The word problem for Pride groups

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Abstract

Pride groups are defined by means of finite (simplicial) graphs and examples include Artin groups, Coxeter groups and generalized tetrahedron groups. Under suitable conditions we calculate an upper bound of the first order Dehn function for a finitely presented Pride group. We thus obtain sufficient conditions for when finitely presented Pride groups have solvable word problems. As a corollary to our main result we show that the first order Dehn function of a generalized tetrahedron group, containing finite generalized triangle groups, is at most cubic.

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1 Introduction

An *Artin group* has a presentation of the form

$$\langle a_1, a_2, \ldots, a_n\,;\underbrace{a_i a_j a_i \ldots}_{\mu_{ij}} = \underbrace{a_j a_i a_j \ldots}_{\mu_{ji}} \text{ for all } i \neq j\rangle,$$

where $\mu_{ij} = \mu_{ji}$ is an integer greater than or equal to 2 or $\mu_{ij} = \infty$, in which case the relation involving a_i and a_j is omitted. Examples of Artin groups include free groups, free abelian groups and braid groups. Indeed Artin groups are sometimes referred to as generalized braid groups, the latter introduced explicitly by Emil Artin in 1925. It is unknown in general whether or not Artin groups have solvable word problems, however some partial results do exist. If we add the defining relation $a_i = a_i^{-1}$ for each $i = 1, \ldots, n$ then we obtain the corresponding *Coxeter group*

$$\langle a_1, a_2, \ldots, a_n \,; a_i^2 = 1, (a_i a_j)^{\mu_{ij}} = 1 \text{ for all } i \neq j \rangle$$

and we say that an Artin group is *spherical* (or of *finite type*) if its corresponding Coxeter group is finite. The word problem is solvable for spherical Artin groups (see [7] and [4]). The word problem is also solvable for *right-angled Artin groups*, where $\mu_{ij} = 2$ for all $i \neq j$. If $\mu_{ij} \geq 3$ for all $i \neq j$ then the Artin group is said to be of *large type*. Following joint work with Schupp, Appel [2] showed that such groups have solvable word problems. Finally, Altobelli [1] proved that Artin groups of FC type have solvable word problem. Artin groups of FC type can be characterized as the smallest class of Artin groups which is closed under free products amalgamated over special subgroups and which contain spherical Artin groups.

Artin groups, and their corresponding Coxeter groups, are but two examples of a much wider class of groups. Notice that in each of their presentations the defining relations involve at most two generators. Finitely presented groups that have such presentations were first studied by Pride in [12] and later in [15] under the title "groups given by presentations in which each defining relator involves at most two types of generators." It is common to refer to such groups as simply *Pride groups*.

Let $\Gamma = \{V, E\}$ be a finite simplicial graph with vertex set V and edge set E. To each vertex $v \in V$ assign a group G_v with a fixed finite presentation. Let $e = \{u, v\} \in E$ and let $\widetilde{G_e}$ denote the free product $G_u * G_v$.

To each edge assign a set $\mathbf{t}_e$ that consists of cyclically reduced elements of $\widetilde{G_e}$, where each element of $\mathbf{t}_e$ involves at least one term from each of G_u and G_v. Associated to this edge is a group $G_e = \widetilde{G_e}/ << \mathbf{t}_e >>$. The *Pride group* associated with the above data is then

$$G = \mathop{\ast}_{v \in V} G_v / << \bigcup_{e \in E} \mathbf{t}_e >> .$$

We call Γ the *underlying graph* of G. The groups G_v $(v \in V)$ are called the *vertex groups* of G and the groups G_e $(e \in E)$ are called the *edge groups* of G. Thus Artin groups are Pride groups with infinite cyclic vertex groups and one-relator edge groups given by the presentation

$$\langle a_i, a_j\, ; \underbrace{a_i a_j a_i \ldots}_{\mu_{ij}} = \underbrace{a_j a_i a_j \ldots}_{\mu_{ji}} \rangle.$$

Coxeter groups have $\mathbb{Z}_2$ vertex groups and dihedral edge groups. Further examples include generalized tetrahedron groups (see Section 5) and groups given by cyclic presentations.

