Stability for a Class of Equilibrium Solutions to the Coagulation–Fragmentation Equation

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Abstract. We consider the behaviour of solutions to the continuous constant-rate coagulation-fragmentation equation in the vicinity of an equilibrium solution. Semigroup methods are used to show that the governing linear equation for a perturbation $\varepsilon(x,t)$ has a unique globally defined solution for suitable initial conditions. In addition, Laplace transforms and the method of characteristics lead to an explicit formula for ε. The long-term behavior of ε is also discussed.

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INTRODUCTION

We examine the continuous coagulation-fragmentation equation (CFE)

$$\frac{\partial u}{\partial t}(x,t) = \frac{F}{2}\left(-x\,u(x,t) + 2\int_x^\infty u(y,t)\,dy\right) + \frac{K}{2}\left(\int_0^x u(x-y,t)u(y,t)dy - 2u(x,t)\int_0^\infty u(y,t)dy\right), \qquad (1)$$

with initial data given by

$$u(x,0) = u_0(x) \geq 0. \qquad (2)$$

Equation (1) describes the time evolution of the concentration of particles $u(x,t) \geq 0$ of size $x \geq 0$ at time $t \geq 0$ whose change in mass is governed by the constant reaction rates $K > 0$ and $F > 0$ which are called, respectively, the coagulation and fragmentation kernels.

Investigations into (1) and (2) have a long history. For example, in 1979 Aizenman and Bak [1] used semigroup methods to establish global existence and uniqueness of nonnegative, mass-conserving solutions satisfying

$$\int_0^\infty (1+x)u(x,t)\,dx < \infty, \quad \text{for all } t > 0. \qquad (3)$$

An alternative approach, involving compactness arguments, has also been used by Stewart [2] to prove the global existence of solutions, satisfying (3), for CFEs in which the kernels are not necessarily constant. Note that condition (3) guarantees that, for all $t \geq 0$, the total number of particles in the system, $N(t) := \int_0^\infty u(x,t)\,dx$, and the total mass in the system, $M(t) := \int_0^\infty x u(x,t)\,dx$, are both finite.

As the long-term behaviour of such solutions is also of interest, there have been a number of studies aimed at determining what happens to $u(x,t)$ as $t \to \infty$. The asymptotic behaviour of solutions to (1) and (2) was considered initially in [1], and then investigated further, via the invariance principle, in a later paper [3] by Stewart and Dubovskiĭ. It was shown that, as $t \to \infty$, each time-dependent solution $u(x,t)$ approaches an equilibrium

$$u_M(x) = \lambda \exp(-x\sqrt{\lambda/M}), \qquad \lambda = F/K, \qquad (4)$$

that is uniquely determined by the initial mass $M = \int_0^\infty x u_0(x)\,dx \; (= \int_0^\infty x u(x,t)\,dx)$.

In [1], where $F = K = 2$, the behaviour of solutions in the vicinity of a typical equilibrium was examined by setting

$$u(x,t) = u_M(x) + \psi(x,t)\exp(-M^{-1/2}x/2),$$

and then obtaining an approximate linear equation for ψ which was studied in the Hilbert space $L_2([0,\infty))$. In the present paper we re-examine the behaviour of solutions in the vicinity of the equilibrium u_M, but work instead within the more physical setting of the Banach space

$$X := \{\phi : \|\phi\|_X := \int_0^\infty (1+x)|\phi(x)|\,dx < \infty\}, \qquad (5)$$

and use a perturbation of the form

$$u(x,t) = u_M(x) + \varepsilon(x,t)\,.$$

The resulting (approximating) linear equation for ε,

$$\frac{\partial \varepsilon}{\partial t}(x,t) = \frac{F}{2}\left(-x\varepsilon(x,t)+2\int_x^\infty \varepsilon(y,t)dy\right) + K\left(\int_0^x u_M(x-y)\varepsilon(y,t)dy - u_M(x)\int_0^\infty \varepsilon(y,t)dy - \varepsilon(x,t)\int_0^\infty u_M(y)dy\right), \tag{6}$$

is shown to have a unique global solution such that $\varepsilon(x,t) \in X$ for each t. An explicit formula for ε is also obtained by using Laplace transform techniques and the method of characteristics.

THE GOVERNING PERTURBATION EQUATION

To obtain the governing perturbation equation (6), we assume that, in the vicinity of of an equilibrium u_M, the solution of (1) and (2) takes the form

$$u(x,t) = u_M(x) + \varepsilon(x,t)\,,$$

where ε is a small perturbation to the equilibrium solution u_M and $\int_0^\infty x u_0(x)\,dx = M$. Substituting into (1), and neglecting terms involving products of ε, leads to the linear equation (6). Consequently, if ε is small and satisfies (6) then $v(x,t) := u_M(x) + \varepsilon(x,t)$ should be a good approximation to the solution u.

