CHAOS **25**, 083107 (2015)

# Synchronizability of random rectangular graphs

Ernesto Estrada[a)] and Guanrong Chen
*Department of Mathematics & Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XQ, United Kingdom and Department of Electronic Engineering, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*

(Received 12 April 2015; accepted 28 July 2015; published online 11 August 2015)

Random rectangular graphs (RRGs) represent a generalization of the random geometric graphs in which the nodes are embedded into hyperrectangles instead of on hypercubes. The synchronizability of RRG model is studied. Both upper and lower bounds of the eigenratio of the network Laplacian matrix are determined analytically. It is proven that as the rectangular network is more elongated, the network becomes harder to synchronize. The synchronization processing behavior of a RRG network of chaotic Lorenz system nodes is numerically investigated, showing complete consistence with the theoretical results. © *2015 AIP Publishing LLC*.
[http://dx.doi.org/10.1063/1.4928333]

**Many real-world complex spatial networks cannot be accurately described by the traditional Erdös–Rényi random graph model but may be well represented by the random geometric graph (RGG) model, which is formulated over a cubic region in space. Even so, for some complex spatial networks, such as rectangle-shaped urban street maps, infrastructure and transportation systems, and various sensor networks, the cubic-shaped RGG model becomes unreasonable and so has to be replaced by the recently developed random rectangular graph (RRG) model, in which the spatial domain of the networks is a rectangle that generalizes the square domain of the RGG. The changes in the structure of the embedding space dramatically affect the Laplacian spectra of such RGGs, which significantly affect some dynamical behavior of the underlying networks. In particular, it will be shown here that the synchronization process and the synchronizability of the networks are very much affected by these changes. Therefore, relevant important network topological and dynamical properties need to be carefully investigated. Motivated by these facts and observations, the present paper addresses the important issue of network synchronizability for the RRGs, analytically determining both upper and lower bounds of their sync-index, which is a key eigenratio of the network Laplacian matrices. Finally, to visualize and also validate the theoretical results, the synchronization processing behavior of a RRG network of chaotic Lorenz system nodes is numerically investigated, showing complete consistence with the mathematical analysis.**

## I. INTRODUCTION

The study of both topological and dynamical properties of the networked skeletons of many real-world complex systems has motivated and stimulated tremendous interests and efforts in pursuing network science today. This has triggered wide-range applications in almost all scientific and technological fields. A fundamental principle in the study of complex networks is the development and analysis of simplified random models that capture the essence of both structural and dynamical properties of the studied real-world networks. The first of these models was described under a unified framework of random graph theory established by Erdüs and Rényi in the late 1950s.[10,19] It was followed by the more recent developments of the small-world network model introduced by Watts and Strogatz[23] and the scale-free random network model formulated by Barabási and Albert.[3] However, many real-world networked systems are embedded into geometrical spaces, but those simplified random models do not capture the spatial constraints in which the networks grow. These spatially embedded networks may represent many different kinds of scenarios, ranging from urban street networks, infrastructure and transportation systems, to biological networks such as the brain neuronal networks, and vascular and cellular networks, just to mention a few. When modeling such spatial networks, the most commonly used random model is the RGG. In a RGG, each node is randomly assigned geographic coordinates and then two nodes are connected if the Euclidean distance between them is smaller than or equal to a certain radial distance threshold.

The RGGs have found important applications in the study of wireless sensor networks (WSNs), where the nodes represent the sensors that are deployed onto a given geographical region and their communications define the connectivity of the nodes. This is analogous to many other communication systems ranging from mobile phones to radios. The sensors may be deployed on a given geographic area either by using a deterministic deployment in which every sensor is located at a specified position, e.g., a grid deployment, or by using a random deployment in which every sensor is randomly located in the space. Although the deterministic deployment appears to have certain advantages compared to the random one, e.g., it may require fewer sensors to achieve a designated degree of coverage and

[a)]Author to whom correspondence should be addressed. Electronic mail: ernesto.estrada@strath.ac.uk. Tel.: +44 (0)141 548 3657, Fax. +44 (0)141 548 3345.

