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SOME REMARKS ON STABILITY FOR A PHASE FIELD MODEL WITH MEMORY

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(Dedicated to Lee A. Segel (1932-2005))

Abstract. In the present paper we treat the system

$$(\mathbf{PFM})\quad \begin{cases} u_t + \dfrac{l}{2}\phi_t = \displaystyle\int_0^t a_1(t-s)\triangle u(s)\,ds, \\ \tau\phi_t = \displaystyle\int_0^t a_2(t-s)[\xi^2\triangle\phi + \frac{1}{\eta}(\phi-\phi^3)+u](s)\,ds, \end{cases}$$

for $(x,\,t)\in\Omega\times(0,\,T)$, $0<T<\infty$, with the boundary conditions

$$\mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0,\quad (x,\,t)\in\partial\Omega\times(0,\,T),$$

and initial conditions $u(x,\,0)=u_0(x)$, $\phi(x,\,0)=\phi_0(x)$, $x\in\Omega$, which was proposed in [36] to model phase transitions taking place in the presence of memory effects which arise as a result of slowly relaxing internal degrees of freedom, although in [36] the effects of past history were also included. This system has been shown to exhibit some intriguing effects such as grains which appear to rotate as they shrink [36]. Here the set of steady states of (PFM) and of an associated classical phase field model are shown to be the same. Moreover, under the assumption that a_1 and a_2 are both proportional to a kernel of positive type, the index of instability and the number of unstable modes for any given stationary state of the two systems can be compared and spectral instability is seen to imply instability. By suitably restricting further the memory kernels, the (weak) $\omega-$limit set of any initial condition can be shown to contain only steady states and linear stability can be shown to imply nonlinear stability.

1. **Background.**

In [36], the following phase field system with memory was proposed:

$$(\mathcal{PFM})\quad \begin{cases} u_t + \frac{l}{2}\phi_t = \displaystyle\int_{-\infty}^t a_1(t-s)\triangle u(s)\,ds, & (x,\,t)\in\Omega_T, \\ \tau\phi_t = \displaystyle\int_{-\infty}^t a_2(t-s)[\xi^2\triangle\phi + \frac{1}{\eta}(\phi-\phi^3)+u](s)\,ds, & (x,\,t)\in\Omega_T, \\ \mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0, & (x,\,t)\in\partial\Omega_T, \\ u(x,\,0)=u_0(x),\quad \phi(x,\,0)=\phi_0(x), & x\in\Omega. \end{cases}$$

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Here Ω is a bounded domain in $R^n, n = 1, 2$, or 3 with a smooth boundary $\partial\Omega$, $\Omega_T = \Omega \times (0, T)$, $\partial\Omega_T = \partial\Omega \times (0, T)$, and $\mathbf{n}$ denotes the outer unit normal to $\partial\Omega$. In these equations, $u = u(x, t)$ represents a dimensionless temperature and $\phi = \phi(x, t)$ is a non-conserved order parameter. l is a dimensionless latent heat which is assumed to be constant, τ is a dimensionless relaxation time, ξ is a dimensionless interaction length, and η is the dimensionless depth of the potential wells. The first equation in $(\mathcal{PFM})$ describes the energy balance in the system. The second equation in $(\mathcal{PFM})$ models relaxation of the system to equilibrium with deviations from equilibrium acting as the driving force. The memory kernels a_1 and a_2 appear in $(\mathcal{PFM})$ since the responses of the system to gradients in the thermal field and to deviations from equilibrium are assumed to be delayed or time averaged over their past values.

The phase field system with memory can be viewed as a phenomenological extension of the classical phase field equations in which memory effects have been taken into account in both fields. Such memory effects could be important for example during phase transition in polymer melts in the proximity of the glass transition temperature where configurational degrees of freedom in the polymer melt constitute slowly relaxing "internal modes" which are difficult to model explicitly. They should be relevant in particular to glass-liquid-glass transitions where re-entrance effects have been recently reported [27]. We note that in numerical studies based on sharp interface equations obtained from $(\mathcal{PFM})$, grains have been seen to rotate as they shrink [35, 36]. While further modelling and numerical efforts are now being undertaken, the present manuscript is devoted to strengthening the analytical underpinnings of the model.

Typically in formulating a well-posed problem for $(\mathcal{PFM})$ the past histories

$$f_1 := \int_{-\infty}^{0} a_1(t-s)\triangle u(s)\, ds, \quad f_2 := \int_{-\infty}^{0} a_2(t-s)[\xi^2\triangle\phi + \frac{1}{\eta}(\phi - \phi^3) + u](s)\, ds,$$

are prescribed. In the present manuscript, for simplicity we shall take f_1 and f_2 to be equal to zero, so that the system $(\mathcal{PFM})$ reduces to the system (PFM) as stated in the Abstract. We remark that some of our results with regard to stability can be extended to include the case of sufficiently small, non-vanishing histories.

Assumptions on the memory kernels. With regard to the memory kernels, various assumptions may be made, and the actual assumptions which we shall make will vary from result to result. The basic setting for our considerations will be that a_1 and a_2 satisfy Hypothesis I,

Hypothesis I: $a \in L^1_{loc}(R^+)$, $a \geq 0$, $a \not\equiv 0$, and a is of *positive type.*

We remind the reader that a kernel a is said to be of *positive type* [16] if

$$\int_0^T \langle \psi,\, a * \psi \rangle\, dt \geq 0, \quad \forall \psi \in L^2(0,\, T; L^2(\Omega)), \quad \forall T > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product and $*$ denotes convolution in time.

Some of our results will be based on either stronger or weaker assumptions than those contained in Hypothesis I. In particular we shall at times consider kernels $a(t)$ that are of *strong positive type* [16], that is, such that there exists a constant $\nu > 0$ such that $a(t) - \nu e^{-t}$ is of positive type. To give some intuition into the above definition, we note that if $a \in L^1_{loc}(R^+)$, and a is nonnegative, decreasing, and convex, then a is of strong positive type.

In studying asymptotic stability we shall also find it useful to consider kernels which satisfy Hypothesis II,

Hypothesis II: $a \in L^1_{loc}(R^+)$ and there exists a constant γ such that

$$\int_0^T \langle \phi, a * \phi \rangle \, ds \geq \gamma ||a * \phi||^2_{L^2(0,\,T;\,L^2(\Omega))}, \tag{1.1}$$

for any $0 < T < \infty$ and for any $\phi \in L^2(0,\, T;\, L^2(\Omega))$.

Kernels satisfying Hypothesis II are known as kernels of anti-coercive type [16]. We note (see [37]) that if

$(*)$ $a,\, a' \in L^1(R^+)$, and a is of strong positive type,

then (1.1) holds with γ depending on $||a||_{L^1(R^+)}$ and on $||a'||_{L^1(R^+)}$, though anti-coercive kernels need not be of strong positive type (see §16.5 in [16]).

Additional hypotheses will be used when we turn our attention to formulating and proving a connection between linear stability and stability.

Existence and uniqueness results for ($\mathcal{PFM}$). Under the assumption that the kernels a_1 and a_2 satisfy Hypothesis I, for initial data $(u_0,\, \phi_0)$ in $L^2(\Omega) \times H^1(\Omega)$, $f_1 \in L^1(0,\, T; L^2(\Omega))$ and $f_2 = 0$, existence of solutions $(u,\, \phi)$ such that $u \in L^\infty(0,\, T; L^2(\Omega)) \cap \mathcal{C}([0,\, T]; H^{-2}(\Omega))$, $u_t \in L^\infty(0,\, T; H^{-2}(\Omega)) + L^1(0,\, T; L^2(\Omega))$ and $\phi \in L^\infty(0,\, T;\, H^1(\Omega)) \cap \mathcal{C}([0,\, T]; L^2(\Omega)) \cap W^{1,\,\infty}(0,\, T; H^{-1}(\Omega))$ was proven by Grasselli in [11]. Shortly later it was demonstrated by the second author in [33] that the assumptions in [11] actually imply the existence of global solutions such that for all $T > 0$, $(u,\, \phi) \in \mathcal{C}([0,\, T];\, L^2(\Omega) \times H^1(\Omega))$ and $(u_t,\, \phi_t) \in L^\infty([0,\, T]; H^{-2}(\Omega) \times H^{-1}(\Omega)) + L^1(0,\, T; L^2(\Omega) \times H^1(\Omega))$. The analysis in [33] allows f_2 to be an arbitrary function in $L^1(0,\, T; H^1(\Omega))$. With regard to the system (PFM), since $f_1 = f_2 = 0$, the analysis in [33] can be seen to imply that in fact $(u_t,\, \phi_t) \in \mathcal{C}(R^+; H^{-2}(\Omega) \times H^{-1}(\Omega))$. We remark that for the related classical phase field system which can be obtained from (PFM) by setting $a_1(t) = \tilde{a}_1\delta(t)$ and $a_2(t) = \tilde{a}_2\delta(t)$, where $\tilde{a}_1$ and $\tilde{a}_2$ are positive constants, existence of global solutions $(u,\, \phi) \in \mathcal{C}(R^+;\, L^2(\Omega) \times H^1(\Omega))$, as well as uniqueness and additional regularity and compactness results were proven by Bates and Zheng [4] for initial data in $L^2(\Omega) \times H^1(\Omega)$. However these additional properties cannot be expected to hold for (PFM) without placing additional restrictions on the memory kernels. For example, under the assumption that the memory kernels are in $W^{1,\,1}(0,\, T)$ and positive at the origin, uniqueness and well-posedness was proven in [11, 14]. We remark that long time asymptotic properties have been proven for related models such as the phase field equations with memory in which memory is included in the energy balance equation only, see for example Aizicovici & Barbu [1], Colli & Laurençot [6, 7], Aizicovici & Feireisl [2], and Grasselli & Pata [12]. Quite recently, long time asymptotics have also been considered for ($\mathcal{PFM}$) by Grasselli & Pata in [13]; however their results rely on making many regularity assumptions on the memory kernels which we shall not be making here.

