PROLIFIC PERMUTATIONS AND PERMUTED PACKINGS: DOWNSETS CONTAINING MANY LARGE PATTERNS

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ABSTRACT. A permutation of n letters is k-prolific if each $(n-k)$-subset of the letters in its one-line notation forms a unique pattern. We present a complete characterization of k-prolific permutations for each k, proving that k-prolific permutations of m letters exist for every $m \geqslant k^2/2 + 2k + 1$, and that none exist of smaller size. Key to these results is a natural bijection between k-prolific permutations and certain "permuted" packings of diamonds.

Keywords: permutation, pattern, pattern poset, downset, prolific permutation, packing, permuted packing

1. INTRODUCTION

The set of permutations of $[n] = \{1, 2, \ldots, n\}$ is denoted S_n. We write a permutation $\sigma \in S_n$ as a word over $[n]$ in one-line notation, $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, and say that such a permutation σ has *size* n. If π_1 and π_2 are words of the same size over $\mathbb{R}$, then we write $\pi_1 \approx \pi_2$ to denote that their letters appear in the same relative order. This prompts the classical notion of pattern containment.

Definition 1.1. Consider $\pi \in S_r$. A permutation $\sigma \in S_n$ *contains* the *pattern* π if there are indices $1 \leqslant i_1 < \cdots < i_r \leqslant n$ such that $\sigma(i_1)\cdots\sigma(i_r) \approx \pi$. If σ contains π, we write $\pi \preceq \sigma$. If σ does not contain π, then σ *avoids* π.

From this, it is natural to define the "pattern poset" on permutations.

Definition 1.2. Let the *pattern poset*, $\mathcal{P}$, be the graded poset over $\bigcup_{k \geqslant 1} S_k$, ordered by the containment relation $\preceq$.

By definition, the elements of rank k in $\mathcal{P}$ are exactly the elements of S_k.

This paper is concerned with principal downsets of this poset, that is, with the sets of patterns which lie below a given permutation. In particular, we examine those permutations whose downset is as large as possible in the upper ranks.

This is related to problems of *pattern packing* [1, 12], which seek to maximize the total number of distinct patterns contained in a permutation, and to problems of *superpatterns* [4, 6, 7, 12], which are concerned with determining the size of the smallest permutations whose downset contains every permutation of some fixed size. Other related work addresses permutation *reconstruction* [5, 13, 14], establishing when permutations are uniquely determined by the (multi)set of large patterns they contain. The reader is referred to the books

by Bóna [2] and Kitaev [11] for an overview of problems related to the permutation pattern poset.

It follows immediately from the definition of $\mathcal{P}$ that, for a permutation $\sigma \in S_n$, there are at most $\binom{n}{k}$ distinct permutations $\pi \preceq \sigma$ that lie exactly k ranks below σ in $\mathcal{P}$, since each such permutation is obtained from σ by the deletion of k letters from the one-line notation for σ. Our interest is in those permutations of size n which contain maximally many patterns of size $n - k$.

Definition 1.3. Fix positive integers $n > k \geqslant 1$. A permutation $\sigma \in S_n$ is *k-prolific* if

$$\left| \left\{ \pi \in S_{n-k} \,:\, \pi \preceq \sigma \right\} \right| \;=\; \binom{n}{k}.$$

Clearly, not every permutation is k-prolific. As a trivial example, the identity permutation $12 \cdots n \in S_n$ contains only one pattern of each size, and thus is never k-prolific for any $k < n$.

Prolific permutations were previously investigated by the second author in [8]. The present work corrects and significantly improves upon the results presented there.
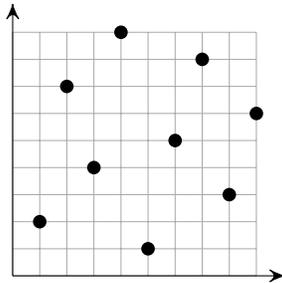
FIGURE 1. The plot of the permutation 274915836.

It is helpful to consider permutations from a graphical perspective.

Definition 1.4. The *plot* of a permutation $\sigma \in S_n$ is the collection of lattice points $\{(i, \sigma(i)) : 1 \leqslant i \leqslant n\}$ in the Euclidean plane $\mathbb{R}^2$. In practice, we tend to identify the ith entry of a permutation σ with the *point* $(i, \sigma(i))$ in its plot, and we linearly order the points in a plot from left to right; that is,

$$(i, \sigma(i)) \;<\; (j, \sigma(j)),$$

if $i < j$.

See Figure 1 for an illustration of a permutation plot.

This viewpoint motivates the following two definitions concerning the distance between entries of a permutation.

Definition 1.5. For a permutation $\sigma \in S_n$ and $i, j \in [n]$, the *distance* $d_\sigma(i, j)$ between the ith and jth entries of σ is given by the L_1 distance (the "taxicab" or "Manhattan" distance) between the corresponding points in the plot of σ:

$$d_\sigma(i, j) \;=\; \left\| (i, \sigma(i)) - (j, \sigma(j)) \right\|_1 \;=\; \left| i - j \right| \;+\; \left| \sigma(i) - \sigma(j) \right|.$$

For example, if $\sigma = 274915836$, as in Figure 1, then $d_\sigma(1, 2) = 1 + 5 = 6$, and $d_\sigma(1, 3) = 2 + 2 = 4$.

Definition 1.6. The *breadth* of $\sigma \in S_n$, denoted $\mathsf{br}(\sigma)$, is the minimum distance between any two entries:

$$\mathsf{br}(\sigma) = \min_{i,j\in[n]} d_\sigma(i,j).$$

For example, $\mathsf{br}(274915836) = 4$, and this is realized by any of the pairs of entries

$$\{i,j\} \in \{\{1,3\}, \{2,3\}, \{2,4\}, \{3,6\}, \{4,7\}, \{6,7\}, \{6,8\}, \{6,9\}, \{7,9\}, \{8,9\}\}.$$

With these definitions in place, we can state our two primary results. First, we have the following complete characterization of k-prolific permutations (Theorem 2.22):

A permutation σ is k-prolific if and only if $\mathsf{br}(\sigma) \geqslant k+2$.

That is, permutations are prolific precisely if their points are not too close together. As a consequence, it is readily seen that k-prolific permutations of size n are in bijection with certain packings of diamonds, which we call *permuted packings*. Section 2 is dedicated to the proof of this theorem. (This result was previously presented in [8], but the short proof given there contains an error.)

It is not possible for small permutations to be k-prolific because their points are too close together. Hence, our second main result is an exact determination of the minimum possible size of a k-prolific permutation (Corollary 4.3):

The smallest k-prolific permutations have size $\lceil k^2/2 + 2k + 1 \rceil$.

In Section 3, we prove that every k-prolific permutation must be at least this big (Theorem 3.4). The argument relies heavily on the interpretation of k-prolific permutations as permuted diamond packings. Then, in Section 4, we present constructions demonstrating that k-prolific permutations do exist of this size (Theorem 4.2), and also of all greater sizes (Theorem 4.4).

In the final section of the paper we discuss possible directions for further research, including some questions concerning permuted packings which may be of independent interest.

2. CHARACTERIZING k-PROLIFIC PERMUTATIONS

We begin by introducing notation to denote the pattern that results from the deletion of specified entries from a permutation.

Definition 2.1. For a permutation $\sigma \in S_n$ and $i \in [n]$, let $\sigma_{\langle i \rangle} \in S_{n-1}$ be the pattern formed by deleting the ith entry from σ. Similarly, if $A = \{i_1, i_2, \ldots, i_k\} \subset [n]$, then let $\sigma_{\langle A \rangle} \in S_{n-k}$ be the pattern formed by deleting the i_1th, i_2th, ..., i_kth entries from σ.