 

connectivity, in practice the random deployment is more preferable. The preference is mainly based on the fact that sensors are cheap and a large number of them can be easily used and also because it is difficult to locate each sensor precisely at some location considering the real geographic constraints of the region in applications. Due to the increasing number of sensors to be deployed in future applications, the random strategy is gaining more importance. In this respect, as Kenniche and Ravelomananana[13] recently argued, "*the modeling with Random Geometric Graphs is the most appropriate*" for the purposes of random sensor deployment.

However, it has been noticed that a large number of real-world complex networks or their sub-networks possess excellent dynamical properties such as high dynamic synchronizability, good controllability, and fast information transmission capability.[2,15,17,19] Synchronization has been studied for RGGs as a general process of importance in communication networks.[8,9,14] In particular, the communication among the sensors in WSNs requires synchronization between the transmitter and receiver.[24] As just one more example, synchronized brain waves can greatly enhance human learning.[20] Here, one problem of fundamental interest for the analysis of synchronization in a RGG is the influence of the geometric shape of the region, where the nodes are located, on the synchronizability of the resulting network. For the sake of simplicity, we consider here an example based on the problem of wireless sensors deployment. Similar motivational problems are easy to formulate in different contexts and scenarios. In a WSN, a large number of cheap sensors are scattered randomly inside a target area, which may be a city, a forest, or a given geographical region. Such an area to be monitored is assumed to be a square region of side length $a$. This is a reasonable approximation for many geographical regions, such as the San Francisco city, which is known as the "seven-by-seven-mile square," due to the mainland city's squared shape of nearly 11 km by 11 km. However, other regions, like Manhattan that is 21.6 km long and 3.7 km wide, are far from being square-like and they are very much of a rectangular shape. Here, we are interested in investigating, both analytically and computationally, how this elongation of the squared region in which the nodes of a RGG are deployed affects the synchronizability of the network constructed on it.

We start by introducing the concept of RRG, which was recently developed by Estrada and Sheerin,[12] and continue with the description of the synchronization model to be considered. Then, we state the main result of this work which proves that for a RRG with a fixed number of nodes and a given connection radius, it is not possible for the network to achieve synchronization independently of the value of the coupling strength when the rectangle is very elongated. We finally support our analytic results with computational simulations on some representative RRGs.

## II. PRELIMINARIES

Some graph-theoretic concepts and notation are first introduced.[11]

A *graph* $G = (V, E)$ is defined by a set of $n$ nodes (vertices) $V$ and a set of $m$ edges (links) $E = \{(u, v)|u, v \in V\}$ between the nodes. An edge is said to be *incident* to a node $u$ if there exists a node $v \neq u$ such that either $(u, v) \in E$ or $(v, u) \in E$. The degree of a node, designated by $k_i$, is the number of edges incident to node $i$. The graph is said to be *undirected* if the edges are formed by unordered pairs of nodes. A *path* of length $k$ in $G$ is a set of nodes, $i_1, i_2, \ldots, i_k, i_{k+1}$, such that for all $1 \leq l \leq k$, $(i_l, i_{l+1}) \in E$, there are no repeated nodes. The graph is *connected* if there is a path connecting every pair of nodes. The shortest of all paths, each of which connects two nodes in the graph, is the *shortest path distance* between the corresponding nodes. The *diameter*, $diam(G)$, of the graph is the maximum of all shortest path distances between pairs of nodes in the graph. A graph is said to be *simple* if it has unweighted edges with no self-loops (edges from a node to itself) and no multiple edges between any pair of nodes. Throughout this work, we will always consider connected undirected simple graphs, also called networks.

In a simple graph, the *local clustering coefficient*, usually known as the *Watts-Strogatz clustering coefficient*, of a node $i$ is defined as[6,23] $C_i = \frac{2t_i}{k_i(k_i-1)}$, where $t_i$ is the number of triangles involving the node $i$ and $k_i$ is the degree of the node $i$. Taking the mean of these values as $i$ varies among the nodes in $G$, one gets the *average clustering coefficient* of the network: $\langle C \rangle = \frac{1}{n}\sum_{i=1}^{n} C_i$.

The Laplacian matrix of a network is defined as the square symmetric matrix $\mathcal{L}$ with entries

$$\mathcal{L}_{uv} = \begin{cases} k_i & \text{if } u = v \\ -1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases} \quad \forall u, v \in V.$$

The Laplacian matrix can be written as $\mathcal{L} = K - A$, where $A = (a_{uv})$ is the *adjacency matrix* of the graph and $K = diag(k_i 71)$, where $k_i$ is the degree of node $i$, i.e., $k_i = \sum_j a_{ij}$. The Laplacian matrix is positive semi-definite with eigenvalues denoted by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If the network is connected, the multiplicity of the zero eigenvalue is equal to one, i.e., $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, and the smallest nontrivial eigenvalue $\lambda_2$ is known as the *algebraic connectivity* of the network.

