}^n \mathbb{E}[D^2_{(i-1)\lambda/n, i\lambda/n}]$$

Use the notation $N_t := \mathrm{card}(\mathcal{P}_t)$, and similarly $N''_t := \mathrm{card}(\mathcal{P}''_t)$; then N_t, N''_t are Poisson distributed with mean t. We write $D_{s,t} = \sum_{i=0}^2 D_{s,t,i}$, where for $i = 0, 1$ we set

$$\begin{aligned} D_{s,t,i} &:= \mathbb{E}[(g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,s,t}))\mathbf{1}_{\{N_t - N_s = i\}}|\mathcal{F}_t]; \\ \text{and} \quad D_{s,t,2} &:= \mathbb{E}[(g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,s,t}))\mathbf{1}_{\{N_t - N_s \geq 2\}}|\mathcal{F}_t]. \end{aligned}$$

Suppose $\lambda > 1$ and $h > 0$ is small, so that $\lambda - h > 1$. By the conditional Hölder inequality, for $0 \leq t \leq t + h \leq \lambda$ we have that for $p \in (1, 2]$

$$\begin{aligned} \mathbb{E}[D^2_{t,t+h,0}] &= \mathbb{E}[(\mathbb{E}[(g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h}))\mathbf{1}_{\{N_{t+h} = N_t\}}\mathbf{1}_{\{N''_h > 0\}}|\mathcal{F}_{t+h}])^2] \\ &\leq \mathbb{E}[(\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^p|\mathcal{F}_{t+h}]^{1/p}\mathbb{P}[N''_h > 0]^{(p-1)/p})^2], \end{aligned}$$

using the fact that N''_h is independent of $\mathcal{F}_{t+h}$. By the conditional Jensen inequality, for $p \in (1, 2]$,

$$\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^p|\mathcal{F}_{t+h}]^{2/p} \leq \mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^2|\mathcal{F}_{t+h}],$$

and hence we obtain

$$\begin{aligned} \mathbb{E}[D^2_{t,t+h,0}] &= O(h^{2(p-1)/p})\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^2] \\ &= O(h^{2(p-1)/p})\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,\lambda-h,\lambda})|^2]. \end{aligned} \tag{4.15}$$

Note that

$$g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,\lambda-h,\lambda}) = (g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})) - (g(\tilde{\mathcal{P}}_{\lambda,\lambda-h,\lambda}) - g(\mathcal{P}_{\lambda-h}))$$

and that $g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})$ and $g(\tilde{\mathcal{P}}_{\lambda,\lambda-h,\lambda}) - g(\mathcal{P}_{\lambda-h})$ are identically distributed. Hence, by Minkowski's inequality, for $q \geq 1$ we have

$$\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,\lambda-h,\lambda})|^q] \leq 2^q\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\mathcal{P}_{\lambda-h})|^q]. \tag{4.16}$$

Suppose that (4.4) holds for some $p > 2$. By (4.15) with the $q = 2$ case of (4.6) and (4.16) we obtain that as $h \downarrow 0$

$$\mathbb{E}[D_{t,t+h,0}^2] = o(h). \tag{4.17}$$

Next, choose $r > 1$ with $2r < p$, with p as in (4.4). By the conditional Jensen inequality

$$\begin{aligned} \mathbb{E}[D_{t,t+h,2}^2] &\leq \mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^2 \mathbf{1}_{\{N_{t+h}-N_t \geq 2\}}] \\ &\leq \mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^{2r}]^{1/r} \mathbb{P}[N_{t+h} - N_t \geq 2]^{(r-1)/r}, \end{aligned}$$

by Hölder's inequality. Then by (4.16) and (4.6) for $q = 2r > 2$ we obtain

$$\mathbb{E}[D_{t,t+h,2}^2] = o(h). \tag{4.18}$$

Finally, we deal with $D_{t,t+h,1}$. We have

$$\begin{aligned} \mathbb{E}[D_{t,t+h,1}^2] &= \mathbb{E}[(\mathbb{E}[(g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h}))\mathbf{1}_{\{N_{t+h}-N_t=1\}}|\mathcal{F}_{t+h}])^2] \\ &= \mathbb{E}[(D_{t,t+h,1,0} + D_{t,t+h,1,1})^2], \end{aligned}$$