Plan of the paper. The present paper is devoted to considering various questions concerning the steady states of (PFM) and their stability under a variety of restrictions on the memory kernels. The basis of our approach is a comparison of the predictions for (PFM) with those of a related (CPF) system in which a_1 and a_2 are replaced by delta functions. More specifically, (CPF) is obtained by setting $a_1(t) = \delta(t)$ and $a_2(t) = \alpha^{-1}\delta(t)$ in (PFM), where α is a positive constant. In §2, we demonstrate that the steady states are the same for (PFM) and for (CPF) under the assumption that a_1 and a_2 satisfy Hypothesis I.

Throughout the remainder of the paper we make the additional assumption that the kernels in (PFM) are **proportional**; i.e., that $a_1(t) = a(t)$ and $a_2(t) = \alpha^{-1}a(t)$, where α is the same constant as in the related (CPF) system and $a(t)$ is a kernel whose properties we shall prescribe. We shall refer to the resultant system as **(PFM′)**, (see Section 3).

In §3, a discussion of linear stability is given. We prove, making use of a change of variables introduced by Bates & Fife [3] in the context of (CPF), that under the assumption that the kernel a satisfies Hypothesis I, the eigenspectrum and eigenfunctions of (PFM′) and (CPF) are identical. However within the context of (PFM′), the growth or decay of the amplitudes are governed by a certain set of integro-differential amplitude equations.

§4 is devoted to a study of these integro-differential amplitude equations which were derived in §3. These results provide a guideline for understanding our later results in §5 with regard to nonlinear stability. After first demonstrating that if a is a kernel of positive type, then the sign of λ_n determines whether stability or neutral stability, or neutral stability or instability is indicated, we turn in §4.1 and §4.2 to consider the qualitative features of the amplitude equations in more detail under various assumptions on the kernels. In particular, we see in §4.1 that if a satisfies Hypothesis I and if $\lambda_n < 0$, then the associated amplitude grows unboundedly as $t \to \infty$. If $a \in L^1(R^+)$, growth is shown to be at most exponential. By bounding the kernel a from below, the growth of the amplitudes can also be bounded from below. Bounds from below are also found for kernels which are of strong positive type. In §4.2 we focus on stability and demonstrate that if $\lambda_n > 0$, $a \in L^1(R^+)$, and a is of positive type, under some additional assumptions on a, asymptotic stability of the associated amplitude equations can be guaranteed. By a Paley-Wiener type argument, it can be seen that indeed some additional condition is necessary. We demonstrate that sufficient additional conditions can be formulated, for example, in terms of strong positivity, Hypothesis II, or directly as an integral condition on the kernel. Thus we see that more must be required of the kernel in order to obtain asymptotic stability than is required to obtain unbounded growth. Finally we consider oscillation and show that under rather minimal assumptions, i.e., $a \in L^1_{loc}(R^+)$, $a \geq 0$, and a is nontrivial, the amplitude associated with λ_n oscillates for all n sufficiently large, by which we mean that its sign changes at least once. Even if the solution does oscillate, some initial control on the rate of decay is possible if $a \geq 0$. In §4.3 for illustration two specific kernels, an exponential kernel and Abel's kernel, are analyzed in detail.

In §5 we consider the asymptotic behavior of (PFM′) and to what extent the linear stability analysis can predict stability or instability for the original problem. Whereas for (CPF) the results for strong gradient systems may be called upon [20, 30], these results are not directly available for (PFM′), since compactness and a Liapunov functional are lacking in general. We nevertheless prove that the same functional which acts as a Liapunov functional for (CPF) can be used to demonstrate instability for (PFM′) in the linearly unstable case when a satisfies Hypothesis I. Under certain additional restrictions on the kernel a, we show that the weak $\omega-$limit set for (PFM′) contains only steady states. Thus we see once more that it is easier to guarantee instability than it is to guarantee stability. Lastly by adopting an integral formulation for (PFM′) based on an analytic resolvent [34] and adapting results from semi-linear parabolic theory [21], we obtain a principle of linear stability when the kernel a is suitably restricted. An explicit example of a suitable restricted kernel is provided. These restrictions are stronger than is necessary to guarantee that the resolvent is analytic.

2. **Steady states.** As explained in the Introduction, the results in [32, 33] imply the existence of solutions $(u, \phi) \in \mathcal{C}(R^+; L^2(\Omega) \times H^1(\Omega))$, for the system (PFM),

$$(\mathbf{PFM}) \quad \begin{cases} u_t + \frac{l}{2}\phi_t = a_1 * \triangle u, & (x, t) \in \Omega \times R^+, \\ \phi_t = a_2 * [\xi^2 \triangle \phi + \frac{1}{\eta}(\phi - \phi^3) + u], & (x, t) \in \Omega \times R^+, \\ \mathbf{n} \cdot \nabla u = \mathbf{n} \cdot \nabla \phi = 0, & (x, t) \in \partial\Omega \times R^+, \end{cases}$$

with initial data $(u_0, \phi_0) \in L^2(\Omega) \times H^1(\Omega)$ and for kernels a_1 and a_2 which satisfy Hypothesis I. Thus it is reasonable to look for steady state solutions to (PFM) in $L^2(\Omega) \times H^1(\Omega)$, although additional regularity of the steady states follows directly by bootstrapping. Furthermore, it is convenient to compare the set of steady state solutions of (PFM) with the steady states of the associated classical phase field model, (CPF),

$$(\mathbf{CPF}) \quad \begin{cases} u_t + \frac{l}{2}\phi_t = \triangle u, & (x, t) \in \Omega \times R^+, \\ \phi_t = \alpha^{-1}\left[\xi^2 \triangle \phi + \frac{1}{\eta}(\phi - \phi^3) + u\right], & (x, t) \in \Omega \times R^+, \\ \mathbf{n} \cdot \nabla u = \mathbf{n} \cdot \nabla \phi = 0, & (x, t) \in \partial\Omega \times R^+, \end{cases}$$

where α is an arbitrary positive constant. We have:

Theorem 2.1. *If a_1 and a_2 satisfy Hypothesis I, then $L^2(\Omega) \times H^1(\Omega)$ steady state solutions of (PFM) correspond to $L^2(\Omega) \times H^1(\Omega)$ steady state solutions of (CPF), and vice versa.*

Proof. Let us set $u_t = \phi_t = 0$ in (PFM), and let us look for $L^2(\Omega) \times H^1(\Omega)$ steady state solutions of (PFM) which we shall denote by (u_s, ϕ_s). Proceeding in this manner,

$$\begin{aligned} 0 &= 1 * a_1 \, [\triangle u_s], \quad (x, t) \in \Omega \times R^+, \\ 0 &= 1 * a_2 \left[\xi^2 \triangle \phi_s + \frac{1}{\eta}(\phi_s - \phi_s^3) + u_s\right], \quad (x, t) \in \Omega \times R^+, \\ \mathbf{n} \cdot \nabla u_s &= \mathbf{n} \cdot \nabla \phi_s = 0, \quad (x, t) \in \partial\Omega \times R^+. \end{aligned}$$

Since by assumption a_1 and a_2 are non-negative and a_1 and a_2 are non-trivial, for t sufficiently large we may divide through by $1 * a_1$ and $1 * a_2$ to obtain

$$\begin{aligned} 0 &= \triangle u_s, \quad x \in \Omega, \\ 0 &= \xi^2 \triangle \phi_s + \frac{1}{\eta}(\phi_s - \phi_s^3) + u_s, \quad x \in \Omega, \\ \mathbf{n} \cdot \nabla u_s &= \mathbf{n} \cdot \nabla \phi_s = 0, \quad x \in \partial\Omega. \end{aligned}$$

Hence steady states of (PFM) correspond to steady states of (CPF). The opposite direction is obvious, once one notes that the steady state solutions of (CPF) belong to $L^2(\Omega) \times H^1(\Omega)$. $\square$

For a discussion of the steady states of (CPF) in one dimension, see [9, 10]. With regard to the steady states of (CPF) in higher dimensions, some results may be inferred from results on the steady states of the Cahn-Hilliard equation [22, 39].

3. **Linear Stability.** In this section, we shall linearize the system (PFM) about a given steady state, $(u_s,\, \phi_s)$, and we shall demonstrate that the linear stability analysis can be reduced to the study of integro-differential amplitude equations considered further in the next section. We shall simplify our analysis by assuming that the memory kernels are **proportional**; i.e.,

$$a_1(t) = a(t), \quad a_2(t) = \alpha^{-1}\, a(t),$$

where $a(t)$ satisfies Hypothesis I and α is a positive constant. Thus, we shall consider the system

$$(\mathbf{PFM'}) \quad \begin{cases} u_t + \dfrac{l}{2}\phi_t = a * \triangle u, & (x,\, t) \in \Omega \times (0,\, T), \\ \phi_t = \alpha^{-1} a * \Big[\xi^2 \triangle \phi + \frac{1}{\eta}(\phi - \phi^3) + u\Big], & (x,\, t) \in \Omega \times (0,\, T), \\ \mathbf{n} \cdot \nabla u = \mathbf{n} \cdot \nabla \phi = 0, & (x,\, t) \in \partial\Omega \times (0,\, T), \end{cases}$$

for $T > 0$. Note that (PFM′) conserves energy:

$$\frac{d}{dt} \int_\Omega \Big[u + \frac{l}{2}\phi\Big]\, dx = 0,$$

hence in perturbing about a given steady state it is reasonable, though not essential (see e.g. [38, Chapter 3]) to consider perturbations $(\tilde{u},\, \tilde{\phi}) \in L^2(\Omega) \times H^1(\Omega)$ such that

$$\int_\Omega \Big[\tilde{u} + \frac{l}{2}\tilde{\phi}\Big]\, dx = 0, \tag{3.1}$$

so that the total energy of the system remains unchanged.