To enable semigroup methods to be applied to (6), we note first that the initial-value problem (1) and (2) can be expressed in abstract form as

$$\frac{d}{dt}u(t) = G[u(t)] + N[u(t)],\ t > 0; \quad u(0) = u_0, \tag{7}$$

where the nonlinear coagulation operator N is given by

$$(N\phi)(x) := \frac{K}{2}\int_0^x \phi(x-y)\phi(y)\,dy - K\phi(x)\int_0^\infty \phi(y)\,dy\,, \quad \phi \in X\,, \tag{8}$$

and G is an operator realisation of the mapping

$$\phi(x) \to -\frac{F}{2}x\phi(x) + F\int_x^\infty \phi(y)\,dy \tag{9}$$

that generates a strongly continuous semigroup of operators, $\{\exp(tG)\}_{t\geq 0}$, on X; see [4] for details. As the Fréchet derivative of N at $f \in X$ is given by

$$(N_f\phi)(x) = K\left(\int_0^x f(x-y)\phi(y)\,dy - f(x)\int_0^\infty \phi(y)\,dy - \phi(x)\int_0^\infty f(y)\,dy\right), \tag{10}$$

for all $\phi \in X$, it follows that the abstract version of the initial-value problem associated with (6) takes the form

$$\frac{d}{dt}\varepsilon(t) = G[\varepsilon(t)] + N_{u_M}[\varepsilon(t)]\,,\ t > 0\ \ \varepsilon(0) = \varepsilon_0\,. \tag{11}$$

From [5, Theorem 2.32] and the fact that N_{u_M} is a bounded linear operator on X, we deduce that $L = G + N_{u_M}$ is the infinitesimal generator of a strongly continuous semigroup $\{\exp(tL)\}_{t\geq 0}$ on X and so, for suitable ε_0, (11) has a unique strongly-differentiable solution $\varepsilon : [0,\infty) \to X$ given by $\varepsilon(t) = \exp(tL)\varepsilon_0$. Moreover, on setting $\varepsilon(x,t) = [\varepsilon(t)](x)$, it can be shown that, when ε_0 satisfies the mass-conserving condition $\int_0^\infty x\,\varepsilon_0(x)\,dx = 0$, then, for each $t > 0$,

$$\int_0^\infty x\,\varepsilon(x,t)\,dx = 0\,. \tag{12}$$

and

$$\int_0^\infty \varepsilon(x,t)\,dx = \left(\int_0^\infty \varepsilon_0(x)\,dx\right)\exp(-Kt\int_0^\infty u_M(x)\,dx)\,. \tag{13}$$

EXPLICIT SOLUTIONS TO THE PERTURBATION EQUATION

Case (i) $K(N(0))^2 = FM$

One interesting feature observed in [3] is that $N(t) = N(0)$, for all $t > 0$, whenever the initial data u_0 satisfies the constraint $K(N(0))^2 = FM$. In this case, it is natural to require that the initial perturbation satisfies

$$\int_0^\infty \varepsilon_0(x)\,dx = \int_0^\infty x\,\varepsilon_0(x)\,dx = 0\,. \tag{14}$$

It then follows from (4) and (13) that the governing linear perturbation equation (6) reduces to

$$\frac{\partial \varepsilon}{\partial t}(x,t) \quad = \quad -2b\varepsilon(x,t)\left[\frac{1}{a}+\frac{x}{2}\right]+2b\int_0^x\left(e^{-a(x-y)}-1\right)\varepsilon(y,t)dy\,, \tag{15}$$

subject to the initial condition

$$\varepsilon(x,0) = \varepsilon_0(x)\,, \tag{16}$$

where

$$a = \sqrt{\frac{F}{KM}}\,, \qquad b = \frac{F}{2}\,. \tag{17}$$

On applying the Laplace transform to equation (15) we obtain

$$\frac{\partial \hat{\varepsilon}}{\partial s}(s,t) - \frac{1}{b}\frac{\partial \hat{\varepsilon}}{\partial t}(s,t) = 2\left(\frac{1}{a}+\frac{1}{s}-\frac{1}{s+a}\right)\hat{\varepsilon}(s,t)\,, \tag{18}$$

where $\hat{\varepsilon}(s,t)$ denotes the Laplace transform of $\varepsilon(x,t)$. The method of characteristics then leads to

$$\begin{aligned}\hat{\varepsilon}(s,t) \quad = \quad \hat{\varepsilon}_0(s+bt)e^{-2bt/a}\Bigg[1+\frac{(abt)^2}{(bt-a)^2}\left(\frac{1}{(s+bt)^2}+\frac{1}{(s+a)^2}\right)\\ +\,2abt\frac{(a^2-abt+b^2t^2)}{(bt-a)^3}\left(\frac{1}{s+bt}-\frac{1}{s+a}\right)\Bigg]\,.\end{aligned} \tag{19}$$

