Asymptotic behaviour of randomly reflecting billiards in unbounded tubular domains

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We study stochastic billiards in infinite planar domains with curvilinear boundaries: that is, piecewise deterministic motion with randomness introduced via random reflections at the domain boundary. Physical motivation for the process originates with ideal gas models in the Knudsen regime, with particles reflecting off microscopically rough surfaces. We classify the process into recurrent and transient cases. We also give almost-sure results on the long-term behaviour of the location of the particle, including a super-diffusive rate of escape in the transient case. A key step in obtaining our results is to relate our process to an instance of a one-dimensional stochastic process with asymptotically zero drift, for which we prove some new almost-sure bounds of independent interest. We obtain some of these bounds via an application of general semimartingale criteria, also of some independent interest.

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1 Introduction

We consider stochastic billiards (Knudsen random walk) in two-dimensional infinite domains ('tubes') of the form $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x > A, |y| < g(x)\}$ where $A \geq 1$ and $g : [1, \infty) \to (0, \infty)$ is a monotone smooth function. Generally speaking, billiards are dynamical systems describing the motion of a particle in a region with reflection rules at the boundary: such systems have been extensively studied in the mathematical and physical literature. When the reflection rule is randomized, we have a stochastic billiard process. The stochastic models have received much less attention; invariant distributions for stochastic billiards in general (mostly bounded) domains were studied in [5, 9]. In the present paper we study the qualitative (recurrence or transience) and quantitative (almost-sure bounds) behaviour of stochastic billiards in *unbounded* domains in the plane.

Physical motivation for the billiards model comes from the dynamics of ideal gas models in the so-called Knudsen regime where intermolecular interactions are neglected. The random reflection law is motivated by the fact that the particle is small and the surface off which it reflects has a complicated (rough) microscopic structure. The behaviour of such Knudsen flows is of interest in several areas of physics, chemistry, and technology. For physical background, see for instance [3, 16]. For more on the motivation of the model in the present paper, see [5] and references therein. We mention some related models at the end of this section.

Informally, the model that we study can be described as follows. A particle moves ballistically inside the domain $\mathcal{D}$ at constant velocity (of unit magnitude, say), and each time that the particle hits the boundary, it is reflected at a random angle $\alpha \in (-\pi/2, \pi/2)$ to the inwards-pointing normal at the boundary curve $\partial\mathcal{D}$, where α does not depend on the initial direction of the particle. So, inside the domain the motion is deterministic: the stochasticity is introduced via the distribution α of the random reflections. In this paper, we take the distribution of α to be symmetric about 0.

We consider two types of problem: (i) the continuous-time motion of the particle in the tube; and (ii) the discrete-time embedded process obtained by observing the instances of collisions on the boundary (in other words, unit time elapses between reflections). The embedded process is, from our point of view, of interest in its own right (and behaves very differently to the continuous-time process), and is also a vital ingredient for the study of the continuous-time process. The asymptotic phenomena that we study for each process are also of two main types: (i) recurrence or transience, and existence of moments for recurrence times; and (ii) almost-sure bounds for the location of the particle at time t, as $t \to \infty$.

What kind of regions $\mathcal{D}$ are of interest? In the case of $g(x) \equiv 1$, our model becomes symmetrically reflecting motion in a strip. Roughly speaking for the moment, the horizontal motion of the particle is then described by a random walk with zero drift, and so the model is null-recurrent. On the other hand, if $g(x) = x$ our tube is a wedge and, with our reflection rule, at any reflection there is a positive chance that the particle will head off to infinity and never return to a bounded set: it is transient. Similarly if $g(x) = \beta x$ for $\beta \in (0, 1)$, it is not difficult to show that the particle will follow nearby the deterministic trajectory which goes to infinity with linear speed. This dichotomy motivates the study of tubes with widths that grow sub-linearly, to probe precisely the transition between recurrence and transience. It is also natural to consider the case where the tube has decreasing width.

The primary family of tubes to bear in mind has $g(x) = x^\gamma$ for some $\gamma < 1$, although we do consider more general forms for the function g. Then there are two main cases: $\gamma \in (0, 1)$ and $\gamma < 0$. See Figure 1.

We obtain criteria for recurrence and transience of our processes. Loosely speaking, these results imply, for example, that for $\gamma \geq 1/2$ the particle always has transient dynamics, but can be recurrent for $\gamma \in (0, 1/2)$ depending on the reflection distribution. Also, for very shallow tubes such as $g(x) = \log x$, the introduction of random reflections ensures that the process is recurrent, while the corresponding deterministic evolution along the normals is transient. (See Theorem 2.1 below.)

We also obtain almost-sure bounds for the horizontal position of the particle, in both discrete- and continuous-time settings. Thus we have information about the asymptotic speed of the processes. For example, when $g(x) = x^\gamma$ for $\gamma \in [1/2, 1)$, Theorem 2.5 below implies that for all large times t the continuous-time process is at position $t^{1/(2-\gamma)}$, ignoring logarithmic terms. In particular, the motion is *super-diffusive*.

In the case $\gamma < 0$, recurrence is evident. In this case we obtain a criterion for the finiteness of the mean recurrence-time (roughly speaking, 'ergodicity'). We also obtain polynomial estimates for the position of the particle; the ergodicity has so-called 'heavy tails'.

A step in our proofs will involve relating the stochastic billiard process to an instance of the so-called Lamperti problem (after [17–19]) of a one-dimensional stochastic process with asymptotically zero mean drift. A crucial ingredient to our proofs for the stochastic billiard process is provided by some new results for general Lamperti-type processes. These results (Theorems 4.1, 4.2, 4.3) are of independent interest, and are in some cases apparently new even for the nearest-neighbour random walk (birth-and-death chain). To obtain our almost-sure bounds for Lamperti-type processes, we state and prove some general results on obtaining almost-sure bounds for stochastic processes via semimartingale-type criteria.

We conclude this section with some remarks upon related models. As previously mentioned, stochastic billiard models have received comparatively little attention. Classical (that is, non-stochastic) billiard models have been extensively studied, particularly in mathematical physics, and the literature is vast; see for instance [28]. A common approach in the classical setting is via dynamical systems for which the reflection rule is elastic. In this setting, we mention that billiards in certain unbounded domains resembling the tubes considered here (at least for $\gamma \leq 0$) have been studied; see for instance [8, 20–22] and references therein. An infinite-tube billiard model with a stochastic component (cf the $\gamma = 0$ case here) is analyzed in [1]. In the dynamical systems setting, studying ergodicity is typically a primary goal. In the stochastic setting, the results of the present paper demonstrate a complete range of behaviours, from positive-recurrence (roughly speaking, 'ergodicity'), through null-recurrence, to transience.

In the next section we give a more formal definition of the model and state our main results. The statement of our auxiliary results on the Lamperti problem is deferred to Section 4.

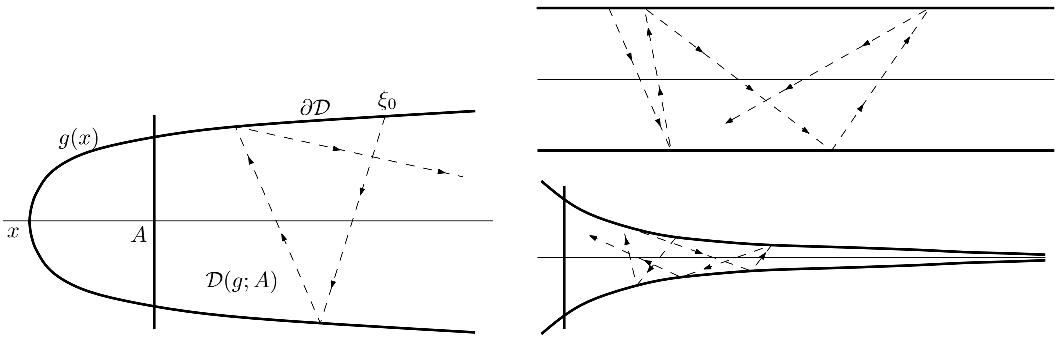

Figure 1: Some examples of the tubes and the trajectories of the process

2 Description of the model and results

2.1 Construction

The rigorous formulation of the stochastic billiard model that we consider is essentially given in [5]. We now describe the construction, which is modified slightly from that in [5] to fit with our context. Let $A \in [1, \infty)$, which will be large but fixed, to be specified later. For a monotone function $g : [1, \infty) \to (0, \infty)$, let

$$\mathcal{D} := \mathcal{D}(g; A) := \{(x, y) \in \mathbb{R}^2 : x > A, |y| < g(x)\}.$$

While the continuous-time process is perhaps most natural to describe, it is more convenient to construct the discrete-time process first, and then to construct the continuous-time process from that. Thus we define a discrete-time Markov chain, informally obtained by recording the locations of the successive hits of the particle on the boundary $\partial\mathcal{D}$. This is a Markov chain with state-space $\partial\mathcal{D} \cup \{\underline{\infty}\}$ that we denote by $\xi = (\xi_n)_{n\in\mathbb{Z}^+}$ ($\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$), where for $\xi_n \in \partial\mathcal{D}$ we write in coordinates $\xi_n = (\xi_n^{(1)}, \xi_n^{(2)}) \in \partial\mathcal{D}$. Thus when $\xi_n^{(1)} > A$, $\xi_n^{(2)} = \pm g(\xi_n^{(1)})$. We call ξ the *collisions process*. Informally, the state $\underline{\infty}$ represents the absorbing state achieved if the particle attains a trajectory that will never intersect with $\partial\mathcal{D}$ again.

Figure 1 illustrates the idea of the construction. We construct ξ formally as follows. Suppose that $\xi_0 \in \partial\mathcal{D}$, with $\xi_0^{(1)} > 2A$. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be i.i.d. random variables with the distribution of a random angle α where $\mathbf{P}[|\alpha| < \pi/2] = 1$. For $k \in \mathbb{Z}^+$, given $\xi_k \in \partial\mathcal{D}\cup\{\underline{\infty}\}$, we perform a step of our process as follows:

(i) If $\xi_k = \underline{\infty}$, set $\xi_{k+1} = \underline{\infty}$;

(ii) Otherwise, α_k specifies a ray Γ_k starting at $\xi_k \in \partial\mathcal{D}$ with angle α_k to the interior normal to $\partial\mathcal{D}$ at ξ_k; we adopt the convention that positive values of the angle correspond to the right of the normal; negative values correspond to the left.

(iii) If the ray Γ_k does not intersect with $\partial\mathcal{D} \setminus \{\xi_k\}$, set $\xi_{k+1} = \underline{\infty}$. Otherwise, let (x_k, y_k) be the first point of intersection of the ray Γ_k with $\partial\mathcal{D}$. Then if $x_k = A$, set $\xi_{k+1} = (2A, g(2A)) \in \partial\mathcal{D}$, else set $\xi_{k+1} = (x_k, y_k) \in \partial\mathcal{D}$.

This defines the discrete-time process ξ. The randomness is introduced through the random draws from the reflection distribution α; if α is degenerate (i.e., equal to a constant almost surely), then ξ is deterministic. Note that the construction ensures that $\xi_k^{(1)} > A$ whenever it is defined.

We need some modified form of 'reflection' away from the vertical boundary at $x = A$, such as that specified by (iii). This is for technical reasons, and in particular ensures that the process does not jump directly to $\underline{\infty}$, since that case will not be of interest to us here. The particular form of this 'reflection' given by (iii) is rather arbitrary; any comparable rule will leave the behaviour unchanged. Moreover, if the distribution of α has no atom at 0, we could instead take the reflection rule to be the same on all boundaries without changing the characteristics of the process.

We obtain the continuous-time stochastic process $X = (X_t)_{t\geq 0}$ with state-space $\mathcal{D}\cup\partial\mathcal{D}\cup\{\underline{\infty}\}$, which we call the *stochastic billiard process*, from the collisions process ξ, essentially by interpolation, as follows. Suppose $X_0 = \xi_0 \in \partial\mathcal{D}$. Define the successive *collision times* of the particle by $\nu_0 := 0$, and if $\xi_k \neq \underline{\infty}$

$$\nu_k := \sum_{j=0}^{k-1} \|\xi_{j+1} - \xi_j\|, \quad (k \in \mathbb{N}), \tag{2.1}$$

where $\|\cdot\|$ is Euclidean distance on $\mathbb{R}^2$ and $\mathbb{N} := \{1, 2, 3, \ldots\}$; if $\xi_k = \underline{\infty}$ set $\nu_k := \infty$. Then for $t \geq 0$ define

$$n(t) := \max\{n \in \mathbb{Z}^+ : \nu_n \leq t\}, \tag{2.2}$$

so that $\nu_{n(t)} \leq t < \nu_{n(t)+1}$ for any $t \geq 0$. Then for any $t \in (0, \infty)$ we define X_t as follows:

$$X_t := \begin{cases} \xi_n & \text{if } t = \nu_n,\ n \in \mathbb{N}; \\ \xi_{n(t)} + \frac{\xi_{n(t)+1}-\xi_{n(t)}}{\|\xi_{n(t)+1}-\xi_{n(t)}\|} \cdot (t - \nu_{n(t)}) & \text{if } t \in (\nu_{n(t)}, \nu_{n(t)+1}) \text{ and } \xi_{n(t)+1} \neq \underline{\infty}; \\ \underline{\infty} & \text{if } t \in (\nu_{n(t)}, \nu_{n(t)+1}) \text{ and } \xi_{n(t)+1} = \underline{\infty}. \end{cases}$$

This defines the process X, and in particular ξ is embedded in X via $\xi_n = X_{\nu_n}$, $n \in \mathbb{Z}^+$ (where $X_\infty := \underline{\infty}$). When $X_t \in \mathcal{D} \cup \partial\mathcal{D}$, write $X_t = (X_t^{(1)}, X_t^{(2)})$ in coordinates.

A realization of the sequence $(\alpha_0, \alpha_1, \ldots)$ therefore specifies the processes ξ and X via the construction just described. We now describe some further assumptions that we make on the function g that specifies the region $\mathcal{D}$ and on the distribution of the angle α.

Suppose that the random variable α for the angle of reflection satisfies for some $\alpha_0 \in (0, \pi/2)$

$$\mathbf{P}[|\alpha| < \alpha_0] = 1, \text{ and } \mathbf{P}[\alpha \geq x] = \mathbf{P}[\alpha \leq -x] \quad (x \in \mathbb{R}), \tag{2.3}$$

so that α is bounded strictly away from $\pm\pi/2$ and the distribution of α is symmetric around 0. Of special nature is the degenerate case where $\mathbf{P}[\alpha = 0] = 1$; this leads to a deterministic evolution (i.e., reflection always occurs along the normals).

We will shortly introduce some assumptions on the function g defining the domain $\mathcal{D}$. If $g(x) = x$, $\mathcal{D}$ is a right-cone, and at any reflection from $\partial\mathcal{D}$ there is positive probability (in fact, $\mathbf{P}[\alpha \geq 0] \geq 1/2$ under condition (2.3)) that the process ξ (hence X) will go to $\underline{\infty}$: this is an obvious case of transience. Moreover, if α is degenerate, then it is clear that for g strictly monotone $\xi_n^{(1)} \to \infty$ if and only if $g(x) \to \infty$; so transience is possible even if $g(x)$ grows arbitrarily slowly, at least in this deterministic case. On the other hand, if $g(x) = c \in (0, \infty)$, we have that if α is non-degenerate then $\xi_n^{(1)}$ performs a one-dimensional random walk on a half-line with zero drift and uniformly bounded jumps: so in this case ξ will be null-recurrent (loosely speaking for the moment). On another hand, it is at least intuitively plausible that if $g(x)$ decreases sufficiently fast, ξ will be positive-recurrent.

In the present paper we will interpolate between these three situations; formally we assume:

(A1) $g : [1, \infty) \to (0, \infty)$ is monotonic, and thrice-differentiable. Moreover, there exists $\gamma \in (-\infty, 1)$ for which, as $x \to \infty$,

$$\begin{aligned} g(x) &= x^{\gamma + o(1)} \\ g'(x) &= [\gamma + o(1)] \frac{g(x)}{x} \\ g''(x) &= [\gamma(\gamma - 1) + o(1)] \frac{g(x)}{x^2} \\ |g'''(x)| &= o(x^{-2}). \end{aligned}$$

Examples of functions satisfying (A1) include x^γ or $x^\gamma (\log x)^\beta$, where $\gamma < 1$. Then in particular we will be interested in the two cases where in addition to (A1) either:

$$g(x) \to \infty \text{ and } x^{-1} g(x) \to 0 \text{ as } x \to \infty; \tag{2.4}$$

or:

$$g(x) \to 0 \text{ as } x \to \infty. \tag{2.5}$$

Given (A1), a necessary condition for (2.4) is that $\gamma \in [0, 1)$; a sufficient condition for (2.5) is $\gamma < 0$. The conditions on the derivatives of g imposed by (A1) are natural smoothness constraints that are necessary for our purposes. A particular case that we will be interested in is where for $\gamma < 1$, $g(x) = x^\gamma$ for $x \geq 1$. Then g satisfies (A1) and moreover satisfies (2.4), (2.5) for $\gamma \in (0, 1)$, $\gamma < 0$ respectively.