where we set

$$\begin{aligned} D_{t,t+h,1,0} &:= \mathbb{E}[(g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h}))\mathbf{1}_{\{N_{t+h}-N_t=1\}}\mathbf{1}_{\{N_h''=0\}}|\mathcal{F}_{t+h}], \\ \text{and } \; D_{t,t+h,1,1} &:= \mathbb{E}[(g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h}))\mathbf{1}_{\{N_{t+h}-N_t=1\}}\mathbf{1}_{\{N_h''\geq 1\}}|\mathcal{F}_{t+h}]. \end{aligned}$$

We have

$$\begin{aligned} &D_{t,t+h,1,0} = \\ &\mathbb{E}[(g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}) \cup \{X\}) - g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h})))\mathbf{1}_{\{N_{t+h}-N_t=1\}}\mathbf{1}_{\{N_h''=0\}}|\mathcal{F}_{t+h}] \\ &= e^{-h}\mathbb{E}[g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})\mathbf{1}_{\{N_{t+h}-N_t=1\}}|\mathcal{F}_{t+h}], \end{aligned}$$

since N_h'' is Poisson with mean h and independent of $\mathcal{F}_{t+h}$. Then using the fact that N_{t+h}, N_t are $\mathcal{F}_{t+h}$-measurable

$$\begin{aligned} D_{t,t+h,1,0} &= e^{-h}\mathbf{1}_{\{N_{t+h}-N_t=1\}}\mathbb{E}[g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}) \cup \{X\}) - g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}))|\mathcal{F}_{t+h}] \\ &= e^{-h}\mathbf{1}_{\{N_{t+h}-N_t=1\}}\mathbb{E}[g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}) \cup \{X\}) - g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}))|\mathcal{F}_t], \end{aligned}$$

recalling that X is $\mathcal{F}_t$-measurable for all $t > 0$. Here $\mathbf{1}_{\{N_{t+h}-N_t=1\}}$ and $\mathbb{E}[g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|\mathcal{F}_t]$ are independent, so squaring and taking expectations we obtain

$$\begin{aligned} &\mathbb{E}[D_{t,t+h,1,0}^2] = \\ &e^{-2h}\mathbb{P}[N_{t+h} - N_t = 1]\mathbb{E}[(\mathbb{E}[g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}) \cup \{X\}) - g(\mathcal{P}_t \cup (\mathcal{P}_\lambda \setminus \mathcal{P}_{t+h}))|\mathcal{F}_t])^2] \\ &= h(1+o(1))\mathbb{E}[(\mathbb{E}[g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|\mathcal{F}_t])^2]. \end{aligned} \tag{4.19}$$

Also we have by Hölder's inequality that for some p with $1 < p \leq 2$

$$\begin{aligned} \mathbb{E}[D_{t,t+h,1,1}^2] &\leq \mathbb{E}[(\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|\mathbf{1}_{\{N_h''>0\}}|\mathcal{F}_{t+h}])^2] \\ &\leq \mathbb{E}[(\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^p|\mathcal{F}_{t+h}]^{1/p}\mathbb{P}[N_h'' > 0]^{(p-1)/p})^2] \\ &= O(h^{2(p-1)/p})\mathbb{E}[|g(\mathcal{P}_\lambda) - g(\tilde{\mathcal{P}}_{\lambda,t,t+h})|^2], \end{aligned} \tag{4.20}$$

which is $o(h)$ by (4.16) and (4.6).