For each $v \in V$ and each $e \in E$ there are natural homomorphisms $G_v \to G$ and $G_e \to G$. In general, these homomorphisms are not injective; however, it was shown in [5] that if G satisfies the *asphericity condition* (see end of this paragraph) then the vertex and edge groups embed in G. For each $e = \{u, v\} \in E$ let ψ_e be the natural epimorphism of $\widetilde{G_e}$ onto G_e and define m_e to be the (free product) length of a shortest non-identity element of $\ker \psi_e$, when this is non-trivial. If $\ker \psi_e$ is trivial then $m_e := \infty$. An edge group G_e (or more precisely a given presentation of G_e) has *property-W_k* if and only if $m_e > 2k$. So, for example, property-W_1 states that no non-empty word of the form $W(u)W(v)$ where $W(u) \in G_u$, $W(v) \in G_v$ is equal to 1 in G_e. Then G satisfies the asphericity condition if and only if G_e has property-W_1 for each $e \in E$, and for any triangle in Γ (with edges

e_1, e_2, e_3, say)

$$\frac{1}{m_{e_1}} + \frac{1}{m_{e_2}} + \frac{1}{m_{e_3}} \leq \frac{1}{2}.$$

Pride groups which satisfy the asphericity condition are said to be *non-spherical.*

Example ([13, Example 2] [8, Example 4.1]). *Each edge group of an Artin group satisfies property-*$W_{\mu_{ij}-1}$ *and each edge group of a Coxeter group always satisfies property-*W_1 *and satisfies property-*W_2 *if* $\mu_{ij} > 2$.

When the natural homomorphisms $G_v \to G$ $(v \in V)$ and $G_e \to G$ $(e \in E)$ are injective the aim is then to describe the structure of G in terms of its vertex and edge groups. We conjecture that a non-spherical Pride group has solvable word problem if each of its edge groups do. With this in mind we prove the following result.

Theorem A. *Let G be a Pride group for which one of the following two conditions hold.*

1. *Each edge group has property-*W_2.
2. *Each edge group has property-*W_1 *and the underlying graph of G is triangle-free.*

Then, if each vertex group is finite, G has a solvable word problem.

We will compute an upper bound for the *first order Dehn function* of G and show that it is given in terms of the maximum of the first order Dehn functions of its edge groups. To do this we will estimate the areas of *simply-connected pictures* over a particular presentation of a Pride group. In particular, we will use the following pictorial version of *van Kampen's Lemma*: Let $\mathcal{P} = \langle \mathbf{x} \,;\mathbf{r}\rangle$ be a finite presentation defining a group G and let

W be a word on $\mathbf{x}^{\pm 1}$. Then W represents the identity element of G if and only if there exists a simply-connected picture for W over $\mathcal{P}$.

2 Pictures

A *closed punctured disc* Π with $n \geq 0$ holes is the closure of

$$D - \bigcup_{i=1}^{n} \operatorname{int}(B_i),$$

where D is a closed disc and $B_1, \ldots, B_n$ are disjoint closed discs in the interior of D. The *boundary* $\partial\Pi$ of Π is

$$\partial D \cup \bigcup_{i=1}^{n} \partial B_i,$$

where ∂D and ∂B_i are the boundaries of D and B_i ($i = 1, \ldots, n$), respectively.

Definition 1. A *picture* $\mathbb{P}$ is a geometric configuration consisting of a finite collection of pairwise disjoint closed discs $\Delta_1, \ldots, \Delta_m$ in a closed punctured disc Π with $n \geq 0$ holes, together with a finite collection of pairwise disjoint compact one-manifolds $\alpha_1, \ldots, \alpha_k$ (the *arcs* of $\mathbb{P}$) properly embedded in

$$\Pi - \bigcup_{i=1}^{m} \operatorname{int}(\Delta_i).$$

The punctured disc Π has a basepoint 0 on $\partial\Pi$, each disc B_i has a basepoint b_i on ∂B_i, and each disc Δ_i has a basepoint 0_i on $\partial\Delta_i$. Each arc is either a simple closed curve having trivial intersection with $\partial\Pi \bigcup_{i=1}^{m} \partial\Delta_i$ or is a simple curve which joins two points of $\partial\Pi \bigcup_{i=1}^{m} \partial\Delta_i$, neither point being a basepoint.

If Π does not contain any holes then we say $\mathbb{P}$ is *simply-connected.* Otherwise $\mathbb{P}$ is *non-simply-connected.* Our definition of a simply-connected picture coincides with the standard definition of a picture (e.g. in [14]) and we shall use the terms "simply-connected picture" and "picture" to mean the same. A picture $\mathbb{P}$ is *spherical* if no arc of $\mathbb{P}$ meets the boundary of $\mathbb{P}$, where the boundary of $\mathbb{P}$ is $\partial\mathbb{P} := \partial\Pi$.

We shall assume the reader is familiar with the theory of pictures over finite presentations (see [14]) but note that a picture $\mathbb{P}$ over a finite presentation $\mathcal{P} = \langle \mathbf{x}\,, \mathbf{r}\rangle$ satisfies the following conditions.