When $x^{-1} g(x) \to 0$, elementary geometry, using the fact that α is bounded strictly away from $\pm \pi/2$, implies that ξ (and hence X) will almost surely never jump directly to $\underline{\infty}$, if A is large enough (see Lemma 5.1 below). Thus for the remainder of the paper we can take the state-space of ξ to be $\partial \mathcal{D}$ and that of X to be $\mathcal{D} \cup \partial \mathcal{D}$.

In the next section we state our main results. Then in Section 2.3 we mention some open problems, and outline the remainder of the paper.

2.2 Main results

Recall the parameter $A \in [1, \infty)$. Set

$$\tau_A := \inf\{t > 0 : X_t^{(1)} \leq 2A\}; \quad \text{and } \sigma_A := \inf\{n \in \mathbb{Z}^+ : \xi_n^{(1)} \leq 2A\};$$

here and throughout the paper we use the convention that $\inf \emptyset := +\infty$. We will say that the process X is recurrent if $\mathbf{P}[\tau_A < \infty] = 1$ and transient otherwise; if recurrent then it is positive- or null-recurrent according to whether $\mathbf{E}[\tau_A] < \infty$ or not. Similarly for the process ξ, but with σ_A instead of τ_A.

By the construction of our processes, we have that

$$\tau_A < \infty \iff \sigma_A < \infty; \tag{2.6}$$

thus the classification of recurrence or transience transfers directly between ξ and X.

Since $|\alpha| < \alpha_0 < \pi/2$, we have that the nonnegative random variable $\mathrm{tg}^2\alpha$ is bounded above by $\mathrm{tg}^2\alpha_0 < \infty$. Thus $\mathbf{E}[\mathrm{tg}^2\alpha] < \infty$, and $\mathbf{E}[\mathrm{tg}^2\alpha] = 0$ if and only if $\mathbf{P}[\alpha = 0] = 1$. As an example, in the case where for $\alpha_0 \in (0, \pi/2)$, α is uniformly distributed on the interval $(-\alpha_0, \alpha_0)$, it is the case that $\mathbf{E}[\mathrm{tg}^2\alpha] = \frac{\mathrm{tg}\alpha_0}{\alpha_0} - 1 > 0$.

Our first result covers the recurrence/transience classification for our two processes ξ and X for the case of a growing tube. Positive-recurrence is clearly ruled out in this case (as our processes dominate the zero-drift process in a strip). So here the first important issue is that of transience versus null-recurrence. Define

$$\gamma_c := \frac{\mathbf{E}[\mathrm{tg}^2\alpha]}{1 + 2\mathbf{E}[\mathrm{tg}^2\alpha]}, \tag{2.7}$$

so that $\gamma_c \in [0, 1/2)$ and $\gamma_c = 0$ if and only if $\mathbf{P}[\alpha = 0] = 1$.

Theorem 2.1 *Suppose that the random variable α satisfies (2.3) and that for $\gamma \in [0, 1)$, g satisfies (A1) and (2.4). Then there exists $A_0 \in (0, \infty)$ such that for all $A > A_0$ and $\xi_0 = X_0$ with $\xi_0^{(1)} > 2A$:*

(i) ξ, X are transient if $\gamma > \gamma_c$; and

(ii) ξ, X are null-recurrent if $\gamma < \gamma_c$.

In particular, since $\gamma_c < 1/2$, Theorem 2.1 says that for $\gamma \geq 1/2$, the processes ξ and X are always transient, regardless of the distribution of α.

A special case of Theorem 2.1 is given by $g(x) = (\log x)^K$ for some $K > 0$; then (A1) holds with $\gamma = 0$ and (2.4) holds. In this case it follows from Theorem 2.1 that ξ is transient if $\mathbf{P}[\alpha = 0] = 1$, otherwise it is null-recurrent. That is, the introduction of *randomness* ensures that the process returns to a neighbourhood of the origin infinitely often (with probability 1), whereas in the deterministic case the process is transient. The same remark applies for other functions g that grow as $x^{o(1)}$.

Next we deal with the case of a narrowing tube, when g is decreasing. Now our processes are dominated by the zero-drift random walk in the strip, so transience is impossible. Thus recurrence is evident, and the question of interest in this case is which moments of τ_A, σ_A exist. The following result is for the collisions process ξ.

Theorem 2.2 *Suppose α satisfies (2.3) and that for $\gamma \leq 0$, g satisfies (A1) and (2.5). Then there exists $A_0 \in (0, \infty)$ such that for all $A > A_0$ and ξ_0 with $\xi_0^{(1)} > 2A$:*

(i) ξ *is positive-recurrent if* $\gamma < -\mathbf{E}[\mathrm{tg}^2\alpha]$*; and*

(ii) ξ *is null-recurrent if* $\gamma > -\mathbf{E}[\mathrm{tg}^2\alpha]$.

It follows from Lemma 5.3(ii) below that the continuous-time process X is also positive-recurrent under the conditions of Theorem 2.2(i). Again, Theorem 2.2 shows that for a sub-polynomial function such as $g(x) = 1/\log x$, a non-degenerate α leads to null-recurrence.

Theorem 2.3 below deals with the important special case of $g(x) = x^\gamma$, $\gamma \in (0, 1)$ or $\gamma < 0$. In particular, when α is non-degenerate, it covers the two critical cases omitted in Theorems 2.1 and 2.2; they are null-recurrent.

Theorem 2.3 *Suppose* α *satisfies (2.3) and is not degenerate, and that* $g(x) = x^\gamma$, $\gamma < 1$.

(i) Suppose $\gamma = \gamma_c > 0$*. Then there exists* $A_0 \in (0, \infty)$ *such that for all* $A > A_0$ *and* ξ_0 *with* $\xi_0^{(1)} > 2A$*,* ξ *is null-recurrent.*

(ii) Suppose $\gamma = -\mathbf{E}[\mathrm{tg}^2\alpha] < 0$*. Then there exists* $A_0 \in (0, \infty)$ *such that for all* $A > A_0$ *and* ξ_0 *with* $\xi_0^{(1)} > 2A$*,* ξ *is null-recurrent.*

We now turn to our results on almost-sure bounds for our processes, that is, how far from the origin the particle will typically be in both the discrete- and continuous-time processes. First we give a basic statement that says that such questions are non-trivial.

Proposition 2.1 *Suppose that* α *satisfies (2.3). Suppose that* g *satisifies (A1) and either (i)* g *satisfies (2.4); or (ii)* g *satisfies (2.5) and* α *is non-degenerate. Then for* A *sufficiently large and any* $\xi_0 = X_0$ *with* $\xi_0^{(1)} = X_0^{(1)} > 2A$*, a.s.,*

$$\limsup_{n\to\infty} \xi_n^{(1)} = +\infty, \text{ and } \limsup_{t\to\infty} X_t^{(1)} = +\infty.$$

If g satisfies (2.5), and $\xi_0^{(1)}$ is not large enough in this last result, then it is clear that for some distributions of α (with small bound α_0) the particle may get 'trapped' in a bounded set. This case is not of interest for us here.

We want to quantify the result of Proposition 2.1. Again we can proceed more generally, but our results are clearer if we take $g(x) = x^\gamma$ ($\gamma < 1$) from now on. Under the more general assumption (A1), we believe that our techniques still apply, with some modifications, and that the polynomial exponents in the theorems below remain valid.

Our first almost-sure bounds are for the collisions process in the case $\gamma \in (0, 1)$:

Theorem 2.4 *Suppose* α *satisfies (2.3) and* $g(x) = x^\gamma$ *with* $\gamma \in (0, 1)$*. Suppose that* $X_0^{(1)} = \xi_0^{(1)} > 2A$ *for* A *sufficiently large. For any* $\varepsilon > 0$*, a.s., for all but finitely many* $n \in \mathbb{Z}^+$*,*

$$\max_{0\le m\le n} \xi_m^{(1)} \le n^{\frac{1}{2(1-\gamma)}} (\log n)^{\frac{1}{2(1-\gamma)}+\varepsilon}, \text{ and} \tag{2.8}$$

$$\max_{0\le m\le n} \xi_m^{(1)} \ge n^{\frac{1}{2(1-\gamma)}} (\log n)^{-\frac{1}{2(1-\gamma)}-\varepsilon}. \tag{2.9}$$

Moreover, if $\gamma > \gamma_c$ (so that by Theorem 2.1 ξ is transient), then there exists $D \in (0, \infty)$ such that a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\xi_n^{(1)} \geq n^{\frac{1}{2(1-\gamma)}} (\log n)^{-D}. \tag{2.10}$$

Theorem 2.4 shows that we have polynomial behaviour, but that on the time-scale of the collisions process ξ, the speed $n^{-1}\xi_n^{(1)}$ can be very large for γ close to 1. The next result gives the corresponding bounds for the stochastic billiard process X. On this time-scale, the behaviour is very different, since the particle can spend much more time between collisions.

Theorem 2.5 *Suppose α satisfies (2.3) and $g(x) = x^\gamma$ for $\gamma \in (0, 1)$. Suppose that $\xi_0^{(1)} > 2A$ for A sufficiently large. For any $\varepsilon > 0$, a.s., for all $t > 0$ large enough,*

$$\sup_{0 \leq s \leq t} X_s^{(1)} \geq t^{\frac{1}{2-\gamma}} (\log t)^{-\frac{1}{(1-\gamma)(2-\gamma)} - \varepsilon}. \tag{2.11}$$

Moreover, if $\gamma > \gamma_c$ (so that by Theorem 2.1 X is transient), then there exists $D \in (0, \infty)$ such that a.s., for all t large enough,

$$t^{\frac{1}{2-\gamma}} (\log t)^{-D} \leq X_t^{(1)} \leq t^{\frac{1}{2-\gamma}} (\log t)^{D}. \tag{2.12}$$

In particular, in the (perhaps most interesting) transient case, (2.12) shows that the asymptotic speed for the particle is zero, i.e., $\lim_{t \to \infty} t^{-1} X_t^{(1)} = 0$ a.s., but the motion is super-diffusive. Moreover, as $\gamma \uparrow 1$, we approach linear growth, while as $\gamma \downarrow 0$, we approach diffusive growth, as expected in view of the remarks just above the statement of (A1). (2.11) also demonstrates super-diffusive growth, even in the recurrent case.

Now we consider the case where $g(x) = x^\gamma$ for $\gamma < 0$. If $\gamma < -\mathbf{E}[\mathrm{tg}^2\alpha]$, we use the notation

$$\rho(\gamma) := \frac{\mathbf{E}[\mathrm{tg}^2\alpha]}{(1-2\gamma)\mathbf{E}[\mathrm{tg}^2\alpha] - \gamma},$$

so that $0 \leq \rho(\gamma) < 1/(2(1-\gamma))$. The next result deals with the collisions process ξ.

Theorem 2.6 *Suppose α satisfies (2.3) and is non-degenerate, and $g(x) = x^\gamma$ for $\gamma < 0$. Suppose that $\xi_0^{(1)} > 2A$ for A sufficiently large.*

(i) Suppose that $\gamma > -\mathbf{E}[\mathrm{tg}^2\alpha]$. Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$, (2.8) and (2.9) hold.

(ii) Suppose that $\gamma < -\mathbf{E}[\mathrm{tg}^2\alpha]$. Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\max_{0 \leq m \leq n} \xi_m^{(1)} \leq n^{\rho(\gamma)} (\log n)^{2\rho(\gamma)+\varepsilon}, \text{ and} \tag{2.13}$$

$$\max_{0 \leq m \leq n} \xi_m^{(1)} \geq n^{\rho(\gamma)} (\log n)^{-2\rho(\gamma)-\varepsilon}. \tag{2.14}$$

Now we state the corresponding result for the stochastic billiard process X.

Theorem 2.7 *Suppose α satisfies (2.3) and is non-degenerate, and $g(x) = x^\gamma$ for $\gamma < 0$. Suppose that $X_0^{(1)} = \xi_0^{(1)} > 2A$ for A sufficiently large.*

(i) Suppose that $\gamma > -\mathbf{E}[\mathrm{tg}^2\alpha]$. Then for any $\varepsilon > 0$, a.s., for all $t > 0$ large enough, (2.11) holds.

(ii) Suppose that $\gamma < -\mathbf{E}[\mathrm{tg}^2\alpha]$. Then for any $\varepsilon > 0$, a.s., for all $t > 0$ large enough,

$$\sup_{0 \le s \le t} X_s^{(1)} \ge t^{\frac{\rho(\gamma)}{1+\gamma\rho(\gamma)}} (\log t)^{-\frac{4\gamma\rho(\gamma)^2+2\rho(\gamma)}{1+\gamma\rho(\gamma)}-\varepsilon}. \tag{2.15}$$

Theorems 2.6(ii) and 2.7(ii) show that even in the positive-recurrent case, the behaviour is essentially polynomial (i.e., heavy-tailed) in nature. Moreover, the exponents depend explicitly on the reflection distribution α, unlike in the other cases.

2.3 Open problems

We briefly mention some open problems for the model studied here.

Of interest is the weak (distributional) limiting behaviour of the horizontal components of ξ, X. We have from Lemma 5.4 and Corollary 5.1 below together with Theorem 4.1 in [19] that Lamperti's invariance principle (4.11) below will hold with $\eta_\bullet = (\xi_\bullet^{(1)})^{1-\gamma}$ and $\gamma \in (-\mathbf{E}[\mathrm{tg}^2\alpha], 1)$, α non-degenerate, provided that Lamperti's 'condition (c)' from [19] is satisfied. If Lamperti's 'condition (c)' were verified, this would then imply that for $\gamma \in (-\mathbf{E}[\mathrm{tg}^2\alpha], 1)$ and α non-degenerate, we would have the weak invariance principle as $n \to \infty$:

$$n^{-1/2}(\xi_{\lfloor n\bullet \rfloor}^{(1)})^{1-\gamma} \Rightarrow \Upsilon_\bullet.$$

Here $(\Upsilon_t)_{t>0}$ is a diffusion process on $[0, \infty)$ with Kolmogorov backwards equation

$$u_t = \frac{a}{x} u_x + \frac{b}{2} u_{xx},$$

where

$$a = 2\gamma(1-\gamma)(1 + \mathbf{E}[\mathrm{tg}^2\alpha]), \text{ and } b = 4(1-\gamma)^2 \mathbf{E}[\mathrm{tg}^2\alpha].$$

A more challenging problem would be to determine whether any corresponding weak limit theory holds for the continuous-time process $X_t^{(1)}$ (for $\gamma > -\mathbf{E}[\mathrm{tg}^2\alpha]$, say).

It may also be of interest to relax some of our assumptions, such as those on the reflection distribution. For instance, one might consider distributions for α with support on all of $[-\pi/2, \pi/2]$; there are several 'natural' distributions relevant in this case [5, 9, 16]. The techniques of the present paper require the assumption that α be bounded away from $\pm\pi/2$. Indeed, if α can take value $\pi/2$, say, with positive probability, the particle will eventually jump to $\underline{\infty}$ in the case of a tube of nondecreasing width. On the other hand, if α has distribution on $[-\pi/2, \pi/2]$ with sufficiently light tails at the endpoints (such that in particular $\mathbf{E}[\mathrm{tg}^2\alpha] < \infty$, say), it may be possible to modify the techniques in the present paper to obtain similar results; one would need to extend various results from processes with jumps that are bounded to processes with jumps satisfying higher-order moment assumptions, for example.

The structure of the remainder of the paper is as follows. In Section 3 we present and prove general semimartingale criteria (of some independent interest) on almost-sure bounds for one-dimensional stochastic processes. In Section 4 we make use of the results in Section 3, and give some results for the so-called Lamperti problem, which are of interest in their own right as well as being a crucial element in the proofs of our main results. In Section 5 we prove the results stated in Section 2.2 on the processes ξ, X.