## III. RANDOM RECTANGULAR GRAPHS

A RGG is defined as follows.[7,19]

First, $n$ nodes are uniformly and independently distributed in the $d$-dimensional unit cube $[0, 1]^d$. Then, two nodes are connected by an edge if their Euclidean distance is at most $r > 0$, which is a given fixed number known as the *radius*.

Now, define a unit hyperrectangle as the Cartesian product $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, where $a_i$, $b_i \in \mathbb{R}$, $a_i \leq b_i$, and $1 \leq i \leq d$. Hereafter, for notational simplicity without loss of generality, we restrict our discussions to the 2-dimensional case, which corresponds to a rectangle of unit area, called the *unit rectangle*. Thus, the RRG is defined by distributing uniformly and independently $n$ nodes inside the unit rectangle $[a, b]$ and then connecting two nodes with an

edge if their Euclidean distance is at most $r > 0$.[12] The rest of the construction process remains the same as for the RGG. This means that $RRG \to RGG$ as $(a/b) \to 1$. In this sense, the RRG is a generalization of the RGG.

Figure 1 illustrates a RGG and a RRG constructed with the same number of nodes and edges.

The following are some of the most important structural properties of RRGs, which have been previously studied analytically.[12]

**Theorem 1.** *(Average node degree):*[12] *Let $G_R(n,a,b,r)$ be a RRG with n nodes embedded in a rectangle of sides with lengths a and b, and connection radius r. Then, the expected average degree, denoted as $\mathbf{E}\bar{k}$, is*

$$\mathbf{E}\bar{k} = \frac{(n-1)f}{(ab)^2}, \tag{1}$$

where

$$f = \begin{cases} 0 \le r \le b & \pi r^2 ab - \frac{4}{3}(a+b)r^3 + \frac{1}{2}r^4, \\ b \le r \le a & -\frac{4}{3}ar^3 - r^2b^2 + \frac{1}{6}b^4 + a\left(\frac{4}{3}r^2 + \frac{2}{3}b^2\right)\sqrt{r^2-b^2} \\ & +2r^2ab\arcsin\left(\frac{b}{r}\right), \\ a \le r \le \sqrt{a^2+b^2} & -r^2(a^2+b^2) + \frac{1}{6}(a^4+b^4) - \frac{1}{2}r^4 \\ & +b\left(\frac{4}{3}r^2 + \frac{2}{3}a^2\right)\sqrt{r^2-a^2} + a\left(\frac{4}{3}r^2 + \frac{2}{3}b^2\right)\sqrt{r^2-b^2} \\ & -2abr^2\left(\arccos\left(\frac{b}{r}\right) - \arcsin\left(\frac{a}{r}\right)\right). \end{cases} \tag{2}$$

**Proposition 2.** *(Connectivity):*[12] *Let $G_R(n,a,b,r)$ be a RRG with n nodes embedded in a rectangle of sides with lengths a and b, and connection radius r. If $\frac{(n-1)f_i}{(ab)^2} = \log n + \alpha$, for a given $\alpha \in \mathbb{R}$, then*

$$\lim_{n\to\infty} P[G_R(n,a,r) \text{ is connected}] = \exp(-\exp(-\alpha)). \tag{3}$$

**Proposition 3.** *(Degree distribution):*[12] *Let $G_R(n,a,b,r)$ be a connected RRG with n nodes embedded in a rectangle of sides with lengths a and b, and connection radius r. If $n \to \infty$, the distribution of the node degrees tends to a Poisson distribution of the form*