(a) Each arc of $\mathbb{P}$ is labelled by an element of $\mathbf{x}$ and has a normal orientation indicated by a short arrow meeting the arc transversely.

(b) If we travel around Δ_i once in an anticlockwise direction starting at 0_i and read off labels on the arcs encountered (with the understanding that we read x if we cross an arc labelled x in the direction of its normal orientation, and we read x^{-1} otherwise) then we obtain a word $r_i^{\varepsilon_i}$, where $r_i \in \mathbf{r}$ and $\varepsilon_i = \pm 1$.

Let W be a word on $\mathbf{x}^{\pm 1}$. We say that $\mathbb{P}$ is a *simply-connected* $\mathbf{r}$-*picture for* W (or simply a *picture for* W) if the boundary label of $\mathbb{P}$, i.e. the word obtained by reading the labels of arcs encountered in an anticlockwise transversal of the boundary beginning and ending at 0, is a cyclic permutation of W. The *area* $A(W)$ of W is then defined to be the minimum of the areas of all pictures for W. Recall that the area $A(\mathbb{P})$ of $\mathbb{P}$ is the number of discs contained in $\mathbb{P}$.

Definition 2. The *first order Dehn function of* $\mathcal{P}$ is the increasing function $\delta_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ given by $\delta_{\mathcal{P}} = \max\{A(W) : l(W) \leq n \text{ and } \overline{W} = 1 \text{ in } G\}$, where $\overline{W}$ denotes the group element defined by W and $l(W)$ denotes the length of W.

If $\mathcal{P}$ is finite then $\delta_{\mathcal{P}}$ is a group invariant up to $\sim$-equivalence and we write δ_G for the *first order Dehn function of* $G = (G(\mathcal{P}))$. The $\sim$-equivalence is defined as follows. For two increasing functions $f, g : \mathbb{N} \to \mathbb{R}^+$ write $f \preceq g$ if there exist constants $c_1, c_2, c_3 > 0$ such that

$$f(n) \leq c_1 g(c_2 n) + c_3.$$

Then $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$. It is well known that a finitely presented group has a solvable word problem if and only if its Dehn function is recursive (see e.g. [9]).

Let $\mathbb{P}$ be a picture. The degree of a disc Δ is defined to be the number of arcs incident with Δ, counted with mutliplicity, and we denote it by $d(\Delta)$. The degree of an interior region R of $\mathbb{P}$ is defined to be the number of arcs contained in the boundary of R counted with multiplicity. We denote the degree of R by $d(R)$.

Lemma 1. *Let $\mathbb{P}$ be a picture in which $d(\Delta) \geq 6$ for all discs Δ of $\mathbb{P}$ and $d(R) \geq 3$ for all interior regions R of $\mathbb{P}$. If n arcs are incident with $\partial\mathbb{P}$, then $d(\Delta)$ is at most $n + 3$ for each $\Delta \in \mathbb{P}$.*

Proof. We proceed by induction on $A(\mathbb{P})$. Let C be a connected component of $\mathbb{P}$. If $A(C) = 1$ then delete C and its arcs incident with $\partial\mathbb{P}$ to obtain a picture $\mathbb{P}'$. Now $A(\mathbb{P}') < A(\mathbb{P})$ and there at most $n - 1$ arcs incident with $\partial\mathbb{P}'$. It follows that $d(\Delta') < n + 3$ for each disc $\Delta' \in \mathbb{P}'$. Furthermore $d(C) < n$.

Now suppose that each connected component contains at least two discs. Let D be the dual diagram of $\mathbb{P}$ so that the discs of $\mathbb{P}$ become regions of D and interior regions of $\mathbb{P}$ become the vertices of D. By our hypotheses D satisfies the conditions of [11, Theorem V 4.3] and so must satisfy

$$\sum_{D}^{*}(4 - i(R)) \geq 6,$$

where the sum is taken over boundary regions R of D such that $\partial D \cup \partial R$ is a consecutive part of ∂D. Thus D must contain at least two regions that are both boundary regions and each contain at most 3 interior edges, and whose boundaries form a consecutive part of ∂D. Translating this back into the language of pictures we deduce that $\mathbb{P}$ contains a connected component C that must contain at least two discs which satisfy the following:

- both discs are boundary discs of C;
- each disc has at most three interior arcs;
- the boundary arcs of each disc form a consecutive part of ∂C.