3 Semimartingale criteria for speeds of stochastic processes

In this section we present and prove general semimartingale criteria for obtaining upper and lower almost sure bounds for discrete-time stochastic processes on the half-line. These results will provide some of our main tools for the study of processes with asymptotically zero mean drifts in Section 4 below, but for the present section we work in some generality.

Let $(\mathcal{F}_n)_{n\in\mathbb{Z}^+}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $Y = (Y_n)_{n\in\mathbb{Z}^+}$ be a discrete-time $(\mathcal{F}_n)$-adapted stochastic process taking values in $[0,\infty)$. Suppose that $\mathbf{P}[Y_0 = x_0] = 1$ for some $x_0 \in [0,\infty)$.

The types of process to which our criteria can be applied are quite general. For instance, due to the semimartingale nature of the results, we do not require that Y be a Markov process. This fact is particularly useful if Y is a process of norms $\|Z_t\|$; even if $(Z_t)_{t\in\mathbb{Z}^+}$ is Markov, $(\|Z_t\|)_{t\in\mathbb{Z}^+}$ will not be, in general. We anticipate that the general results in this section are widely applicable. Moreover, continuous-time processes may be treated via embedded discrete-time processes. One condition that we need for some of our results is that jumps of the process Y are uniformly bounded *above*. Elements of the proofs in the present section extend ideas used in [4] and [25].

Our conditions will involve the existence of suitable functions f such that the process $f(Y)$ satisfies an appropriate 'drift' condition. Our criteria will involve only first-order conditions (i.e. expectations); no variance results are required. Our results will yield almost sure bounds for $\max_{0\leq m\leq n} Y_m$ (and hence Y_n) in terms of the function f and simple functions a, v that control our bounds. The functions a, v will belong to classes of eventually increasing functions defined as follows.

Definition 3.1 *We say function $a : [1,\infty) \to [1,\infty)$ satisfies condition (C1) and $v : [1,\infty) \to [1,\infty)$ satisfies condition (C2) if:*

(C1) $a(x) \to \infty$ as $x \to \infty$, there exists $n_a \in \mathbb{N}$ such that $x \mapsto a(x)$ is increasing for all $x \geq n_a$, and $\sum_{n=1}^{\infty} \frac{1}{a(n)} < \infty$;

(C2) $v(x) \to \infty$ as $x \to \infty$, there exists $n_v \in \mathbb{N}$ such that $x \mapsto v(x)$ is increasing for all $x \geq n_v$, and $\sum_{n=1}^{\infty} \frac{1}{nv(n)} < \infty$.

Note that the summability condition in (C2) is equivalent to the condition that $\sum_{n=1}^{\infty} \frac{1}{v(r^n)} < \infty$ for some (hence all) $r > 1$. Throughout this section we interpret $\log x$ as $\max\{1, \log x\}$. With this convention, condition (C1) is satisfied, for example, by $a(x) = x^{1+\varepsilon}$ or $x(\log x)^{1+\varepsilon}$, where $\varepsilon > 0$, and (C2) is satisfied, for example, by $v(x) = (\log x)^{1+\varepsilon}$ or $(\log x)(\log \log x)^{1+\varepsilon}$, $\varepsilon > 0$.

The first result in this section is a submartingale criterion for an upper bound.

Theorem 3.1 *Let $(Y_n)_{n\in\mathbb{Z}^+}$ be a discrete-time $(\mathcal{F}_n)$-adapted stochastic process taking values in $[0, \infty)$. Suppose that there exists $f : [0, \infty) \to [0, \infty)$ a nondecreasing function such that,*

$$\mathbf{E}[f(Y_{n+1}) - f(Y_n) \mid \mathcal{F}_n] \geq 0 \ a.s., \tag{3.1}$$

for all $n \in \mathbb{Z}^+$. Also suppose that there exists $B \in (0, \infty)$ such that for all $n \in \mathbb{N}$

$$\mathbf{E}[f(Y_n)] \leq Bn. \tag{3.2}$$

Define the nondecreasing function f^{-1} for $x > 0$ by

$$f^{-1}(x) := \sup\{y \geq 0 : f(y) < x\}. \tag{3.3}$$

Let a satisfy (C1). Then, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\max_{0\leq m\leq n} Y_m \leq f^{-1}(a(2n)). \tag{3.4}$$

A sufficient condition for (3.2) is clearly that there exists $B' \in (0, \infty)$ such that

$$\mathbf{E}[f(Y_{n+1}) - f(Y_n) \mid \mathcal{F}_n] \leq B' \text{ a.s.}, \tag{3.5}$$

for all $n \in \mathbb{Z}^+$. We next present a variant of Theorem 3.2, which relaxes the condition (3.1) at the expense of this slightly stronger version of (3.2).

Theorem 3.2 *Theorem 3.1 holds with conditions (3.1) and (3.2) replaced by the lone condition (3.5).*

For the proof of Theorem 3.2, we need the following:

Lemma 3.1 *Let $(Z_n)_{n\in\mathbb{Z}^+}$ be an $(\mathcal{F}_n)$-adapted process on $[0, \infty)$ and $z_0 \in [0, \infty)$ such that $\mathbf{P}[Z_0 = z_0] = 1$ and for some $B \in (0, \infty)$ and all $n \in \mathbb{Z}^+$*

$$\mathbf{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] \leq B \text{ a.s..} \tag{3.6}$$

Then for any $x > 0$ and any $n \in \mathbb{N}$

$$\mathbf{P}\left[\max_{0\leq m\leq n} Z_m \geq x\right] \leq (Bn + z_0)x^{-1}. \tag{3.7}$$

Proof. Similarly to Doob's decomposition (see e.g. [32], p. 120), set $W_0 := Z_0$, and for $m \in \mathbb{N}$ let $W_m := Z_m + A^-_{m-1} + A^-_{m-2} + \cdots + A^-_1$, where $A_m = \mathbf{E}[Z_{m+1} - Z_m \mid \mathcal{F}_m]$, $A^-_m = \max\{-A_m, 0\} \geq 0$, $A^+_m = \max\{A_m, 0\} \geq 0$. Then

$$\mathbf{E}[W_{m+1} - W_m \mid \mathcal{F}_m] = \mathbf{E}[Z_{m+1} - Z_m + A^-_m \mid \mathcal{F}_m] = A_m + A^-_m = A^+_m \in [0, B]$$

so that (W_m) is a nonnegative $(\mathcal{F}_m)$-submartingale with $W_m \geq Z_m$ for all m, and $\mathbf{E}[W_n] \leq W_0 + Bn = z_0 + Bn$. Hence by Doob's submartingale inequality (see e.g. [32], p. 137)

$$\mathbf{P}\left[\max_{0 \leq m \leq n} Z_m \geq x\right] \leq \mathbf{P}\left[\max_{0 \leq m \leq n} W_m \geq x\right] \leq x^{-1}\mathbf{E}[W_n] \leq (Bn + z_0)x^{-1},$$

as required. ∎

Proof of Theorems 3.1 and 3.2. First we prove Theorem 3.1. Since, by (3.1), $(f(Y_n))$ is a nonnegative submartingale, Doob's submartingale inequality implies that, for any $n \in \mathbb{N}$,

$$\mathbf{P}\left[\max_{0 \leq m \leq n} f(Y_m) \geq a(n)\right] \leq (a(n))^{-1}\mathbf{E}[f(Y_n)] \leq Bn(a(n))^{-1}, \tag{3.8}$$

using (3.2). Also, for $n \in \mathbb{N}$,

$$\mathbf{P}\left[\max_{0 \leq m \leq n} f(Y_m) \geq a(n)\right] = \mathbf{P}\left[f\left(\max_{0 \leq m \leq n} Y_m\right) \geq a(n)\right], \tag{3.9}$$

since f is nondecreasing. With f^{-1} defined by (3.3), let E_n denote the event

$$E_n := \left\{\max_{0 \leq m \leq n} Y_m > f^{-1}(a(n))\right\}.$$

Then since $z > f^{-1}(r)$ implies $f(z) > r$, we obtain from (3.8) and (3.9) that for all $n \in \mathbb{N}$

$$\mathbf{P}[E_n] \leq \mathbf{P}\left[f\left(\max_{0 \leq m \leq n} Y_m\right) \geq a(n)\right] \leq Bn(a(n))^{-1}. \tag{3.10}$$

Now

$$\sum_{\ell=0}^{\infty} \frac{2^\ell}{a(2^\ell)} < \infty \iff \sum_{\ell=1}^{\infty} \frac{1}{a(\ell)} < \infty. \tag{3.11}$$

Hence by (3.10) and (3.11), along the subsequence $n = 2^\ell$ for $\ell = 0, 1, 2, \ldots$, (C1) and the Borel-Cantelli lemma imply that, a.s., the event E_n occurs only finitely often, and in particular there exists $\ell_0 < \infty$ such that for all $\ell \geq \ell_0$

$$\max_{0 \leq m \leq 2^\ell} Y_m \leq f^{-1}(a(2^\ell)).$$

Every $n \in \mathbb{N}$ sufficiently large has $2^{\ell_n} \leq n < 2^{\ell_n+1}$ for some $\ell_n \geq \ell_0$; then, a.s.,

$$\max_{0 \leq m \leq n} Y_m \leq \max_{0 \leq m \leq 2^{\ell_n+1}} Y_m \leq f^{-1}(a(2^{\ell_n+1})),$$

for all but finitely many n. Now since $2^{\ell_n+1} \leq 2n$ and f^{-1} is nondecreasing, (3.4) follows.

To obtain Theorem 3.2, in the previous argument we replace (3.8) by an application of Lemma 3.1 with $Z_n = f(Y_n)$ and $x = a(n)$. ■

We now work towards obtaining a lower bound for $\max_{0 \leq m \leq n} Y_m$. For $\ell \in \mathbb{N}$ let σ_ℓ denote the *first passage time* of ℓ for Y, that is

$$\sigma_\ell := \min\{n \in \mathbb{Z}^+ : Y_n \geq \ell\}.$$

Then $\sigma_1, \sigma_2, \ldots$ is a nondecreasing sequence of stopping times for the process Y; under the condition $\limsup_{n\to\infty} Y_n = +\infty$ a.s., $\sigma_\ell < \infty$ a.s. for every ℓ.

For $\ell \in \mathbb{N}$ and all $n \in \mathbb{Z}^+$, let $Y_n^\ell := Y_{n \wedge \sigma_\ell}$ (where $a \wedge b := \min\{a, b\}$), the process stopped at σ_ℓ; then $Y_n^\ell = Y_n$ if $n < \sigma_\ell$ and $Y_n^\ell = Y_{\sigma_\ell} \geq \ell$ for $n \geq \sigma_\ell$. We have the following result, which is a 'reverse Foster's criterion' (compare Theorem 2.1.1 in [11]).

Lemma 3.2 *Let $(Y_n)_{n\in\mathbb{Z}^+}$ be a discrete-time $(\mathcal{F}_n)$-adapted stochastic process taking values in $[0, \infty)$, such that for some $b \in \mathbb{N}$*

$$\mathbf{P}[Y_{n+1} \leq Y_n + b] = 1, \tag{3.12}$$

for all $n \in \mathbb{Z}^+$. Fix $\ell \in \mathbb{N}$ and $Y_0 = x_0 \in [0, \ell)$. Suppose that there exists $f : [0, \infty) \to [0, \infty)$ a nondecreasing function for which, for some $\varepsilon > 0$,

$$\mathbf{E}[f(Y_{n+1}^\ell) - f(Y_n^\ell) \mid \mathcal{F}_n] \geq \varepsilon \mathbf{1}_{\{\sigma_\ell > n\}} \text{ a.s.} \tag{3.13}$$

for all $n \in \mathbb{Z}^+$. Then

$$\mathbf{E}[\sigma_\ell] \leq \frac{1}{\varepsilon} f(\ell + b).$$

Proof. Let $\ell \in \mathbb{N}$. Taking expectations in (3.13) we have for all $m \in \mathbb{Z}^+$

$$\mathbf{E}[f(Y_{m+1}^\ell)] - \mathbf{E}[f(Y_m^\ell)] \geq \varepsilon \mathbf{P}[\sigma_\ell > m].$$

Summing for m from 0 up to n we obtain

$$\mathbf{E}[f(Y_{n+1}^\ell)] - f(Y_0^\ell) \geq \varepsilon \sum_{m=0}^{n} \mathbf{P}[\sigma_\ell > m].$$

Since $f(Y_0^\ell) = f(Y_0) \geq 0$ we obtain

$$\mathbf{E}[\sigma_\ell] = \lim_{n\to\infty} \sum_{m=0}^{n} \mathbf{P}[\sigma_\ell > m] \leq \frac{1}{\varepsilon} \limsup_{n\to\infty} \mathbf{E}[f(Y_{n+1}^\ell)] \leq \frac{1}{\varepsilon} f(\ell + b),$$

using the fact that, by (3.12), $f(Y_n^\ell) \leq f(\ell + b)$ for all $n \in \mathbb{Z}^+$ since f is nondecreasing. ■

Now we have the following lower bound:

Theorem 3.3 *Let $(Y_n)_{n\in\mathbb{Z}^+}$ be a discrete-time $(\mathcal{F}_n)$-adapted stochastic process taking values in $[0,\infty)$, such that condition (3.12) holds. Suppose that there exists $f:[0,\infty)\to[0,\infty)$ a nondecreasing function and $\varepsilon>0$ for which*

$$\mathbf{E}[f(Y_{n+1})-f(Y_n)\mid\mathcal{F}_n]\geq\varepsilon \text{ a.s.} \tag{3.14}$$

for all $n\in\mathbb{Z}^+$. Let v satisfy (C2). For $x\geq 0$ define the function r_v by

$$r_v(x):=\inf\{y\geq 0:\varepsilon^{-1}v(y)f(y+b)\geq x\}. \tag{3.15}$$

Then, a.s., for all but finitely many $n\in\mathbb{Z}^+$,

$$\max_{0\leq m\leq n}Y_m\geq r_v(n)-b. \tag{3.16}$$

Proof. First note that by (C2) and the fact that f is nondecreasing, we have that $x\mapsto r_v(x)$ is nondecreasing for all x sufficiently large. Fix $K>1$. By Markov's inequality,

$$\mathbf{P}[\sigma_\ell>v(\ell)\mathbf{E}[\sigma_\ell]]\leq(v(\ell))^{-1}. \tag{3.17}$$

Given (3.17) and (C2), the Borel-Cantelli lemma implies that, a.s., $\sigma_{\lfloor K^\ell\rfloor}>v(\lfloor K^\ell\rfloor)\mathbf{E}[\sigma_{\lfloor K^\ell\rfloor}]$ for only finitely many $\ell\in\mathbb{Z}^+$. Moreover, given that f satisfies (3.14), we have that (3.13) holds for all $\ell\in\mathbb{N}$ and all $n\in\mathbb{Z}^+$. Then Lemma 3.2 with (3.12) in this case implies that $\mathbf{E}[\sigma_\ell]\leq\varepsilon^{-1}f(\ell+b)$ for all ℓ. Thus we have that a.s., for some $\ell_0<\infty$ and all $\ell\geq\ell_0$, $\sigma_{\lfloor K^\ell\rfloor}\leq\varepsilon^{-1}v(\lfloor K^\ell\rfloor)f(\lfloor K^\ell\rfloor+b)$. Hence with the definition of r_v at (3.15) we have, a.s., for all ℓ sufficiently large,

$$r_v(\sigma_{\lfloor K^\ell\rfloor})\leq r_v(\varepsilon^{-1}v(\lfloor K^\ell\rfloor)f(\lfloor K^\ell\rfloor+b))\leq\lfloor K^\ell\rfloor\leq Y_{\sigma_{\lfloor K^\ell\rfloor}}. \tag{3.18}$$

Since $\mathbf{E}[\sigma_\ell]<\infty$ for all ℓ, we have that $\sigma_\ell<\infty$ a.s. for all ℓ. Moreover, the jumps bound (3.12) implies that, for all $\ell\geq Y_0$, $\sigma_{\ell+b}\geq 1+\sigma_\ell$ a.s., so that $\lim_{\ell\to\infty}\sigma_\ell=\infty$ a.s.. Thus, a.s., every $n\in\mathbb{Z}^+$ satisfies $\sigma_{\lfloor K^{\ell_n-1}\rfloor}\leq n<\sigma_{\lfloor K^{\ell_n}\rfloor}$ for some $\ell_n\in\mathbb{N}$. Then, a.s.,

$$\max_{0\leq m\leq n}Y_m\geq\max_{0\leq m\leq\sigma_{\lfloor K^{\ell_n-1}\rfloor}}Y_m\geq Y_{\sigma_{\lfloor K^{\ell_n-1}\rfloor}}\geq\lfloor K^{\ell_n-1}\rfloor\geq\frac{\lfloor K^{\ell_n-1}\rfloor}{\lfloor K^{\ell_n}\rfloor}(Y_{\sigma_{\lfloor K^{\ell_n}\rfloor}}-b), \tag{3.19}$$

for all n sufficiently large, since $Y_{\sigma_{\lfloor K^{\ell_n}\rfloor}}\leq\lfloor K^{\ell_n}\rfloor+b$. Then (3.19) and (3.18) imply that, a.s., for all but finitely many $n\in\mathbb{Z}^+$,

$$\max_{0\leq m\leq n}Y_m\geq\frac{\lfloor K^{\ell_n-1}\rfloor}{\lfloor K^{\ell_n}\rfloor}\left(r_v(\sigma_{\lfloor K^{\ell_n}\rfloor})-b\right)\geq\frac{\lfloor K^{\ell_n-1}\rfloor}{\lfloor K^{\ell_n}\rfloor}\left(r_v(n)-b\right),$$

since $\sigma_{\lfloor K^{\ell_n}\rfloor}>n$ and $r_v(n)$ is nondecreasing for n sufficiently large. Now taking a sequence of values for K converging down to 1 we obtain (3.16). ■

4 Almost-sure bounds for the Lamperti problem

4.1 Introduction and results

In this section we give upper and lower almost-sure bounds for the so-called Lamperti problem of a stochastic process on the half-line with mean drift asymptotically zero. We will need these results for our almost-sure bounds for the stochastic billiard model; our results for the Lamperti problem appear to be new in the generality given here, and in some cases even for a nearest-neighbour random walk.