The goal of this section is to prove that a permutation is k-prolific if and only if its breadth is at least $k+2$. Specifically, we need to demonstrate that, given a permutation $\sigma \in S_n$, there exist distinct k-sets of indices $A, B \subset [n]$ such that $\sigma_{\langle A \rangle} = \sigma_{\langle B \rangle}$ if and only if $\mathsf{br}(\sigma) < k+2$.

The proof of the "if" direction is straightforward, as is the argument in the "only if" direction when there is an index common to A and B; we present these later. The next several pages, leading up to Lemma 2.20 are thus concerned with characterizing the situation when $\sigma_{\langle A \rangle} = \sigma_{\langle B \rangle}$ with A and B disjoint. To this end, we introduce a plane graph associated with such a scenario and determine its structure.

To define this graph, we first need to define what it means for an entry in a permutation to "fulfill" an entry in a pattern that it contains.

Definition 2.2. Suppose $\sigma \in S_n$ and $A \subset [n]$. Let $[n] \setminus A = \{i_1, i_2, \ldots, i_r\}$, where $i_1 < i_2 < \cdots < i_r$. For each $j \in [r]$, say that the i_jth entry of σ *fulfills* the jth entry of $\sigma_{\langle A \rangle}$.

Our graph joins the points of σ that fulfill the "same" point in $\sigma_{\langle A \rangle}$ and $\sigma_{\langle B \rangle}$.

Definition 2.3. Given a permutation $\sigma \in S_n$, and disjoint k-sets of indices $A, B \subset [n]$, such that $\sigma_{\langle A \rangle} = \sigma_{\langle B \rangle}$, the *chain graph* of σ for A and B is a plane graph on the points in the plot of σ. For each index $i \in [n-k]$, an edge is added between the point of σ that fulfills the ith entry of $\sigma_{\langle A \rangle}$ and the point of σ that fulfills the ith entry of $\sigma_{\langle B \rangle}$. If $\sigma_{\langle A \rangle}(i)$ and $\sigma_{\langle B \rangle}(i)$ are fulfilled by the same point, p, of σ, then we call p a *fixed point*, and no edge is added.

To facilitate the discussion, we let the vertices corresponding to elements of A be coloured *red*, and those corresponding to elements of B be coloured *blue*. The remaining vertices are *uncoloured*.

Note that this definition implies that no vertex of a chain graph has degree greater than two. See Figure 2 for an illustration of a chain graph; its vertex set contains eight red points (in A), eight blue points (in B), six fixed points, and seventeen other uncoloured points.
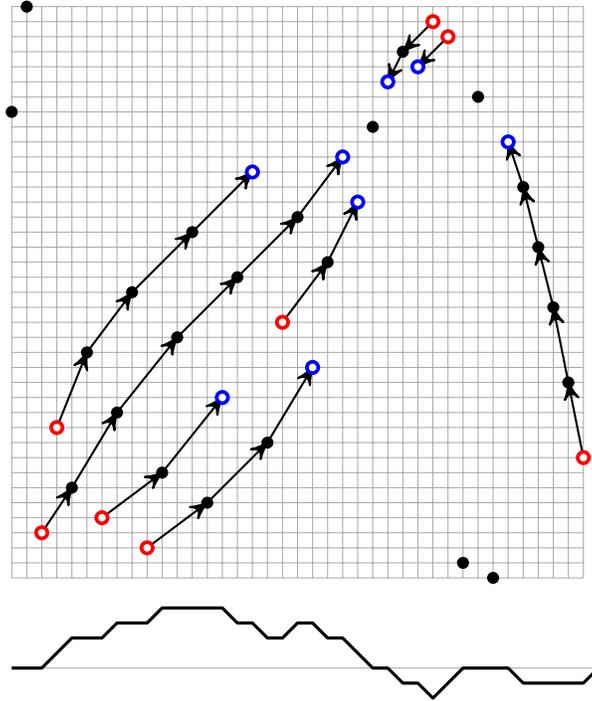
FIGURE 2. An oriented chain graph (Definition 2.3), and a plot of the discrepancy (Definition 2.4) of its vertices. The edges of each chain are oriented away from its red end-vertex towards its blue end-vertex.

Recall the comment in Definition 1.4 about identifying points in the plot of a permutation with their x-coordinates, and ordering them from left to right. In a chain graph, these points are the vertices. Thus, the vertices of a chain graph are also identified by their x-coordinates, and considered to be ordered from left to right.

Note that we restrict the definition of a chain graph to settings in which $\sigma_{\langle A \rangle} = \sigma_{\langle B \rangle} = \pi$, say. In such cases, if the ath entry of σ fulfills the ith entry of $\sigma_{\langle A \rangle}$ and the bth entry of σ fulfills the ith entry of $\sigma_{\langle B \rangle}$, then it is also the case that the $\sigma(a)$th entry of σ^{-1} fulfills the $\pi(i)$th entry of $\sigma_{\langle A \rangle}^{-1}$ and the $\sigma(b)$th entry of σ^{-1} fulfills the $\pi(i)$th entry of $\sigma_{\langle B \rangle}^{-1}$. Hence, properties of the chain graph are preserved under permutation inversion, so symmetry may be invoked to convert "horizontal" arguments into "vertical" ones.

We are interested in determining the properties of chain graphs, with the ultimate goal of proving that their vertices cannot be too far apart. To investigate their structure, we introduce the concept of the "discrepancy" of a vertex in a chain graph.

Definition 2.4. For each point p in a chain graph, let $\mathsf{nred}(p)$ be the number of red points to the left of p and $\mathsf{nblue}(p)$ be the number of blue points to the left of p. We define the *discrepancy* of p, which we denote $\delta(p)$, to be the difference, $\delta(p) = \mathsf{nred}(p) - \mathsf{nblue}(p)$.

A plot showing the discrepancy of the vertices in a chain graph is exhibited at the bottom of Figure 2.

Discrepancy has the properties outlined in the following two observations.

Observation 2.5. *If we consider the points from left to right, the discrepancy either increases by one (after a red point), decreases by one (after a blue point), or stays the same (after an uncoloured point). Since there are equally many red points as blue points, the discrepancy returns to zero after the last point.*

Thus a plot of the discrepancy is a *Motzkin bridge* — similar to a Motzkin path, but permitted to wander both above and below its start point.

Observation 2.6. *If the point p (the pth entry of σ) is not red, then it fulfills the entry of $\sigma_{\langle A \rangle}$ that has index $a := p - \mathsf{nred}(p)$. Similarly, if p is not blue, then it fulfills the entry of $\sigma_{\langle B \rangle}$ that has index $b := p - \mathsf{nblue}(p)$. Hence, p is a fixed point if and only if $\delta(p) = 0$.*

If p is not red and $\delta(p) > 0$, then the ath entry of $\sigma_{\langle B \rangle}$ is fulfilled by the $\delta(p)$th non-blue point to the left of p. Similarly, if p is not blue and $\delta(p) \geqslant 0$, then the bth entry of $\sigma_{\langle A \rangle}$ is fulfilled by the $\delta(p+1)$th non-red point to the right of p. The case for negative $\delta(p)$ is analogous; indeed, it is equivalent to the positive case applied to the reverse of the permutation.

We now show that the structure of a chain graph is tightly constrained, and that its name is justified.

Proposition 2.7. *The chain graph of a permutation σ for k-sets A and B consists of k monotone paths, which we call either* increasing *or* decreasing chains, *based on their left-to-right behaviour, together with one isolated vertex for each fixed point. Each chain has one red end-vertex and one blue.*

Suppose ℓ and r are the left and right end-vertices, respectively, of a chain C. If ℓ is red, then $\delta(\ell) \geqslant 0$ and for every point q of σ such that $\ell < q \leqslant r$, we have $\delta(q) > 0$. Analogously, if ℓ is blue, then $\delta(\ell) \leqslant 0$ and for every point q of σ such that $\ell < q \leqslant r$, we have $\delta(q) < 0$.