Thus combining (4.17), (4.18), (4.19), and (4.20), and using Cauchy-Schwarz, we obtain

$$\mathbb{E}[D^2_{t,t+h}] = h\mathbb{E}[(\mathbb{E}[g(\mathcal{P}_{\lambda-h} \cup \{X\}) - g(\mathcal{P}_{\lambda-h})|\mathcal{F}_t])^2] + o(h),$$

as $h \downarrow 0$. In particular, for $\lambda > 1$,

$$\sum_{i=1}^{n} \mathbb{E}[D^2_{(i-1)\lambda/n, i\lambda/n}] = \lambda n^{-1} \sum_{i=1}^{n} \mathbb{E}[(\mathbb{E}[g(\mathcal{P}_{\lambda(1-n^{-1})} \cup \{X\}) - g(\mathcal{P}_{\lambda(1-n^{-1})})|\mathcal{F}_{(i-1)\lambda/n}])^2] + o(1),$$

as $n \to \infty$. Given (4.4), the right-hand side of the last equation converges as $n \to \infty$ to the integral in (4.5). $\square$

Proof of Theorem 2.2. For $0 \leq t \leq \lambda$ we write $\mathbb{E}_t$ for conditional expectation given $\mathcal{F}_t$ and set

$$G_{\lambda,t} := H_\lambda^{\xi,f}(\mathcal{P}'_t \cup \{X\}) - H_\lambda^{\xi,f}(\mathcal{P}'_t).$$

Recall that we are assuming (2.6), and that K and m, the set $E \subset B_K^m$, and the corresponding set E' of point configurations, were fixed earlier (see (4.2)). We use the notation

$$R_{\lambda,t} := R_K(\lambda^{1/d}(-X + \mathcal{P}'_t)).$$

Given the value of X, there is a non-zero probability that $\mathcal{H}_{\kappa(X)}$ lies in E'. If this happens then by definition (4.2) the value of $\Delta(\mathcal{H}_{\kappa(X)}; R_K(\mathcal{H}_{\kappa(X)}))$ is non-zero. Therefore, since there is a non-zero probability that $f(X) \neq 0$, there is also a non-zero probability that $f(X)\Delta(\mathcal{H}_{\kappa(X)}; R_K(\mathcal{H}_{\kappa(X)})) \neq 0$. Hence, we can we can (and do) choose $\delta > 0$ such that

$$\mathbb{P}[|f(X)\Delta(\mathcal{H}_{\kappa(X)}; R_K(\mathcal{H}_{\kappa(X)}))| > 3\delta] > 4\delta. \tag{4.21}$$

By Lemma 4.3, for large enough λ we have

$$\mathbb{P}[\{|G_{\lambda,\lambda}| > 2\delta\} \cap \{R_{\lambda,\lambda} = K\}] > 3\delta.$$

Given $\varepsilon > 0$, let $\tilde{A}_{\lambda,\varepsilon}$ be the event that $\mathcal{P}'_\lambda \cap B_{\lambda^{-1/d}K}(X) = \mathcal{P}'_{(1-\varepsilon)\lambda} \cap B_{\lambda^{-1/d}K}(X)$. Then we can find $\varepsilon_1 > 0$ such that for large λ,

$$\mathbb{P}[\{|G_{\lambda,\lambda}| > 2\delta\} \cap \{R_{\lambda,\lambda} = K\} \cap \tilde{A}_{\lambda,\varepsilon_1}] > 2\delta.$$

If $R_{\lambda,\lambda} = K$ and $\tilde{A}_{\lambda,\varepsilon}$ occurs, then for $(1-\varepsilon)\lambda \leq t \leq \lambda$, we have both $R_{\lambda,t} = R_{\lambda,\lambda}$ and $G_{\lambda,t} = G_{\lambda,\lambda}$. Hence for large λ and $(1-\varepsilon_1)\lambda \leq t \leq \lambda$,

$$\mathbb{P}[\{|G_{\lambda,t}| > 2\delta\} \cap \{R_{\lambda,t} = K\}] > 2\delta. \tag{4.22}$$