Let $(\mathcal{F}_n)_{n\in\mathbb{Z}^+}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\eta = (\eta_n)_{n\in\mathbb{Z}^+}$ be a discrete-time time-homogeneous stochastic process adapted to $(\mathcal{F}_n)_{n\in\mathbb{Z}^+}$ and taking values in an unbounded subset $\mathcal{S}$ of $[0, \infty)$.

We suppose that jumps of η are uniformly bounded, that is there exists $B \in (0, \infty)$ such that for all $n \in \mathbb{Z}^+$ and all $x \in \mathcal{S}$

$$\mathbf{P}[|\eta_{n+1} - \eta_n| > B \mid \mathcal{F}_n] = 0, \text{ a.s..} \tag{4.1}$$

Under the jumps condition (4.1), the jump-moment functions $\mu_1 : \mathcal{S} \to [-B, B]$ and $\mu_2 : \mathcal{S} \to [0, B^2]$ given for $k \in \{1, 2\}$ by

$$\mu_k(x) := \mathbf{E}[(\eta_{n+1} - \eta_n)^k \mid \eta_n = x], \quad (n \in \mathbb{Z}^+) \tag{4.2}$$

are well-defined; in particular $\mu_2(x)$ is bounded above. Our basic assumption for this section will be (A2) below.

(A2) Let η be a discrete-time stochastic process on $[0, \infty)$ satisfying (4.1) with μ_1, μ_2 as given by (4.2).

For some of the results in this section we also assume that there exists $v > 0$ such that, for all $x \in \mathcal{S}$

$$\mu_2(x) \geq v. \tag{4.3}$$

We will sometimes make the further assumption that

$$\mathbf{P}\left[\limsup_{n\to\infty} \eta_n = +\infty\right] = 1. \tag{4.4}$$

The model that we concentrate on is a particular case of the so-called Lamperti problem (see [17–19, 23]) of a stochastic process on $[0, \infty)$ with mean drift asymptotically zero: that is $\mu_1(x) \to 0$ as $x \to \infty$. Our results do not actually assume $\mu_1(x) \to 0$, but it is in this case (in fact, in the case $|\mu_1(x)| = O(1/x)$) that they are of most interest. As mentioned by Lamperti [17], and as is also true for the results in this section, only the first two moments μ_1, μ_2 of the jump distribution are important: there is some form of 'invariance principle' at work. We will be mainly concerned with the case that, from the point of view of recurrence, turns out to be critical; that is where $x|\mu_1(x)|$ remains bounded away from zero and from infinity.

In the particular case of a nearest-neighbour random walk, where η is supported on $\mathbb{Z}^+$, the problem reduces to that of a simple random walk with asymptotically zero perturbation:

that model was studied by Harris [13] and by Hodges and Rosenblatt [14], and is amenable to special methods for so-called birth-and-death chains. Thus many results are present in the literature for the nearest-neighbour case: in Section 4.2 below we briefly mention some of these results and their relation to the results given in this section. For the applications in the present paper, however, we cannot use these nearest-neighbour results. Thus we need to prove results in the general Lamperti setting. In fact, as we will point out below, some of the results that we give in the present section seem to be new even for the nearest-neighbour situation. Thus the results that we state in this section are of independent interest.

As well as being of interest in their own right, stochastic processes on the half-line with mean drift asymptotically zero are important for the study of multidimensional processes by the method of Lyapunov-type functions (see e.g. [11, 23]). For example, if $(Z_n)_{n\in\mathbb{Z}^+}$ is a zero-drift process with bounded jumps on $\mathbb{Z}^d$, $d \geq 2$, the process $(\|Z_n\|)_{n\in\mathbb{Z}^+}$ is supported on the half-line and has mean drift asymptotically zero. Clearly, the process $\|Z_n\|$ will not be nearest-neighbour; thus the generality of results like Lamperti's [17, 19] and those in the present section is valuable.

Fix $H > 0$, which will need to be large for some of our results. We say that η is *recurrent* or *transient* according to whether the return time $\inf\{n \in \mathbb{Z}^+ : \eta_n \leq 2H\}$ is almost surely finite or not; in the former case we distinguish positive- and null-recurrence according to whether the return time has finite or infinite expectation.

The recurrence and transience properties of η were studied by Lamperti, who proved the following result (see Theorems 3.1 and 3.2 of [17] with Theorem 2.1 of [19]; also Theorem 3 of [23] for a finer result). Note that all of the conditions that we state in this section involving evaluating $\mu_1(x)$, $\mu_2(x)$ only need apply for $x \in \mathcal{S}$.

Proposition 4.1 *[17, 19] Suppose that (A2), (4.3), and (4.4) hold.*

(i) If for all $x > 2H$, $2x|\mu_1(x)| \leq \mu_2(x)$, then η is null-recurrent for any $\eta_0 > 2H$.

(ii) Suppose that there exists $\delta > 0$ such that for all $x > 2H$

$$2x\mu_1(x) - \mu_2(x) > \delta. \tag{4.5}$$

Then η is transient for any $\eta_0 > 2H$.

(iii) Suppose that there exists $\delta > 0$ such that for all $x > 2H$, $2x\mu_1(x) + \mu_2(x) < -\delta$. Then η is positive-recurrent for any $\eta_0 > 2H$.

For our almost-sure lower bounds, we impose an additional 'reflection' condition that ensures that we can avoid getting trapped in a bounded set. Condition (A3) below is the most appropriate way of doing this for our applications to the stochastic billiard model, but is clearly stronger than is necessary for the results in the present section.

(A3) Given $\eta_n = x > H$, if a jump would take $\eta_{n+1} < H$ we replace the jump with $\eta_{n+1} = 2H$ instead.

We now state our results on almost-sure bounds. In the general setting of the present section, the only almost-sure bound result that we could find in the literature is also due to Lamperti [18]; see Proposition 4.2 in our discussion of the literature below. Lamperti's bound is a weaker version of our result Theorem 4.1(i) below, while Theorem 4.1(ii) gives a complementary lower bound.

Theorem 4.1 *Suppose that Assumption (A2) holds.*

(i) Suppose that there exists $C \in (0, \infty)$ such that for all $x \geq 0$

$$2x\mu_1(x) \leq C.$$

Then for any η_0, for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\max_{0 \leq m \leq n} \eta_m \leq n^{1/2} (\log n)^{(1/2)+\varepsilon}.$$

(ii) Suppose that (A3) holds, and that there exists $\delta > 0$ such that for all $x > H$

$$2x\mu_1(x) + \mu_2(x) \geq \delta.$$

Then for any $\eta_0 > H$, for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\max_{0 \leq m \leq n} \eta_m \geq n^{1/2} (\log n)^{-(1/2)-\varepsilon}.$$

In the transient case, we prove the following lower bound that strengthens, in some sense, Theorem 4.1(ii) in this case.

Theorem 4.2 *Suppose that (A2) and (A3) hold. If (4.5) holds for some $\delta > 0$ and all $x > H$, then there exists $D \in (0, \infty)$ such that, for any $\eta_0 > H$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,*

$$\eta_n \geq n^{1/2} (\log n)^{-D}.$$

We could find no reference for a result like Theorem 4.2, even in the nearest-neighbour case. We prove Theorem 4.2 in Section 4.4; it may be possible to extract bounds for the exponent D in the logarithmic term in terms of the constants B, δ by keeping track of the constants in our proofs. Note that the transient critical Lamperti problem for which (A2) and (A3) hold, and

$$\delta < 2x\mu_1(x) - \mu_2(x) < C$$

for some $0 < \delta < C < \infty$ and all x large enough, satisfies the conditions of each of the results in Theorems 4.1 and 4.2.

The following discussion suggests that without taking into account finer behaviour (such as the smallest value of $\delta > 0$ for which (4.5) is satisfied), one cannot expect to improve upon Theorem 4.2.

Let $(S_n^d)_{n\in\mathbb{Z}^+}$ be the symmetric simple random walk on $\mathbb{Z}^d$, $d \geq 2$, so that for $x, y \in \mathbb{Z}^d$, $\mathbf{P}[S_{n+1}^d = y \mid S_n^d = x] = (2d)^{-1}$ if and only if $\|y - x\| = 1$. Then elementary calculations show that

$$2\|x\|\mathbf{E}[\|S_{n+1}^d\| - \|S_n^d\| \mid S_n^d = x] - \mathbf{E}[(\|S_{n+1}^d\| - \|S_n^d\|)^2 \mid S_n^d = x] \to 1 - \frac{2}{d}$$

as $\|x\| \to \infty$; thus the process $(\|S_n^d\|)_{n\in\mathbb{Z}^+}$ is in precisely the critical Lamperti situation, and for $d > 2$ satisfies (4.5). For $d > 2$, a classical result of Dvoretzky and Erdős [7] says that for any $\varepsilon > 0$, a.s.,

$$\|S_n^d\| > n^{1/2}(\log n)^{-\frac{1}{d-2}-\varepsilon} \tag{4.6}$$

for all but finitely many $n \in \mathbb{Z}^+$, and that this bound is sharp in that for $\varepsilon = 0$ the inequality (4.6) fails infinitely often with probability 1. In particular, it is at least informally reasonable to argue that by letting $d \downarrow 2$ in (4.6) one might not expect to improve in general upon the arbitrary logarithmic factor in Theorem 4.2.

Now we state results on almost-sure bounds that we will need for our stochastic billiard model when $g(x) = x^\gamma$ with $\gamma < 0$. Again these results seem to be new in the generality given here.

Theorem 4.3 below is further demonstration of the phenomenon of *polynomial ergodicity* for the critical Lamperti problem, as also evidenced by results of [24] in the context of stationary measures.

Theorem 4.3 *Suppose that (A2) and (4.3) hold.*

(i) Suppose that there exists $\kappa > 1$ such that for all x sufficiently large

$$-2\kappa\mu_2(x) + o(1) \leq 2x\mu_1(x) \leq -\kappa\mu_2(x) + o((\log x)^{-1}). \tag{4.7}$$

Then for any η_0, any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\max_{0\leq m\leq n} \eta_m \leq n^{1/(1+\kappa)}(\log n)^{(2/(1+\kappa))+\varepsilon}.$$

(ii) Suppose that (A3) holds, and that there exists $\kappa \geq 1$ such that for all x sufficiently large

$$2x\mu_1(x) + \kappa\mu_2(x) \geq o((\log x)^{-1}). \tag{4.8}$$

Then there exists $H \in (0, \infty)$ such that for any $\eta_0 > H$, for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$\max_{0\leq m\leq n} \eta_m \geq n^{1/(1+\kappa)}(\log n)^{-(2/(1+\kappa))-\varepsilon}.$$

Note that the $\kappa = 1$ case of Theorem 4.3(ii) gives a weaker form of the lower bound in Theorem 4.1(ii) under a somewhat weaker condition.

Before we prove the results stated in Section 4.1, we briefly discuss the existing literature related to the present section. This is done in Section 4.2 below. Then we give the proofs of Theorems 4.1 and 4.3 in Section 4.3 and Theorem 4.2 in Section 4.4.

4.2 Remarks on the literature

In the general setting of the so-called Lamperti problem, when (A2) holds, there seem to be few almost-sure bounds known. The next result, due to Lamperti himself (see Theorems 2.1, 2.2, 4.2, and 5.1 in [18]), includes an almost-sure upper bound in a particular case where the conditions of (i) or (ii) in Proposition 4.1 hold.

Proposition 4.2 *[18] Suppose that (A2) holds, and that for any finite interval* $I \subset [0, \infty)$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} \mathbf{P}[\eta_m \in I] = 0. \tag{4.9}$$

Suppose that for $a, b \in \mathbb{R}$

$$\lim_{x \to \infty} \mu_2(x) = b > 0, \quad \lim_{x \to \infty} (x \mu_1(x)) = a > -(b/2).$$

Then for any η_0*, for any* $\varepsilon > 0$*, a.s., for all but finitely many* $n \in \mathbb{Z}^+$

$$\eta_n \leq n^{(1/2)+\varepsilon}. \tag{4.10}$$

In addition, suppose that (4.9) holds uniformly in η_0*. Then as* $n \to \infty$ *the following invariance principle applies:*

$$n^{-1/2} \eta_{\lfloor n \bullet \rfloor} \Rightarrow \Upsilon_\bullet, \tag{4.11}$$

where $(\Upsilon_t)_{t>0}$ *is a diffusion process on* $[0, \infty)$ *with Kolmogorov backwards equation* $u_t = (a/x)u_x + (b/2)u_{xx}$ *(see Section 3 in [18] for details).*

Thus Theorem 4.1(i) improves upon the bound in (4.10). Note that the condition (4.9), which does not distinguish between null-recurrence and transience, implies (4.4).

A special case of the Lamperti problem on $[0, \infty)$ is the case where the process η is supported on $\mathbb{Z}^+$ and only nearest-neighbour jumps are allowed. This special case has received much more attention than the general case described in Section 4.1 above; now we briefly describe known results in the nearest-neighbour case.

When they exist, these nearest-neighbour results are sharper than the general results that we give in Section 4.1. Thus it may be possible to sharpen the bounds in the results in Section 4.1.

In the nearest-neighbour case, η is sometimes known as a *birth-and-death* chain (or birth-and-death random walk). Precisely, suppose that there exists a sequence $(p_x)_{x \in \mathbb{Z}^+}$ with $p_x \in (0, 1)$ such that for all $x \in \mathbb{N}$ and $n \in \mathbb{Z}^+$

$$\mathbf{P}[\eta_{n+1} = x - 1 \mid \eta_n = x] = 1 - \mathbf{P}[\eta_{n+1} = x + 1 \mid \eta_n = x] = p_x,$$

with reflection from 0 governed by

$$\mathbf{P}[\eta_{n+1} = 0 \mid \eta_n = 0] = 1 - \mathbf{P}[\eta_{n+1} = 1 \mid \eta_n = 0] = p_0.$$

(In the literature, the slightly more general model where the walk is allowed to stay in the current position also appears. This introduces no new essential features however.)

Such processes have been extensively studied in various contexts. They are often amenable to explicit computation; one particularly fruitful approach is via orthogonal polynomials, dating back at least to Karlin and McGregor [15] and subsequently employed for instance by Voit [29,31]. A recent reference is [6], to which we are indebted for bringing to our attention several of the papers cited in this section.

The one-step mean drift of this walk is for $x \in \mathbb{N}$

$$\mathbf{E}[\eta_{n+1} - \eta_n \mid \eta_n = x] = 1 - 2p_x.$$

We are in the Lamperti situation if we assume $\lim_{x\to\infty} p_x = 1/2$. A case of particular interest is when $p_x \in (0,1)$ satisfies

$$p_x = \frac{1}{2} + \frac{\kappa}{4} x^{-\alpha} + o(x^{-\alpha}), \tag{4.12}$$

for $\alpha > 0$ and $\kappa \in \mathbb{R} \setminus \{0\}$. Then with the notation at (4.2), $\mu_1(x) = -(\kappa/2)x^{-\alpha} + o(x^{-\alpha})$ and $\mu_2(x) = 1$.