$$p(k) \simeq \frac{\bar{k}^k \exp(-\bar{k})}{k!}. \tag{4}$$

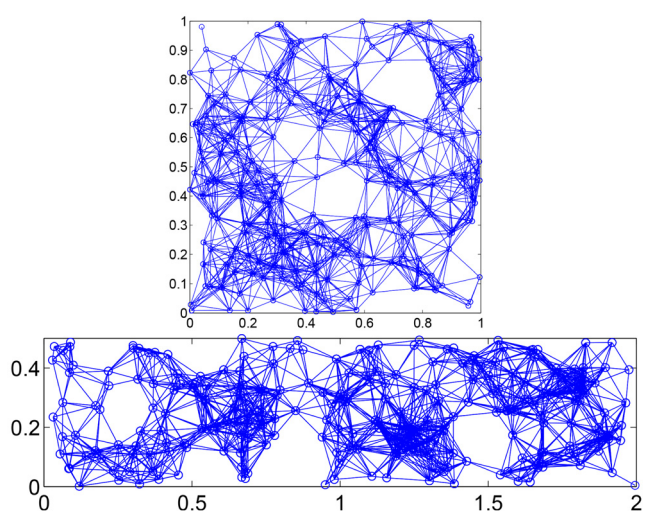

FIG. 1. Illustration of a RGG created with 250 nodes embedded into a unit square with $a = 1$ (top) and RRG into a unit rectangle with $a = 2$ (bottom). In both cases, the nodes are connected if they are at a Euclidean distance smaller than or equal to $r = 0.15$.

In the following, the important unit rectangular case with $ab = 1$ (so, $b = a^{-1}$) is discussed in more detail.

**Proposition 4.** *(Average path length):*[12] *Let $G_R(n,a,r)$ be a connected RRG with n nodes embedded in a rectangle of sides with lengths a and $b = a^{-1}$, and connection radius r. Then, the average path length $\bar{l}$ is bounded as*

$$\bar{l} \le \frac{a^2 + ar + 1}{2ar}. \tag{5}$$

Let $C_i$ be the clustering coefficient of node $i$ in the graph, defined by

$$C_i = \frac{2t_i}{k_i(k_i-1)},$$

and let $\bar{C}$ be the average of the clustering coefficient for all nodes in the graph. Then, we have the following result.

**Proposition 5.** *(Clustering coefficient):*[12] *Let $G_R(n,a,r)$ be a connected RRG with n nodes embedded in a rectangle of sides with lengths a and $b = a^{-1}$, and connection radius r. Then, the average clustering coefficient of the nodes of the RRG is*

$$\bar{C} = \frac{2r^2\arccos\left(\frac{\delta}{2r}\right) - \frac{1}{2}\delta\sqrt{4r^2-\delta^2}}{\pi r^2}, \tag{6}$$

where $\delta$ is the expected Euclidean distance between any pair of connected nodes in $G_R(n,a,r)$.

## IV. DYNAMICAL NETWORK SYNCHRONIZATION

As mentioned above, complex network synchronization finds many important applications in the natural and physical worlds, such as in human learning[20] and in wireless sensor networks.[24] In this section, as an important application, we study the network synchronization problem on some representative RRGs.

### A. Complex dynamical network model

Consider a dynamic network of $n$ nodes interconnected in a certain topology, described by the following dynamical systems on an undirected unweighted simple graph:[6]

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}H(x_j), \quad i = 1,\ldots,N, \tag{7}$$

where $x_i \in R^m$ is the state vector of node $i$, $f(\cdot)$ is a (usually nonlinear) function, $c>0$ is the constant coupling coefficient, $H \in R^{m\times m}$ is the constant inner-coupling matrix, and $A = [a_{ij}]$ is the outer-coupling matrix (i.e., the adjacency matrix), in which $a_{ii}=0$ and $a_{ij}=a_{ji}$, with $a_{ij}=1$ if nodes $i$ and $j$ are connected but $=0$ otherwise, $i,j = 1,\ldots,N$.

The mathematical notion of (*complete*) *synchronization* of the node states, denoted by $x_i(t)$, $i = 1,\ldots,N$, refers to the following asymptotic dynamical behavior:

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, \quad i,j = 1,\ldots,N, \tag{8}$$

where $||\cdot||$ is the Euclidean norm. Physically, this means that the dynamics of all node states approach each other as time approaches infinity.