Let $(u_s,\, \phi_s)$ now denote a given steady state of (PFM′) which belongs to $L^2(\Omega) \times H^1(\Omega)$. Linearization of (PFM′) about $(u_s,\, \phi_s)$ yields

$$(\mathbf{LPFM'}) \quad \begin{cases} \tilde{u}_t + \dfrac{l}{2}\tilde{\phi}_t = a * \triangle \tilde{u}, & (x,\, t) \in \Omega \times (0,\, T), \\ \tilde{\phi}_t = \alpha^{-1}\, a * \Big[(\xi^2 \triangle - q(x) I)\tilde{\phi} + \tilde{u}\Big], & (x,\, t) \in \Omega \times (0,\, T), \\ \mathbf{n} \cdot \nabla \tilde{u} = \mathbf{n} \cdot \nabla \tilde{\phi} = 0, & (x,\, t) \in \partial\Omega \times (0,\, T), \end{cases}$$

where

$$q(x) := \frac{1}{\eta}(-1 + 3\phi_s^2(x)). \tag{3.2}$$

With regard to the system (LPFM′), by the arguments in [32, 33] one can readily ascertain the existence of a solution $(\tilde{u},\, \tilde{\phi})$ such that $(\tilde{u},\, \tilde{\phi}) \in \mathcal{C}([0,\, T];\, L^2(\Omega) \times H^1(\Omega))$ and $(\tilde{u}_t,\, \tilde{\phi}_t) \in \mathcal{C}([0,\, T];\, H^{-2}(\Omega) \times H^{-1}(\Omega))$. Whereas in the nonlinear case uniqueness was difficult to prove, for (LPFM′) it is quite straightforward as we shall demonstrate shortly.

Following [3], we express (LPFM′) in a more convenient self-adjoint form. Define

$$\tilde{e} = \tilde{u} + \frac{l}{2}\tilde{\phi}, \tag{3.3}$$

and note that (LPFM′) may be expressed in terms of the variables $(\tilde{e},\, \tilde{\phi})$ and the initial conditions $(\tilde{e}(x,\, 0),\, \tilde{\phi}(x,\, 0)) = (\tilde{e}_0,\, \tilde{\phi}_0) \in L_0^2(\Omega) \times H^1(\Omega)$, where $L_0^2(\Omega) := \{v \in L^2(\Omega) \mid \int_\Omega v\, dx = 0\}$. The restriction on the integral follows from assumption (3.1). We shall use the notation

$$H_0^1(\Omega) := H^1(\Omega) \cap L_0^2(\Omega).$$

Let us now make a further change of variables by defining $\widetilde{\psi} = \sqrt{2/(\alpha l)}A^{-1/2}\tilde{e}$, where A is the self-adjoint extension in $H_0^1(\Omega)$ of the operator $-\triangle$ acting on sufficiently smooth functions v in $H_0^1(\Omega)$ such that $\mathbf{n}\cdot\nabla v = 0$ on $\partial\Omega$.

Setting $\widetilde{\Phi} = (\tilde{\phi},\, \widetilde{\psi})$, this yields the problem

$$\widetilde{\Phi}_t = -a * \mathcal{L}\widetilde{\Phi}, \quad t > 0, \qquad \tilde{\Phi}(x,\, 0) = \tilde{\Phi}_0 \in V, \tag{3.4}$$

where $V := H^1(\Omega) \times H_0^1(\Omega)$ and where $\mathcal{L}$ is defined by

$$\mathcal{L} = \begin{pmatrix} \alpha^{-1}(B + \frac{l}{2}I) & -\beta A^{1/2} \\ -\beta A^{1/2}P & A \end{pmatrix}, \tag{3.5}$$

where $\beta = \sqrt{l/2\alpha}$ and P is the projection of $H^1(\Omega)$ on $H_0^1(\Omega)$. Here B is defined to be the self-adjoint extension in $H^1(\Omega)$ of the operator $-\xi^2\triangle + q(x)I$ under homogeneous Neumann boundary conditions where $q(x)$ is given by (3.2). We remark here that the existence of solutions $(\tilde{u},\, \tilde{\phi}) \in \mathcal{C}([0,\, T];\, L^2(\Omega) \times H^1(\Omega))$ for (LPFM$'$) implies the existence of solutions $\widetilde{\Phi} \in \mathcal{C}([0,\, T];\, V)$ for (3.4). Moreover, uniqueness now readily follows for solutions to (3.4), since if $\bar{\Psi}$ denotes the difference of two solutions, then $\bar{\Psi}$ again satisfies (3.4) with $\bar{\Psi}(x,\, 0) = 0$. Then taking the inner product of (3.4) with $\bar{\Psi}$, integrating over the interval $[0,\, T]$, and recalling that by Hypothesis I, a is a kernel of positive type, we obtain that

$$||\bar{\Psi}(t)|| \leq ||\bar{\Psi}(0)|| = 0,$$

from which uniqueness for $\widetilde{\Phi}$ follows. This then in turn implies uniqueness for the solutions of (LPFM$'$).

We note that the operator $\mathcal{L}$ also appeared in the linear stability analysis of Bates and Fife [3] for (CPF). There, the following problem was obtained

$$\widetilde{\Phi}_t = -\mathcal{L}\widetilde{\Phi}, \quad t > 0, \qquad \widetilde{\Phi}(x,\, 0) = \widetilde{\Phi}_0. \tag{3.6}$$

In [3], it was demonstrated that $\mathcal{L}$ was self-adjoint, although there $\widetilde{\Phi}_0$ was taken to belong to the space $X := L^2(\Omega) \times L_0^2(\Omega)$ and $\mathcal{L}$ was considered as an operator $\mathcal{L} : X \to X$. Nevertheless, it is straightforward to check [28] that $\mathcal{L}$ is also self-adjoint as an operator from V to V^*, and has a countable set of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of finite multiplicity such that

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \text{ and } \lim_{n\to\infty} \lambda_n = \infty, \tag{3.7}$$

with associated orthonormal eigenfunctions $\{\bar{\Phi}_n\}_{n=1}^{\infty}$ which satisfy

$$\lambda_n\bar{\Phi}_n = \mathcal{L}\bar{\Phi}_n, \quad n = 1, 2, 3, \dots, \tag{3.8}$$

and span X and V.

Since we have shown that there exists a unique solution of (3.4) in $\mathcal{C}([0,\, T],\, V)$, we may seek solutions for (3.4) which have the explicit form $\widetilde{\Phi} = \sum_{n=1}^{\infty} b_n(t)\bar{\Phi}_n(x)$. The coefficients b_n are then readily seen to satisfy

$$b_n{}' = -\lambda_n\, a * b_n,\ b_n(0) = \langle\langle\bar{\Phi}_n,\, \widetilde{\Phi}_0\rangle\rangle_{V^*,V}\ (= (\bar{\Phi}_n,\, \widetilde{\Phi}_0)_X\ ),\ n = 1, 2, 3, \dots. \tag{3.9}$$

It now follows from [29] that since the memory kernel a has been assumed to satisfy Hypothesis I, $b_n \in \mathcal{C}^1([0,\, T])$, $n = 1, 2, 3, \dots$ for any $0 < T < \infty$. Thus we have proven

Theorem 3.1. *The perturbations* $(\tilde{u}, \tilde{\phi})$ *around a steady state solution* $(\tilde{u}_s, \tilde{\phi}_s)$ *of (PFM′), such that* $(\tilde{u}_0, \tilde{\phi}_0) \in L^2(\Omega) \times H^1(\Omega)$, $\int_\Omega [\tilde{u} + \frac{1}{2} l \tilde{\phi}]\, dx = 0$, *can be expressed as*

$$\tilde{u} = -\frac{l}{2} \widetilde{\Phi}^1 + \sqrt{l\alpha/2}\, A^{1/2} \widetilde{\Phi}^2, \quad \tilde{\phi} = \widetilde{\Phi}^1, \tag{3.10}$$

where $\widetilde{\Phi}(x,\, t) = \begin{bmatrix} \widetilde{\Phi}^1(x,\, t) \\ \widetilde{\Phi}^2(x,\, t) \end{bmatrix} = \begin{bmatrix} \tilde{\phi}(x,\, t) \\ \tilde{\psi}(x,\, t) \end{bmatrix}$, *with*

$$\widetilde{\Phi}(x,\, t) = \textstyle\sum_{n=1}^{\infty} b_n(t) \bar{\Phi}_n(x),$$

$$b_n' = -\lambda_n\, a * b_n, \quad b_n(0) = (\bar{\Phi}_n,\, \widetilde{\Phi}_0)_X,\ n = 1, 2, 3, \ldots. \tag{3.11}$$

Here $\{\lambda_n\}_{n=1}^{\infty}$, $\{\bar{\Phi}_n\}_{n=1}^{\infty}$ *correspond to the eigenvalues and eigenvectors of the linearization of the associated classical phase field system (CPF) about the same steady state written in the equivalent form* $\widetilde{\Phi}_t = -\mathcal{L}\widetilde{\Phi}$, *and* $\widetilde{\Phi}_0$ *can be found from* $(\tilde{u}_0, \tilde{\phi}_0)$ *using (3.10).*

We remark that Bates and Fife used (3.6) and its associated eigenvalues and eigenvectors to make a comparison between the spectrum of the phase field equations and that of the bistable reaction-diffusion equation; see [3] for details. It follows therefore from Theorem 3.1 that the analogous comparison is valid between the spectrum of (PFM′) and that of the bistable reaction-diffusion equation, to the extent that we can interpret $\lambda_n > 0$ as a stable mode, $\lambda_n = 0$ as a neutral mode, and $\lambda_n < 0$ as a growing mode. We turn to address this and similar questions in the next section. The stability properties of the linearized system (LPFM′) hinge upon the amplitude equations (3.9), and how they are effected by the properties of the memory kernel, $a(t)$.