Since C can contain at most one disc whose boundary is a consecutive part of ∂C but not of $\partial \mathbb{P}$, it follows that $\mathbb{P}$ must contain at least one disc Δ that satisfies the above conditions. Delete Δ and its boundary arcs from $\mathbb{P}$ to obtain a picture $\mathbb{P}'$ with $A(\mathbb{P}') < A(\mathbb{P})$. Since Δ contained at most three interior arcs and since $d(\Delta') \geq 6$ for each disc Δ' of $\mathbb{P}$, we deduce that $\partial \mathbb{P}'$ contains at most n arcs. The induction hypothesis applies to $\mathbb{P}'$ so $d(\Delta') \leq n+3$ for each disc $\Delta' \in \mathbb{P}'$. Moreover, $d(\Delta) \leq n+3$. The result now follows. $\square$

A very similar argument proves the following lemma.

Lemma 2. *Let $\mathbb{P}$ be a picture in which $d(\Delta) \geq 4$ for all discs Δ of $\mathbb{P}$ and $d(R) \geq 4$ for all interior regions R of $\mathbb{P}$. If n arcs are incident with $\partial \mathbb{P}$, then $d(\Delta)$ is at most $n+2$ for each $\Delta \in \mathbb{P}$.*

Let I be a fixed set whose elements shall be referred to as *colours* and let $\{m_{ij} : i, j \in I\}$ be a fixed family of elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{ij} = m_{ji}$ and $m_{ij} \geq 4$ for $i \neq j$. Following [5], we say that a triple of distinct colours $i, j, k \in I$ is a *spherical triple* if

$$\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ki}} > \frac{1}{2},$$

where $1/\infty := 0$, and a *colouring of a picture* $\mathbb{P}$ *by* I is an I-valued function on the set of arcs of $\mathbb{P}$. A picture together with a colouring function into I is called an *I-coloured picture.*

Lemma 3 ([5, Lemma 2.2])**.** *Suppose I does not contain any spherical triples and let $\mathbb{P}$ be a non-spherical simply-connected I-coloured picture satisfying:*

(i) *no arc is a floating circle nor has both endpoints on the same disc enclosing a region of degree* 1*;*

(ii) *associated to each disc Δ are two distinct colours $i, j \in I$ (with $m_{ij} \neq \infty$) such that each arc incident with Δ is coloured either i or j and there are at least m_{ij} corners of Δ joining one arc of each colour;*

(iii) *no interior region has more than one corner in its boundary joining arcs of the same two distinct colours.*

If under the above conditions some arc of $\mathbb{P}$ is coloured k, then some arc meeting $\partial\mathbb{P}$ is coloured k.

3 Pictures over presentations of Pride groups

Let G be a Pride group in which each vertex group is finite. We will now fix a presentations for G that will be used for the remainder of this paper. For each $v \in V$ let $\mathbf{x}_v$ be a set that is in one-to-one correspondence with the elements of G_v and define

$$\mathbf{r}_v = \{x_1 x_2 x_3^{-1}\,; x_1, x_2, x_3 \in \mathbf{x}_v\},$$

where each x_i $(i = 1, 2, 3)$ corresponds to a group element $g_i \in G_v$ such that $g_3 = g_1 g_2$. Then $\mathcal{P}_v = \langle \mathbf{x}_v \,; \mathbf{r}_v \rangle$ is a finite presentation for G_v. Note that $\mathbf{r}_v$ is simply the multiplication table of G_v. For each $e = \{u, v\} \in E$ let $\mathbf{x}_e = \mathbf{x}_u \cup \mathbf{x}_v$ and let

$$\mathbf{r}_e = \mathbf{r}_u \cup \mathbf{r}_v \cup \mathbf{s}_e,$$

where $\mathbf{s}_e$ is a set of cyclically reduced words on $\mathbf{x}_e^{\pm 1}$ that represent the elements of $\mathbf{t}_e$. Then $\mathcal{P}_e = \langle \mathbf{x}_e \,; \mathbf{r}_e \rangle$ is a presentation for G_e and

$$\mathcal{P} = \langle \mathbf{x} \,; \mathbf{r} \rangle$$

is a finite presentation for G, where

$$\mathbf{x} = \bigcup_{v \in V} \mathbf{x}_v \quad \text{and} \quad \mathbf{r} = \bigcup_{e \in E} \mathbf{r}_e.$$

Let $\mathbb{P}$ be a picture over $\mathcal{P}$. There exists an obvious colouring of $\mathbb{P}$ by V: arcs labelled by an element of $\mathbf{x}_v$ $(v \in V)$ are coloured v. For each pair of vertices $u, v \in V$ define

$$m_{uv} = \begin{cases} m_e & \text{if } \{u, v\} = e \in E; \\ \infty & \text{if } \{u, v\} \notin E. \end{cases}$$

Clearly $m_{uv} = m_{vu}$ and if G is non-spherical, then each G_e has property-W_1 and so $m_e > 2$. It follows that $m_{uv} \geq 4$ for $u \neq v$ since m_e is even for each $e \in E$. Thus we may view $\mathbb{P}$ as a V-coloured picture and it is easy to see that, when viewed as a set of colours, V cannot contain a spherical triple.