In the nearest-neighbour case, partial versions of the recurrence result Proposition 4.1 were given in e.g. [13, 14]. When (4.12) holds, Proposition 4.1 implies that if $\alpha > 1$, η is null-recurrent, while if $\alpha \in (0,1)$, η is transient, positive-recurrent according to $\kappa < 0$, $\kappa > 0$. In the critical case $\alpha = 1$, η is transient if $\kappa < -1$, null-recurrent if $|\kappa| < 1$, and positive-recurrent if $\kappa > 1$.

Almost-sure results are known for certain null-recurrent and transient situations. The following two propositions collect some known results of this type; special cases were dealt with in [2, 10, 26, 27]. The next result follows from Theorem 2.11 of [30]; it deals with the transient case with $\alpha \in (0,1)$. (There appears to be a typo in the statement of Theorem 2.11 in [30].)

Proposition 4.3 *[30] Suppose that (4.12) holds for $\alpha \in (0,1)$ and $\kappa < 0$. Then for any η_0, a.s.*

$$\lim_{n\to\infty} \frac{\eta_n}{n^{1/(1+\alpha)}} = \left(\frac{|\kappa|}{2}(1+\alpha) \right)^{1/(1+\alpha)}.$$

Theorem 7.1 of Lamperti [18] gives a corresponding result for more general processes (in the manner of Section 4.1), but with convergence in probability only.

The following iterated logarithm result that applies in the $\kappa < 0$, $\alpha = 1$ case of (4.12) follows from Theorem 4(b) of [12] (compare also Theorem 1.3 of [29]).

Proposition 4.4 *[12] Suppose that there exist c, C with $0 < c < C < \infty$ such that for all x sufficiently large*

$$\frac{1}{2} - Cx^{-1} \le p_x \le \frac{1}{2} - cx^{-1}.$$

Then for any η_0, a.s.

$$\limsup_{n\to\infty} \frac{\eta_n}{\sqrt{2n \log\log n}} = 1.$$

Apparently missing, for example, is the complete result in the case where (4.12) holds with $\alpha = 1$ and $\kappa \in (0,1)$, for which the walk is null-recurrent; a one-sided result is given in Theorem 4(a) of [12]. Also, there seem to be no (sharp) existing results in the critical positive-recurrent case where (4.12) holds with $\alpha = 1$ and $\kappa > 1$. Thus Theorems 4.1, 4.2, and 4.3 add to known results even in the nearest-neighbour case.

4.3 Proofs of Theorems 4.1 and 4.3

In order to prove Theorems 4.1 and 4.3, we apply the general results on almost-sure bounds for discrete-time stochastic processes given in Section 3.

Proof of Theorem 4.1. For $x \geq 0$ and $n \in \mathbb{Z}^+$ we have

$$\begin{aligned} \mathbf{E}[\eta_{n+1}^2 - \eta_n^2 \mid \eta_n = x] &= 2x\mathbf{E}[\eta_{n+1} - \eta_n \mid \eta_n = x] + \mathbf{E}[(\eta_{n+1} - \eta_n)^2 \mid \eta_n = x] \\ &= 2x\mu_1(x) + \mu_2(x). \end{aligned} \tag{4.13}$$

Under the conditions of part (i) of the theorem, the right-hand side of (4.13) is uniformly bounded above. Hence we can apply Theorem 3.2, with $Y_n = \eta_n$, taking $f(x) = x^2$ and $a(x) = x(\log x)^{1+\varepsilon}$. This proves part (i).

Under the conditions of part (ii) of the theorem, we have that the right-hand side of (4.13) is strictly positive for all $x > H$. Under condition (A3) it suffices to consider the process on $[H, \infty)$. Hence we can apply Theorem 3.3, with $Y_n = \eta_n$, taking $f(x) = x^2$ and $v(x) = (\log x)^{1+\varepsilon}$. This proves part (ii). ■

Now we prepare for the proof of Theorem 4.3. We need two more lemmas, to identify suitable functions f with which to apply the criteria of Section 3.

Lemma 4.1 *Suppose that (A2) and (4.3) hold. Suppose that there exists $\kappa > 1$ such that (4.7) holds for all x sufficiently large. Then there exists $C \in (0, \infty)$ such that for all $x \geq 0$*

$$\mathbf{E}[\eta_{n+1}^{1+\kappa}(\log(1+\eta_{n+1}))^{-1} - \eta_n^{1+\kappa}(\log(1+\eta_n))^{-1} \mid \eta_n = x] \leq C.$$

Proof. It follows from Taylor's theorem applied to the function $x \mapsto x^{1+\kappa}(\log(1+x))^{-1}$ that for $|\theta(x)| = O(1)$ as $x \to \infty$

$$\begin{aligned} &(x+\theta(x))^{1+\kappa}(\log(1+x+\theta(x)))^{-1} - x^{1+\kappa}(\log(1+x))^{-1} \\ &= \frac{x^{1+\kappa}}{\log(1+x)}\left[\frac{1+\kappa}{2x^2}(2x\theta(x) + \kappa\theta(x)^2) - \frac{1}{2x^2\log(1+x)}(2x\theta(x) + (2\kappa+1)\theta(x)^2 + o(1))\right]. \end{aligned}$$

Now conditioning on $\eta_n = x$ and setting $\theta(x) = \eta_{n+1} - \eta_n$, taking expectations in the last displayed equation gives

$$\begin{aligned} &\mathbf{E}[\eta_{n+1}^{1+\kappa}(\log(1+\eta_{n+1}))^{-1} - \eta_n^{1+\kappa}(\log(1+\eta_n))^{-1} \mid \eta_n = x] \\ &= \frac{x^{1+\kappa}}{\log(1+x)}\left[\frac{1+\kappa}{2x^2}(2x\mu_1(x) + \kappa\mu_2(x)) - \frac{1}{2x^2\log(1+x)}(2x\mu_1(x) + (2\kappa+1)\mu_2(x) + o(1))\right] \end{aligned}$$

$$\le \frac{x^{\kappa-1}}{(\log(1+x))^2}\left[-\frac{1}{2}\mu_2(x)+o(1)\right],$$

using (4.7). Then with (4.3) this yields the result. ■

Lemma 4.2 *Suppose that (A2) and (4.3) hold. Suppose that there exists $\kappa \ge 1$ such that (4.8) holds for all x sufficiently large. Then there exist $H \in (0,\infty)$, $\varepsilon > 0$ such that for all $x > H$*

$$\mathbf{E}[\eta_{n+1}^{1+\kappa}\log(1+\eta_{n+1}) - \eta_n^{1+\kappa}\log(1+\eta_n) \mid \eta_n = x] \ge \varepsilon.$$

Proof. This time, it follows from Taylor's theorem that for $|\theta(x)| = O(1)$ as $x \to \infty$

$$\begin{aligned}
&(x+\theta(x))^{1+\kappa}\log(1+x+\theta(x)) - x^{1+\kappa}\log(1+x)\\
&= x^{1+\kappa}\left[\frac{1+\kappa}{2x^2}(\log(1+x))(2x\theta(x)+\kappa\theta(x)^2) + \frac{1}{2x^2}(2x\theta(x)+(2\kappa+1)\theta(x)^2+o(1))\right].
\end{aligned}$$

Now conditioning on $\eta_n = x$ and setting $\theta(x) = \eta_{n+1} - \eta_n$, taking expectations in the last displayed equation gives

$$\begin{aligned}
&\mathbf{E}[\eta_{n+1}^{1+\kappa}\log(1+\eta_{n+1}) - \eta_n^{1+\kappa}\log(1+\eta_n) \mid \eta_n = x]\\
&= x^{1+\kappa}\left[\frac{1+\kappa}{2x^2}(\log(1+x))(2x\mu_1(x)+\kappa\mu_2(x)) + \frac{1}{2x^2}(2x\mu_1(x)+(2\kappa+1)\mu_2(x)+o(1))\right]\\
&\ge x^{\kappa-1}\left[\frac{\kappa+1}{2}\mu_2(x)+o(1)\right] \ge \varepsilon > 0,
\end{aligned}$$

for all x large enough, using (4.8), (4.3), and the fact that $\kappa \ge 1$. ■

Proof of Theorem 4.3. By Lemma 4.1, we can apply Theorem 3.2 with $f(x) = x^{1+\kappa}(\log(1+x))^{-1}$, $a(x) = x(\log x)^{1+\varepsilon}$, and $Y_n = \eta_n$ to obtain part (i) of the theorem. On the other hand, by Lemma 4.2 we can apply Theorem 3.3 with $f(x) = x^{1+\kappa}\log(1+x)$, $v(x) = (\log x)^{1+\varepsilon}$, and $Y_n = \eta_n$ to obtain part (ii) of the theorem. ■

4.4 Proof of Theorem 4.2

Suppose that (4.5) holds for some $\delta > 0$. Our strategy for the proof of Theorem 4.2 is to construct a scale on which the process η is transient with mean drift that is positive and bounded uniformly away from 0. Fix $\beta > 0$. Denote $I_0 := \emptyset$ and for $r \in \mathbb{N}$ define the intervals

$$I_r := [(1+\beta)^r - B, (1+\beta)^r + B],$$

where B is as in the jumps bound (4.1); then for β sufficiently large, $I_1, I_2, \ldots$ do not overlap.

We look at the process η at the moments at which it enters an interval I_r different from the last one visited. We define the random times $\ell_1, \ell_2, \ldots$ inductively as follows. Set $\ell_1 := \min\{n \in \mathbb{Z}^+ : \eta_n \in \cup_r I_r\}$; for $k \in \mathbb{N}$, if $\eta_{\ell_k} \in I_r$, let $\ell_{k+1} := \min\{\ell > \ell_k : \eta_\ell \in I_{r-1} \cup I_{r+1}\}$. Consider the embedded process $\tilde{\eta} = (\tilde{\eta}_k)_{k\in\mathbb{N}} = (\eta_{\ell_k})_{k\in\mathbb{N}}$. The conditions of Theorem 4.2

imply those of Theorem 4.1(ii), so in particular (4.4) holds. This together with the jumps bound (4.1) implies that

$$\mathbf{P}\Big[\bigcup_{m>n}\{\eta_m \in I_{r-1}\cup I_{r+1}\} \mid \eta_n \in I_r\Big] = 1$$

for any $r \in \mathbb{N}$, so that the process $\tilde{\eta}$ is well-defined and the random times ℓ_k are almost surely finite for each k. Denote

$$p_r := \mathbf{P}[\eta_{\ell_{k+1}} \in I_{r+1} \mid \eta_{\ell_k} \in I_r], \quad (k \in \mathbb{N}). \tag{4.14}$$

The next result says that the process $\tilde{\eta}$ has uniformly positive mean drift.

Lemma 4.3 *Under the conditions of Theorem 4.2, there exist $\beta > 0$ and $\varepsilon_0 > 0$ such that for all $r \in \mathbb{N}$*

$$p_r > \frac{1}{2} + \varepsilon_0.$$

Proof. Let $\lambda < 0$. Taylor's theorem implies that as $x \to \infty$

$$\mathbf{E}[\eta_{n+1}^\lambda - \eta_n^\lambda \mid \eta_n = x] = x^{\lambda-2}\left[\lambda x \mu_1(x) + \frac{\lambda(\lambda-1)}{2}\mu_2(x) + O(x^{-1})\right],$$

using (4.1) and (4.2). Then by (4.5) and (4.1) again, we have

$$\mathbf{E}[\eta_{n+1}^\lambda - \eta_n^\lambda \mid \eta_n = x] \le x^{\lambda-2}\left[(\lambda/2)(\delta + \lambda B^2) + O(x^{-1})\right],$$

where B is the jumps bound in (4.1). It follows that for $\lambda \in (-\delta/B^2, 0)$

$$\mathbf{E}[\eta_{n+1}^\lambda - \eta_n^\lambda \mid \eta_n = x] \le 0,$$

for all x sufficiently large. Hence for the stopping times ℓ_k it is also the case that

$$\mathbf{E}[\eta_{\ell_{k+1}}^\lambda - \eta_{\ell_k}^\lambda \mid \eta_{\ell_k} = x] \le 0, \tag{4.15}$$

for all x sufficiently large. Then the supermartingale property (4.15) implies that for β large enough and all $r \in \mathbb{N}$

$$\mathbf{E}[\eta_{\ell_{k+1}}^\lambda \mid \eta_{\ell_k} \in I_r] \le [(1+\beta)^r - B]^\lambda. \tag{4.16}$$

Also, we have that

$$\begin{aligned}\mathbf{E}[\eta_{\ell_{k+1}}^\lambda \mid \eta_{\ell_k} \in I_r] &= p_r \mathbf{E}[\eta_{\ell_{k+1}}^\lambda \mid \eta_{\ell_k} \in I_r, \eta_{\ell_{k+1}} \in I_{r+1}] \\ &\quad + (1-p_r)\mathbf{E}[\eta_{\ell_{k+1}}^\lambda \mid \eta_{\ell_k} \in I_r, \eta_{\ell_{k+1}} \in I_{r-1}] \\ &\ge p_r[(1+\beta)^{r+1} + B]^\lambda + (1-p_r)[(1+\beta)^{r-1} + B]^\lambda. \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17) we see that one can choose $\beta > 0$, $\varepsilon_0 > 0$ such that $p_r > (1/2) + \varepsilon_0$ for all r. ∎

Consider, for $k \in \mathbb{N}$,

$$Z_k := \sum_{r \in \mathbb{N}} r \mathbf{1}\{\eta_{\ell_k} \in I_r\}. \tag{4.18}$$

Then the process $Z = (Z_k)_{k \in \mathbb{N}}$ is a stochastic process on $\mathbb{N}$ with nearest-neighbour transitions that tracks which interval I_r the embedded process $\tilde{\eta}$ is in. With p_r as defined by (4.14), we have

$$\mathbf{P}[Z_{k+1} = r+1 \mid Z_k = r] = 1 - \mathbf{P}[Z_{k+1} = r-1 \mid Z_k = r] = p_r. \tag{4.19}$$

Let κ_r^Z denote the time of the first visit of Z to $r \in \mathbb{N}$, i.e.

$$\kappa_r^Z := \min\{k \in \mathbb{N} : Z_k = r\}. \tag{4.20}$$

For $r \in \mathbb{N}$, $s \in \mathbb{Z}^+$ define

$$\gamma(r, s) := \mathbf{P}[\kappa_r^Z < \infty \mid Z_0 = r + s]. \tag{4.21}$$

In words, $\gamma(r, s)$ is the probability that the process Z hits r in finite time, given that it starts at $r + s$.

The next result will enable us to show, loosely speaking, that the process Z leaves each state for good only shortly after its first visit.

Lemma 4.4 *Under the conditions of Theorem 4.2, there exists $C \in (0, \infty)$ such that*

$$\sum_{r \in \mathbb{N}} \gamma(r, \lfloor C \log r \rfloor) < \infty.$$

Proof. To estimate the required hitting probability we introduce an auxiliary process. Fix $C_0 \in (0, \infty)$, which we will eventually take to be large. Let $r \in \mathbb{N}$. Define the nonnegative process $(W_t)_{t \in \mathbb{Z}^+}$ for $t \in \mathbb{Z}^+$ by

$$W_t := \exp\left\{-C_0^{-1}(Z_t - r)\right\}.$$

Then we have for $n \in \mathbb{N}$ and $t \in \mathbb{Z}^+$

$$\mathbf{E}[W_{t+1} - W_t \mid Z_t = n] = \exp\{-C_0^{-1}(n - r)\}\mathbf{E}[\exp\{-C_0^{-1}(Z_{t+1} - Z_t)\} - 1 \mid Z_t = n]. \tag{4.22}$$

We will make use of the fact that there exists $C_1 \in (0, \infty)$ such that $\exp(-x) - 1 \leq -x + C_1 x^2$ for all x with $|x| \leq 1$. Since Z has nearest-neighbour jumps, it follows that

$$\begin{aligned} &\mathbf{E}[\exp\{-C_0^{-1}(Z_{t+1} - Z_t)\} - 1 \mid Z_t = n] \\ \leq &- C_0^{-1}\mathbf{E}[Z_{t+1} - Z_t \mid Z_t = n] + C_1 C_0^{-2}\mathbf{E}[(Z_{t+1} - Z_t)^2 \mid Z_t = n] \\ \leq &- C_0^{-1}(2p_n - 1) + C_1 C_0^{-2}, \end{aligned} \tag{4.23}$$

using (4.19) and the fact that Z has nearest-neighbour jumps again. It then follows from (4.22), (4.23), and Lemma 4.3 that we can choose C_0 sufficiently large, not depending on r, so that for any $n \in \mathbb{N}$

$$\mathbf{E}[W_{t+1} - W_t \mid Z_t = n] \leq 0, \tag{4.24}$$

for all $t \in \mathbb{Z}^+$.