4. The integro-differential amplitude equations.

We now focus on the equations:

$$\frac{d}{dt} b_n = -\lambda_n\, a * b_n, \quad n = 1,\, 2,\, \ldots. \tag{4.1}$$

Note that (3.7) implies that

$$-\infty < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m < 0 \leq \lambda_{m+1} \leq \lambda_{m+2} \cdots,$$

where the set $\{\lambda_1,\, \lambda_2,\, \ldots, \lambda_m\}$ is finite and possibly empty. With regard to existence and uniqueness for (4.1), we have the following:

Proposition 4.1. [16, Theorem 2.3.1] *If* $a \in L^1_{loc}(R^+)$, *then there exists a unique solution to (4.1) which belongs to* $W^{2,1}_{loc}(R^+)$ *and which can be written explicitly as*

$$b_n(t) = b_n(0)(1 - 1 * R_n), \tag{4.2}$$

where the resolvent $R_n \in W^{1,1}_{loc}(R^+)$ *satisfies*

$$R_n = \lambda_n 1 * a - \lambda_n 1 * a * R_n. \tag{4.3}$$

Let us consider the behavior of (4.1) for the various possible values of λ_n. If $\lambda_n = 0$, integrating, we have (4.1),

$$b_n(t) = b_n(0). \tag{4.4}$$

Thus if $\lambda_n = 0$, then $\bar{\Phi}_n$ acts as a neutral mode, as it does in the context of the corresponding classical phase field system. Let us now suppose that $\lambda_n < 0$. To ascertain the implications of equation (4.1) when $\lambda_n < 0$ with respect to stability, we note that if a satisfies Hypothesis I, then the regularity of the solution implied by Proposition 4.1 allows us to take the $L^2(0,\,t)$ inner product of (4.1) with $b_n(t)$ for any $t > 0$. This gives

$$\frac{1}{2}[b_n^2(t) - b_n^2(0)] = -\lambda_n \int_0^t b_n(\tau)(a * b_n)(\tau)\, d\tau \geq 0. \tag{4.5}$$

Thus we see that $\lambda_n < 0$ implies either neutral stability or instability, although (4.5) does not yield sufficient information to indicate the nature of the stability in a more precise sense. Similarly if $\lambda_n > 0$ and if a satisfies Hypothesis I, then taking the inner product of (4.1) with $b_n(t)$, we now obtain that

$$\frac{1}{2}[b_n^2(t) - b_n^2(0)] = -\lambda_n \int_0^t b_n(\tau)(a * b_n)(\tau)\, d\tau \leq 0. \tag{4.6}$$

Therefore, we see that $\lambda_n > 0$ implies either neutral stability or stability. In the terminology of functional differential equations [23], we may say that "uniform stability" though not necessarily "asymptotic stability" is implied. Thus we may conclude that if a satisfies Hypothesis I, then the number of unstable modes is no greater than for the associated classical phase field system. Noting that in the discussion above we have not made any use of the assumption in Hypothesis I that $a \geq 0$, we may state in summary,

Corollary 4.2. *For the phase field system (PFM′) with memory kernels in $L^1_{loc}(R^+)$ of positive type which are proportional, the number of unstable modes for a given steady state is no greater than for the corresponding classical phase field system.*

A more precise understanding of the stability of the linearized system requires a more careful study of equation (4.1). This is undertaken in §4.1 and §4.2.

4.1. **The unstable case $\lambda_n < 0$.** A natural question to ask is under what conditions the predicted growth is actually exponential. A first result in this direction is

Lemma 4.3. *Let $b_n(t)$ satisfy (4.1) and assume that $a \in L^1(R^+)$. Then the growth of $|b_n(t)|$ is at most exponential.*

Proof. Taking the $L^2(0,\,t)$ inner product of (4.1) with b_n, for any $t > 0$, we get

$$b_n^2(t) = b_n^2(0) - 2\lambda_n \int_0^t b_n(\tau)(a * b_n)(\tau)\, d\tau.$$

By using the Cauchy-Schwartz inequality followed by Young's inequality, we obtain that

$$b_n^2(t) \leq b_n^2(0) - 2\lambda_n ||a||_{L^1(0,t)} \int_0^t b_n^2(s)\, ds,$$

which implies that

$$|b_n(t)|^2 \leq |b_n(0)|^2 e^{-2\lambda_n t\, ||a||_{L^1(0,t)}}\,, \tag{4.7}$$

and hence $|b_n(t)| \leq |b_n(0)| e^{-\lambda_n t ||a||_{L^1(R^+)}}$. □

Note that the proof of Lemma 4.3 does not require the kernel a to be either nonnegative or of positive type. If $a \in L^1_{loc}(R^+)$ but $a \notin L^1(R^+)$, an estimate of the form (4.7) is nevertheless valid. Relying simply on the assumptions that $a \geq 0$ and $a \in L^1_{loc}(R^+)$, one has the following:

Proposition 4.4. *If $a \in L^1_{loc}(R^+)$ and $a \geq 0$, then the resolvent R_n which satisfies (4.3) is nonpositive.*

Proof. By Proposition 4.1, R_n is unique. Moreover, it is possible to construct R_n via the iterative process given by

$$R_n^0 = \lambda_n 1 * a, \quad R_n^m = \lambda_n 1 * a - \lambda_n 1 * a * R_n^{m-1}, \quad m = 1, 2, \ldots.$$

See Theorem 2.3.1 in [16] for details. Clearly since $a \geq 0$ and $\lambda_n < 0$, the conclusion of the lemma follows. □

Note that Proposition 4.4 implies the following:

Corollary 4.5. *Suppose that $a \in L^1_{loc}(R^+)$ and $a \geq 0$, and let $b_n(t)$ denote the unique solution to (4.1), then*

$$\frac{d}{dt}|b_n(t)| \geq 0. \tag{4.8}$$

Proof. It follows from (4.2) that $\frac{d}{dt}b_n(t) = -\lambda_n b_n(0) R_n$. Relying on the negativity of λ_n and the nonpositivity of R_n which was proven in Proposition 4.4, (4.8) follows. □

In the theorem which follows, we prescribe rather minimal conditions which guarantee unbounded growth; i.e., instability.

Theorem 4.6. *If $a(t)$ satisfies Hypothesis I, then $|b_n(t)| \to \infty$ as $t \to \infty$.*

Proof. Multiplying (4.1) by $b_n(t)$, and integrating over time, and noting that by virtue of Corollary 4.5,

$$b_n(s) b_n(q) \geq b_n^2(0) \ \text{ for } \ s \geq q \geq 0, \tag{4.9}$$

we obtain that

$$b_n^2(t) - b_n^2(0) \geq -2\lambda_n b_n^2(0) \int_0^t \int_0^s a(s-q)\, dq ds = -2\lambda_n b_n^2(0) \int_0^t (t-s) a(s)\, ds. \tag{4.10}$$

By Hypothesis I, we have that $a \in L^1_{loc}(R^+)$ and $a \geq 0$. Since we have further assumed that $a \not\equiv 0$, there exists $\delta > 0$ and $r > 0$ such that $a \geq \delta > 0$ on B, where B is a measurable set, $|B| \neq 0$, and $B \subset (0, r)$. Therefore it follows from (4.10) that for $t > r$

$$b_n^2(t) - b_n^2(0) \geq -2\lambda_n b_n^2(0)\, \delta \int_0^t (t-s)\chi_B\, ds = -2\lambda_n b_n^2(0) \delta (t-r)|B|.$$

Noting that $-\lambda_n b_n(0)^2(0)\delta(t-r)|B| \to \infty$ as $t \to \infty$, the claim of the Theorem follows. □

From Corollary 4.2 and Theorem 4.6 it follows that

Corollary 4.7. *If a satisfies Hypothesis I, then the number of unstable modes for (PFM′) is identical to the dimension of the unstable manifold of (CPF) when linearized about the same steady state.*

We now give a condition on the kernel which provides bounds from below on the rate of growth.

Lemma 4.8. *Suppose that $a \in L^1_{loc}(R^+)$ and $a(t) \geq \zeta e^{-\nu t}$ for $t \in (0, \infty)$, where ζ and ν are positive constants. Then*

$$|b_n(t)| \geq \frac{|b_n(0)|}{r_+ - r_-}\Big[-r_-\, e^{r_+ t} + r_+\, e^{r_- t}\Big], \tag{4.11}$$

where $r_\pm = \frac{1}{2}[-\nu \pm \sqrt{\nu^2 - 4\lambda_n \zeta}\,]$.