This paper is concerned only with this type of (complete) state synchronization. Obviously, the synchronizability of a dynamic network depends on the node dynamics, the coupling strength, and the network topology. As usual, assume that the node dynamics $\dot{x}_i = f(x_i)$ satisfy the (local) Lipschitz condition

$$||f(x) - f(y)|| \le L||x-y||, \quad \forall x,y \in \mathcal{D}, \tag{9}$$

where $\mathcal{D} \subseteq R^m$ is the domain of interest and $L>0$ is the Lipschitz constant. For example, the chaotic Lorenz system satisfies this condition locally in a bounded region containing its attractor,[16] which will be used for simulation below.

Therefore, the global dynamics of the network (7) will be dominated by the summation term for appropriately chosen constant coupling coefficient $c>0$ and constant inner coupling matrix $H$. For simplicity, in this paper, the constant matrix $H$ will be fixed.

### B. Network synchronizability

It is now well known that there are two types of networks with bounded and unbounded synchronization regions in the parameter space.

One large class of dynamic networks has an unbounded synchronized region specified by

$$c\lambda_2 > \alpha_1 > 0, \tag{10}$$

where constant $\alpha_1$ depends only on the node dynamics, and a bigger spectral gap $\lambda_2$ implies a better network synchronizability, namely, a smaller coupling strength $c>0$ is needed.[5,21,22]

Another large class of dynamic networks has a bounded synchronized region specified by

$$c\lambda_2/\lambda_n \in (\alpha_2,\alpha_3) \subset (0,\infty), \tag{11}$$

where constants $\alpha_2$, $\alpha_3$ depend only on the node dynamics as well, and a bigger eigenratio $\lambda_2/\lambda_n$ implies a better network synchronizability, which likewise means a smaller coupling strength is needed.[4]

In this paper, only the latter criterion is considered, while the former can be discussed similarly. Thus, in the rest of the paper, the following *sync-index* will be adopted:

$$Q = \frac{\lambda_2}{\lambda_n}. \tag{12}$$

## V. EIGENRATIO OF RRGs

As seen above, a key parameter for measuring the synchronizability of a connected simple network is the eigenratio $Q$ given by (12). To address this important issue, in this section, we derive our main result on the eigenratio of a RRG.

**Theorem 6.** *Let $G_R(n,a,r)$ be a connected RRG with $n$ nodes embedded in a rectangular lattice with side lengths $a>0$ and $b=a^{-1}$ and with a radius $r>0$. Then, the eigenratio $Q$ is bounded by*

$$\frac{1}{(n-1)n^2} \le Q \le \frac{8(ar)^2}{a^4+1}\log_2^2 n. \tag{13}$$

First, it is quite easy to prove the lower bound in (13). Recall (see, e.g., Ref. 18, Theorem 4.1.1) that for a connected simple graph $G$ with diameter $diam(G)$, its algebraic connectivity satisfies

$$\lambda_2 \ge \frac{1}{n\cdot diam(G)}.$$

Since, in the worst case all nodes are uniformly located on the diagonal of the rectangle, within the radius $r$, the diameter of the graph is the longest, $n-1$, one has

$$\lambda_2 \ge \frac{1}{n\cdot diam(G)} \ge \frac{1}{(n-1)n}.$$

Moreover, because $\lambda_n \le n$ for any connected graph with $n$ nodes (see, e.g., Ref. 18, Corollary 1.3.8), one has

$$Q \ge \frac{1}{(n-1)n^2}. \tag{14}$$

Second, it is quite tedious to prove the upper bound in (13). To do so, the following result is first needed.

**Theorem 7.** *Let $G_R(n,a,r)$ be a connected RRG with $n$ nodes embedded in a rectangular lattice with side lengths $a>0$ and $b=a^{-1}$ and with a radius $r>0$. Then, two nodes are connected if and only if they are at a Euclidean distance smaller than or equal to $r$. Let $diam(G_R)$ be the diameter of the corresponding RRG. Then,*

$$diam(G_R) \geq \left\lfloor \frac{\sqrt{a^4+1}}{ar} \right\rfloor. \tag{15}$$

*Proof.* Since the nodes of the RRG are uniformly and independently distributed in the unit rectangle, one may assume that the $n$ nodes are equally spaced in the area of the rectangle separated by the Euclidean distance $r$. In this case, a largest possible number of nodes are connected along the main diagonal of the rectangle. If the length of the main diagonal is $d$, then there are $\lfloor \frac{d}{r} \rfloor$ connected nodes in this line. Thus, the maximum shortest path distance in the RRG is $\lfloor \frac{d}{r} \rfloor$ with $d = \lfloor \frac{\sqrt{a^4+1}}{a} \rfloor$. For a connected RRG, this is the shortest the diameter can be, because if two nodes in the main diagonal are separated at a Euclidean distance larger than $r$, then the diameter of $G_R$ will be larger than $\lfloor \frac{d}{r} \rfloor$. This proves the result. □

Next, recall the following bound[1] for the algebraic connectivity of any simple graph.