Now let $s \in \mathbb{Z}^+$ and take $Z_0 = r + s$. Recall the definition of the stopping time κ_r^Z from (4.20). From (4.24) we have that $(W_t)_{t\in\mathbb{Z}^+}$ is a nonnegative supermartingale, so that $W_\infty := \lim_{t\to\infty} W_t$ exists a.s., and $\mathbf{E}[W_0] \geq \mathbf{E}[W_{\kappa_r^Z}]$. It follows that for any $r \in \mathbb{N}$, $s \in \mathbb{Z}^+$

$$\mathbf{E}[W_0] = \exp\{-C_0^{-1}s\} \geq \mathbf{E}[W_{\kappa_r^Z}\mathbf{1}_{\{\kappa_r^Z<\infty\}}] = \gamma(r, s),$$

using (4.21). In particular, it follows that for some $C \in (0, \infty)$ large enough

$$\sum_{r\in\mathbb{N}} \gamma(r, \lfloor C \log r\rfloor) \leq \sum_{r\in\mathbb{N}} \exp\{-C_0^{-1}\lfloor C \log r\rfloor\} \leq \sum_{r\in\mathbb{N}} r^{-2} < \infty;$$

completing the proof of the lemma. ∎

To complete the proof of Theorem 4.2, we need to translate our results for the embedded process $\tilde{\eta}$ back to the underlying process η.

Proof of Theorem 4.2. Consider the nearest-neighbour process $Z = (Z_k)_{k\in\mathbb{N}}$ on $\mathbb{N}$, as defined by (4.18). Recall the definition of κ_r^Z from (4.20). Let ω_r^Z denote the time of the *last* visit of Z to $r \in \mathbb{N}$, i.e.

$$\omega_r^Z := \max\{k \in \mathbb{N} : Z_k = r\}.$$

Similarly for the process η set for $x > 0$

$$\kappa_x^\eta := \min\{n \in \mathbb{Z}^+ : \eta_n \geq x\}, \quad \omega_x^\eta := \max\{n \in \mathbb{Z}^+ : \eta_n \leq x\}.$$

With Lemma 4.4, the Borel-Cantelli lemma implies that, a.s., for only finitely many $r \in \mathbb{N}$ does Z return to r after visiting $r + \lfloor C \log r\rfloor$. So, a.s., for all but finitely many $r \in \mathbb{N}$,

$$\omega_r^Z \leq \kappa_{r+\lfloor C \log r\rfloor}^Z. \tag{4.25}$$

Set

$$r(x) := \left\lfloor \frac{\log x}{\log(1 + \beta)} \right\rfloor.$$

Observe that by definition of the process Z we have that a.s. for some $C \in (0, \infty)$ and all x large enough

$$\omega_x^\eta \leq \omega_{r(x)+1}^Z \leq \kappa_{r(x)+\lfloor C \log r(x)\rfloor}^Z,$$

by (4.25). Again by the definition of Z, it now follows that a.s. for some $C' \in (0, \infty)$

$$\omega_x^\eta \leq \kappa_{r(x)+\lfloor C \log r(x)\rfloor}^Z \leq \kappa_{\lfloor x(\log x)^{C'}\rfloor}^\eta, \tag{4.26}$$

for all x large enough.

The conditions of Theorem 4.2 imply those of Theorem 4.1(ii). Hence the lower bound in Theorem 4.1(ii) applies, and so for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$

$$\sup\{x \geq 0 : \kappa_x^\eta \leq n\} \geq \max_{0\leq m\leq n} \eta_m \geq n^{1/2}(\log n)^{-(1/2)-\varepsilon}.$$

It follows that for any $\varepsilon > 0$, a.s., for all but finitely many $x \in \mathbb{Z}^+$

$$\kappa_x^\eta \leq x^2 (\log x)^{1+\varepsilon}. \tag{4.27}$$

So by (4.26) and (4.27) there exist $C, x_0 \in (0, \infty)$ such that, a.s., for all $x \geq x_0$

$$\omega_x^\eta \leq x^2 (\log x)^C.$$

By the transience of η, we have that a.s. $\eta_n \geq x_0$ for all but finitely many $n \in \mathbb{Z}^+$. Hence a.s., for all but finitely many $n \in \mathbb{Z}^+$ we have

$$n \leq \omega_{\eta_n}^\eta \leq \eta_n^2 (\log \eta_n)^C \leq \eta_n^2 (\log n)^{C'},$$

using the jumps bound (4.1) for the final inequality. This proves the theorem. ∎

5 Proofs for stochastic billiards

5.1 Preliminaries

To prove our main theorems on the stochastic billiard model, we start by studying the properties of the process between successive collisions; i.e., the jumps of the process ξ.

Suppose that at time $n \in \mathbb{Z}^+$ we have $\xi_n = (\xi_n^{(1)}, \xi_n^{(2)}) = (x, \pm g(x))$ for $x > A$, and then ξ_n is reflected at angle α to the normal. Denote $\Delta(x, \alpha) := \xi_{n+1}^{(1)} - \xi_n^{(1)}$, the jump of the horizontal component of ξ. Also set $\theta := \mathrm{arctg} g'(x)$, so $\mathrm{tg}\theta = g'(x)$.

We now proceed to obtain estimates for $\Delta(x, \alpha)$ and its moments. The next lemma gives an upper bound on $\Delta(x, \alpha)$ that follows from the fact that for large enough x our tube will be almost flat, while α is bounded strictly away from $\pm\pi/2$.

Lemma 5.1 *Let $\alpha_0 \in (0, \pi/2)$, and suppose that g satisfies (A1), and also (2.4) or (2.5). Then there exist $A, C \in (0, \infty)$ such that for all $x > A$ and all $\alpha \in (-\alpha_0, \alpha_0)$,*

$$|\Delta(x, \alpha)| \leq C g(x).$$

Proof. Fix $\alpha_0 \in (0, \pi/2)$. Assuming (A1), we have that $g'(x) \to 0$ and hence $\theta \to 0$ as $x \to \infty$; in particular we can choose A large enough so that for all $x \geq A$, $|\theta| < \min\{\alpha_0, (\pi/2) - \alpha_0\}$ and $\mathrm{tg}(\alpha_0 + \theta) < c_0$ for some $c_0 < \infty$.

By symmetry, it suffices to suppose that we start on the positive half of the curve, i.e., at $(x, g(x))$. We have that $|\Delta(x, \alpha)|$ is bounded by $\max\{|\Delta(x, \alpha_0)|, |\Delta(x, -\alpha_0)|\}$. First suppose (2.4) holds. Then g is nondecreasing, so $\theta \geq 0$. Consider $\Delta(x, \alpha_0)$, represented as Δ_+ when $\alpha = \alpha_0$ in Figure 2. The reflected ray at angle α_0 to the normal from $(x, g(x))$ has equation in (x_0, y_0) given by

$$x_0 - x = -(y_0 - g(x))\mathrm{tg}(\alpha_0 + \theta).$$

Take $a \in (0, 1/c_0)$. As $g'(x) \to 0$, we have $|g'(x)| < a$ for all x large enough. Consider the line

$$y_0 + g(x) = -a(x_0 - x).$$

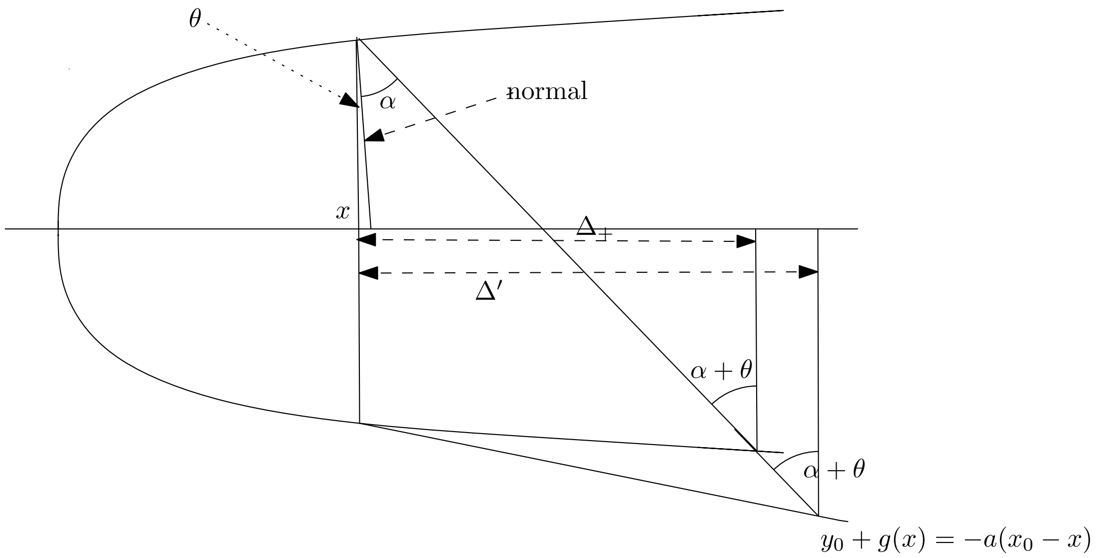

Figure 2: Auxiliary construction for the proof of Lemma 5.1

This line intersects $\partial\mathcal{D}$ at $(x, -g(x))$ and it intersects the reflected ray at angle α_0 to the normal from $(x, g(x))$ at $x_0 \geq x$ with

$$x_0 - x = \frac{2g(x)\mathrm{tg}(\alpha_0 + \theta)}{1 - a\mathrm{tg}(\alpha_0 + \theta)}. \tag{5.1}$$

Since $|g'(x_0)| < a$ for $x_0 \geq x$, the curve $y_0 = -g(x_0)$ remains above the line $y_0 + g(x) = -a(x_0 - x)$ for all $x_0 \geq x$. Thus $|\Delta(x, \alpha_0)|$ is bounded by $x_0 - x$ as given by (5.1); this is Δ' in Figure 2.

Thus, for some $C \in (0, \infty)$, $|\Delta(x, \alpha_0)| \leq Cg(x)$, for all x large enough. A similar argument applies for $\Delta(x, -\alpha_0)$, and with slight modification when (2.5) holds. ∎

A related geometrical argument also provides the proof of Proposition 2.1:

Proof of Proposition 2.1. Under condition (2.4), so that $g(x)$ is strictly increasing, the deterministic path in the case $\mathbf{P}[\alpha = 0] = 1$ tends to infinity. It is then clear that for any distribution for α in this case that there exists $\varepsilon > 0$ for which

$$\mathbf{P}[\xi^{(1)}_{n+1} - \xi^{(1)}_n > \varepsilon \mid \xi^{(1)}_n = x] > \varepsilon \tag{5.2}$$

for all x large enough. The stated result for ξ then follows in this case, and hence the result for X also.

Now suppose that condition (2.5) holds and that α is non-degenerate. Then there exists $\varepsilon_1 > 0$ for which $\mathbf{P}[\alpha > \varepsilon_1] > \varepsilon_1$. Under (A1), $g'(x) \to 0$ as $x \to \infty$, so we can choose A big enough so that the angle θ to the normal satisfies $|\theta| < \varepsilon_1/2$ for all $x > A$. From (A1), we have that g is monotone and $g(x) > 0$. Then it follows that for all x in any bounded interval (A, C), (5.2) holds for some $\varepsilon > 0$. Thus with positive

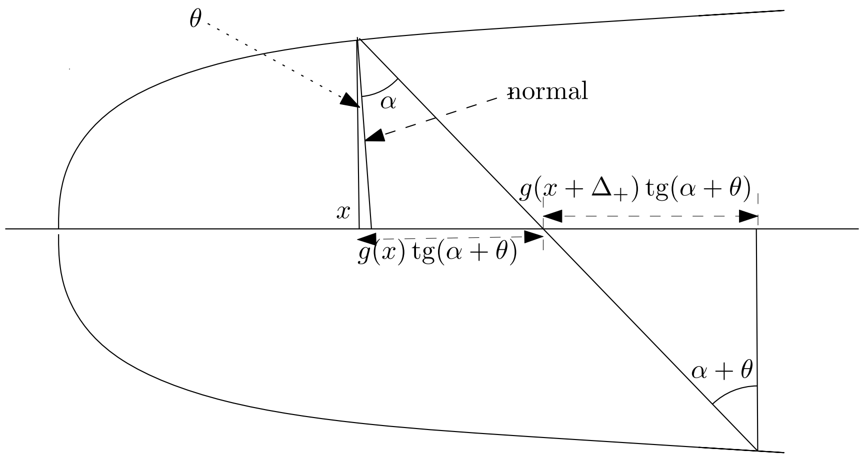

Figure 3: Δ_+: $\alpha > 0$

probability ξ reaches any finite horizontal distance, and the result follows in this case also. ■

The next lemma gives crucial estimates for the first two moments of $\Delta(x, \alpha)$. These require fairly lengthy computations.

Lemma 5.2 *Suppose that α satisfies (2.3) and $g(x)$ satisfies (A1), and also* (2.4) *or* (2.5). *Then as $x \to \infty$*

$$\mathbf{E}[\Delta(x,\alpha)] = 2g'(x)g(x)(1 + 2\mathbf{E}[\mathrm{tg}^2\alpha]) + O(g(x)^3/x^2); \tag{5.3}$$

and

$$\mathbf{E}[\Delta(x,\alpha)^2] = 4g(x)^2\mathbf{E}[\mathrm{tg}^2\alpha] + O(g(x)^3/x). \tag{5.4}$$

Proof. First suppose that (2.4) holds (the case of an increasing tube). Then $\theta \geq 0$. Take x sufficiently large so that $\alpha_0 + \theta < \pi/2$. Consider the jump $\Delta(x, \alpha)$. We need to consider 3 cases. In the first case, $\alpha > 0$ (see Figure 3),

$$\Delta(x,\alpha) = \Delta_+ = (g(x) + g(x + \Delta_+))\mathrm{tg}(\alpha + \theta).$$

In the second case, (see Figure 4), $\alpha < 0, |\alpha| < \theta$,

$$\Delta(x,\alpha) = \Delta'_+ = (g(x) + g(x + \Delta'_+))\mathrm{tg}(\alpha + \theta).$$

In the third case, (see Figure 5), $\alpha < 0, |\alpha| \geq \theta$,

$$-\Delta(x,\alpha) = \Delta_- = (g(x) + g(x - \Delta_-))\mathrm{tg}(-\alpha - \theta).$$

We start with the first of the three cases. Using Taylor's theorem, we can write

$$\Delta_+ = \mathrm{tg}(\alpha + \theta)\Big[g(x) + g(x) + g'(x)\Delta_+ + \frac{g''(x + \phi\Delta_+)}{2}\Delta_+^2\Big],$$