Proof. Firstly, each fixed point has degree 0, by definition.

Secondly, each red vertex has degree 1 since it fulfills a point of $\sigma_{\langle B \rangle}$, but does not fulfill a point of $\sigma_{\langle A \rangle}$. Analogously, each blue vertex also has degree 1.

Finally, let p be a non-fixed uncoloured vertex. By Observation 2.6, we know that $\delta(p) \neq 0$ and that p is adjacent to two other vertices, one to its left (which we denote p^-), and the other to its right (denoted p^+). So p is a medial vertex in a path whose vertices are ordered from left to right. By symmetry, p is also medial in a path whose vertices are ordered from bottom to top. Thus, from left to right, the path is either monotonically increasing or monotonically decreasing.

Suppose that $\delta(p) > 0$. If there are n_r red and n_b blue points in $[p^-, p)$, then $\delta(p^-) = \delta(p) - n_r + n_b$. Since p^- is the $\delta(p)$th non-blue point to the left of p, we know that $n_r \leqslant \delta(p)$. Thus, $\delta(p^-) \geqslant 0$. Moreover, $\delta(p^-) = 0$ if and only if the $\delta(p)$ points immediately to the left of p, including p^-, are all red. Furthermore, for all $q \in (p^-, p)$, we have $\delta(q) > 0$. By repeating this argument as long as p^- is uncoloured, we see that the left end-vertex, ℓ, of the chain containing p, is red, and $\delta(q) > 0$ for all $q \in (\ell, p]$.

An analogous argument shows that $\delta(p^+) > 0$, and for all $q \in (p, p^+)$, we have $\delta(q) > 0$. By iterating this, we see that the right end-vertex, r, of the chain containing p, is blue, and $\delta(q) > 0$ for all $q \in (p, r]$.

As before, the argument for negative $\delta(p)$ is equivalent to the positive case applied to the reverse of the permutation. $\qquad\square$

As an example, the chain graph illustrated in Figure 2 consists of seven increasing chains, one decreasing chain and six isolated fixed points.

As we go forward, in referring to edges in a chain, we make use of the following notation, as used in the proof of Proposition 2.7.

Definition 2.8. Given a point p in a chain, but not at its rightmost end, write p^+ for the point adjacent to p on its right. Thus, every edge of a chain is pp^+ for some point p.

Before stating further properties of chain graphs, we introduce the "span" and "central span" of a pair of points, and the idea of a point "cutting" an edge.

Definition 2.9. Fix $\sigma \in S_n$. For any $i, j \in [n]$, the *span* of i and j in σ is the set of entries of σ whose positions lie strictly between i and j *or* whose values lie strictly between $\sigma(i)$ and $\sigma(j)$. The *central span* of i and j in σ is the set of entries of σ whose positions lie strictly between i and j *and* whose values lie strictly between $\sigma(i)$ and $\sigma(j)$.

Thus, the span of two points consists of the points in a cross-shaped region, and the central span of two points consists of the points in a rectangular region.

Definition 2.10. Let q be in the span of points p and p'. This q *cuts pp' from the left* (respectively, *right*) if q's position is to the left (respectively, right) of both p and p'. This q *cuts pp' from below* (respectively, *above*) if q's value is less (respectively, greater) than both $\sigma(p)$ and $\sigma(p')$. Cuts from the left or right are *horizontal*; cuts from below or above are *vertical*. Points in the central span of p and p' are considered to cut pp' both horizontally and vertically.

There is a close relationship between cutting and the distance between two points.

Observation 2.11. *The distance $d_\sigma(p, p')$ is two greater than the number of times that pp' is cut.*

With these definitions in place, we can state two elementary corollaries of Proposition 2.7. These are the first of several results characterizing how edges of chains may be cut, which we use later to prove that some pair of points in a chain graph must be close together.

Corollary 2.12. *An edge of a chain C cannot be cut by another point from C.*

Corollary 2.13. *An edge of a chain cannot be cut by a fixed point.*

In light of the fact that one end-vertex of each chain is red and the other is blue, it makes sense to orient the edges of a chain graph. We choose to orient the edges of each chain away from its red end-vertex and towards its blue end-vertex. In Figure 2, five chains are oriented leftwards and three rightwards, while six are oriented upwards and two downwards.

We now show that chains in an oriented chain graph are further constrained by having to satisfy a "consistent orientation" property.

Proposition 2.14. *Suppose that C and C' are chains in the chain graph of a permutation σ, with left end-vertices ℓ and ℓ' and right end-vertices r and r', respectively. If $\ell < r'$ and $\ell' < r$ (that is, if C and C' overlap horizontally), then either both chains are oriented leftwards or both chains are oriented rightwards. Analogously, if two chains overlap vertically, then either they are both oriented upwards or they are both oriented downwards.*

Proof. Without loss of generality, suppose that ℓ is red. By Proposition 2.7, for every point q of σ that lies strictly between ℓ and r, we have $\delta(q) > 0$. Now, if ℓ' were blue, then, similarly, for every point q' of C', we would have $\delta(q') \leqslant 0$. But, since C and C' overlap horizontally, there is some point of C' between the end-vertices of C, so ℓ' must in fact be red.

The vertical case follows by symmetry. $\square$

This consistent orientation property has consequences for how edges of chains may be cut.

Corollary 2.15. *If C is an increasing chain and C' is a decreasing chain, then it is not possible for C and C' to overlap both horizontally and vertically. Thus, an edge of an increasing chain C cannot be cut both horizontally and vertically by points from a decreasing chain C'.*

Corollary 2.16. *If C_1 and C_2 are increasing chains that overlap (either horizontally or vertically), and C' is a decreasing chain, then it is not possible for there to be points q_1 and q_2 of C' such that q_1 cuts an edge of C_1 horizontally and q_2 cuts an edge of C_2 vertically.*

Chains in a chain graph also satisfy an "interleaving" property, which implies that two chains cannot cross. Recall the linear ordering of points from Definition 1.4.

Proposition 2.17. *If pp^+ and qq^+ are edges in distinct chains such that $p < q$, then $p^+ < q^+$.*

Proof. If $p^+ < q$, then the result follows trivially. Assume that $q < p^+$. Without loss of generality, suppose that $\delta(p^+) > 0$, and hence $\delta(q^+) > 0$.

Suppose p^+ were to the right of q^+. Let d be the difference between their x-coordinates, and let n_b be the number of blue points in the interval $[q^+, p^+)$.

Then, $\delta(p^+) \leqslant \delta(q^+) + d - 2n_b$, with a strict inequality if $n_b = 0$, since q^+ cannot be red.

Point p is the $\delta(p^+)$th non-blue point to the left of p^+. Since there are $d - n_b$ non-blue points in $[q^+, p^+)$, it is the case that p is no further to the left than the $(\delta(q^+) - n_b)$th non-blue point to the left of q^+.

But q is the $\delta(q^+)$th non-blue point to the left of q^+, which means that p is to the right of q, a contradiction. $\square$

Corollary 2.18. *By symmetry, if p is above and to the left of q, then p^+ is above and to the left of q^+. Thus chains cannot cross.*

The interleaving property further restricts the ways in which an edge of a chain may be cut by points from another chain.

Corollary 2.19. *An edge of a chain C can be cut at most once horizontally and at most once vertically by points from some other chain C'.*

Observe that if C is an increasing chain, and e is an edge of another chain, then it is only possible for points of C to cut e either from the left and from above, or else from the right and from below.

