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Decomposing recurrent states of the abelian sandpile model [1]

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Abstract

The recurrent states of the Abelian sandpile model (ASM) are those states that appear infinitely often. For this reason they occupy a central position in ASM research. We present several new results for classifying recurrent states of the Abelian sandpile model on graphs that may be decomposed in a variety of ways. These results allow us to classify, for certain families of graphs, recurrent states in terms of the recurrent states of its components. We use these decompositions to give recurrence relations for the generating functions of the level statistic on the recurrent configurations. We also interpret our results with respect to the sandpile group.

Keywords: Abelian sandpile model, recurrent states, graph decomposition, level polynomial, sandpile group.

1 Introduction

One fundamental aspect of ASM research concerns the classification of the recurrent states of the model; those states that appear infinitely often in the

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long-time running of the model. In this abstract we present several new results classifying recurrent states of the ASM on graphs which may be decomposed in a variety of ways. The level statistic of a sandpile configuration, a quantity that is equal (up to an additive constant) to the sum of its heights, is studied with respect to these decompositions. We then show the effect of these graph decompositions on the sandpile group, which is the set of recurrent states equipped with an abelian addition operation. More detailed proofs and additional examples can be found in [7].

Let $G = (V, E)$ be a finite, connected, loop-free, undirected multigraph with vertex set $V = \{v_0, \dots, v_n\}$. Let $d_i = d_i^G = \deg(v_i)$ be the degree of the vertex v_i in G. Given $i, j \in \{0, \dots, n\}$ with $i \leq j$, we let $V_{[i,j]} = \{v_i, v_{i+1}, \dots, v_j\}$. For $W \subseteq V$, we let $G[W]$ be the subgraph of G with vertex set W and edge set the edges of G with both endpoints in W. We will consider the sandpile model on the graph G in which a distinguished vertex, v_0 say, acts as a *sink*. We indicate this by writing it as a pair, e.g. (G, v_0).

Let $\mathbb{Z}_+$ be the set of non-negative integers. A *configuration* on (G, v_0) is a vector $c = (c_1, \dots, c_n) \in \mathbb{Z}_+^n$ that assigns the number c_i to vertex v_i. We think of c_i as the number of 'grains of sand' at the vertex v_i. $\mathrm{Config}_{v_0}(G)$ is the set of all configurations on (G, v_0). Let $\alpha_i \in \mathbb{Z}_+^n$ be the vector with 1 in the i-th position and 0 elsewhere.

We say that a vertex v_i in a configuration $c = (c_1, \dots, c_n) \in \mathrm{Config}_{v_0}(G)$ is *stable* if $c_i < d_i$. Otherwise it is *unstable*. A configuration is stable if all its non-sink vertices are stable, and we denote by $\mathrm{Stable}_{v_0}(G)$ the set of all stable configurations on (G, v_0).

Unstable vertices may topple. We define the toppling operator T_i corresponding to the toppling of an unstable vertex $v_i \in V$ in a configuration $c \in \mathrm{Config}_{v_0}(G)$ by $T_i(c) := c - d_i\alpha_i + \sum_{j:\{v_i,v_j\}\in E} \alpha_j$, where the sum is over all vertices adjacent to v_i, counted with multiplicity.

Performing this toppling may cause other vertices to become unstable, and we may topple these also. One can show that starting from some unstable configuration c and toppling successively unstable vertices, we eventually reach a stable configuration c' (think of the sink as absorbing grains). Moreover, this configuration c' does not depend on the sequence in which vertices are toppled. We write $c' = \sigma(c)$ and call it the *stabilisation* of c.