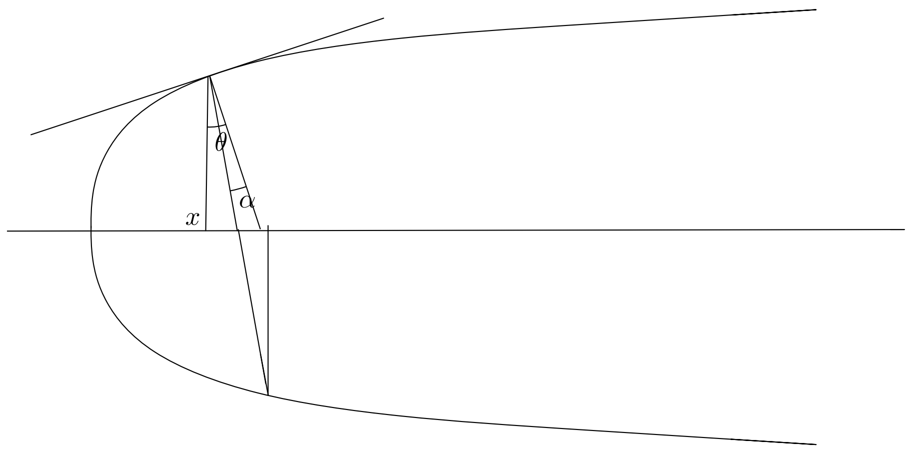

Figure 4: Δ'_+: $-\theta < \alpha < 0$

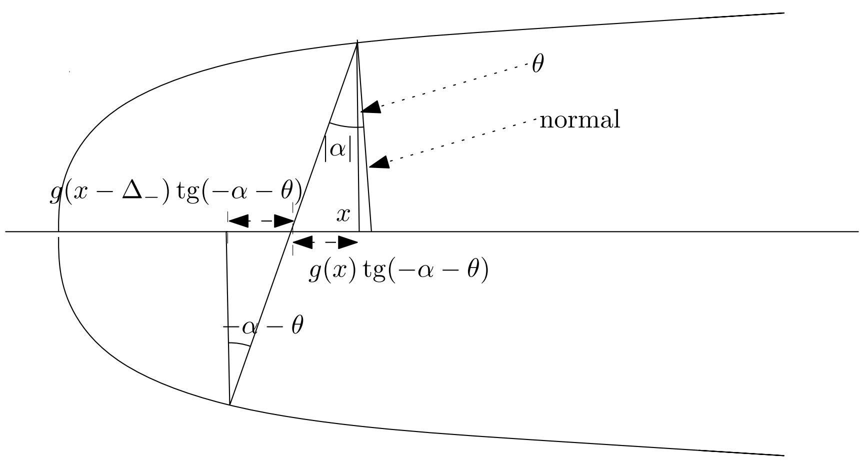

Figure 5: Δ_-: $\alpha < -\theta$

where $\phi \in [0,1]$. Thus, writing $x_+ = x + \phi\Delta_+$,

$$\begin{aligned}
\Delta_+ &= 2g(x)\frac{\mathrm{tg}(\alpha+\theta)}{1-g'(x)\mathrm{tg}(\alpha+\theta)} + \frac{g''(x_+)\mathrm{tg}(\alpha+\theta)}{2(1-g'(x)\mathrm{tg}(\alpha+\theta))}\Delta_+^2 \\
&= 2g(x)\frac{\mathrm{tg}\alpha+\mathrm{tg}\theta}{1-\mathrm{tg}\alpha\mathrm{tg}\theta-g'(x)(\mathrm{tg}\alpha+\mathrm{tg}\theta)} \\
&\quad + \frac{g''(x_+)(\mathrm{tg}\alpha+\mathrm{tg}\theta)}{2(1-\mathrm{tg}\alpha\mathrm{tg}\theta-g'(x)(\mathrm{tg}\alpha+\mathrm{tg}\theta))}\Delta_+^2 \\
&= 2g(x)\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha-g'(x)^2} + \frac{g''(x_+)}{2}\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha-g'(x)^2}\Delta_+^2,
\end{aligned}$$

where we have used the fact that

$$\mathrm{tg}(u \pm v) = \frac{\mathrm{tg}u \pm \mathrm{tg}v}{1 \mp \mathrm{tg}u\mathrm{tg}v}.$$

(Recall that $\mathrm{tg}\theta = g'(x)$). Analogously, in the second case,

$$\begin{aligned}
\Delta'_+ &= 2g(x)\frac{\mathrm{tg}(\alpha+\theta)}{1-g'(x)\mathrm{tg}(\alpha+\theta)} + \frac{g''(x'_+)}{2}\frac{\mathrm{tg}(\alpha+\theta)}{(1-g'(x)\mathrm{tg}(\alpha+\theta))}(\Delta'_+)^2 \\
&= 2g(x)\frac{g'(x)-\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha|-g'(x)^2} + \frac{g''(x'_+)}{2}\frac{g'(x)-\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha|-g'(x)^2}(\Delta'_+)^2,
\end{aligned}$$

where $x'_+ = x + \phi\Delta'_+$, $\phi \in [0,1]$; and in the third case

$$\begin{aligned}
\Delta_- &= 2g(x)\frac{\mathrm{tg}(-\alpha-\theta)}{1+g'(x)\mathrm{tg}(-\alpha-\theta)} + \frac{g''(x_-)\mathrm{tg}(-\alpha-\theta)}{2(1+g'(x)\mathrm{tg}(-\alpha-\theta))}\Delta_-^2 \\
&= 2g(x)\frac{-g'(x)+\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha|-g'(x)^2} + \frac{g''(x_-)}{2}\frac{-g'(x)+\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha|-g'(x)^2}\Delta_-^2,
\end{aligned}$$

where $x_- = x - \phi\Delta_-$, $\phi \in [0,1]$.

By Lemma 5.1, we have that $\max\{\Delta_+, \Delta'_+, \Delta_-\} = O(g(x))$ a.s., and so

$$\max\{|x_+ - x|, |x'_+ - x|, |x_- - x|\} = O(g(x)) = o(x).$$

So in particular $g''(x_+) = O(g''(x))$, and similarly for x'_+, x_-. Thus,

$$\Delta_+ = 2g(x)\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha-g'(x)^2} + O(g''(x)g(x)^2)$$

$$\Delta'_+ = 2g(x)\frac{g'(x)-\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha|-g'(x)^2} + O(g''(x)g(x)^2),$$

and

$$\Delta_- = 2g(x)\frac{-g'(x)+\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha|-g'(x)^2} + O(g''(x)g(x)^2).$$

Let us now estimate the first two moments of $\Delta(x,\alpha)$. For convenience write $F(x) := \mathbf{P}[\alpha \le x]$. For the first moment we obtain, using the symmetry of F,

$$
\begin{aligned}
\mathbf{E}[\Delta(x,\alpha)] &= \int_0^{\alpha_0} \Delta_+ \mathrm{d}F(\alpha) + \int_{-\theta}^{0} \Delta'_+ \mathrm{d}F(\alpha) - \int_{-\alpha_0}^{-\theta} \Delta_- \mathrm{d}F(\alpha) \\
&= \int_0^{\theta} 2g(x)\Big[\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha - g'(x)^2} + \frac{g'(x)-\mathrm{tg}\alpha}{1+2g'(x)\mathrm{tg}\alpha - g'(x)^2}\Big]\mathrm{d}F(\alpha) \\
&\quad + \int_{\theta}^{\alpha_0} 2g(x)\Big[\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha - g'(x)^2} - \frac{-g'(x)+\mathrm{tg}\alpha}{1+2g'(x)\mathrm{tg}\alpha - g'(x)^2}\Big]\mathrm{d}F(\alpha) \\
&\quad + O(g''(x)g(x)^2) \\
&= \int_0^{\alpha_0} 2g(x)\Big[\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha - g'(x)^2} + \frac{g'(x)-\mathrm{tg}\alpha}{1+2g'(x)\mathrm{tg}\alpha - g'(x)^2}\Big]\mathrm{d}F(\alpha) \\
&\quad + O(g''(x)g(x)^2) \\
&= \int_0^{\alpha_0} 2g(x)\Big[\frac{2g'(x)+4g'(x)\mathrm{tg}^2\alpha - 2g'(x)^3}{(1-2g'(x)\mathrm{tg}\alpha - g'(x)^2)(1+2g'(x)\mathrm{tg}\alpha - g'(x)^2)}\Big]\mathrm{d}F(\alpha) \\
&\quad + O(g''(x)g(x)^2).
\end{aligned}
$$

The denominator in the term in square brackets in the last integrand is $1+O(g'(x)^2)$. Hence

$$
\mathbf{E}[\Delta(x,\alpha)] = 2g(x)g'(x)\Big(1+2\mathbf{E}[\mathrm{tg}^2\alpha]\Big) + O(g''(x)g(x)^2 + g'(x)^3 g(x)).
$$

Then using (A1) to bound the error terms, we obtain (5.3).

For the second moment, we have, in a similar fashion,

$$
\begin{aligned}
\mathbf{E}[\Delta(x,\alpha)^2] &= \int_0^{\alpha_0} \Delta_+^2 \mathrm{d}F(\alpha) + \int_{-\theta}^{0} (\Delta'_+)^2 \mathrm{d}F(\alpha) + \int_{-\alpha_0}^{-\theta} \Delta_-^2 \mathrm{d}F(\alpha) \\
&= \int_0^{\alpha_0} \Big[2g(x)\frac{g'(x)+\mathrm{tg}\alpha}{1-2g'(x)\mathrm{tg}\alpha - g'(x)^2} + O(g''(x)g(x)^2)\Big]^2 \mathrm{d}F(\alpha) \\
&\quad + \int_{-\theta}^{0} \Big[2g(x)\frac{g'(x)-\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha| - g'(x)^2} + O(g''(x)g(x)^2)\Big]^2 \mathrm{d}F(\alpha) \\
&\quad + \int_{-\alpha_0}^{-\theta} \Big[2g(x)\frac{-g'(x)+\mathrm{tg}|\alpha|}{1+2g'(x)\mathrm{tg}|\alpha| - g'(x)^2} + O(g''(x)g(x)^2)\Big]^2 \mathrm{d}F(\alpha)
\end{aligned}
$$

$$= \int_{0}^{\alpha_0} \Big[2g(x)\mathrm{tg}\alpha + O(g(x)g'(x)) + O(g''(x)g(x)^2)\Big]^2 \mathrm{d}F(\alpha)$$

$$+ \int_{-\theta}^{0} \Big[2g(x)\mathrm{tg}\alpha + O(g(x)g'(x)) + O(g''(x)g(x)^2)\Big]^2 \mathrm{d}F(\alpha)$$

$$+ \int_{-\alpha_0}^{-\theta} \Big[2g(x)(-\mathrm{tg}\alpha) + O(g(x)g'(x)) + O(g''(x)g(x)^2)\Big]^2 \mathrm{d}F(\alpha)$$

$$= 2\int_{0}^{\alpha_0} 4g(x)^2(\mathrm{tg}^2\alpha)\mathrm{d}F(\alpha) + O(g'(x)g(x)^2) + O(g''(x)g(x)^3)$$

$$= 4g(x)^2\mathbf{E}[\mathrm{tg}^2\alpha] + O(g'(x)g(x)^2) + O(g''(x)g(x)^3).$$

Then (5.4) follows, again using (A1) to bound the error terms.

Now suppose that (2.5) holds (the case of a decreasing tube). In this case the argument follows similar lines to the previous one, and we only sketch the details. Now $\theta \leq 0$.

Analogously to the case where (2.4) holds, we define Δ_+, Δ_-, Δ'_-. For $\alpha > |\theta|$, we have

$$\Delta(x,\alpha) = \Delta_+ = (g(x) + g(x+\Delta_+))\mathrm{tg}(\alpha - |\theta|).$$

In the second case, when $0 \leq \alpha \leq |\theta|$,

$$-\Delta(x,\alpha) = \Delta'_- = (g(x) + g(x-\Delta'_-))\mathrm{tg}(|\theta| - \alpha).$$

In the third case, when $\alpha < 0$,

$$-\Delta(x,\alpha) = \Delta_- = (g(x) + g(x-\Delta_-))\mathrm{tg}(|\alpha| + |\theta|).$$

In the same way as before, we obtain

$$\Delta_+ = 2g(x)\frac{g'(x) + \mathrm{tg}\alpha}{1 - 2g'(x)\mathrm{tg}\alpha - g'(x)^2} + O(g''(x)g(x)^2),$$

$$\Delta'_- = -2g(x)\frac{g'(x) + \mathrm{tg}\alpha}{1 - 2g'(x)\mathrm{tg}\alpha - g'(x)^2} + O(g''(x)g(x)^2),$$

and

$$\Delta_- = 2g(x)\frac{-g'(x) + \mathrm{tg}|\alpha|}{1 + 2g'(x)\mathrm{tg}|\alpha| - g'(x)^2} + O(g''(x)g(x)^2).$$

Similar computations to before then yield the same expressions (5.3) and (5.4). Lemma 5.2 is proved. ■

The next result will allow us to compare the recurrence times σ_A and τ_A.

Lemma 5.3 *Suppose that α satisfies (2.3) and g satisfies (A1).*

(i) If g satisfies (2.4) then for all A sufficiently large $\tau_A \geq \sigma_A$ a.s..

(ii) If g satisfies (2.5) then for all A sufficiently large $\tau_A \leq \sigma_A$ a.s..

Proof. First we observe that when $\xi_n^{(1)} = x$ is large enough (such that $|g'(x)| < \mathrm{tg}(\frac{\pi}{2} - \alpha_0)$), we have $\xi_{n+1}^{(2)} \xi_n^{(2)} < 0$, that is, successive collisions are a.s. on different sides of the tube. Thus, for $\xi_n^{(1)} = x \geq A$, a.s.,

$$\|\xi_{n+1} - \xi_n\| \geq |\xi_{n+1}^{(2)} - \xi_n^{(2)}| \geq g(x). \tag{5.5}$$

Now suppose that (2.4) holds. Then for x large enough, under (2.4), (5.5) implies $\|\xi_{n+1} - \xi_n\| \geq 1$. In other words, the time between collisions for the process X is no less than that for the process ξ, and part (i) follows.

Now suppose that (2.5) holds. By the triangle inequality, we have

$$\|\xi_{n+1} - \xi_n\| \leq |\xi_{n+1}^{(1)} - \xi_n^{(1)}| + |\xi_{n+1}^{(2)} - \xi_n^{(2)}|.$$

Thus Lemma 5.1 with (2.5) implies that for some $A, C \in (0, \infty)$, given $\xi_n^{(1)} = x \geq A$,

$$\|\xi_{n+1} - \xi_n\| \leq C g(x) \leq 1,$$

for all x large enough. Then part (ii) follows. ∎

5.2 Proofs for recurrence classification

Our approach to studying the horizontal component of the collisions process ξ is to consider a rescaled version of the process in such a way that we get exactly an instance of the Lamperti problem. The key is to find a scale on which the process has uniformly bounded jumps.

Define the function $h : [1, \infty) \to (0, \infty)$ via $h(x) := x/g(x)$. Under assumption (A1) on g, it follows that

$$\begin{aligned} h'(x) &= \frac{1}{g(x)} - \frac{x g'(x)}{g(x)^2} = [1 - \gamma + o(1)] \frac{1}{g(x)}, \\ h''(x) &= -\frac{2g'(x)}{g(x)^2} - \frac{x g''(x)}{g(x)^2} + \frac{2x g'(x)^2}{g(x)^3} = [\gamma(\gamma - 1) + o(1)] \frac{1}{x g(x)}, \\ h'''(x) &= o\Big(\frac{1}{x g(x)^2}\Big). \end{aligned}$$

Now for $n \in \mathbb{Z}^+$ set $\zeta_n := h(\xi_n^{(1)})$. The process $\zeta = (\zeta_n)_{n \in \mathbb{Z}^+}$ is then covered by the Lamperti problem (cf Section 4), as the following result shows.

Lemma 5.4 *Suppose that (A1) holds. Suppose that α satisfies (2.3). Then there exists $B \in (0, \infty)$ such that for all $n \in \mathbb{Z}^+$ and all $y \geq 0$*

$$\mathbf{P}[|\zeta_{n+1} - \zeta_n| \leq B \mid \zeta_n = y] = 1. \tag{5.6}$$

Also, for all $n \in \mathbb{Z}^+$, as $y \to \infty$

$$m_1(y) := \mathbf{E}[\zeta_{n+1} - \zeta_n \mid \zeta_n = y] = \frac{2\gamma(1-\gamma)(1+\mathbf{E}[\mathrm{tg}^2\alpha])}{y} + o(y^{-1}); \text{ and} \tag{5.7}$$

$$m_2(y) := \mathbf{E}[(\zeta_{n+1} - \zeta_n)^2 \mid \zeta_n = y] = 4(1-\gamma)^2\mathbf{E}[\mathrm{tg}^2\alpha] + o(1). \tag{5.8}$$

Proof. Given $\xi_n^{(1)} = x$, denote $\zeta_n = y = h(x) > 0$. If the reflection is at angle α, we have from Taylor's theorem that as $x \to \infty$

$$\begin{aligned}\zeta_{n+1} - \zeta_n &= h(x + \Delta(x,\alpha)) - h(x) \\ &= h'(x)\Delta(x,\alpha) + \frac{h''(x)}{2}\Delta(x,\alpha)^2 + O(h'''(x)\Delta(x,\alpha)^3) \\ &= (1-\gamma+o(1))\frac{\Delta(x,\alpha)}{g(x)} + \frac{\gamma(\gamma-1)+o(1)}{2}\frac{\Delta(x,\alpha)^2}{xg(x)} + o(g(x)/x), \end{aligned} \tag{5.9}$$

using Lemma 5.1. By Lemma 5.1 we have that $|\Delta(x,\alpha)| = O(g(x))$, and then (5.6) is immediate.