Corollary 2.19, together with Corollaries 2.12, 2.13, 2.15, and 2.16, completes our characterization of how edges of chains may be cut. We are now able to prove that the points in a chain graph cannot be very far apart.

Lemma 2.20. *Suppose we have a permutation σ, and disjoint k-sets of indices A and B, such that $\sigma_{\langle A \rangle} = \sigma_{\langle B \rangle}$. Then $\mathsf{br}(\sigma) < k + 2$.*

Proof. Assume, to the contrary, that $\mathsf{br}(\sigma) \geqslant k + 2$.

Let G be the chain graph of σ for A and B, and let $e = pp^+$ be an edge of a chain C of G. Without loss of generality, suppose that C is an increasing chain.

Since $d_\sigma(p, p^+) \geqslant k+2$, the edge e is cut by at least k vertices. By Corollaries 2.12 and 2.13, e is not cut by a point of C or by a fixed point. There are only $k - 1$ chains in G distinct from C, so, by the pigeonhole principle, there is at least one chain, C', whose points cut e twice. By Corollary 2.15, C' is an increasing chain, and by Corollary 2.19, one cut must be horizontal and the other vertical.

First, suppose that a point q of C' cuts e both horizontally and vertically; that is, this q is in the central span of p and p^+. Now, e can be cut by at most $k - 1$ points horizontally, one from each chain, and similarly by at most $k - 1$ points vertically, so $d_\sigma(p, p^+) \leqslant 2k$. Thus, since $d_\sigma(p, p^+) = d_\sigma(p, q) + d_\sigma(q, p^+)$, either $d_\sigma(p, q) \leqslant k$ or $d_\sigma(q, p^+) \leqslant k$, and so $\mathsf{br}(\sigma) \leqslant k$, a contradiction. Thus, there is no point in the central span of p and p^+, and e is cut by two distinct points of C'.

Suppose now that G contains m increasing chains and $d := k - m$ decreasing chains. Without loss of generality, we may assume that $m \geqslant 1$. For an illustration of the argument that follows, see Figure 3. Note that this figure includes no decreasing chains. In fact, decreasing chains can essentially be ignored, although proving this is more work than accounting for them explicitly. For brevity in what follows, we call a point of an increasing chain an *increasing point*, and a point of a decreasing chain a *decreasing point*.

Let p_1 be the lowermost increasing point in G, and let e_1 be the edge $p_1 p_1^+$. Then, for each $j = 2, \ldots, m$, let p_j be the lowermost increasing point that cuts e_{j-1} from the left (we show that such a point always exists), and set $e_j := p_j p_j^+$. Our goal is to prove that, for each j, the edge e_j is not cut from below by any increasing point. We proceed by induction on j.

For the base case, it follows from the definition of p_1 and the fact that the central span of p_1 and p_1^+ is empty that e_1 is not cut from below by any increasing point.

Now, fix $j \geqslant 1$ and assume that e_j is not cut from below by any increasing point.
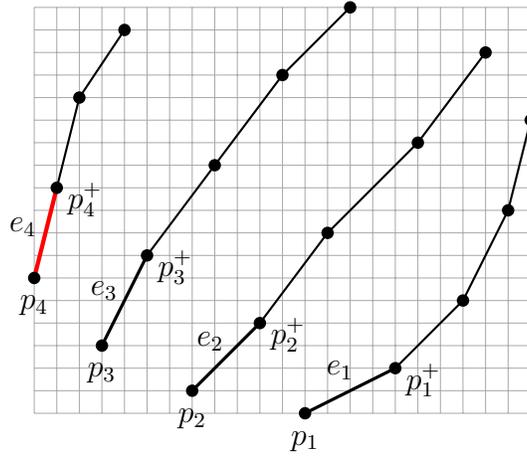
FIGURE 3. An illustration of the second half of the proof of Lemma 2.20, for $k = m = 4$.

Since e_j is not cut from below by an increasing point, points of some increasing chain must cut e_j from both the left and from above. So there exists a lowermost increasing point that cuts e_j from the left, namely p_{j+1}.

Suppose that e_j is cut from the right by r_j increasing points and cut horizontally by h_j decreasing points. By the definition of p_{j+1}, these are the only points that can occur above p_j and below p_{j+1}, so $\sigma(p_{j+1}) - \sigma(p_j) \leqslant r_j + h_j + 1$. Hence, since $d_\sigma(p_j, p_{j+1}) \geqslant k+2$, we have

$$p_j - p_{j+1} \;\geqslant\; (k+2) - (r_j + h_j + 1) \;=\; k + 1 - r_j - h_j.$$

Now suppose that e_{j+1} is cut vertically by v_{j+1} decreasing points. By Proposition 2.16, we know that $h_j + v_{j+1} \leqslant d$. Observe that e_{j+1} is cut from above by at most $m - 1 - r_j$ increasing points, the r_j chains cutting e_j from the right also being to the right of e_{j+1}. Hence,

$$p_{j+1}^+ - p_{j+1} \;\leqslant\; m - 1 - r_j + v_{j+1} + 1 \;\leqslant\; m - r_j + d - h_j \;=\; k - r_j - h_j.$$

Thus,

$$p_j - p_{j+1}^+ \;\geqslant\; (k + 1 - r_j - h_j) - (k - r_j - h_j) \;=\; 1.$$

Thus, p_{j+1}^+ is to the left of p_j, so e_{j+1} is not cut from below by p_j.

Moreover, since the central span of p_{j+1} and p_{j+1}^+ is empty, the edge e_{j+1} is not cut from below by any other increasing point. This is because, otherwise, either p_1 would not be the lowermost increasing point in G, or else $p_2, \ldots, p_{j+1}$ would not be the lowermost increasing points cutting $e_1, \ldots, e_j$ from the left.

Now consider the final edge $e_m = p_m p_m^+$. By the same inductive argument, it too is cut from the left by some increasing point p_{m+1}. But this is impossible, because there are only m increasing chains in G. Hence, our initial assumption is false, and thus $\mathsf{br}(\sigma) < k + 2$. $\square$

Having established the desired result when A and B are disjoint, we now have almost all we need to establish the relationship between the breadth of a permutation and whether that permutation is k-prolific or not. The final ingredient is the following proposition, adapted from [8].

Proposition 2.21. *Deleting a single entry from a permutation decreases the breadth by at most one.*

Proof. Let σ be a permutation having breadth b, and let i and j be indices such that $d_\sigma(i, j) = b$. The only way to decrease the distance between these entries through deletion is to delete an element of their span. To decrease the distance by two would require deleting a point, p, from their central span. But if there were such an point, then we would have $d_\sigma(i, p) < d_\sigma(i, j)$, and $\mathsf{br}(\sigma)$ would be less than b. □

Here, finally, is our first main result: a complete characterization of k-prolific permutations.

Theorem 2.22. *A permutation σ is k-prolific if and only if $\mathsf{br}(\sigma) \geqslant k + 2$.*

Proof. First suppose that $i \neq j$ are such that $d_\sigma(i, j) < k + 2$. Let S be the set of elements of σ in the span of i and j. It follows that $|S| < k$. But we have $\sigma_{\langle S \cup \{i\}\rangle} = \sigma_{\langle S \cup \{j\}\rangle}$, so σ is not k-prolific.

In the other direction, we proceed by induction on k.