A (u, v)-*subpicture* of $\mathbb{P}$ is a subpicture in which each arc has colour u or colour v $(u, v \in V)$ only.

Definition 3. A *federation* $\mathbb{F}$ is a (u, v)-subpicture that satisfies the following conditions: $\{u, v\} = e$ for some edge $e \in E$; $\mathbb{F}$ contains at least one disc whose label is an element of $\mathbf{s}_e$ (i.e. $\mathbb{F}$ does not consist solely of discs whose labels are from $\mathbf{r}_u \cup \mathbf{r}_v$); $\mathbb{F}$ is *maximal* in the sense that $\partial\mathbb{F}$ cannot be extended to include any other disc of $\mathbb{P}$ whose label is an element of $\mathbf{s}_e$.

Define $\Sigma(\mathbb{F}) = e$ if $\mathbb{F}$ contains an $\mathbf{s}_e$-disc. A federation is *simply-connected* if it is a simply-connected (u, v)-subpicture, otherwise it is *non-simply-connected.* If $\mathbb{F}$ is simply-connected and $\Sigma(\mathbb{F}) = e$, then the boundary label $W(\mathbb{F})$ of $\mathbb{F}$ is a word on $\mathbf{x}_e^{\pm 1}$ that represents the identity element of G_e. Equivalently, $W(\mathbb{F})$ represents an element of $\ker \psi_e$.

Let $\mathbb{F}_1$ be a federation of $\mathbb{P}$. If $\mathbb{F}_1 \neq \mathbb{P}$, then construct a federation $\mathbb{F}_2$ of $\mathbb{P}\backslash\mathbb{F}_1$. If $\mathbb{F}_2 \neq \mathbb{P}\backslash\mathbb{F}_1$, then construct a federation $\mathbb{F}_3$ of $\mathbb{P}_1\backslash(\mathbb{F}_1 \cup \mathbb{F}_2)$, and so on. Eventually we end up with a collection of federations $\mathbb{F}_1, \ldots, \mathbb{F}_n$ that cover $\mathbb{P}$ and satisfy the property that $\mathbb{F}_{i+1}$ is a federation of

$$\mathbb{P} \setminus \left(\bigcup_{j=1}^{i} \mathbb{F}_j \right) \quad (i = 1, \ldots, n-1).$$

As in [15] we call this collection of federations a *federal subdivision* of $\mathbb{P}$ and denote it by $\mathcal{F}$.

For each $e \in E$ let Ω_e denote the set of all words on $\mathbf{x}_e^{\pm 1}$ that represent a *non-identity* element of $\ker \psi_e$. Denote the union of $\{\Omega_e : e \in E\}$ by Ω. Let $\mathbb{P}$ be a connected picture and suppose $\mathcal{F}$ is a federal subdivision of $\mathbb{P}$ that satisfies the following two conditions:

(I) each $\mathbb{F} \in \mathcal{F}$ is simply-connected;

(II) $W(\mathbb{F}) \in \Omega$ for each $\mathbb{F} \in \mathcal{F}$.

The *derived picture* $\widehat{\mathbb{P}}$ *of* $\mathbb{P}$ *corresponding to* $\mathcal{F}$ (or simply the *derived picture*) is obtained from $\mathbb{P}$ by deleting the arcs and discs contained in each federation $\mathbb{F} \in \mathcal{F}$. If α is a boundary arc of $\mathbb{F}$, then we delete only that portion of α contained in $\mathbb{F}$. The boundary of $\mathbb{F}$ is then identified as the boundary of a disc of $\widehat{\mathbb{P}}$. It is clear that $\widehat{\mathbb{P}}$ is a V-coloured picture and that $W(\widehat{\mathbb{P}})$ is identically equal to $W(\mathbb{P})$, where $W(\mathbb{P})$ is the boundary label of $\mathbb{P}$. Furthermore, the label of a disc Δ obtained from a federation $\mathbb{F}$ is identical to $W(\mathbb{F})$ and so is an element of Ω_e.

The following result appears in [6, Lemma 3.3]. The presentation in the present paper differs from that in [6], however this does not affect the proof of the lemma.