The inverse Laplace transform of (19) can now be obtained using standard formulae for convolutions and shifts, and results in

$$\begin{aligned}\varepsilon(x,t) \quad = \quad e^{-2bt/a}\Bigg[e^{-btx}\varepsilon_0(x)+\frac{(abt)^2}{(bt-a)^2}\int_0^x(x-y)(e^{-bt(x-y)}+e^{-a(x-y)})e^{-bty}\varepsilon_0(y)dy\\ +\,2abt\frac{(a^2-abt+b^2t^2)}{(bt-a)^3}\int_0^x\left(e^{-bt(x-y)}-e^{-a(x-y)}\right)e^{-bty}\varepsilon_0(y)dy\Bigg]\,.\end{aligned} \tag{20}$$

It can be verified that $\varepsilon(x,t)$ is finite and continuous at $t = a/b$, so that the solution in (20) is continuous for all $x \geq 0$ and $t \geq 0$ if ε_0 is continuous. It is also seen from (20) that, pointwise in x,

$$\varepsilon(x,t) \to 0 \quad \text{as} \quad t \to \infty. \tag{21}$$

Hence the equilibrium solution is linearly stable provided (14) holds.

Case (ii) $K(N(0))^2 \neq FM$

In this case, $N(t)$ is no longer constant, but instead is given by the formula

$$N(t) = \frac{2aM(N(0)+aM)}{N(0)+aM+(aM-N(0))e^{-2bt/a}} - aM\,. \tag{22}$$

The Laplace transform of equation (6) gives

$$\frac{\partial \varepsilon}{\partial s}(s,t) - \frac{1}{b}\frac{\partial \varepsilon}{\partial t}(s,t) = 2\left(\frac{1}{a} + \frac{1}{s} - \frac{1}{s+a}\right)\varepsilon(s,t) + 2\left(\frac{1}{s+a} - \frac{1}{s}\right)g(t), \tag{23}$$

where

$$g(t) = \int_0^\infty \varepsilon(y,t)dy. \tag{24}$$

This equation is similar to that in (18) except for the additional terms on the right-hand side, and as before, we can obtain a solution by the method of characteristics. In this case, we arrive at

$$\begin{aligned}\hat{\varepsilon}(s,t) &= \hat{\varepsilon}_0(s+bt)e^{-2bt/a}\frac{(s+a+bt)^2}{(s+bt)^2}\frac{s^2}{(s+a)^2} \\ &\quad + 2a\frac{s^2}{(s+a)^2}\int_0^{bt} g\left(\frac{bt-y}{b}\right)e^{-2y/a}\frac{(y+s+a)}{(y+s)^3}\,dy. \end{aligned} \tag{25}$$

From (13), $g(t) = B\exp(-2bt/a)$, where $B = \int_0^\infty \varepsilon_0(x)\,dx$, and therefore equation (25) becomes

$$\begin{aligned}\hat{\varepsilon}(s,t) &= \hat{\varepsilon}_0(s+bt)e^{-2bt/a}\frac{(s+a+bt)^2}{(s+bt)^2}\frac{s^2}{(s+a)^2} \\ &\quad + 2aBe^{-2bt/a}\frac{s^2}{(s+a)^2}\left[\frac{a}{2}\left(\frac{1}{s^2} - \frac{1}{(s+bt)^2}\right) + \frac{1}{s} - \frac{1}{s+bt}\right]. \end{aligned} \tag{26}$$

Consequently,

$$\begin{aligned}\varepsilon(x,t) &= e^{-2bt/a}\left[e^{-btx}\varepsilon_0(x) + \frac{(abt)^2}{(bt-a)^2}\int_0^x (x-y)(e^{-bt(x-y)} + e^{-a(x-y)})e^{-bty}\varepsilon_0(y)dy \right. \\ &\quad + 2abt\frac{(a^2 - abt + b^2t^2)}{(bt-a)^3}\int_0^x \left(e^{-bt(x-y)} - e^{-a(x-y)}\right)e^{-bty}\varepsilon_0(y)dy \\ &\quad + aB\frac{bt}{(a-bt)^3}\left\{e^{-ax}\left(2bt(a-bt) - 2a^2 - abtx(a-bt)\right)\right. \\ &\quad \left.\left. + e^{-btx}\left(2bt(bt-a) + 2a^2 - abtx(a-bt)\right)\right\}\right]. \end{aligned} \tag{27}$$

Once again it can be shown that $\varepsilon(x,t)$ has a finite limit as $t \to a/b$, and $\varepsilon(x,t) \to 0$ (pointwise in x) as $t \to \infty$.

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