For the base case, suppose that $\mathsf{br}(\sigma) \geqslant 3$, but that σ is not 1-prolific; that is, there exists $i \neq j$ such that $\sigma_{\langle i \rangle} = \sigma_{\langle j \rangle}$. Assume without loss of generality that $i < j$ and $\sigma(i) < \sigma(j)$. The $(j-1)$th entry of $\sigma_{\langle i \rangle}$ is $\sigma(j) - 1$, while the $(j-1)$th entry of $\sigma_{\langle j \rangle}$ is $\sigma(j-1)$. But if $\sigma(j) - 1 = \sigma(j-1)$, then

$$\mathsf{br}(\sigma) \;\leqslant\; d_\sigma(j, j-1) \;=\; \big|j - (j-1)\big| + \big|\sigma(j) - \sigma(j-1)\big| \;=\; 2,$$

a contradiction. Therefore σ must be 1-prolific.

Now fix $k > 1$ and assume that, for any permutation τ, if $\mathsf{br}(\tau) \geqslant k + 1$, then τ is $(k-1)$-prolific. Suppose that the breadth of σ is at least $k + 2$, but that σ is not k-prolific; that is, there are distinct k-sets A and B such that $\sigma_{\langle A \rangle} = \sigma_{\langle B \rangle}$.

If there is an index $c \in A \cap B$, then $\sigma' = \sigma_{\langle c \rangle}$ is $(k-1)$-prolific, by Proposition 2.21 and the induction hypothesis. But

$$\sigma'_{\langle A \setminus \{c\}\rangle} \;=\; \sigma_{\langle A \rangle} \;=\; \sigma_{\langle B \rangle} \;=\; \sigma'_{\langle B \setminus \{c\}\rangle},$$

so σ' cannot be $(k-1)$-prolific. Thus A and B must be disjoint.

The result then follows by Lemma 2.20. □

As a consequence of this characterization, we see that any permutation of size n containing maximally many patterns of size $n - k$ also contains maximally many larger patterns.

Corollary 2.23. *If σ is k-prolific, then σ is also j-prolific for all $1 \leqslant j < k$.*

3. Bounding the size of k-prolific permutations from below

In this section, we determine a lower bound on the size of k-prolific permutations. We use the following notation to denote the size of the smallest k-prolific permutation.

Definition 3.1. Given a positive integer k, let $\mathsf{minprol}(k)$ be the minimum value n for which there exists a k-prolific permutation in S_n.

Clearly, for a k-prolific permutation of size n to exist, we need $(n-k)! \geqslant \binom{n}{k}$. This inequality yields a very weak lower bound on $\mathsf{minprol}(k)$, which can, using Stirling's approximation, be shown to grow like $k + e\sqrt{k}$ for large k. Nevertheless, in conjunction with the fact that both

2413 and 3142 cover all four non-monotone permutations in S_3, it is sufficient to determine that $\mathsf{minprol}(1) = 4$.

We now establish a much tighter lower bound on $\mathsf{minprol}(k)$ by recasting Theorem 2.22 in terms of packings of diamonds. Recall that a *translational packing* with a tile (a compact non-empty subset of $\mathbb{R}^2$) is a collection of translates of the tile whose interiors are mutually disjoint (see [3]). We are interested in translational packings in which the tiles are centered on the points of a permutation.

Definition 3.2. A translational packing Π, consisting of n translates of a tile T, is a *permuted packing* if there exists a permutation $\sigma \in S_n$ such that $\Pi = \{T + (i, \sigma(i)) : 1 \leqslant i \leqslant n\}$.

The following proposition establishes the relationship between k-prolific permutations and permuted packings.

Proposition 3.3. *Let D be a diamond whose diagonal has length $k + 2$. The family of k-prolific permutations of size n is equinumerous to the family of permuted packings that consist of n translates of D.*

Proof. By Theorem 2.22, a permutation is k-prolific if and only if the minimum L_1 distance between two points in the plot of the permutation is at least $k + 2$. Thus, if we center a ball of radius $k/2 + 1$ (under the L_1 metric) at each point of the plot, the interiors of these balls are mutually disjoint. Since, in $\mathbb{R}^2$, an L_1 ball of radius $k/2 + 1$ is a diamond whose diagonal has length $k + 2$, it is readily seen that this construction yields a bijection between k-prolific permutations of size n and the specified family of permuted packings. $\qquad\square$

Figure 4 depicts a 6-prolific permutation and the corresponding permuted diamond packing. Using this characterization of k-prolific permutations, we now bound $\mathsf{minprol}(k)$ from below.

Theorem 3.4. *For each positive integer k, $\mathsf{minprol}(k) \geqslant \lceil k^2/2 + 2k + 1 \rceil$.*

Proof. Suppose that $\sigma \in S_n$ is k-prolific. Let $s = k/2 + 1$ be the length of the semidiagonals of the diamonds in the associated permuted packing. The area of each of the n diamonds in the packing is $2s^2$.

Consider the square box $[s - 1, n + 2 - s]^2$, centered over the packing, as illustrated in Figure 4. The margins around this box have width k. Note that when k is even, s is an integer and the sides of the box pass through the centers of four of the diamonds; when k is odd, they do not.

The total area of the diamond tiles is bounded above by the area of this square box plus the total area of the parts of the diamonds that "overflow" into the margins outside the box. The overflowing parts are shaded more darkly in Figure 4. The area of the parts of the diamonds in a given margin can be calculated exactly, as follows.

Consider the region to the left of the left side of the box, including the top left and bottom left corners. In this margin are parts of each of the diamonds centered at the first k points of σ. For each $j < k/2$, the overflowing part of the jth diamond from the left and the overflowing part of the $(k - j)$th diamond can be glued together to form a complete diamond. For example, in Figure 4, the two parts labelled a form a complete diamond, as do the two labelled b. If k is even, then the overflowing part of the $(k/2)$th diamond is exactly half a diamond. Finally, the overflowing part of the kth diamond is a triangle with area 1. Thus,
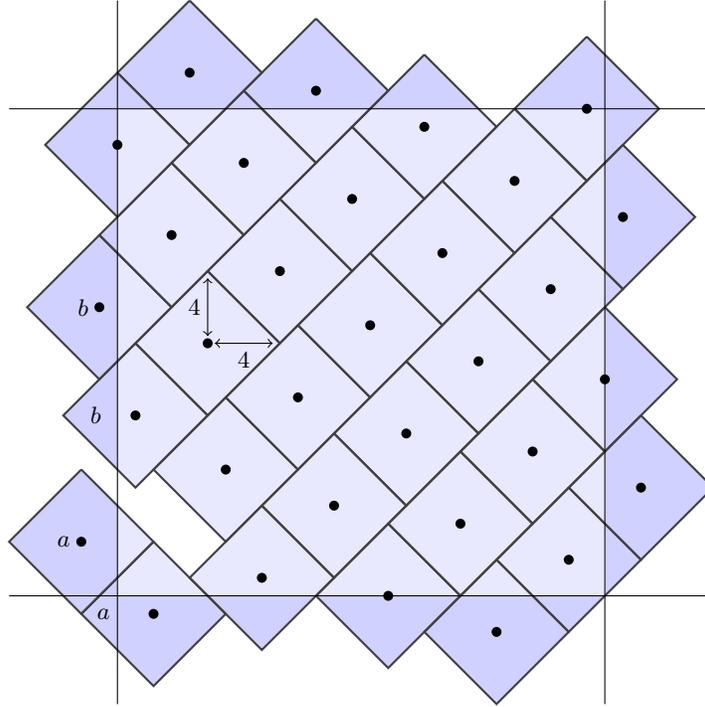
FIGURE 4. The permuted diamond packing corresponding to a 6-prolific permutation, showing the square box used in the proof of Theorem 3.4.

the total area of the parts of the diamonds that overflow into any one margin is given by

$$\left(\tfrac{k-1}{2}\right) \cdot 2s^2 + 1 \;=\; (k-1)s^2 + 1.$$

The total area of all of the parts of the diamonds that overflow is no more than four times this, a value which counts the contributions from the corners twice. Therefore, we have the inequality

$$2s^2 n \;\leqslant\; (n+3-2s)^2 \;+\; 4\big((k-1)s^2 + 1\big).$$

After substitution for s and application of the quadratic formula, this yields

$$n \;\geqslant\; \frac{k^2 + 8k + \sqrt{k^4 + 32k - 16}}{4},$$

an expression which exceeds $k^2/2 + 2k$ for all $k > 1/2$.