Lemma 4. *Let $\mathbb{P}$ be a connected, non-spherical, simply-connected picture over $\mathcal{P}$ and let $\widehat{\mathbb{P}}$ be the derived picture corresponding to a federal subdivision of $\mathbb{P}$. Then $\widehat{\mathbb{P}}$ satisfies Conditions* (i), (ii) *and* (iii) *of Lemma 3.*

Let

$$\mathbf{s} = \bigcup_{e \in E} \mathbf{s}_e$$

and let $\mathbb{P}$ be a picture for a word W that contains k $\mathbf{s}$-discs, i.e. each of the k discs is labelled by an element of $\mathbf{s}$. We shall say that $\mathbb{P}$ is *minimal with respect to* $\mathbf{s}$ if, for any picture $\mathbb{P}'$ for W that contains l $\mathbf{s}$-discs, then $k \leq l$.

Lemma 5. *Let W be a word on $\mathbf{x}_e^{\pm 1}$ that represents a non-identity element of $\ker \psi_e$ and let $\mathbb{P}$ be a picture for W that is minimal with respect to $\mathbf{s}$. Then $\mathbb{P}$ is a picture over $\mathcal{P}_e$.*

Proof. Suppose $\mathbb{P}$ is not a picture over $\mathcal{P}_e$ and let $\mathcal{F}$ be a federal subdivision of $\mathbb{P}$. If each $\mathbb{F} \in \mathcal{F}$ is simply-connected then the boundary label of $\mathbb{F}$ cannot be freely equal to the empty word nor represent the trivial element of $\widetilde{G}_e$ for some $e \in E$. Otherwise, we would replace $\mathbb{F}$ with either the empty picture, by performing bridge moves on $\partial\mathbb{F}$, or a picture over $\widetilde{\mathcal{P}}_e$ to obtain a picture $\mathbb{P}'$ for W that contains strictly less $\mathbf{s}$-discs than $\mathbb{P}$, thus contradicting the minimality of $\mathbb{P}$. It follows that the boundary label of each federation must be an element of Ω_e. Thus we may construct the derived picture $\widehat{\mathbb{P}}$, which, by Lemma 4, satisfies the conditions of Lemma 3. Since $\mathbb{P}$ is not a picture over $\mathcal{P}_e$, $\widehat{\mathbb{P}}$ must contain an arc coloured k such that this colour does not appear on the boundary of $\widehat{\mathbb{P}}$. However, this contradicts the conclusion of Lemma 3. We deduce that $\mathbb{P}$ must be a picture over $\mathcal{P}_e$.

Now suppose that $\mathcal{F}$ contains at least one non-simply-connected federation $\mathbb{F}$. We may assume that $\mathbb{F}$ is *innermost*, meaning that no other non-simply-connected federation is bounded by $\mathbb{F}$. The simply-connected picture bounded by $\mathbb{F}$ is minimal with respect to $\mathbf{s}$ (by the minimality of $\mathbb{P}$) and must have two colours appearing on its boundary. We deduce that this simply-connected picture and $\mathbb{F}$ are both pictures over $\mathcal{P}_e$ (for some $e \in E$) contradicting the fact that $\mathbb{F}$ is a maximal subpicture. The result now follows. $\square$

The next result follows easily from the previous lemma.

Lemma 6. *Let W be a word on $\mathbf{x}^{\pm 1}$ that represents the identity element of G and let $\mathbb{P}$ be a simply-connected picture for W that is minimal with respect to $\mathbf{s}$. Then any federal subdivision of $\mathbb{P}$ contains only simply-connected federations.*

Proof. The second part of the proof of Lemma 5 applies directly. $\square$

We end this section with a description of some modifications that can be carried out on a derived picture. These are pictorial versions of diagrammatic modifications that originally appeared in [12]. The first is the standard modification of removing all interior regions of degree 2 (see Figure 1). Both arcs of an interior region of degree 2 must have the same colour, i.e. $x_i, x_j \in \mathbf{x}_v$ for some $v \in V$. Replace these arcs with a single arc labelled $x \in \mathbf{x}_v$ such that

$$\overline{x} = \overline{x_i^{\varepsilon_i} x_j^{\varepsilon_j}},$$

where $\varepsilon_i, \varepsilon_j = \pm 1$ depending on the orientation of the corresponding arcs x_i, x_j.

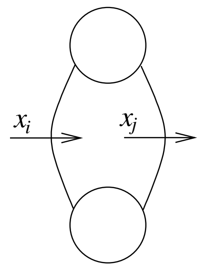

Figure 1: Interior region of degree 2

The second modification removes interior regions of degree 3 whose boundary arcs are each of the same colour.

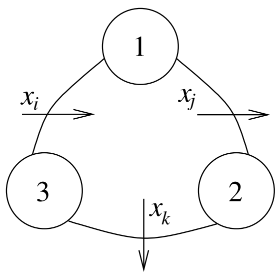

Figure 2: Interior region of degree 3

Adjoin new arcs between discs 1 and 2, and 1 and 3 as in Figure 3. Label these arcs with the label on the arc incident with discs 2 and 3, and assign the same orientation as this arc. Finally delete the arc joining discs 2 and 3. The label of disc 1 has changed, however it remains an element of Ω. Also we have created new interior regions of degree 2 which must be removed.