Definition 1.1 *A configuration $c \in \mathrm{Config}_{v_0}(G)$ is* recurrent *if there exists some configuration $a \in \mathrm{Config}_{v_0}(G)$, satisfying $a_i \geq d_i$ for all $i \in \{1, \dots, n\}$, such that $c = \sigma(a)$. Given a graph G, we let $\mathrm{Rec}_{v_0}(G)$ be the set of recurrent states on the graph (G, v_0).*

Given a recurrent configuration $c \in \mathrm{Rec}_{v_0}(G)$, define the *level* of c to be $\mathrm{level}_{v_0}(c) := d_{v_0} - |E| + \sum_{v \in V_{[1,n]}} c_v$, where $|E|$ denotes the number of elements in the set E. From [8, Thm. 3.5] we have that if $G = (V, E)$ is a graph and $c \in \mathrm{Rec}_{v_0}(G)$, then $0 \leq \mathrm{level}_{v_0}(c) \leq |E| - |V|$. Thus the level of a recurrent configuration is always a non-negative integer. We define the *level polynomial* of a graph G to be the generating function of the level statistic over the set of recurrent configurations on that graph: $\mathrm{Level}_{G,v_0}(x) := \sum_{c \in \mathrm{Rec}_{v_0}(G)} x^{\mathrm{level}_{v_0}(c)}$.

2 Recurrent states under edge duplication

Let $G = (V, E)$ be a graph. For a positive integer k we define $G^{(k)}$ to be the multigraph G where every edge of E is replaced with k copies of itself. That is $G^{(k)} = \left(V, E^{(k)}\right)$, where $E^{(k)}$ is the multiset $\bigcup_{e \in E} \{e_1, \ldots, e_k\}$, with $e_j = e$ for all $j \in \{1, \ldots, k\}$.

Theorem 2.1 *Let $c = (c_1, \ldots, c_n) \in \mathrm{Config}_{v_0}(G)$. Then the following are equivalent:*

(a) $c \in \mathrm{Rec}_{v_0}\left(G^{(k)}\right)$.

(b) $\tilde{c} := (\lfloor c_1/k \rfloor, \ldots, \lfloor c_n/k \rfloor) \in \mathrm{Rec}_{v_0}(G)$.

Theorem 2.1 can be re-stated as follows: $c = (c_1, \ldots, c_n) \in \mathrm{Rec}_{v_0}\left(G^{(k)}\right)$ iff there exists a unique pair (γ, α) with $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathrm{Rec}_{v_0}(G)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, \ldots, k-1\}^n$ such that $c_i = k\gamma_i + \alpha_i$ for all $1 \leq i \leq n$. A straightforward computation then yields the following.

Corollary 2.2 *Let k be a positive integer. We have:*

$$\mathrm{Level}_{G^{(k)},v_0}(x) = \left(1 + x + \ldots + x^{k-1}\right)^{|V|-1} \cdot \mathrm{Level}_{G,v_0}\left(x^k\right).$$

In particular, $\left|\mathrm{Rec}_{v_0}\left(G^{(k)}\right)\right| = k^{(|V|-1)} |\mathrm{Rec}_{v_0}(G)|$.

3 Cut vertices in graphs with an underlying tree-like structure

A cut vertex in a graph G is a vertex whose removal increases the number of its connected components. Consider a tree (T, ρ), rooted at some vertex ρ. For clarity, we will refer to vertices of the tree as *nodes*, to distinguish them from vertices of the graph G. Let $\mathsf{nodes}(T)$ denote the node set of T. Since T

is rooted, every non-root node $t \in \mathsf{nodes}\,(T) \setminus \{\rho\}$ will have an adjacent node on the (unique) path from t to the root ρ called the *parent* of t and denoted by $\mathsf{parent}(t)$. Any other nodes adjacent to t is called a *descendant* of t. If a node has no descendants then it is called a *leaf*.