Moreover, by the Cauchy-Schwarz inequality and (2.7), and an argument similar to (4.7), we can find $\varepsilon_2 \in (0, \varepsilon_1)$ such that for large enough λ and for $(1-\varepsilon_2)\lambda \leq t \leq \lambda$ we have

$$\mathbb{E}[|G_{\lambda,\lambda} - G_{\lambda,t}|\mathbf{1}_{\{R_{\lambda,t}=K\}}] \leq (\mathbb{E}[|G_{\lambda,\lambda} - G_{\lambda,t}|^2])^{1/2}\mathbb{P}[\tilde{A}^c_{\lambda,\varepsilon_2}]^{1/2} \leq \delta^2,$$

and therefore by the Jensen and Markov inequalities,

$$\begin{aligned}\mathbb{P}[|\mathbb{E}_t[(G_{\lambda,\lambda} - G_{\lambda,t})\mathbf{1}_{\{R_{\lambda,t}=K\}}]| > \delta] &\leq \mathbb{P}[\mathbb{E}_t[|G_{\lambda,\lambda} - G_{\lambda,t}|\mathbf{1}_{\{R_{\lambda,t}=K\}}] > \delta] \\ &\leq \delta^{-1}\mathbb{E}[|G_{\lambda,\lambda} - G_{\lambda,t}|\mathbf{1}_{\{R_{\lambda,t}=K\}}] < \delta.\end{aligned} \tag{4.23}$$

Since $G_{\lambda,t}$ and $R_{\lambda,t}$ are both $\mathcal{F}_t$-measurable,

$$\mathbb{E}_t[G_{\lambda,\lambda}\mathbf{1}_{\{R_{\lambda,t}=K\}}] = G_{\lambda,t}\mathbf{1}_{\{R_{\lambda,t}=K\}} + \mathbb{E}_t[(G_{\lambda,\lambda} - G_{\lambda,t})\mathbf{1}_{\{R_{\lambda,t}=K\}}]$$

and therefore we may deduce from (4.22) and (4.23) that for large enough λ and for $(1-\varepsilon_2)\lambda \leq t \leq \lambda$ we have

$$\mathbb{P}[|\mathbb{E}_t[G_{\lambda,\lambda}\mathbf{1}_{\{R_{\lambda,t}=K\}}]| \geq \delta] \geq \delta,$$

and since $\mathbb{E}_t[G_{\lambda,\lambda}\mathbf{1}_{\{R_{\lambda,t}=K\}}] = \mathbf{1}_{\{R_{\lambda,t}=K\}}\mathbb{E}_t[G_{\lambda,\lambda}]$ almost surely, it follows that $\mathbb{P}[|\mathbb{E}_t[G_{\lambda,\lambda}]| \geq \delta] \geq \delta$. Then by Lemma 4.4 with (2.7) we have for $\varepsilon \in (0, \varepsilon_2)$ that

$$\mathrm{Var}[H_\lambda^{\xi,f}(\mathcal{P}_\lambda)] \geq \int_{\lambda(1-\varepsilon)}^{\lambda} \mathbb{E}[(\mathbb{E}_t(G_{\lambda,\lambda}))^2]\mathrm{d}t \geq \delta^3\varepsilon\lambda. \ \square$$

5 Example: the k-nearest neighbour graph

The arguments indicated in Section 2.3 are spelled out for the particular case of the k-nearest neighbour graph in Section 3.1 of [21]. Recall that for $k \in \mathbb{N}$ and a locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, the k-nearest neighbour (undirected) graph on $\mathcal{X}$ (denoted kNG($\mathcal{X}$)) is the graph with vertex set $\mathcal{X}$ obtained by including $\{x, y\}$ as an edge whenever $y \in \mathcal{X}$ is one of the k nearest neighbours of $x \in \mathcal{X}$, or vice versa (or both). For some applications of the k-nearest neighbour graph and its relatives, see e.g. [24].

Let $\xi(x; \mathcal{X})$ be one half the sum of the lengths in kNG($\mathcal{X}$) incident to x. Thus (for example) we have that the total length of kNG($\mathcal{X}$) is given by

$$\sum_{x\in\mathcal{X}} \xi(x; \mathcal{X}).$$

Suppose $\Gamma_1, \ldots, \Gamma_n$ are disjoint convex or polyhedral regions. We give two examples of conditions on $\{f_i\}$ and κ which, by known results together with Theorem 2.1, yield (2.3) for this case.