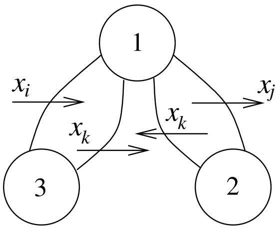

Figure 3: Adjoing new arcs.

A consequence of these modifications is that if W is the label of a transverse path α in $\widehat{\mathbb{P}}$ then $l(W)$ is equal to the number of arcs intersected by α (counted with multiplicity). In particular, *the boundary label of each disc of* $\widehat{\mathbb{P}}$ *has length equal to the degree of the disc.*

Note that if Γ is triangle-free then $\widehat{\mathbb{P}}$ cannot contain any interior regions of degree 3 whose boundary arcs each have distinct colours from one another. In such a region we would have $x_i \in \mathbf{x}_u$, $x_j \in \mathbf{x}_v$, $x_k \in \mathbf{x}_w$ for some distinct $u, v, w \in V$, i.e. $\{x_i, x_j, x_k\}$ would form a triangle in Γ.

4 First order Dehn function

Let G be a Pride group in which each vertex group is finite and define

$$\delta_E = \max\{\delta_{G_e} : e \in E\},$$

where δ_{G_e} is the first order Dehn function of the edge group G_e.

Proposition B. *If each edge group has property-*W_2*, or has property-*W_1 *and the underlying graph of* G *is triangle-free, then*

$$\delta_G(n) \preceq n^2 \delta_E(n)$$

for all $n \in \mathbb{N}$*.*

Proof. Suppose that each G_e has property-W_2. Let W be a non-empty word of length n on $\mathbf{x}^{\pm 1}$ that represents the identity element of G. We may assume, without loss of generality, that W is cyclically reduced. By the van Kampen Lemma, there exists a simply-connected picture $\mathbb{P}$ for W that is minimal with respect to $\mathbf{s}$. Let k denote the number of $\mathbf{s}$-discs of $\mathbb{P}$.

If $k = 0$ or $k = 1$ then $\mathbb{P}$ is either a picture over $\mathcal{P}_v$ or a picture over $\mathcal{P}_e$ for some $v \in V$ or $e \in E$. In both cases $A(W) \leq \delta_E(n)$. Assume that $k > 1$ and let $\mathcal{F}$ be a federal subdivision of $\mathbb{P}$. If $\mathcal{F}$ contains only one federation then $A(W) \leq \delta_E(n)$, so assume that $\mathcal{F}$ contains at least two federations. Each federation of $\mathcal{F}$ must, by Lemma 6, be simply-connected and the boundary label must be an element of Ω. Otherwise, we would argue as in the proof of Lemma 5 to contradict the minimality of $\mathbb{P}$ with respect to $\mathbf{s}$.

Let $\widehat{\mathbb{P}}$ be the derived picture corresponding to $\mathcal{F}$. Since each G_e has property-W_2 each disc of $\widehat{\mathbb{P}}$ must have degree at least 6. Furthermore, we may assume that each interior region has degree at least 3 by removing all interior regions of degree 2 as detailed at the end of Section 3. It follows from [3, Theorem 3.6 p. 182] that there exists a constant $c > 0$ such that

$$A(\widehat{\mathbb{P}}) \leq c \cdot d(\widehat{\mathbb{P}})^2,$$

where $d(\widehat{\mathbb{P}})$ is the degree of $\widehat{\mathbb{P}}$. Since $d(\widehat{\mathbb{P}}) \leq l(W)$ we have

$$A(\widehat{\mathbb{P}}) \leq c \cdot n^2.$$

The degree of each disc Δ of $\widehat{\mathbb{P}}$ is at most $n + 3$ by Lemma 1. It follows that if $U \in (\mathbf{x}_e^{\pm 1})^*$ is the label of Δ, then $|U| \leq n + 3$ by the italicized comment appearing at the end of Section 3. Since U represents a non-identity element of $\ker \psi_e$, there exists a minimal simply-connected