Definition 3.1 *We will say that the graph-sink pair (G, v_ρ) has an underlying* cv-tree structure *(T, ρ) if the following holds. The graph G can be written as*

$$G = \bigcup_{t \in \mathsf{nodes}(T)} H_t,$$

where the pair (T, ρ) is a rooted tree and associated to every node $t \in \mathsf{nodes}\,(T)$ is a connected, loop-free, graph $H_t = (V_t, E_t)$. Each graph H_t has a distinguished vertex $v_t \in V_t$ such that:

(a) for all $t \in \mathsf{nodes}\,(T) \setminus \{\rho\}$, we have $V_t \cap V_{\mathsf{parent}(t)} = \{v_t\}$;

(b) if (t, t') is not an edge of T, then $V_t \cap V_{t'} = \emptyset$.

See Figure 1 for an illustration of the construction in Definition 3.1. Note that for all $t \in \mathsf{nodes}\,(T) \setminus \{\rho\}$, v_t is a cut vertex for the graph G. In particular, the so-called *block decomposition*, where the blocks are the maximal two-connected components of a separable graph G, gives a cv-tree structure for G.

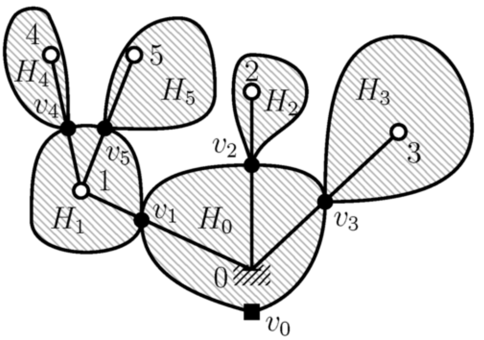

Fig. 1. A graph with its underlying tree structure.

Theorem 3.2 *Let $G = \bigcup_{t \in \mathsf{nodes}(T)} H_t$ have an underlying cv-tree structure (T, ρ). Let $c \in \mathrm{Config}_{v_\rho}(G)$. For any node $t \in \mathsf{nodes}\,(T)$, we let $t_1, \ldots, t_k$ be the descendants of t in T. Define the configurations $c^{(t)}$ on $\mathrm{Config}_{v_t}(H_t)$ by:*

$$c^{(t)}(v) := \begin{cases} c(v) & \text{if } v \notin \{v_{t_1}, \ldots, v_{t_k}\} \\ c(v) - d^{H_{t_i}}(v) & \text{if } v = v_{t_i} \end{cases}$$

Then $c \in \mathrm{Rec}_{v_\rho}(G)$ *iff* $c^{(t)} \in \mathrm{Rec}_{v_t}(H_t)$ *for all* $t \in \mathsf{nodes}(T)$*.*

The proof is by induction on the number of nodes in the cv-tree $|\mathsf{nodes}(T)|$, which essentially reduces the proof to the case where G has a single cut vertex. This characterisation allows us to compute the following expression for the level polynomial of a graph with a cv-tree structure.

Corollary 3.3 *Let* $G = \bigcup_{t \in \mathsf{nodes}(T)} H_t$ *be constructed on a rooted tree* (T, ρ)*. Then we have*

$$\mathrm{Level}_{G,v_\rho}(x) = \prod_{t \in \mathsf{nodes}(T)} \mathrm{Level}_{H_t,v_t}(x). \tag{1}$$

Remark 3.4 *The level polynomial is a specification of the well-studied Tutte polynomial [8]. Since the Tutte polynomial satisfies an identical decomposition to Equation* (1) *[9, Prop. (iv)], Corollary 3.3 follows, although our Theorem 3.2 provides a new combinatorial explanation of this equality.*

4 Decomposition of the sandpile group

Let (G, v_0) be a graph, with vertex set $V = \{v_0, \ldots, v_n\}$ and edge set E. Recall that $\sigma : \mathrm{Config}_{v_0}(G) \to \mathrm{Stable}_{v_0}(G)$ denotes the stabilisation operator. We define a binary operation $\oplus$ on $\mathrm{Stable}_{v_0}(G)$ by:

$$\forall c, c' \in \mathrm{Stable}_{v_0}(G),\ c \oplus c' := \sigma(c + c'), \tag{2}$$

where $+$ denotes pointwise addition in $\mathrm{Stable}_{v_0}(G)$. Dhar showed in [5] that $(\mathrm{Rec}_{v_0}(G), \oplus)$ is an abelian group, called the *sandpile group*, and we denote it by $S(G, v_0)$.