First, suppose that κ is bounded away from 0 on $\cup_i\Gamma_i$. Then ξ is exponentially stabilizing and has moments of all orders. If f_i is continuous on Γ_i, then (2.5) holds with $\sigma_i^2 > 0$ (see [21], Section 3.1). Hence Theorem 2.1 applies in this case. The conditions on f_i and κ may be relaxed (see [15]), but then extra work (such as making use of Theorem 2.2 in the present paper) is needed to show that $\sigma_i^2 > 0$.

Alternatively, suppose that κ is equal to a positive constant κ_i on each Γ_i, so that $\mathcal{P}_\lambda$ is a homogeneous Poisson point process with intensity $\lambda\kappa_i > 0$ on Γ_i. Suppose that $f_i = \mathbf{1}_{\Gamma_i}$, the indicator of Γ_i, for each i. Then by the results of [18], we again have that (2.5) holds with $\sigma_i^2 > 0$, and so Theorem 2.1 holds. In this case, T_i is the total length of kNG($\mathcal{P}_\lambda \cap \Gamma_i$).

We conclude this section by presenting an explicit multivariate CLT of this type, derived from Theorem 2.1, for the case of the nearest-neighbour (directed) graph in one dimension. The nearest-neighbour (directed) graph on a locally finite point set $\mathcal{X}$ is the graph with vertex set $\mathcal{X}$ obtained by including (x, y) as a (directed) edge from $x \in \mathcal{X}$ to $y \in \mathcal{X}$ when y is the nearest neighbour of x (arbitrarily breaking any ties). The required moments, regularity and stabilization conditions all follow from previous work (particularly [18]), and the fact that the limiting variance is non-zero follows from an explicit calculation (which we give below) based on the general results of [15, 16].

For a finite set $\mathcal{X} \subset (0, 1)$ and a Borel set $\Gamma \subseteq (0, 1)$, let $\mathcal{L}^\alpha(\mathcal{X}; \Gamma)$ denote the total weight of the nearest-neighbour (directed) graph on $\mathcal{X}$, with α-power weighted edges, counting only edges originating from points of $\mathcal{X} \cap \Gamma$. That is, if $d(x;\mathcal{X}) := d_2(x;\mathcal{X} \setminus \{x\})$ denotes the (Euclidean) distance from x to its nearest neighbour in $\mathcal{X}$, take

$$\xi(x;\mathcal{X}) = (d(x;\mathcal{X}))^\alpha, \tag{5.1}$$

for some fixed parameter $\alpha \in (0,\infty)$. Then

$$\mathcal{L}^\alpha(\mathcal{X};\Gamma) = \sum_{x \in \mathcal{X} \cap \Gamma} \xi(x;\mathcal{X}).$$

For $m \in \mathbb{N}$, let $\Gamma_1, \ldots, \Gamma_m$ be disjoint, finite, non-null interval subsets of $\mathbb{R}$. In particular, let $\pi_i = |\Gamma_i| \in (0,\infty)$ be the length of the interval Γ_i. Take $f_i = \mathbf{1}_{\Gamma_i}$. Let the underlying density κ be piecewise Borel-measurable, bounded away from 0 and from ∞, on each interval Γ_i; in particular, for each i set $\kappa(x) = \kappa_i(x)$ for $x \in \Gamma_i$, where $\kappa_i \in \mathcal{B}(\Gamma_i)$ and $\kappa_i(x) > 0$ for all $x \in \Gamma_i$. Consider the unmarked case (so $\mathcal{M} = \{1\}$). Then for $\lambda > 0$, $\mathcal{P}_\lambda$ is a Poisson point process with intensity $\kappa_i(x)\lambda$ on each Γ_i. Using the notation of Theorem 2.1, in this set-up we have that