Remark 4.9. It is easy to check that (4.11) implies that

$$|b(t)| \geq |b(0)|\Big[1 + \frac{1}{2} r_+^2 t^2\Big]. \tag{4.12}$$

Proof. Let us suppose that $b_n(0) > 0$. From Corollary 4.5 we obtain that $b_n(t) > 0$ for all $t > 0$. Therefore

$$\frac{d}{dt} b_n(t) \geq -\lambda_n \int_0^t a(t-s) b_n(s)\, ds \geq -\lambda_n \zeta \int_0^t e^{-\nu(t-s)} b_n(s)\, ds. \tag{4.13}$$

Let us now define $g(t) := \zeta \int_0^t e^{-\nu(t-s)} b_n(s)\, ds$, and let us note that $g(t)$ satisfies

$$g_t = \zeta b_n - \nu g \ \text{ for } \ t > 0, \quad g(0) = 0.$$

Differentiating the above equation with respect to t and using (4.13)

$$g_{tt} + \nu g_t + \lambda_n \zeta g \geq 0 \ \text{ for } \ t > 0, \quad g(0) = 0, \ g_t(0) = \zeta\, b_n(0). \tag{4.14}$$

Setting $r_\pm = \frac{1}{2}(-\nu \pm \sqrt{\nu^2 - 4\lambda_n \zeta}\,)$ and making the substitution $g(t) = c(t) e^{r_+ t}$ in (4.14), we get

$$c_{tt} + (r_+ - r_-) c_t \geq 0.$$

Integration of this equation yields

$$c_t + (r_+ - r_-) c \geq \zeta\, b_n(0) \ \text{ for } \ t > 0, \quad c(0) = 0.$$

From this differential inequality we obtain a bound from below on $c(t)$ which provides an obvious bound from below on $g(t)$, which can then be used in the differential inequality $\frac{d}{dt} b_n \geq -\lambda_n g(t)$ to obtain the bound from below given in the statement of the lemma.

The case $b_n(0) < 0$ can be treated by similar arguments, and the case $b_n(0) = 0$ is trivial. □

The condition in Lemma 4.8 on the form of the kernel may, roughly speaking, be replaced by the condition that a be of strong positive type.

Lemma 4.10. *Suppose that a satisfies Hypothesis I and that a is of strong positive type. Then a bound from below on $|b_n(t)|$ can be obtained which is analogous to the bound obtained in Lemma 4.8; namely*

$$|b_n(t)|^2 \geq \frac{|b_n(0)|^2}{r_+ - r_-}\Big[-r_-\, e^{r_+ t} + r_+\, e^{r_- t}\Big], \tag{4.15}$$

where $r_\pm = \frac{1}{2}[-1 \pm \sqrt{1 - 8\lambda_n \zeta}]$, where $\zeta > 0$ is chosen so that $a(t) - \zeta e^{-t}$ is of positive type.

Proof. Multiplying (4.1) by $b_n(t)$ and integrating over time, one obtains by virtue of the assumption that a is of strong positive type that

$$\frac{1}{2}[b_n^2(t) - b_n^2(0)] = -\lambda_n \int_0^t b_n(s)(a * b_n)(s)\, ds \geq -\lambda_n \zeta \int_0^t b_n(s) \int_0^s e^{-(s-q)} b_n(q)\, dq\, ds,$$

for some positive constant ζ. By virtue of Corollary 4.5

$$b_n(q) b_n(s) \geq b_n^2(q) \ \text{ for } \ s \geq q \geq 0, \tag{4.16}$$

and hence

$$\frac{1}{2}[b_n^2(t) - b_n^2(0)] \geq -\lambda_n \zeta \int_0^t \int_0^s e^{-(s-q)} b_n^2(q)\, dq\, ds. \tag{4.17}$$

We may now identify $g(t) := 2\zeta \int_0^t e^{-(t-s)} b_n^2(s)\, ds$, set $\int_0^t g(\tau)\, d\tau = c(t) e^{r_+ t}$ where $r_+ = \frac{1}{2}[-1 + \sqrt{1 - 8\lambda_n \zeta}\,]$, and then proceed roughly as in Lemma 4.8. □

Note that here also a bound similar to the bound obtained in (4.12) from Lemma 4.8 is again implied by Lemma 4.10, namely

$$|b_n(t)|^2 \geq |b_n(0)|^2 \Big[1 + \frac{1}{2} r_+^2 t^2\Big]. \tag{4.18}$$

Lastly we state a result which gives a short time bound from below which does not require that $a \geq 0$.

Lemma 4.11. *Suppose that $a \in L^1_{loc}(R^+)$ is of strong positive type. Then $|b_n(t)| \geq \frac{|b(0)|}{\sqrt{3}} e^{[\frac{-\lambda_n \zeta}{3}]^{\frac{1}{2}} t}$ for $0 \leq t \leq 2$, where ζ denotes the positive constant arising in the definition of a kernel of strong positive type.*

Note that the bound given in the statement of the lemma falls slightly short of what one should really like to obtain in terms of an exponential bound from below.

Proof. Let us suppose that $b_n(0) > 0$. From (4.1) and the assumption that a is of strong positive type it follows that there exists a positive constant ζ such that

$$\frac{1}{2}[b_n^2(t) - b_n^2(0)] = -\lambda_n \int_0^t b_n(s)(a * b_n)(s)\, ds \geq -\lambda_n \zeta \int_0^t b_n(s) \int_0^s e^{-(s-q)} b_n(q)\, dq\, ds. \tag{4.19}$$

Let us now define $h(t) := \zeta \int_0^t e^{-(t-s)} b_n(s)\, ds$ and note that $h(t)$ satisfies

$$h_t + h = \zeta b_n \ \text{ for } \ t > 0, \quad h(0) = 0. \tag{4.20}$$

Using (4.20) in (4.19), we obtain that

$$b_n^2(t) \geq b_n^2(0) - \lambda_n \zeta^{-1} h^2(t) - 2\lambda_n \zeta^{-1} \int_0^t h^2(s)\, ds.$$

From Jensen's inequality we have that

$$\frac{1}{t}\Big[\int_0^t h(s)\, ds\Big]^2 \leq \int_0^t h^2(s)\, ds.$$

Combining the above inequalities yields

$$b_n^2(t) \geq b_n^2(0) - \lambda_n \zeta^{-1} h^2(t) - \frac{2\lambda_n}{t\zeta}\Big[\int_0^t h(s)\, ds\Big]^2.$$

We wish now to take the square root of both sides of this inequality. By considering (4.5), we see that the assumption that $b_n(0) > 0$ implies that $b_n(t) > 0$ for $t > 0$, and hence $h(t) > 0$ for $t > 0$. Therefore we obtain that

$$b_n(t) \geq \frac{1}{\sqrt{3}}\bigg(b_n(0) + \sqrt{-\lambda_n \zeta^{-1}}\, h(t) + \sqrt{\frac{-2\lambda_n}{t\zeta}} \int_0^t h(s)\, ds\bigg).$$

Integrating (4.20),

$$h(t) + \int_0^t h(s)\, ds = \zeta \int_0^t b_n(s)\, ds,$$

and therefore

$$b_n(t) \geq \frac{1}{\sqrt{3}}\bigg(b_n(0) + \sqrt{-\lambda_n \zeta} \int_0^t b_n(s)\, ds + \sqrt{-\lambda_n/\zeta}\,\Big(\sqrt{2/t} - 1\Big) \int_0^t h(s)\, ds\bigg).$$

Noting that $\sqrt{2/t} \geq 1$ for $0 \leq t \leq 2$, we may conclude that

$$b_n(t) \geq \frac{1}{\sqrt{3}}\Big(b_n(0) + \sqrt{-\lambda_n \zeta} \int_0^t b_n(s)\, ds\Big)$$

for $0 \leq t \leq 2$, which implies the bound given in the statement of the lemma. □

4.2. **The stable case $\lambda_n > 0$.** We have seen in §3 in equation (4.6) that if $\lambda_n > 0$ and a satisfies Hypothesis I (or more simply that $a \in L^1_{loc}(R^+)$ and is of positive type), then the zero solution for (4.1) is stable. It is of interest to determine conditions on the kernel which guarantee that the solution, $b_n(t)$, will in fact decay to zero as $t \to \infty$. An assortment of such results are given in the lemmas which follow. Throughout this subsection, $\hat{f}(z)$ will denote the Laplace transform of $f(t)$.

The first such result relies on the classical approach of Paley–Wiener.

Lemma 4.12. *If $a \in L^1(R^+)$, and*

$$z + \hat{a}(z) \neq 0, \quad Re\, z \geq 0, \tag{4.21}$$

then $b_n(t) \to 0$ as $t \to \infty$, where $b_n(t)$ is the solution to (4.1).

Proof. It follows from the results of Paley-Wiener (see Theorem 3.3.5 in [16]) that if $a \in L^1(R^+)$ and (4.21) holds, then $b_n(t) \in L^1(R^+)$, where $b_n(t)$ is the solution to (4.1). Referring now to (4.1), we obtain from Young's inequality that moreover $\frac{d}{dt}b_n(t) \in L^1(R^+)$. Therefore we may conclude as claimed in the lemma that $b_n(t) \to 0$ as $t \to \infty$. □

Numerous results in the literature are based on establishing conditions under which (4.21) holds. One such result appears in Krisztin [24, Theorem 1] which implies in the context of (4.1) the following:

Lemma 4.13. *Suppose that $a \in L^1(R^+)$ and $\int_0^\infty a(s)\, ds > 0$, and suppose that*

$$\int_0^\infty sa(s)\, ds \leq \frac{1}{\lambda_n}\left[1 + \left[\frac{\int_0^\infty a(s)\, ds}{\int_0^\infty |a(s)|\, ds}\right]^2\right]^{1/2}, \tag{4.22}$$

then $b_n(t) \to 0$ as $t \to \infty$, where $b_n(t)$ is the solution to (4.1).

We remark that Lemma 4.13 is rather unsatisfactory in our context since condition (4.22) will eventually fail as $\lambda_n \to \infty$, even though one would expect larger values of λ_n to imply greater stability. A condition with a similar drawback based on the construction of a Liapunov functional can be found, for example, in [40, Theorem 3.1].

Another approach is to assume that $a \in L^1(R^+)$ and to establish that (4.21) holds by placing additional restrictions on the kernel. Two such results are stated in Lemmas 4.14 and 4.15 which follow.