$\mathbf{r}_e$-picture $\mathbb{P}_\Delta$ for U such that

$$A(\mathbb{P}_\Delta) \leq \delta_{G_e}(|U|) = \delta_{G_e}(n+3).$$

We now replace Δ with $\mathbb{P}_\Delta$ as follows. First, surround Δ with a circle S^1 in such a way that S^1 is transverse to the arcs incident with Δ and does not intersect any other arc of $\mathbb{P}$. Next delete Δ and the arcs contained in S^1 which are incident with Δ and in their place add $\mathbb{P}_\Delta$. Finally, join together the boundary arcs of $\mathbb{P}_\Delta$ and the arcs of $\mathbb{P}$ which meet S^1 in such a way that no two arcs cross each other and only arcs of like label and orientation are joined. We proceed to replace the remaining discs of $\widehat{\mathbb{P}}$ in the same way. In doing so we obtain a simply-connected picture $\mathbb{P}'$ for W with

$$A(\mathbb{P}') \leq M \cdot c \cdot n^2,$$

where $M = \max\{A(\mathbb{P}_\Delta)\}$ with the maximum taken over all discs Δ of $\widehat{\mathbb{P}}$. Since $M = \delta_E(n+3)$ we have

$$A(W) \leq A(\mathbb{P}') \leq c \cdot n^2 \delta_E(n+3).$$

If each G_e satisfies property-W_1 and the underlying graph is triangle-free, then we modify $\widehat{\mathbb{P}}$ to remove all interior regions of degree 3 and then use Lemma 2 to show that

$$A(W) \leq c \cdot n^2 \delta_E(n+2).$$

The statement of the result then follows. $\square$

5 Generalized tetrahedron groups

Generalized tetrahedron groups are defined by the presentation

$$\langle x, y, z\,;\, x^l, x^m, x^n, w_1(x,y)^p, w_2(y,z)^q, w_3(z,x)^r \rangle,$$

where each $w_i(a,b)$ $(i = 1,2,3)$ is a cyclically reduced word involving both a and b and all powers are integers greater than 1. We observe that a generalized tetrahedron group T is a Pride group with finite vertex groups (cyclic of orders l, m, n) and generalized triangle edge groups. It was shown in [10, Theorem 3.2] that the generalized triangle group

$$\triangle = \langle x, y\,; x^l, y^m, w(x,y)^P \rangle,$$

where $w \equiv x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_k} y^{\beta_k}$ $(1 \leq \alpha_i < l, 1 \leq \beta_i < M)$, satisfies $m_e = pk$. Thus, considered as an edge group of T, $\triangle$ has property-W_1 if $k \geq 2$ and property-W_1 if $k > 2$. It follows that $\triangle$ embeds in T if $k > 2$ and, in this case, T has a solvable word problem precisely when each of its generalized triangle groups do. In particular, if each $\triangle$ is finite, then $\delta_T(n) \preceq n^3$.

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References

[1] Joseph A. Altobelli, *The word problem for artin groups of FC type*, J. Pure Appl. Algebra **129** (1998), no. 1, 1–22.

[2] Kenneth I. Appel, *On Artin groups and Coxeter groups of large type*, Contributions to group theory, Contemp. Math., vol. 33, Amer. Math. Soc., Providence, RI, 1984, pp. 50–78.

[3] Noel Brady, Tim Riley, and Hamish Short, *The geometry of the word problem for finitely generated groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2007, Papers from the Advanced Course held in Barcelona, July 5–15, 2005.

[4] Ruth Charney, *Artin groups of finite type are biautomatic*, Math. Ann. **292** (1992), no. 4, 671–683.

[5] J. M. Corson, *Amalgamated sums of groups*, Proc. Edinburgh Math. Soc. (2) **39** (1996), no. 3, 561–570.

[6] Peter J. Davidson, *Second order Dehn functions of Pride groups*, J. Group Theory, to appear.

[7] Pierre Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302.

[8] M. Edjvet, *On a certain class of group presentations*, Math. Proc. Cambridge Philos. Soc. **105** (1989), no. 1, 25–35.

[9] Steve M. Gersten, *Isoperimetric and isodiametric functions of finite presentations*, Geometric group theory, Vol. 1 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 181, pp. 79–96.

[10] James Howie and Natalia Kopteva, *The Tits alternative for generalized tetrahedron groups*, J. Group Theory **9** (2006), no. 2, 173–189.

[11] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition.

[12] Stephen J. Pride, *Groups with presentations in which each defining relator involves exactly two generators*, J. London Math. Soc. (2) **36** (1987), no. 2, 245–256.

[13] ———, *The diagrammatic asphericity of groups given by presentations in which each defining relator involves exactly two types of generators*, Arch. Math. (Basel) **50** (1988), no. 6.

[14] ———, *Identities among relations of group presentations*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 687–717.

[15] ———, *The (co)homology of groups given by presentations in which each defining relator involves at most two types of generators*, J. Austral. Math. Soc. Ser. A **52** (1992), no. 2, 205–218.