**Theorem 8.** *(Alon-Milman) The second smallest eigenvalue of the Laplacian matrix of any graph is bounded as*

$$\lambda_2(G) \leq \frac{8k_{max}}{(diam(G))^2} \log_2^2 n. \tag{16}$$

Now, by substituting (15) into (16), one obtains

$$\lambda_2(G) \leq \frac{8k_{max}}{(diam(G))^2} \log_2^2 n \leq \frac{8k_{max}(ar)^2}{a^4+1} \log_2^2 n. \tag{17}$$

Furthermore, because $\lambda_n \geq k_{max}+1$, one has

$$Q = \frac{\lambda_2(G)}{\lambda_n(G)} \leq \frac{8k_{max}}{\lambda_n(G)(diam(G))^2} \log_2^2 n \tag{18}$$

$$\leq \frac{8k_{max}(ar)^2}{(k_{max}+1)(a^4+1)} \log_2^2 n \leq \frac{8(ar)^2}{(a^4+1)} \log_2^2 n, \tag{19}$$

which proves the result.

Theorem 6 proves that for a RRG with a fixed number of nodes and a connection radius, the eigenratio $Q \to 0$ as $a \to \infty$. Therefore, for very elongated rectangles, it is not possible for the network to achieve synchronization, independently of the value of the coupling strength $c>0$.

## VI. SIMULATION RESULTS

In this section, the simulation results on a RRG of chaotic Lorenz systems, in various numerical settings, are reported and analyzed. All simulations are performed using MatLab on a RRG with $n=200$. No size effect is expected here, thus we can avoid the tedious calculations using larger-scale networks. Choose coupling strength $c=12.0$ while the connection radius $r$ is selected to guarantee the connectivity of the generated RRGs and that synchronization can be achieved in all simulations. Every node is a 3-dimensional chaotic Lorenz system

$$\dot{x} = 10(y-x); \quad \dot{y} = x(28-z)-y; \quad \dot{z} = xy-(8/3)z. \tag{20}$$

In all the simulations, the synchronization measure $Q$ defined in (12) is adopted.

Simulations are performed for various side lengths $a \in [1, 40]$ with a fixed step size 0.5 and different connection radii $r \in [0.1, 3.5]$ with a fixed step size 0.1. The synchronizing process is terminated whenever for both $t=t^*$ and $t=t^*+1$, it achieves

$$\max_{1\leq i,j\leq n} ||x_i - x_j|| \leq \delta = 10^{-3}. \tag{21}$$

Then, the time $t^*$ is recorded as the synchronized time.

One typical case of synchronization performance is shown in Figure 2, which displays only the $x$-variables of 20 randomly selected nodes from the network of 200 nodes, with $a=50$, $c=12$, and $r=2.5$, plotted on the time interval $[0, 1]$. The total sync error is defined by

$$E = \sum_{i,j} ||x_i - x_j||_2. \tag{22}$$

The most interesting situations emerge when we study the synchronization time as a function of both the connection radius $r$ and the rectangle side length $a$. In Figure 3, we show the results where the synchronization time is contour plotted as a function of $r$ and $a$. The first clear observation is the existence of a region (in white color in Figure 3) where synchronization of the system is not achievable. This region corresponds to very elongated rectangles for a given connection radius, as predicted by the theory above. For instance, the synchronization process for a RRG with connection radius $r=2.0$ and rectangle side length $a>10$ is not achievable. The main cause for the divergence in the synchronization time is that the RRG becomes disconnected for a corresponding