Taking expectations in (5.9) and using Lemma 5.2, we obtain

$$\begin{aligned}&\mathbf{E}[\zeta_{n+1} - \zeta_n \mid \zeta_n = h(x)] \\ &= 2(1-\gamma+o(1))(1+2\mathbf{E}[\mathrm{tg}^2\alpha])g'(x) + (2\gamma(\gamma-1)+o(1))\mathbf{E}[\mathrm{tg}^2\alpha]\frac{g(x)}{x} \\ &= \frac{2\gamma(1-\gamma)(1+\mathbf{E}[\mathrm{tg}^2\alpha])g(x)}{x} + o(g(x)/x).\end{aligned}$$

This yields (5.7). Similarly, squaring both sides of (5.9) and taking expectations gives (5.8). ■

This last result, together with Proposition 2.1, immediately implies the following:

Corollary 5.1 *Suppose that g satisfies (A1), and that α satisfies (2.3). Then $(\zeta_n)_{n\in\mathbb{Z}^+}$ is a Lamperti-type problem as discussed in Section 4, satisfying (A2) and (A3). Moreover, (4.4) holds if g satisfies (2.4), and also, if α is non-degenerate, if g satisfies (2.5). Finally, (4.3) holds provided that α is non-degenerate.*

In the special case where $g(x) = x^\gamma$, $\gamma < 1$, so that $\zeta_n = (\xi_n^{(1)})^{1-\gamma}$, we will need a more precise version of Lemma 5.4. This is Lemma 5.5 below. Not only will this enable us to deal with the critical case in the recurrence classification (Theorem 2.3), it will also be crucial for our proofs of the almost-sure bounds carried out in Section 5.3.

Lemma 5.5 *Suppose that $g(x) = x^\gamma$ where $\gamma < 1$, and that α satisfies (2.3). Then for all $n \in \mathbb{Z}^+$, as $y \to \infty$*

$$m_1(y) := \mathbf{E}[\zeta_{n+1} - \zeta_n \mid \zeta_n = y] = \frac{2\gamma(1-\gamma)(1+\mathbf{E}[\mathrm{tg}^2\alpha])}{y} + o(y^{-1}(\log y)^{-1}); \tag{5.10}$$

$$m_2(y) := \mathbf{E}[(\zeta_{n+1} - \zeta_n)^2 \mid \zeta_n = y] = 4(1-\gamma)^2\mathbf{E}[\mathrm{tg}^2\alpha] + o((\log y)^{-1}). \tag{5.11}$$

Proof. We can apply Taylor's theorem to obtain, conditional on $\zeta_n = y = x^{1-\gamma} > 0$,

$$\zeta_{n+1} - \zeta_n = (1-\gamma)x^{-\gamma}\Delta(x,\alpha) - \frac{\gamma(1-\gamma)}{2}x^{-\gamma-1}\Delta(x,\alpha)^2 + O(x^{2\gamma-2}).$$

Then taking expectations and using (5.3) and (5.4) we obtain (5.10). Similarly we obtain (5.11) after squaring the last displayed expression. ■

Proof of Theorem 2.1. Under the conditions of Theorem 2.1, Corollary 5.1 holds. We apply Lamperti's result Proposition 4.1 to the process ζ described by Lemma 5.4, noting that ζ is null-recurrent, positive-recurrent, or transient exactly when ξ is. From Lemma 5.4, we have that

$$2ym_1(y) + m_2(y) = 4(1-\gamma)(\gamma + \mathbf{E}[\mathrm{tg}^2\alpha] + o(1)), \tag{5.12}$$

and also

$$\begin{aligned} 2ym_1(y) - m_2(y) &= 4(1-\gamma)\left(\gamma(1+2\mathbf{E}[\mathrm{tg}^2\alpha]) - \mathbf{E}[\mathrm{tg}^2\alpha] + o(1)\right) \\ &= 4(1-\gamma)\left((\gamma-\gamma_c)(1+2\mathbf{E}[\mathrm{tg}^2\alpha]) + o(1)\right), \end{aligned} \tag{5.13}$$

where γ_c is given by (2.7).

For part (i) of the theorem, if $\gamma \in (\gamma_c, 1)$ we have from (5.13) that there exists $\delta > 0$ such that $2ym_1(y) - m_2(y) \geq \delta$ for all y sufficiently large, and hence by Proposition 4.1(ii), ζ is transient.

For part (ii) of the theorem, it suffices to consider the case where α is non-degenerate, so $\gamma_c > 0$. Then from (5.13) we have that for $0 \leq \gamma < \gamma_c$, $2ym_1(y) \leq m_2(y)$ for all y sufficiently large. Also, in this case $\gamma + \mathbf{E}[\mathrm{tg}^2\alpha] > 0$, so we have from (5.12) that $2ym_1(y) \geq -m_2(y)$ for all y sufficiently large. Hence by Proposition 4.1(i), ζ is null-recurrent.

This proves Theorem 2.1 for the process ξ, and the statement for the process X follows from (2.6) and Lemma 5.3(i). ■

Proof of Theorem 2.2. For part (i), if $\gamma + \mathbf{E}[\mathrm{tg}^2\alpha] < 0$, we have from (5.12) that $2ym_1(y) + m_2(y) < -\delta$ for some $\delta > 0$ and all y sufficiently large. Then (noting Corollary 5.1) it follows from Proposition 4.1(iii) that ζ and hence ξ is positive-recurrent.

For part (ii), it suffices to suppose that α is non-degenerate. Then the $\gamma \leq 0$ case of (5.13) implies that $2ym_1(y) \leq m_2(y)$ for all y large enough. On the other hand, if $\gamma + \mathbf{E}[\mathrm{tg}^2\alpha] > 0$, we have from (5.12) that $2ym_1(y) + m_2(y) \geq 0$ for all y sufficiently large. Then null-recurrence follows from Proposition 4.1(i). ■

In order to complete the proof of Theorem 2.3, we need a sharper form of Lamperti's recurrence classification result presented in Proposition 4.1. Fine results in this direction are given in [23]. We will only need the following consequence of Theorem 3 of [23].

Lemma 5.6 *[23] For η a Lamperti-type problem satisfying (A2), (4.3), and (4.4), η is null-recurrent if, for all x large enough,*

$$2x|\mu_1(x)| \leq \left(1 + \frac{1}{\log x}\right)\mu_2(x).$$

Proof of Theorem 2.3. This now follows from Lemma 5.6 with Lemma 5.5. ■

5.3 Proofs for almost-sure bounds

The key to the proof of our almost-sure bounds for the stochastic billiard model is to apply our almost-sure bound results from Section 4 to the rescaled process ζ that we studied in Lemma 5.5. This will allow us to obtain Theorems 2.4 and 2.6. We will then derive the results for the continuous-time process X, Theorems 2.5 and 2.7, from the corresponding results for ξ. Recall that for $n \in \mathbb{Z}^+$, $\zeta_n := (\xi_n^{(1)})^{1-\gamma}$.

Proof of Theorem 2.4. The idea here is to apply Theorem 4.1 to the process ζ. By Lemma 5.5, we have that (5.12) holds. Then since $\gamma > 0$ this implies that the conditions of Theorem 4.1(i) and (ii) are satisfied for the process ζ (using Corollary 5.1). Thus for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$,

$$n^{1/2}(\log n)^{-(1/2)-\varepsilon} \leq \max_{0\leq m\leq n} \zeta_m \leq n^{1/2}(\log n)^{(1/2)+\varepsilon},$$

and then (2.8) and (2.9) follow, since $\zeta_n = (\xi_n^{(1)})^{1-\gamma}$.

Also by Lemma 5.5, we have that (5.13) holds. If $\gamma > \gamma_c$, it follows (using Corollary 5.1) that we can apply Theorem 4.2 with $\eta = \zeta$ to obtain that for some $D \in (0, \infty)$, a.s., for all but finitely many $n \in \mathbb{Z}^+$, $\zeta_n \geq n^{1/2}(\log n)^{-D}$. Then (2.10) follows. ■

Proof of Theorem 2.6. This time we will apply Theorems 4.1 and 4.3. It follows from Lemma 5.5 that

$$2ym_1(y) = -\kappa m_2(y) + o((\log y)^{-1}),$$

where

$$\kappa = \frac{-\gamma(1 + \mathbf{E}[\mathrm{tg}^2\alpha])}{(1-\gamma)\mathbf{E}[\mathrm{tg}^2\alpha]}.$$

Hence for $\gamma < -\mathbf{E}[\mathrm{tg}^2\alpha]$, $\kappa > 1$ and (using Corollary 5.1) the conditions of parts (i) and (ii) of Theorem 4.3 are satisfied for ζ. Then part (ii) of the theorem follows.

On the other hand, for $\gamma > -\mathbf{E}[\mathrm{tg}^2\alpha]$, we have $\kappa < 1$ so that (using Corollary 5.1) the conditions of parts (i) and (ii) of Theorem 4.1 are satisfied. This yields part (i) of the theorem, in the same way as in the proof of the corresponding results in Theorem 2.4. ■

The following lemma will enable us to derive our 'infinitely-often' lower bounds for $X_t^{(1)}$ from bounds for $\xi_n^{(1)}$.

Lemma 5.7 *Suppose that $\gamma < 1$. Suppose that there exist $a, b > 0$ with $a\gamma > -1$, such that for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{Z}^+$*

$$n^a(\log n)^{-b-\varepsilon} \leq \max_{0\leq m\leq n} \xi_m^{(1)} \leq n^a(\log n)^{b+\varepsilon}. \tag{5.14}$$

Then for any $\varepsilon > 0$, a.s., for all t sufficiently large,

$$\sup_{0\leq s\leq t} X_s^{(1)} \geq t^{\frac{a}{1+\gamma a}}(\log t)^{-\frac{2\gamma ab+b}{1+\gamma a}-\varepsilon}.$$

Proof. Recall the definition of the collision times ν_k from (2.1). We have from the triangle inequality and Lemma 5.1 that for some $C \in (0, \infty)$, for all $k \in \mathbb{N}$,

$$\nu_k = \sum_{j=0}^{k-1} \|\xi_{j+1} - \xi_j\| \le \sum_{j=0}^{k-1} \left(|\xi_{j+1}^{(1)} - \xi_j^{(1)}| + |\xi_{j+1}^{(2)} - \xi_j^{(2)}| \right) \le C \sum_{j=0}^{k} (\xi_j^{(1)})^\gamma.$$

Thus by the upper bound in (5.14), for any $\varepsilon > 0$, a.s., for some $C \in (0, \infty)$ and all $k \in \mathbb{Z}^+$,

$$\nu_k \le C k^{1+\gamma a} (\log k)^{\gamma b + \varepsilon}. \tag{5.15}$$

Let $\varepsilon > 0$, and for $t > 1$, set

$$k_\varepsilon(t) := \left\lfloor t^{\frac{1}{1+\gamma a}} (\log t)^{-\frac{\gamma b}{1+\gamma a} - \varepsilon} \right\rfloor.$$

Also recall the definition of $n(t)$ from (2.2). Then by (5.15) we have that for any $\varepsilon > 0$, there exist $C \in (0, \infty)$ and $\varepsilon' > 0$ for which, a.s., for all t large enough

$$\nu_{k_\varepsilon(t)} \le C t (\log t)^{-\varepsilon'} \le t;$$

hence for any $\varepsilon > 0$, a.s., for all t sufficiently large

$$k_\varepsilon(t) \le n(t), \text{ and } \nu_{k_\varepsilon(t)} \le \nu_{n(t)} \le t < \nu_{n(t)+1}. \tag{5.16}$$

Now from (5.16) we have that, a.s., for all t large enough

$$\sup_{0 \le s \le t} X_s^{(1)} \ge \max_{0 \le m \le k_\varepsilon(t)} X_{\nu_m}^{(1)} = \max_{0 \le m \le k_\varepsilon(t)} \xi_m^{(1)}. \tag{5.17}$$

Now applying the lower bound in (5.14) we obtain for any $\varepsilon > 0$, a.s., for all t large enough

$$\sup_{0 \le s \le t} X_s^{(1)} \ge (k_\varepsilon(t))^a (\log k_\varepsilon(t))^{-b-\varepsilon} \ge C t^{\frac{a}{1+\gamma a}} (\log t)^{-\frac{\gamma a b}{1+\gamma a} - \varepsilon a} (\log t)^{-b-\varepsilon},$$

using the definition of $k_\varepsilon(t)$. Simplifying leads to the desired result. ■

We will use Lemma 5.7 in the proofs of Theorems 2.5 and 2.7 below. We will apply the lemma taking either $a = 1/(2(1-\gamma))$ with $\gamma < 1$ or $a = \rho(\gamma)$ with $\gamma < 0$. Note that for $\gamma \le 0$,

$$\gamma \rho(\gamma) \ge \frac{\gamma}{2(1-\gamma)} \ge -\frac{1}{2},$$

so that the hypothesis $a\gamma > -1$ in Lemma 5.7 is satisfied in either case.

Proof of Theorem 2.5. First of all, from Theorem 2.4 we have that (2.8) and (2.9) hold. Thus we can apply the $a = b = 1/(2(1-\gamma))$ case of Lemma 5.7, which yields (2.11). It remains to prove (2.12).

By the construction of the process, we have that a.s., for $t \ge 0$,

$$X_t^{(1)} \in [\min\{\xi_{n(t)}^{(1)}, \xi_{n(t)+1}^{(1)}\}, \max\{\xi_{n(t)}^{(1)}, \xi_{n(t)+1}^{(1)}\}]. \tag{5.18}$$

Suppose that $\gamma > \gamma_c$, so that we have transience. First we prove the lower bound in (2.12). We have from (5.18) that a.s., for all t large enough,

$$X_t^{(1)} \geq \min\{\xi_{n(t)}^{(1)}, \xi_{n(t)+1}^{(1)}\}.$$

Hence from (2.10) we have that for some $D \in (0, \infty)$, a.s., for all t large enough

$$X_t^{(1)} \geq (n(t))^{\frac{1}{2(1-\gamma)}} (\log n(t))^{-D} \geq (k_\varepsilon(t))^{\frac{1}{2(1-\gamma)}} (\log k_\varepsilon(t))^{-D},$$

the final inequality using (5.16) and the fact that the function $z \mapsto z^{\frac{1}{2(1-\gamma)}} (\log z)^{-D}$ is eventually increasing in z. Now (2.12) follows by the definition of $k_\varepsilon(t)$.

Now we prove the upper bound in (2.12). From (2.1) and (5.5) we have that $\nu_k \geq \sum_{j=0}^{k-1} (\xi_j^{(1)})^\gamma$. Then using (2.10) we have that for some $D \in (0, \infty)$, a.s., for all but finitely many $k \in \mathbb{N}$,

$$\nu_k \geq k^{\frac{2-\gamma}{2(1-\gamma)}} (\log k)^{-D}. \tag{5.19}$$

Let $D > 0$, and for $t > 1$ set

$$k'_D(t) := \lfloor t^{\frac{2(1-\gamma)}{2-\gamma}} (\log t)^D \rfloor.$$

Then by (5.19), we have that for D large enough, a.s., for all t sufficiently large,

$$\nu_{k'_D(t)} \geq t, \text{ and } n(t) \leq k'_D(t).$$

Now from (5.18) and (2.8) we have that for some $C \in (0, \infty)$, a.s., for all t sufficiently large,

$$X_t^{(1)} \leq \xi_{n(t)}^{(1)} + \xi_{n(t)+1}^{(1)} \leq n(t)^{\frac{1}{2(1-\gamma)}} (\log n(t))^C.$$

Now using the fact that $n(t) \leq k'_D(t)$ a.s., and the definition of $k'_D(t)$, the result follows. ■

Proof of Theorem 2.7. We apply Lemma 5.7 again. For part (i), we have from part (i) of Theorem 2.6 that (2.8) and (2.9) hold, so we can apply the $a = b = 1/(2(1-\gamma))$ case of Lemma 5.7 to obtain (2.11) in this case.

For part (ii), we have from part (ii) of Theorem 2.6 that (2.13) and (2.14) hold. So we can apply the $2a = b = 2\rho(\gamma)$ case of Lemma 5.7, which yields (2.15). ■

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