$$T_i = \langle \mathbf{1}_{\Gamma_i}, \mu_\lambda^\xi \rangle = \sum_{x \in \mathcal{P}_\lambda \cap \Gamma_i} \xi_\lambda(x;\mathcal{P}_\lambda) = \sum_{x \in \mathcal{P}_\lambda \cap \Gamma_i} \xi(\lambda x; \lambda \mathcal{P}_\lambda),$$

the final equality by translation-invariance. By the scaling properties ('homogeneity') of ξ as given by (5.1), we have

$$T_i = \sum_{x \in \mathcal{P}_\lambda \cap \Gamma_i} \xi(\lambda x; \lambda \mathcal{P}_\lambda) = \lambda^\alpha \sum_{x \in \mathcal{P}_\lambda \cap \Gamma_i} \xi(x;\mathcal{P}_\lambda) = \lambda^\alpha \mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i).$$

All relevant stabilization, regularity and moments conditions are satisfied. Let $\mathcal{H}_1$ denote a homogeneous Poisson point process of unit intensity on $(0,1)$, and let $\mathcal{U}_n$ denote a binomial point process consisting of n independent uniform random points on $(0,1)$. Then by Theorems 2.1 and 2.3 of [15], for $\alpha > 0$

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}[T_i] &= \lim_{\lambda\to\infty} \lambda^{2\alpha-1}\mathrm{Var}[\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i)] \\ &= V_\alpha \int_{\Gamma_i} \kappa_i(x)\mathrm{d}x + \left(\delta_\alpha \int_{\Gamma_i} \kappa_i(x)\mathrm{d}x\right)^2, \end{aligned} \tag{5.2}$$

where

$$V_\alpha := \lim_{n\to\infty} n^{2\alpha-1}\mathrm{Var}[\mathcal{L}^\alpha(\mathcal{U}_n;(0,1))], \tag{5.3}$$

and

$$\delta_\alpha := \mathbb{E}[d(\mathbf{0};\mathcal{H}_1)^\alpha] + \int_{\mathbb{R}} \mathbb{E}[d(\mathbf{0};\mathcal{H}_1 \cup \{y\})^\alpha - d(\mathbf{0};\mathcal{H}_1)^\alpha]\mathrm{d}y. \tag{5.4}$$

Let $\Gamma(\cdot)$ denote the (Euler) Gamma function, and let ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ denote the (Gauss) hypergeometric function (see e.g. [1], Chapter 15). By (5.3) and equations (20) and (22) in [17], we have that for $\alpha > 0$

$$\begin{aligned} V_\alpha = {} & (4^{-\alpha} + 2 \cdot 3^{-1-2\alpha})\Gamma(1+2\alpha) - 4^{-\alpha}(3+\alpha^2)\Gamma(1+\alpha)^2 \\ & + 8 \cdot \frac{6^{-\alpha-1}\Gamma(2+2\alpha)}{(1+\alpha)}\, {}_2F_1(-\alpha, 1+\alpha; 2+\alpha; 1/3). \end{aligned} \tag{5.5}$$

We now compute δ_α. By standard properties of the Poisson process, $D := d(\mathbf{0}; \mathcal{H}_1)$ is distributed as an exponential random variable with parameter 2. So we have that for $\alpha > 0$

$$\mathbb{E}[D^\alpha] = \int_0^\infty 2r^\alpha \exp(-2r) \mathrm{d}r = 2^{-\alpha}\Gamma(1+\alpha),$$

(using Euler's Gamma integral; see e.g. 6.1.1 in [1]). By Fubini's theorem and (5.4) we have

$$\begin{aligned} \delta_\alpha = \mathbb{E}\left[D^\alpha - 2\int_0^D (D^\alpha - t^\alpha)\mathrm{d}t\right] &= \mathbb{E}[D^\alpha + ((2/(1+\alpha)) - 2)D^{1+\alpha}] \\ = 2^{-\alpha}\Gamma(1+\alpha) - \frac{2\alpha}{1+\alpha}2^{-1-\alpha}\Gamma(2+\alpha) &= 2^{-\alpha}\Gamma(1+\alpha)(1-\alpha), \end{aligned} \tag{5.6}$$

using the functional relation $\Gamma(x) = x^{-1}\Gamma(1+x)$ (see e.g. 6.1.15 in [1]) for the final equality. Of note is the fact that $\delta_1 = 0$, so that in the $\alpha = 1$ case the constant in the limiting (scaled) variance is the same in the Poisson and binomial cases. For $\alpha \neq 1$, $\delta_\alpha^2 > 0$ and the variance in the Poisson case is greater than that in the binomial case, as one expects (the Poisson process introduces additional randomness).