Lemma 4.14. *Suppose that $a \in L^1(R^+)$, and $a(t)$ is of positive type, and suppose additionally that*

$$\int_0^\infty a(s) \cos \omega s\, ds \neq 0, \quad \forall\, \omega \in R^+, \tag{4.23}$$

then $b_n(t) \to 0$ as $t \to \infty$.

Proof. Since by assumption $a(t) \in L^1(R^+)$ and $a(t)$ is of positive type, it follows from Theorem 16.2.4 in [16] that

$$Re\, \hat{a}(z) \geq 0 \text{ for } Re\, z > 0. \tag{4.24}$$

Thus condition (4.21) holds when $Re\, z > 0$. Condition (4.23) then ensures that (4.21) also holds when $Re\, z = 0$. □

Lemma 4.15. *If $a \in L^1(R^+)$ and $a(t)$ is of strong positive type, then $b_n(t) \to 0$ as $t \to \infty$.*

Proof. Since by assumption $a(t)$ is a kernel of strong positive type, it may be written in the form $a(t) = \nu(t) + \epsilon e^{-t}$, where $\nu(t)$ is a kernel of positive type and $\epsilon > 0$. Noting that $\widehat{(e^{-t})}(z) = (1+z)^{-1}$ for $Re\, z \geq 0$, and since $\nu(t)$ is a kernel of positive type, $Re\widehat{(\nu(t))}(z) \geq 0$ for $Re\, z \geq 0$, it is readily seen that condition (4.21) is satisfied. □

Lemma 4.16. *Suppose that $a \in L^1(R^+)$, $\int_0^\infty a(s)\, ds \neq 0$, and a satisfies Hypothesis II. Then $\lim_{t\to\infty} b_n(t) = 0$.*

Remark 4.17. Note that if $a(t)$ is of strong positive type, then asymptotic stability is already implied by Lemma 4.15. However, strong positivity is not a necessary condition for Hypothesis II to hold, see [16].

Proof. It follows from (4.1) and (1.1) that there exists a $\gamma > 0$ such that for any $0 < T < \infty$,

$$\lambda_n \gamma ||a * b_n||^2_{L^2(0,\,T)} + \frac{1}{2}(b_n^2(t) - b_n^2(0)) \leq 0. \tag{4.25}$$

From (4.25) we obtain that

$$||b_n||_{L^\infty(R^+)} \leq |b_n(0)|, \tag{4.26}$$

$$||a * b_n||_{L^2(R^+)} \leq \frac{1}{\sqrt{2\lambda_n \gamma}}\, |b_n(0)|. \tag{4.27}$$

Therefore, referring back to (4.1), we get from (4.27) that

$$||b_{nt}||_{L^2(R^+)} \leq \sqrt{\frac{\lambda_n}{2\gamma}}\, |b_n(0)|, \tag{4.28}$$

and from (4.26) and Young's inequality we obtain that

$$||b_{nt}||_{L^\infty(R^+)} \leq \lambda_n |b_n(0)| ||a||_{L^1(R^+)}.$$

Let us now consider translates of $b_n(t)$ which we define by

$$b^N(t) := b_n(t^N + t), \qquad t \in (0,\, \infty),$$

where $\{t^N\}_{N=1}^\infty$ denotes an increasing sequence such that $\lim_{N\to\infty} t^N = \infty$. By the estimates above we find that

$$||b^N||_{L^\infty(R^+)} \leq |b_n(0)|, \quad ||b_t^N||_{L^\infty(R^+)} \leq \lambda_n |b_n(0)|\, ||a||_{L^1(R^+)}, \tag{4.29}$$

and since $a \in L^1(R^+)$, it is readily follows that for any $T > 0$, $\frac{d}{dt} b^N$ is equicontinuous on the interval $[0,\, T]$. Referring to (4.28), we obtain that

$$\frac{d}{dt} b^N \to 0 \text{ as } N \to \infty \quad \text{ in } \; L^2(0,\, T). \tag{4.30}$$

Therefore by the Ascoli-Arzela theorem, along subsequences there exists a constant β such that for any $0 < T < \infty$

$$b^N(t) \to \beta \text{ and } b_t^N \to 0, \text{ as } N \to \infty \quad \text{ uniformly on } \; [0,\, T]. \tag{4.31}$$

We shall now show that in fact $\beta = 0$. To show this we proceed somewhat as in [6, 5]. Namely, let us note that

$$(a * \beta)(t) = (a * b)^N(t) - \{(a * b)^N - (a * b^N)\}(t) - \{(a * b^N) - (a * \beta)\}(t),$$

where $(a*b)^N(t) := (a*b)(t^N+t)$. We now demonstrate that the three terms on the right tend to zero as $t \to \infty$ and $N \to \infty$. To treat the middle term, we note that

$$\begin{aligned}\{(a*b)^N - (a*b^N)\}(t) &= \int_0^{t^N+t} a(s)b(t^N+t-s)\,ds - \int_0^t a(s)b(t^N+t-s)\,ds \\ &= \int_t^{t^N+t} a(s)b(t^N+t-s)\,ds.\end{aligned}$$

Therefore relying on (4.29)

$$|\{(a*b)^N - (a*b^N)\}(t)| \le |b_n(0)| \int_t^{t^N+t} |a(s)|\,ds,$$

which tends to zero as $t \to \infty$ since by assumption $a \in L^1(R^+)$. Thus having taken t sufficiently large in order to make the middle term arbitrarily small, we consider $T > t$ and examine the other two terms. Noting that $(a*b)^N(t) = -\frac{1}{\lambda_n} b_t^N(t)$, (4.31) implies that the first term tends to zero as $N \to \infty$. The third term tends to zero as $N \to \infty$ by virtue of (4.31) and Young's inequality.

Thus we have obtained that $\lim_{t\to\infty}(a*\beta)(t) = 0$, However since β is constant,

$$\lim_{t\to\infty}(a*\beta)(t) = \beta \int_0^\infty a(s)\,ds,$$

and the conclusion of the lemma follows upon recalling our assumptions on a. □

Remark 4.18. We remark that the results in Lemmas 4.12-4.16 require that $a(t) \in L^1(R^+)$. This requirement can in fact be replaced by the weaker requirement that $a(t) \in L^1_{loc}(R^+)$ if, for example, sufficiently strong monotonicity assumptions are imposed on $a(t)$. Results in this direction are demonstrated for Abel's kernel, $a(t) = \gamma t^{-1/2}$, $\gamma > 0$ in § 4.3.

We now consider some qualitative properties of the solutions to the stable amplitude equations with regard to oscillations and rates of decay. Following Györi & Lada [17] and Kolmanovskii & Myshkis [23], we shall use the following definition:

Definition 4.19. *A function $f : R^+ \to R$ is said to oscillate if $f(t)$ is neither positive nor negative for all $t \ge 0$.*

Lemma 4.20. *If $a \in L^1_{loc}(R^+)$, $a \ge 0$, and a is nontrivial, then for n sufficiently large, the solution to (4.1) oscillates.*

Proof. Our proof relies on the following claim:

Claim 4.21. *If $\alpha \in L^1_{loc}(R^+)$ and*

$$-\gamma + \int_0^\infty e^{\gamma s}\alpha(s)\,ds > 0 \quad \textit{for all } \gamma > 0, \tag{4.32}$$

then the equation

$$b_t + \alpha * b = 0$$

has no positive (or negative) solutions on $[0, \infty)$.

Proof. The claim can be readily proved along the lines of the proof of Theorem 9.1.2 in Györi & Ladas [17]. □

In the context of (4.1) with $\lambda_n > 0$, condition (4.32) may be written as

$$G(\gamma, \lambda_n) := \gamma\left(-1 + \lambda_n \int_0^\infty \gamma^{-1} e^{\gamma s} a(s)\, ds\right) > 0 \qquad \text{for all} \quad \gamma > 0.$$

Noting that $a \geq 0$ and

$$\gamma^{-1} e^{\gamma s} \geq s \;\text{ for all } \;\; \gamma > 0 \;\text{ and }\; s \geq 0,$$

we obtain that

$$G(\gamma,\, \lambda_n) \geq \gamma\left(-1 + \lambda_n \int_0^\infty s\, a(s)\, ds\right). \tag{4.33}$$

Since $a \geq 0$, if $t\,a(t) \notin L^1(R^+)$, then (4.32) holds trivially for all $\lambda_n > 0$. If $t\,a(t) \in L^1(R^+)$, then since by (3.7) $\lim_{n\to\infty} \lambda_n = \infty$, it follows from (4.33) that $G(\gamma,\, \lambda_n) > 0$ whenever $\gamma > 0$ for n sufficiently large. Relying now on Claim 4.21, the lemma is proven. □

Note that Lemma 4.20 allows for the possibility that there might be a finite number of non-oscillatory solutions.

Even if the solution does oscillate, it is possible under appropriate assumptions to have some control on the initial rate of decay. Such a result is given in the following lemma.

Lemma 4.22. *Suppose that $a \in L^1_{loc}(R^+)$ and $a \geq 0$. If $|b_n(t)| \neq 0$ for $t \in [0,\, t_0]$, then $|b_n(t)| \leq |b_n(0)| e^{-\lambda_n (1*1*a)(t)}$ for $t \in [0,\, t_0]$.*

Proof. Let us suppose that $b_n(0) > 0$. If $|b(t)| \neq 0$ for $t \in [0,\, t_0]$, then clearly $b_n(t) > 0$ for $t \in [0,\, t_0]$. Hence it follows from (4.1), the positivity of λ_n, and the non-negativity of a that $b_{n\,t} \leq 0$ for $t \in [0,\, t_0]$. Therefore

$$(a * b_n)(t) \geq (a * 1)(t)\, b_n(t), \quad t \in [0,\, t_0]. \tag{4.34}$$

From (4.1) and (4.34), we obtain that

$$b_{n\,t} \leq -\lambda_n (1 * a)\, b_n(t), \quad t \in [0,\, t_0],$$

which yields the estimate in the statement of the lemma. The case $b_n(0) < 0$ is proven similarly. □

4.3. **Examples.** For the sake of illustration, we demonstrate the behavior of the integro-differential amplitude equations for two specific kernels.