Theorem 4.1 *Let* $G = \bigcup_{t \in \mathsf{nodes}(T)} H_t$ *have an underlying cv-tree structure* (T, ρ)*. We have:*

$$S(G, v_\rho) \cong \prod_{t \in \mathsf{nodes}(T)} S(H_t, v_t),$$

where $\cong$ *denotes group isomorphism, and* $\prod$ *the direct product of the groups.*

Proof sketch. As with Theorem 3.2, an induction argument reduces the proof to the case where $|\mathsf{nodes}(T)| = 2$. Denote by H_1, H_2 the two components of the graph G in this case. Given $c = (c_1, \ldots, c_{n+m}) \in \mathrm{Config}_{v_0}(G)$, let $c^{(1)} := (c_1, \ldots, c_{n-1}, c_n - d_n^{H_2})$ and $c^{(2)} := (c_{n+1}, \ldots, c_{n+m})$. The main step of the proof is to understand the behaviour of the operator $\oplus$ restricted to the components H_1, H_2 of the graph G. This yields Lemmas 4.2 and 4.3. Theorem 3.2 then follows from several group theory results [3, Chapter IV].

Lemma 4.2 *For any configurations $c, c' \in \mathrm{Rec}_{v_0}(G)$, we have*

$$(c \oplus c')^{(2)} = c^{(2)} \oplus c'^{(2)}. \tag{3}$$

Lemma 4.3 *For any configurations $c, c' \in \mathrm{Rec}_{v_0}(G)$, we have*

$$(c \oplus c')^{(1)} = c^{(1)} \oplus c'^{(1)} \oplus \kappa\alpha_n, \tag{4}$$

where α_n is the configuration with one grain at the vertex v_n and none elsewhere, and $\kappa = \kappa\left(c^{(2)}, c'^{(2)}\right) := d_n^{H_2} + |c^{(2)}| + |c'^{(2)}| - |c^{(2)} \oplus c'^{(2)}|$, with $|c|$ designating the total number of grains of a configuration c. □

References

[1] J.-C. Aval, M. D'Adderio, M. Dukes, A. Hicks, and Y. L. Borgne. Statistics on parallelogram polyominoes and a q,t-analogue of the Narayana numbers. *J. Comb. Theory Ser. A*, 123(1):271 – 286, 2014.

[2] J.-C. Aval, M. D'Adderio, M. Dukes, and Y. L. Borgne. Two operators on sandpile configurations, the sandpile model on the complete bipartite graph, and a cyclic lemma. *Adv. Appl. Math.*, 73:59 – 98, 2016.

[3] K. Brown. Cohomology of groups, *Graduate Texts in Mathematics*, Vol. 87, Springer-Verlag, New York-Berlin, 1982.

[4] R. Cori and D. Rossin. On the sandpile group of a graph. *Eur. J. Combin.*, 21:447–459, 2000.

[5] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64:1613–1616, Apr 1990.

[6] M. Dukes and Y. Le Borgne. Parallelogram polyominoes, the sandpile model on a complete bipartite graph, and a q,t-Narayana polynomial. *J. Comb. Theory Ser. A*, 120(4):816 – 842, 2013.

[7] M. Dukes and T. Selig. Decomposing recurrent states of the abelian sandpile model. *Preprint*, 2016.

[8] C. Merino. Chip firing and the Tutte polynomial. *Ann. Comb.* 1(1):253–259, 1997. Elsevier, 2006.

[9] W.T. Tutte. A contribution to the theory of chromatic polynomials. *Can. J. Math.* 6:80–91, 1954.