Since $\mathsf{minprol}(k)$ is an integer, for even k we thus have $\mathsf{minprol}(k) \geqslant k^2/2 + 2k + 1$, as required, and for odd k, $\mathsf{minprol}(k) \geqslant k^2/2 + 2k + 1/2$.

A marginally greater lower bound can be established for odd k by using a slightly different shape of tile. If k is odd, then the length of the semidiagonal of the diamonds, $s = k/2 + 1$, is a half-integer. Since, in the packing, each diamond is centered on an integer lattice point, it is not possible for three of these diamonds to meet at a point. Thus, either above or below the rightmost corner of each of these diamonds is a small diamond-shaped region, of semidiagonal length $1/2$, not covered by any diamond tile. We thus extend the tiles by the addition of these regions, which we call *extensions*, and consider permuted packings of these extended diamonds. See Figure 5 for an illustration.
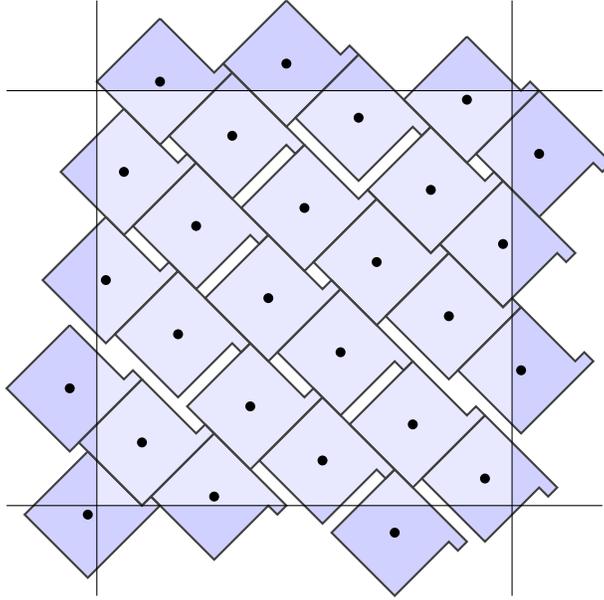
FIGURE 5. A permuted packing of extended diamonds corresponding to a 5-prolific permutation.

For these extended diamonds, we now repeat our analysis of the parts of the tiles that overflow into the margins. The total area of the parts of the diamond-shaped extensions to the right of the box is $k/2 + 1/4$, made up of k complete extensions, each of area $1/2$, and half an extension from the $(k + 1)$th diamond from the right. The contribution from the extensions that overflow into the top margin is at most $k/4$, made up of $(k - 1)/2$ complete extensions and half an extension. The bottom margin is analogous. Since no extension can overflow into the left margin, the total area of the parts of the extensions that overflow is no more than $k + 1/4$. Therefore, accounting for the additional area of each tile, we have the inequality

$$(2s^2 + 1/2)n \leqslant (n + 3 - 2s)^2 + 4\big((k - 1)s^2 + 1\big) + k + 1/4.$$

After substitution for s and the application of the quadratic formula, this yields

$$n \geqslant \frac{k^2 + 8k + 1 + \sqrt{k^4 + 2k^2 + 32k - 19}}{4},$$

an expression which exceeds $k^2/2 + 2k + 1/2$ for all $k > 5/8$.

Since $\mathsf{minprol}(k)$ is an integer, for odd k we thus have $\mathsf{minprol}(k) \geqslant k^2/2 + 2k + 3/2$, as required. $\qquad\square$

4. CONSTRUCTIONS OF k-PROLIFIC PERMUTATIONS

In this section, we establish that there is a k-prolific permutation of every size greater than or equal to the lower bound of Theorem 3.4, by construction.

Let $m(k) = \lceil k^2/2 + 2k + 1 \rceil$ be the lower bound function from Theorem 3.4. Our initial constructions enable us to prove that $\mathsf{minprol}(k) = m(k)$.

Definition 4.1. For each $k \geqslant 1$, define σ_k as follows. For $i = 1, \ldots, m(k)$, let

$$\sigma_k(i) \;=\; \begin{cases} i(k+2) \mod m(k) + 1, & \text{if } k \text{ is odd, and} \\ i(k+1) \mod m(k) + 1, & \text{if } k \text{ is even.} \end{cases}$$

See Figure 6 for an illustration of σ_5 and σ_6. We claim that σ_k is a k-prolific permutation of size $m(k)$.

First, we must prove that σ_k is indeed a permutation; that is, that $\sigma_k(i)$ takes a distinct value for each i. To do so, it is sufficient to show that $k+2$ is coprime to $m(k)+1$ when k is odd, and that $k+1$ is coprime to $m(k)+1$ when k is even. Observe that for odd k, we have

$$2\big(k^2/2 + 2k + 5/2\big) \;+\; (-k-2)\big(k+2\big) \;=\; 1,$$

and for even k,

$$2\big(k^2/2 + 2k + 2\big) \;+\; (-k-3)\big(k+1\big) \;=\; 1.$$

Thus, by Bézout's Lemma, the relevant terms are coprime.

To demonstrate that each σ_k is k-prolific, we use an alternative characterization of the permutations, in terms of two interlocking grids of lattice points as illustrated in Figure 6.
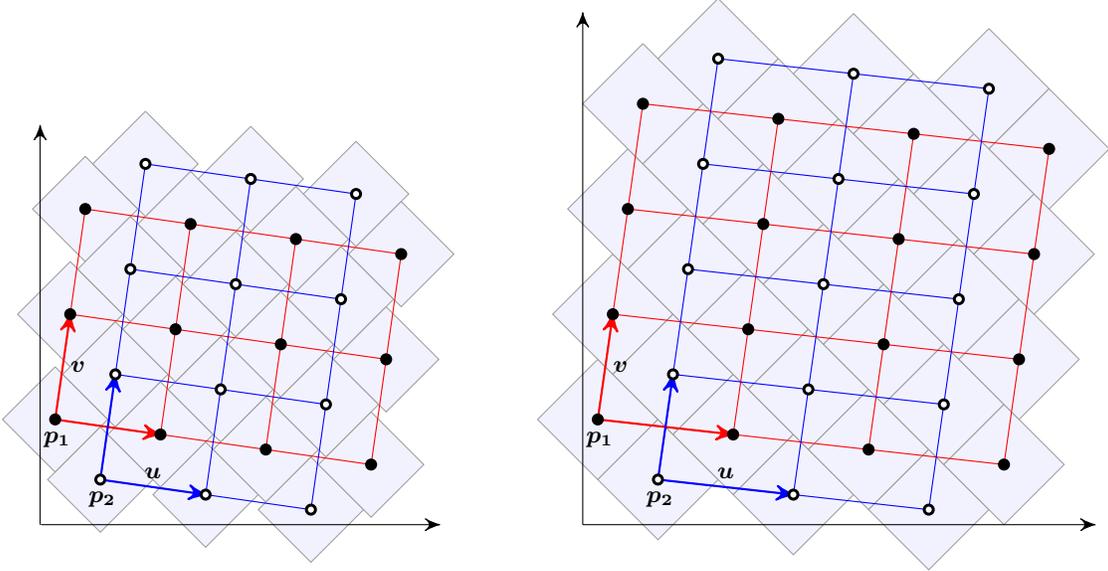
FIGURE 6. Plots of the permutations $\sigma_5 \in S_{24}$ and $\sigma_6 \in S_{31}$, showing their construction from two interlocking grids of lattice points.