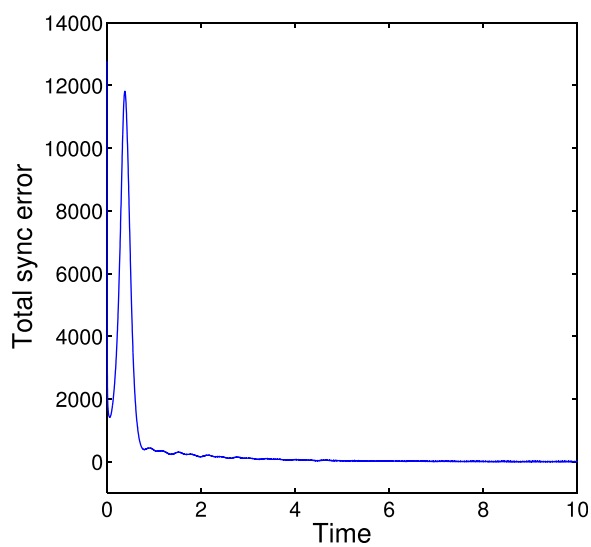


FIG. 2. Total network synchronization error versus time.

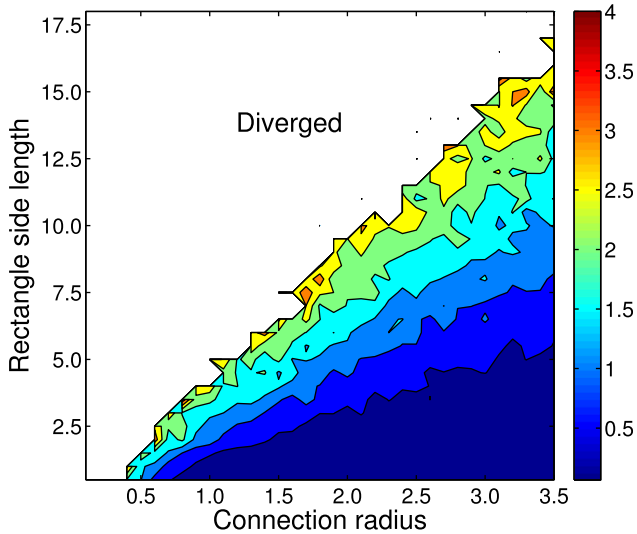


FIG. 3. Dependence of the synchronization time on both the connection radius and the rectangle side length. The region of good synchronization (deep blue, lower right triangular region) is separated from that of bad synchronization by the line $a = 2r - 0.83$. The upper left triangular region represents the region in which synchronization is not achievable due to network disconnectedness.

radius as the rectangle becomes extremely elongated (see (3) and Ref. 12).

Another interesting observation is that for a given side length $a > 1$, which generates a rectangle, the synchronization time decays as the connection radius increases. This is an expected result because increasing the connection radius makes the RRG more densely connected, approaching a complete graph in the limit. However, as we have found analytically (see (19)), the elongation of the rectangle makes the network less synchronizable. That is, for a given node number $n$ and connection radius $r$, the eigenratio $Q$ depends only on $a$, and because $Q \leq \frac{8(ar)^2}{(a^4+1)} \log_2^2 n$, as $a \to \infty$, the eigenratio $Q \to 0$, which implies that the network is poorly synchronizable. In Figure 3, it can be seen that for a given value of $r$, the synchronization time increase as $a$ increases, i.e., going from bottom to top of the plot, and at a certain point the synchronization time diverges due to the network disconnection. The structural causes, explaining this increase in the synchronization time with the increase of the rectangular side length, can be resumed as follows. As $a \to \infty$,

- The average degree of the nodes in a RRG decays according to the results given in Theorem 1;
- The average path length in a RRG grows to infinity according to the results given in Proposition 4. Also, the diameter of the graph increases as we have seen in Theorem 7;
- The probability that a RRG becomes disconnected increases according to the results given in Proposition 2.

These three structural properties have significant influences on the synchronizability of the resulting RRGs. These results show that as the rectangle becomes more elongated, the resulting RRG becomes less densely connected, less "small-world"-like, and more prone to be disconnected. All of which make the synchronization process more difficult to converge.

As we have reported previously (see Ref. 12), the average clustering coefficient follows a non-monotonic behavior with the increase of $a$. A RRG first becomes more clustered as the rectangle elongates, and after a certain critical value, the clustering decays almost linearly with the subsequent increment of $a$. Thus, it is possible that this initial increase of the clustering attenuates the loss of the synchronizability of a RRG for relatively small values of $a$.