Also by Theorem 2.1 of [14] (see also [16]) and equation (21) in [17], we have that for $\alpha > 0$

$$\lambda^{\alpha-1}\mathbb{E}[\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i)] \to 2^{-\alpha}\Gamma(1+\alpha)\int_{\Gamma_i}\kappa_i(x)\mathrm{d}x,$$

as $\lambda \to \infty$. Thus we have the following application of Theorem 2.1.

Theorem 5.1 *For $m \in \mathbb{N}$, let $\Gamma_1, \ldots, \Gamma_m$ be disjoint intervals in $\mathbb{R}$ with $|\Gamma_i| = \pi_i \in (0,\infty)$. Let $\kappa(x) = \sum_{i=1}^m \kappa_i(x)\mathbf{1}_{\Gamma_i}(x)$ where, for each i, $\kappa_i \in \mathcal{B}(\Gamma_i)$ and $\kappa_i(x) > 0$ for all $x \in \Gamma_i$. Suppose $\alpha \in (0,\infty)$.*

(i) For $1 \leq i \leq m$,

$$\lim_{\lambda\to\infty} \lambda^{\alpha-1}\mathbb{E}[\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i)] = 2^{-\alpha}\Gamma(1+\alpha)\int_{\Gamma_i}\kappa_i(x)\mathrm{d}x.$$

(ii) For $1 \leq i \leq m$,

$$\lim_{\lambda\to\infty} \lambda^{2\alpha-1}\mathrm{Var}[\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i)] = V_\alpha\int_{\Gamma_i}\kappa_i(x)\mathrm{d}x + \left(\delta_\alpha\int_{\Gamma_i}\kappa_i(x)\mathrm{d}x\right)^2 =: \sigma_i^2,$$

where V_α and δ_α are given by (5.5) and (5.6) respectively.

(iii) Given $\varepsilon > 0$, there exists $C \in (0,\infty)$ such that for all $\lambda \geq 1$,

$$\sup_{t_1,\ldots,t_m\in\mathbb{R}}\left|\mathbb{P}\left[\bigcap_{i=1}^m\left\{\frac{\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i) - \mathbb{E}[\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i)]}{(\mathrm{Var}[\mathcal{L}^\alpha(\mathcal{P}_\lambda;\Gamma_i)])^{1/2}} \leq t_i\right\}\right] - \prod_{i=1}^m\Phi(t_i)\right| \leq C\lambda^{\varepsilon-(1/2)}.$$

Part (iii) of Theorem 5.1 is our multivariate CLT. In the particular case of piecewise constant κ, that is $\kappa_i(x) = \kappa_i \in (0,\infty)$ for all $x \in \Gamma_i$, we have that

$$\int_{\Gamma_i}\kappa_i(x)\mathrm{d}x = \kappa_i|\Gamma_i| = \kappa_i\pi_i,$$

and so, for example, $\sigma_i^2 = V_\alpha\kappa_i\pi_i + \delta_\alpha^2\kappa_i^2\pi_i^2$. Table 1 gives some values of the constants V_α, given by (5.5), and δ_α^2, given by (5.6).

α	$1/2$	1	2	3	4
V_α	$\frac{1}{2} + \sqrt{2}\arcsin(1/\sqrt{3}) - \frac{13\pi}{32} \approx 0.094148$	$\frac{1}{6}$	$\frac{85}{108}$	$\frac{149}{18}$	$\frac{135793}{972}$
δ_α^2	$\frac{\pi}{32}$	0	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{81}{4}$

Table 1: Some values of V_α and δ_α^2.

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