For odd k, define the four vectors

$$\boldsymbol{p_1} \;=\; (1, k+2), \quad \boldsymbol{p_2} \;=\; \left(\tfrac{k+3}{2}, \tfrac{k+1}{2}\right), \quad \boldsymbol{u} \;=\; (k+2, -1), \quad \boldsymbol{v} \;=\; (1, k+2).$$

Now let

$$\Gamma_1 \;=\; \big\{\boldsymbol{p_1} + q\boldsymbol{u} + r\boldsymbol{v} \;:\; 0 \leqslant q \leqslant \tfrac{k+1}{2}, \; 0 \leqslant r \leqslant \tfrac{k-1}{2}\big\},$$
$$\Gamma_2 \;=\; \big\{\boldsymbol{p_2} + q\boldsymbol{u} + r\boldsymbol{v} \;:\; 0 \leqslant q \leqslant \tfrac{k-1}{2}, \; 0 \leqslant r \leqslant \tfrac{k+1}{2}\big\}$$

be two finite grids of lattice points.

We claim that $\Gamma_1 \cup \Gamma_2$ is the plot of σ_k. Indeed, for $1 \leqslant i \leqslant m(k)$, if

$$q = \lfloor (i-1)/(k+2) \rfloor \quad \text{and} \quad r = (i-1) \bmod (k+2),$$

then

$$(i, \sigma_k(i)) = \begin{cases} \boldsymbol{p_1} + q\boldsymbol{u} + r\boldsymbol{v}, & \text{if } r \leqslant \frac{k-1}{2}, \text{ and} \\ \boldsymbol{p_2} + q\boldsymbol{u} + \left(r - \frac{k+1}{2}\right)\boldsymbol{v}, & \text{otherwise.} \end{cases}$$

Similarly, for even k, define the four vectors

$$\boldsymbol{p_1} = (1, k+1), \quad \boldsymbol{p_2} = \left(\tfrac{k}{2} + 2, \tfrac{k}{2}\right), \quad \boldsymbol{u} = (k+3, -1), \quad \boldsymbol{v} = (1, k+1),$$

and let

$$\Gamma_1 = \left\{\boldsymbol{p_1} + q\boldsymbol{u} + r\boldsymbol{v} : 0 \leqslant q \leqslant \tfrac{k}{2}, \ 0 \leqslant r \leqslant \tfrac{k}{2}\right\},$$

$$\Gamma_2 = \left\{\boldsymbol{p_2} + q\boldsymbol{u} + r\boldsymbol{v} : 0 \leqslant q \leqslant \tfrac{k}{2} - 1, \ 0 \leqslant r \leqslant \tfrac{k}{2} + 1\right\}.$$

Now, for $1 \leqslant i \leqslant m(k)$, if

$$q = \lfloor (i-1)/(k+3) \rfloor \quad \text{and} \quad r = (i-1) \bmod (k+3),$$

then

$$(i, \sigma_k(i)) = \begin{cases} \boldsymbol{p_1} + q\boldsymbol{u} + r\boldsymbol{v}, & \text{if } r \leqslant \frac{k}{2}, \text{ and} \\ \boldsymbol{p_2} + q\boldsymbol{u} + \left(r - \frac{k}{2} - 1\right)\boldsymbol{v}, & \text{otherwise.} \end{cases}$$

Thus, for each k, the plot of σ_k is given by $\Gamma_1 \cup \Gamma_2$. We now use this characterization to bound $\mathsf{minprol}(k)$ from above.

Theorem 4.2. *For each positive integer k, the permutation σ_k is k-prolific.*

Proof. By Theorem 2.22, we need only show that the breadth of σ_k is at least $k + 2$. Let x and y be distinct points of σ_k.

Recall that the points in the plot of σ_k are partitioned into two sets, Γ_1 and Γ_2. If x and y both lie in the same set, then their positions in the plot differ by a nonzero integer linear combination $q\boldsymbol{u} + r\boldsymbol{v}$. Thus, the L_1 distance between these points is given by

$$d_{\sigma_k}(x, y) = |qa + r| + |rb - q|,$$

where

$$(a, b) = \begin{cases} (k+2, k+2) & \text{if } k \text{ is odd, and} \\ (k+3, k+1) & \text{if } k \text{ is even.} \end{cases}$$

If $q = 0$, then $|r| \geqslant 1$, because $x \neq y$. In that case, $d_{\sigma_k}(x, y) = |r| + |rb| \geqslant 1 + b$. Similarly, if $r = 0$, then $|q| \geqslant 1$ and $d_{\sigma_k}(x, y) = |qa| + |q| \geqslant a + 1$. Suppose now that both $|q|$ and $|r|$ are nonzero. Without loss of generality, we may assume that $r \geqslant 1$. If $q \geqslant 1$, then $d_{\sigma_k}(x, y) \geqslant |qa + r| \geqslant a + 1$. If, on the other hand, $q \leqslant -1$, then $d_{\sigma_k}(x, y) \geqslant |rb - q| \geqslant b + 1$. In each case, we have $d_{\sigma_k}(x, y) \geqslant k + 2$.

Now suppose, without loss of generality, that $x \in \Gamma_1$ and $y \in \Gamma_2$. In this case, their positions in the plot differ by a vector $\boldsymbol{p_1} - \boldsymbol{p_2} + q\boldsymbol{u} + r\boldsymbol{v}$, for some integers q and r. Thus, the L_1 distance between these points is given by

$$d_{\sigma_k}(x, y) = |qa - c + r| + |rb + d - q|,$$

where

$$(c, d) \; = \; \begin{cases} \left(\frac{k+1}{2}, \frac{k+3}{2}\right) & \text{if } k \text{ is odd, and} \\ \left(\frac{k}{2} + 1, \frac{k}{2} + 1\right) & \text{if } k \text{ is even.} \end{cases}$$

The possible distances can be partitioned into the following five cases:

$$\begin{aligned} d_{\sigma_k}(x, y) &= k + 2, & \text{if } 0 \leqslant q \leqslant 1 \text{ and } -1 \leqslant r \leqslant 0, \\ d_{\sigma_k}(x, y) &\geqslant |qa - c + r| \geqslant 2a - c - 1, & \text{if } q \geqslant 2 \text{ and } r \geqslant -1, \\ d_{\sigma_k}(x, y) &\geqslant |rb + d - q| \geqslant b + d - 1, & \text{if } q \leqslant 1 \text{ and } r \geqslant 1, \\ d_{\sigma_k}(x, y) &\geqslant |qa - c + r| \geqslant a + c, & \text{if } q \leqslant -1 \text{ and } r \leqslant 0, \\ d_{\sigma_k}(x, y) &\geqslant |rb + d - q| \geqslant 2b - d, & \text{if } q \geqslant 0 \text{ and } r \leqslant -2. \end{aligned}$$

Again, in each case, we have $d_{\sigma_k}(x, y) \geqslant k + 2$.

Therefore, the permutation σ_k has breadth $k + 2$, and so, by Theorem 2.22, it is k-prolific. $\qquad\square$

This construction determines an upper bound on the size of the smallest k-prolific permutation. Together with the lower bound of Theorem 3.4, it establishes the value of $\mathsf{minprol}(k)$ exactly.

Corollary 4.3. *For each positive integer k, the smallest k-prolific permutations have size* $\lceil k^2/2 + 2k + 1 \rceil$.

The first few terms of this sequence are $4, 7, 12, 17, 24, 31, 40, 49$. It is sequence A074148 in [16].