The region of better synchronizability for the RRGs here appears to be demarcated by an approximate straight line: $a = 2r - 0.83$ (see the deep blue lower triangle in Figure 3). This means that for a network with the size studied here, a good synchronization is reached if

$$\frac{a + 0.83}{r} < 2.$$

## VII. CONCLUSIONS

This paper studies the synchronizability of a recently proposed RRG network model, deriving analytically the upper and lower bounds of the eigenratio of the network Laplacian matrix. RRGs account for the spatial distribution of nodes allowing the variation of the shape of the unit square commonly used in RGGs. The paper also investigates the synchronization processing of representative RRG networks with nodes being chaotic Lorenz systems, showing complete consistence with the theoretical results. The new RRG model has some attractive theoretical and practical features that deserve further investigation in the near future, including its controllability, observability, identifiability, and potential real-world applications.

## ACKNOWLEDGMENTS

This research was supported by a Wolfson Research Merit Award from the Royal Society (EE) and by the Hong Kong Research Grants Council under the GRF Grant CityU-11201414 (GC).

[1]N. Alon and V. D. Milman, "$\lambda_1$ isoperimetric inequalities for graphs and superconcentrators," J. Comb. Theory B **38**, 73–88 (1985).
[2]A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, "Synchronization in complex networks," Phys. Rep. **469**, 93–153 (2008).
[3]A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," Science **286**, 509–512 (1999).
[4]M. Barahona and L. M. Pecora, "Synchronization in small-world systems," Phys. Rev. Lett. **89**, 054101 (2002).
[5]G. Chen and Z. Duan, "Network synchronizability analysis: A graph-theoretic approach," Chaos **18**(3), 037102 (2008).
[6]G. Chen, X. F. Wang, and X. Li, *Fundamentals of Complex Networks: Models, Structures and Dynamics* (Wiley, 2015).
[7]J. Dall and M. Christensen, "Random geometric graphs," Phys. Rev. E **66**, 016121 (2002).
[8]A. Díaz-Guilera, J. Gómez-Gardeñes, Y. Moreno, and M. Nekovee, "Synchronization in random geometric graphs," Int. J. Bifurcation Chaos **19**, 687–693 (2009).
[9]F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," Proc. Natl. Acad. Sci. U. S. A. **110**, 2005–2010 (2013).
[10]P. Erdös and A. Rényi, "On the evolution of random graphs," Pub. Math. Inst. Hung. Acad. Sci. **5**(1), 17–60 (1960).

[11]E. Estrada, *The Structure of Complex Networks: Theory and Applications* (Oxford University Press, 2011).
[12]E. Estrada and M. Sheerin, "Random rectangular graphs," Phys. Rev. E **91**, 042805 (2015); e-print arXiv:1502.0257.
[13]H. Kenniche and V. Ravelomananana, "Random geometric graphs as model of wireless sensor networks," in the 2nd IEEE International Conference on Computer and Automation Engineering (ICCAE, 2010), Vol. 4.
[14]C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems* (World Scientific Publishing Company, 2007).
[15]Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," Nature **473**, 167–173 (2011).
[16]E. N. Lorenz, "Deterministic nonperiodic flow," J. Atmos. Sci. **20**, 130–141 (1963).
[17]A. E. Motter, C. Zhou, and J. Kurths, "Network synchronization, diffusion, and the paradox of heterogeneity," Phys. Rev. E **71**, 016116 (2005).
[18]M. W. Newman, "The Laplacian spectrum of graphs," M.S. thesis (University of Manitoba, 2000).
[19]M. D. Penrose, *Random Geometric Graphs* (Oxford University Press, 2003).
[20]A. Trafton, "Synchronized brain waves enable rapid learning," MIT News, June 2014; see http://newsoffice.mit.edu/2014/synchronized-brain-waves-enable-rapid-learning-0612.
[21]X. F. Wang and G. Chen, "Synchronization in small-world dynamical networks," Int. J. Bifurcation Chaos **12**, 187–192 (2002).
[22]X. F. Wang and G. Chen, "Synchronization in scale-free dynamical networks: Robustness and fragility," IEEE Trans. Circuits Syst. **49**, 54–62 (2002).
[23]D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," Nature **393**, 440–442 (1998).
[24]Y.-C. Wu, Q. Chaudhari, and E. Serpedin, "Clock synchronization of sensor networks," IEEE Signal Process. Mag. **28**, 124–138 (2011).