Exponential kernels.

Let us consider the predictions for exponential kernels; i.e.,

$$a(t) = r e^{-r\,t}, \quad r > 0. \tag{4.35}$$

Note that $\int_0^\infty r e^{-r\,t}\, dt = 1$ for all $r > 0$, hence such kernels are in $L^1(R^+)$, and clearly these kernels are of strongly positive type. Moreover, equation (4.1) is now readily solvable using Laplace transforms. One may distinguish between the following cases:

(i) if $\lambda_n > r/4$, then $b_n(t) = b_n(0)[e^{-rt/2} \cos \beta t + \frac{r}{2\beta}\, e^{-rt/2} \sin \beta t]$, where $\beta = \sqrt{\lambda_n r - r^2/4}$,

(ii) if $\lambda_n = r/4$, then $b_n(t) = b_n(0)[e^{-rt/2} + \frac{rt}{2}\, e^{-rt/2}]$, and

(iii) if $\lambda_n < r/4$, then $b_n(t) = b_n(0)[A_+\, e^{s_+\,t} + A_-\, e^{s_-\,t}]$, with $s_\pm = (-r \pm \gamma)/2$ and $A_\pm = (r \pm \gamma)/(2r)$, where $\gamma = \sqrt{r^2 - 4\lambda_n r}$.

Note that for any $r > 0$, if n is sufficiently large, then $\lambda_n > r/4$ and damped oscillations follow from (i). If $\lambda_n > 0$ but $\lambda_n \leq r/4$, then it follows from (ii) and (iii) that there is monotone decay to zero. If $\lambda_n < 0$, exponential growth is predicted by (iii).

Abel's kernel.

We turn now to consider kernels of Abel's type; i.e. the algebraic kernel

$$a(t) = \gamma\, t^{-1/2} \tag{4.36}$$

with $\gamma > 0$, which appears in Abel's equation. Clearly $a(t) \in L^1_{loc}(R^+)$, but $a(t) \notin L^1(R^+)$ so that some but not all of the results which we presented earlier apply. Note that $a(t)$ is non-negative, non-increasing, and convex, and hence by Proposition 16.3.1 in [16] $a(t)$ constitutes a kernel of *positive type*. Moreover, $\frac{d^2}{dt^2}a(t) > 0$ for $t > 0$. Therefore by Proposition 16.4.3 in [16] it is also a kernel of *strong positive type*. Indeed it is easy to check that $a(t)$ satisfies

$$(-1)^j \frac{d^j}{dt^j} a(t) \geq 0, \qquad j = 0,\, 1,\, 2\ldots, \tag{4.37}$$

and thus also constitutes [16, Def. 5.2.1] a kernel of *completely monotone type*. Computing the Laplace transform of b_n, $\hat{b}_n(s)$, expanding it in a Taylor series in $s^{-1/2}$, and inverting term by term, one finds (M. Krush-Bram [25]) that the solution of (4.1) can be written as

$$b_n(t) = b_n(0) \sum_0^\infty \frac{(\gamma \pi^{1/2} \lambda_n)^n}{\Gamma(\frac{3n}{2}+1)}\, t^{\frac{3n}{2}}, \quad t \in [0,\, \infty), \tag{4.38}$$

for all values of $\lambda_n \in R$.

In the unstable case, we may use Lemma 4.4, Corollary 4.5, and Theorem 4.6 to find for (4.1) that the resolvent is non-positive and that $|b_n(t)|$ exhibits monotone growth with $|b_n(t)| \to \infty$ as $t \to \infty$. Moreover, Lemmas 4.8, 4.10, and 4.11 and Remark 4.9 may be used to obtain certain bounds from below on the growth. Lemma 4.3 does not apply since $a(t) \notin L^1(R^+)$, however the bound from above

$$|b_n(t)| \leq \frac{4}{3} |b_n(0)| \exp^{\gamma \pi^{1/2} \lambda_n t^{3/2}}$$

follows readily from the explicit formula (4.38) given above.

In the stable case, we may use Theorem 5.4.1 in [16] to conclude immediately that

Lemma 4.23. *Let $b_n(t)$ be the solution to*

$$\frac{d}{dt} b_n = -a * b_n, \quad b_n(0) \neq 0,$$

where $a(t)$ is the Abel's kernel given in (4.36). Then $b_n(t) \in L^1(R^+)$, $b_n(t) \to 0$ as $t \to 0$, and $b_n(t)$ can be expressed as the sum of an exponentially decaying function and a completely monotone function (i.e., a function which satisfies (4.37)) which decays to zero as t tends to infinity.

5. **Nonlinear stability.** From Theorem 2.1 and Corollary 4.7, we know that if a satisfies Hypothesis I, then
(i) the steady states are the same for (PFM′) as they are for (CPF),
(ii) the index of instability of the steady states is the same for (PFM′) as it is for (CPF).

If we were to know that the dynamics of (PFM$'$) produced a strong gradient system in the sense of Hale [18] which was asymptotically smooth, then it would follow that there was an attractor for (PFM$'$). Moreover in the one-dimensional case, the results of Hattori & Mischaikow and Mischaikow [20, 30] would imply that the flow on this attractor was semi-conjugate to the flow on the attractor of the Chaffee-Infante equation in the parameter regimes where the bifurcation diagram for (PFM$'$) in $1/\eta$ is of Chaffee-Infante type. However in order to have a strong gradient system in the sense of Hale, one must verify that (a) the dynamics of the system yield a strongly continuous $\mathcal{C}^r$-semigroup with $r \geq 1$, (b) the bounded positive orbits are precompact, and (c) there exists a Liapunov function for the semiflow.

However, within the framework of our assumptions, the solution operator for (PFM$'$) (a resolvent in the terminology of Prüss [34]) in general does not have a semigroup structure. It is not obvious how to prove compactification properties of the resolvent. Hence both (a) and (b) in this approach encounter severe difficulties. One could conceivably work in a weaker setting [19] or perhaps adopt the non-autonomous process approach which has been used to prove the existence of an attractor for various related functional differential equations [12, 18]. Another approach is to suitably restrict the memory kernel. In [13], by requiring that the kernels be differentiable with derivatives in $\mathcal{C}(R^+) \cap L^1(R^+)$ and of a specific form, existence of an attractor is proven.

We note that even under weaker assumptions on the kernels, it is possible to obtain some characterization of the long time behavior as the following result indicates.

Theorem 5.1. *If a, $a' \in L^1(R^+)$, $\int_0^\infty a(s)\,ds \neq 0$, and a is a kernel of strong positive type, then for arbitrary initial data $(u_0, \phi_0) \in L^2(\Omega) \times H^1(\Omega)$ the weak $\omega-$limit set of (u_0, ϕ_0) contains only steady states.*

The proof of Theorem 5.1 is given in Appendix A. We remark that if $a \in L^1(R^+)$, $\int_0^\infty a(s)\,ds \neq 0$ and a satisfies Hypothesis II, then the conclusions of Theorem 5.1 are still true.

With regard to (c), the existence of a Liapunov functional, note that

$$\mathcal{F} = \int_\Omega \left\{ \frac{\xi^2}{2}|\nabla\phi|^2 + \frac{1}{4\eta}(\phi^2-1)^2 + \frac{1}{l}u^2 \right\} dx \tag{5.1}$$

acts as a Liapunov functional for (CPF), since within the context of (CPF)

$$\mathcal{F}_t = -\alpha^{-1}||\xi^2\triangle\phi + \eta^{-1}(\phi-\phi^3) + u||^2_{L^2(\Omega)} - 2l^{-1}||\nabla u||^2_{L^2(\Omega)}. \tag{5.2}$$

The analogous result in the context of (PFM$'$) is

$$\mathcal{F}_t = -\alpha^{-1}\langle g(u,\phi), a * g(u,\phi)\rangle - 2l^{-1}\langle \nabla u, a * \nabla u\rangle, \tag{5.3}$$

where

$$g(u,\phi) = \xi^2\triangle\phi + \eta^{-1}(\phi-\phi^3) + u. \tag{5.4}$$

Note that (5.3) is weaker than (5.2), since from (5.2) it follows that

$$\mathcal{F}(t_2) \leq \mathcal{F}(t_1), \quad \forall t_2 \geq t_1 \geq 0, \tag{5.5}$$

whereas the analogous results for (5.3) when a satisfies Hypothesis I is only that

$$\mathcal{F}(t) \leq \mathcal{F}(0), \quad \forall t \geq 0. \tag{5.6}$$

We now point out some explicit conclusions with regard to stability and instability, which can be obtained from (5.3).

The unstable case:

Let us now focus on some specific steady state, $(u_s,\, \phi_s)$, which is unstable according to linear theory; i.e., $\lambda_1 < 0$, and let us consider perturbations $(\tilde{u},\, \tilde{\phi})$ of $(u_s,\, \phi_s)$, setting

$$(u,\, \phi) = (u_s,\, \phi_s) + (\tilde{u},\, \tilde{\phi}). \tag{5.7}$$

As in §3, we will assume the perturbations to preserve the original energy of the system. This implies that $\int_\Omega \tilde{e}\, dx = \int_\Omega [\tilde{u} + (l/2)\tilde{\phi}]\, dx = 0$, as we saw in (3.1). From §2 it follows that $(u_s,\, \phi_s)$ satisfies

$$g(u_s,\, \phi_s) = 0, \quad \Delta u_s = 0, \tag{5.8}$$

where g is defined in (5.4).