Clearly, any symmetry of σ_k is k-prolific. However, for even $k \geqslant 6$, these permutations are not the only k-prolific permutations of minimal size.

See Figure 7 for an illustration of another 6-prolific permutation of size 31.
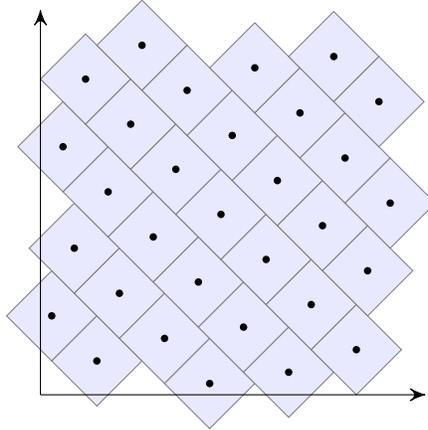
FIGURE 7. An alternative 6-prolific permutation in S_{31}.

It is not immediately obvious that k-prolific permutations exist of every size greater than or equal to $\mathsf{minprol}(k)$. We conclude this section by briefly presenting a construction that demonstrates that this is, in fact, the case.

Theorem 4.4. *There is a k-prolific permutation of every size greater than or equal to $\lceil k^2/2 + 2k + 1 \rceil$.*

Proof. We construct a k-prolific permutation, σ_k^{+j}, of size $\mathsf{minprol}(k) + j$, for each $k \geqslant 1$ and $j \geqslant 0$.

Let $\sigma_k^{+0} = \sigma_k$. For $j \geqslant 0$, the permutation σ_k^{+j+1} is constructed by inserting a new first entry immediately above the $(k+2)$th entry of σ_k^{+j}, if k is odd, or immediately above the $(k+3)$th entry of σ_k^{+j}, if k is even. See Figure 8 for an illustration.

We leave as an exercise for the reader the rather tedious details of the proof that this construction never leads to a reduction in the breadth of the permutation.

Furthermore, it can be shown that the breadth eventually increases: if k is odd, then σ_k^{+k+2} is $(k+1)$-prolific, and, if k is even, then σ_k^{+k+3} is $(k+1)$-prolific. $\square$
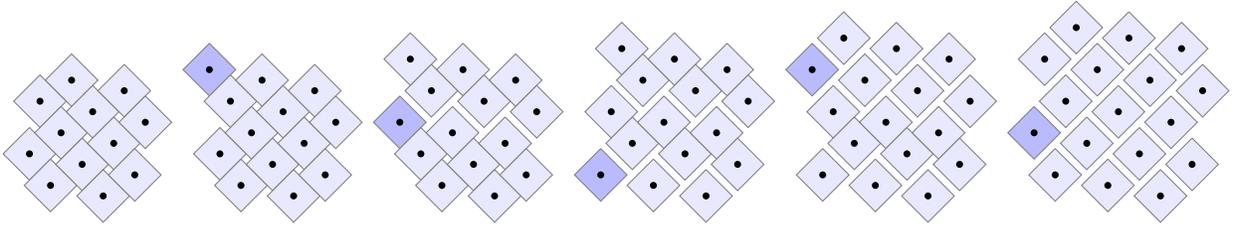
FIGURE 8. Plots of the 3-prolific permutations σ_3^{+j}, for $j = 0, \ldots, 5$; note that σ_3^{+5} is, in fact, 4-prolific.

5. DIRECTIONS FOR FURTHER RESEARCH

In Section 4, we noted that σ_k and its symmetries were not necessarily the only k-prolific permutations of minimal size. However, for odd k, no additional k-prolific permutations of size $\mathsf{minprol}(k)$ are known. This prompts the following conjecture.

Conjecture 5.1. *For each odd k, the permutation σ_k (described in Definition 4.1) and its symmetries are the only k-prolific permutations of minimal size.*

More generally, we wonder whether it is possible to enumerate and characterize all minimal k-prolific permutations.

Question 5.2. *For each k, how many distinct k-prolific permutations of minimal size are there, and what are they?*

Another topic of potential interest concerns the presence of k-prolific permutations in specific permutation classes (sets closed downwards in the pattern poset $\mathcal{P}$). For example, there appear to be no 1-prolific permutations avoiding 132, and no 2-prolific permutations avoiding 123. This motivates the following question.

Question 5.3. *For each k, which principal permutation classes (those avoiding a single pattern) contain k-prolific permutations?*

In various guises, the enumeration of 1-prolific permutations (sequence A002464 in [16]) has been well-studied ever since Kaplansky's 1944 paper addressing the "n king problem" [9, 10]. Tauraso [15] presents complete asymptotics. For large n, the proportion of permutations of size n which are 1-prolific is

$$e^{-2}\left(1 - \frac{2}{n^2} - \frac{10}{3n^3} - \frac{6}{n^4} - \frac{154}{15n^5} + O\left(\frac{1}{n^6}\right)\right).$$

However, nothing specific appears to be known about the enumeration of k-prolific permutations for larger k, although elementary considerations suggest that the number of k-prolific permutations of size n is asymptotically $e^{-k^2-k}n!$.

Question 5.4. *For a given $k > 1$, how does the number of k-prolific permutations of size n grow with n?*

The notion of being k-prolific can also be transferred to the context of other graded posets, an element of rank n being k-prolific if it has maximally many children of rank $n - k$. The characterization of the k-prolific elements of various combinatorial posets, perhaps most obviously those relating to the various subgraph orders, may be of interest.

Finally, permuted packings also invite further investigation. In addition to the permuted diamond packings studied here, one might consider permuted packings of other regular tiles. Permuted packings of axis-parallel squares appear uninteresting. On the other hand, permuted circle packings raise some intriguing questions. See Figure 9 for an illustration.
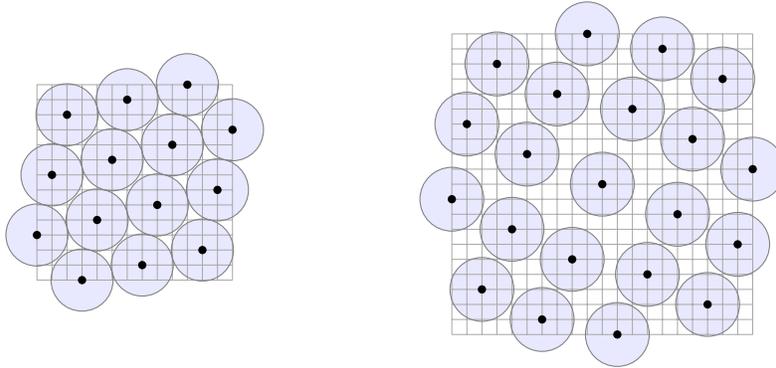
FIGURE 9. Minimal permuted packings of circles of diameter $\sqrt{17}$ and $\sqrt{18} + \varepsilon$.

Recall that the *density* of a packing Π relative to a bounded domain D is defined as

$$d(\Pi, D) = \frac{\sum_{T \in \Pi} \mu(T \cap D)}{\mu(D)},$$

where $\mu(X)$ is the area of X (see [3]).

Let us call a permuted packing of minimal cardinality a *minimal* permuted packing. Among other problems, one that is particularly attractive would be to determine how poor a minimal permuted packing can be, asymptotically as the radius of the circular tiles tends to infinity.

Question 5.5. *What is the value of*

$$\liminf_{\rho \to \infty} d(\Pi_\rho, [1, n_\rho]^2),$$

where Π_ρ is a minimal permuted packing of circles of radius ρ, and n_ρ is the number of circles in such a packing?

Similar questions might be asked about permuted packings of regular hexagons.

Acknowledgements

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