Noting (5.7) and (5.8), we may readily calculate that

$$\mathcal{J}(t) := \mathcal{F}(t) - \mathcal{F}_0 = \int_\Omega \left\{ \frac{\xi^2}{2}|\nabla\tilde{\phi}|^2 + \frac{1}{2\eta}[3\phi_s^2 - 1]\tilde{\phi}^2 + \frac{1}{\eta}\phi_s\tilde{\phi}^3 + \frac{1}{4\eta}\tilde{\phi}^4 + \frac{1}{l}\tilde{u}^2 \right\} dx, \tag{5.9}$$

where

$$\mathcal{F}_0 = \int_\Omega \left\{ \frac{\xi^2}{2}|\nabla\phi_s|^2 + \frac{1}{4\eta}(\phi_s^2 - 1)^2 + \frac{1}{l}u_s^2 \right\} dx$$

denotes the energy of the unperturbed steady state, $(u_s,\, \phi_s)$. Expressing the perturbations in terms of the variables from §3, we obtain that

$$\mathcal{J}(t) = \frac{\alpha}{2}\langle \widetilde{\Phi},\, \mathcal{L}\widetilde{\Phi} \rangle + \frac{1}{4\eta}\int_\Omega \{4\phi_s\tilde{\phi}^3 + \tilde{\phi}^4\}\, dx, \tag{5.10}$$

where

$$\widetilde{\Phi} = (\tilde{\phi},\, \tilde{\psi}), \text{ and } \tilde{\psi} := \sqrt{2/(\alpha l)}A^{-1/2}(\tilde{u} + (l/2)\tilde{\phi}). \tag{5.11}$$

It follows from (3.7) and (3.8) that there exists an eigenfunction $\bar{\Phi}_1 = (\bar{\phi}_1,\, \bar{\psi}_1) \in L^2(\Omega) \times L_0^2(\Omega)$ such that

$$\lambda_1 ||\bar{\Phi}_1||^2_{L^2(\Omega)} = \langle \bar{\Phi}_1,\, \mathcal{L}\bar{\Phi}_1 \rangle,$$

and

$$\lambda_1 ||\tilde{\Phi}||^2_{L^2(\Omega)} \le \langle \tilde{\Phi},\, \mathcal{L}\tilde{\Phi} \rangle, \quad \forall \tilde{\Phi} \in L^2(\Omega) \times L_0^2(\Omega).$$

Taking $(\tilde{\phi}_0,\, \tilde{\psi}_0) = \zeta(\bar{\phi}_1,\, \bar{\psi}_1)$ as our initial perturbation yields

$$\mathcal{J}(0) = \lambda_1\zeta^2 ||\bar{\Phi}_1||^2_{L^2(\Omega)} + G(\bar{\Phi}_1;\, \zeta),$$

where

$$G(\bar{\Phi}_1;\, \zeta) = \mathcal{O}(\zeta^3).$$

Since by assumption $\lambda_1 < 0$, taking $0 < \zeta$ sufficiently small, one obtains that

$$\mathcal{J}(0) < \frac{\lambda_1\zeta^2}{2} ||\bar{\Phi}_1||^2_{L^2(\Omega)} = -\delta,$$

where $\delta > 0$. From (5.6) and (5.9), it follows that

$$\mathcal{J}(t) \le \mathcal{J}(0) = -\delta, \quad \delta > 0,$$

or equivalently

$$\mathcal{F}(t) \le \mathcal{F}_0 - \delta, \quad \delta > 0. \tag{5.12}$$

Thus the solution stays bounded (in an energetic sense) away from $(u_s,\, \phi_s)$ for $t \geq 0$, and in this sense linear instability implies instability. In terms of the original variables, we may conclude that

Theorem 5.2. *If $\lambda_1 < 0$ for a given steady state $(u_s,\, \phi_s)$ of (PFM′), then there exists an initial perturbation $(\tilde{u}_0,\, \tilde{\phi}_0) \in L^2(\Omega) \times H^1(\Omega)$ satisfying (3.1), such that the solution to the perturbed problem satisfies*

$$\mathcal{F}(t) \leq \mathcal{F}_0 - \delta, \quad t \geq 0,$$

for some $\delta > 0$, where $\mathcal{F}(t)$ denotes the functional $\mathcal{F}$ evaluated along the solution to the perturbed problem at time t and $\mathcal{F}_0$ denotes the functional $\mathcal{F}$ evaluated on the unperturbed steady state.

<u>The stable case:</u>

Suppose now that $\lambda_1 \geq 0$ for a given steady state $(u_s,\, \phi_s)$, and thus that according to linear stability analysis the steady state under consideration is stable. By coercivity and weak lower semi-continuity, there exists a global minimizer $(\bar{u}_s,\, \bar{\phi}_s)$ of $\mathcal{F}$ in $L^2(\Omega) \times H^1(\Omega)$, which may or may not be unique, such that

$$\mathcal{F}_m := \mathcal{F}|_{(\bar{u}_s,\, \bar{\phi}_s)} \leq \mathcal{F}|_{(u,\, \phi)}, \quad \forall (u,\, \phi) \in L^2_0(\Omega) \times H^1(\Omega),$$

where $\mathcal{F}|_{(\bar{u},\, \bar{\phi})}$ denotes the functional $\mathcal{F}$ evaluated on $(\bar{u},\, \bar{\phi})$. Therefore we obtain that for arbitrary initial conditions $(u(0),\, \phi(0)) \in L^2(\Omega) \times H^1(\Omega)$ which conserve the original internal energy,

$$\mathcal{F}_m \leq \mathcal{F}(t) \leq \mathcal{F}(0), \quad t \geq 0.$$

In terms of $\mathcal{J}$, this implies that

$$\mathcal{J}_m \leq \mathcal{J}(t) \leq \mathcal{J}(0), \quad t \geq 0,$$

where $\mathcal{J}_m = \mathcal{F}_m - \mathcal{F}_0$, and $\mathcal{F}_0$ again denotes the functional $\mathcal{F}$ evaluated on the unperturbed steady state, $(u_s,\, \phi_s)$. Using arguments similar to those used in the linearly unstable case, it can be shown that for $||\widetilde{\Phi}(0)||^2_{L^2(\Omega)}$ sufficiently small, $\mathcal{J}(0)$ is positive. In terms of the original variables, this implies that

$$\mathcal{F}_m \leq \mathcal{F}(t) \leq \mathcal{F}_0 + \delta, \quad t \geq 0, \tag{5.13}$$

with $\delta > 0$.

Suppose now that $\mathcal{F}_0 = \mathcal{F}_m$. For example, in one-dimension when the average of e, the internal energy, is zero, it is known that linearly stable states also minimize the free energy, $\mathcal{F}$ [41, 15]. It then follows from (5.13) that

$$\mathcal{F}_0 \leq \mathcal{F}(t) \leq \mathcal{F}_0 + \delta, \quad t \geq 0,\ \delta > 0,$$

and $(u_s,\, \phi_s)$ is seen to be energetically stable.

If $\mathcal{F}_0 > \mathcal{F}_m$, and if $(u(t),\, \phi(t))$ approaches a steady state as $t \to \infty$, then (5.13) constitutes an energetic restriction on the set of steady states which may be approached. Such a statement, though, does not imply stability of the steady state $(u_s,\, \phi_s)$. This, however, may under certain circumstances be possible to verify using additional tools.

We now provide a result which demonstrates that if the memory kernel is suitably restricted, then linear stability implies stability. For $\omega \in R$, let $\sum(\omega,\, \theta)$ denote the open sector defined by

$$\sum(\omega,\, \theta) := \{\lambda \in \mathbb{C} \mid |\arg(\lambda - \omega)| < \theta,\ \lambda \neq \omega\}.$$

Let $\sigma = \{\lambda_i\}_{i=1}^{\infty}$ refer to the spectrum of $\mathcal{L}$ about a given steady state, and set $\rho = \mathbb{C} \setminus \sigma$.

Theorem 5.3. *Suppose that $(u_s,\, \phi_s)$ constitutes a steady state of (PFM′) which is linearly stable in the context of (CPF). Suppose also that the memory kernel a satisfies Hypothesis I and*

(A1) *$\hat{a}(\lambda)$ admits a meromorphic extension to $\sum(-\lambda_1,\, \pi)$,*

(A2) *$\lambda^{-1}\hat{a}(\lambda) \neq 0$ and $\lambda\hat{a}(\lambda)^{-1} \in \rho(-\mathcal{L})$ for all $\lambda \in \sum(-\lambda_1,\, \pi)$,*

(A3) *For each $\omega < \lambda_1$ and $0 < \theta < \frac{\pi}{2}$, there is a constant $C_1 = C_1(\omega,\, \theta)$ such that $H(\lambda) := (\lambda + \hat{a}(\lambda)\mathcal{L})^{-1}$ satisfies*

$$|H(\lambda)| < C_1 \frac{1}{|\lambda + \omega|}, \;\; \textit{for all } \lambda \in \sum(-\omega,\, \theta + \frac{\pi}{2}),$$

(A4) *There are constants $0 < \beta < \lambda_1$, $0 < \theta_0 < \frac{\pi}{2}$, $c > 0$, and $0 < \alpha < 2$ such that*

$$|\hat{a}(\lambda)| \geq c|\lambda|(|\lambda + \beta|^{\alpha} + 1)^{-1} \;\; \textit{for all } \lambda \in \sum(-\beta,\, \theta_0 + \frac{\pi}{2}).$$

Then for any $(u_0,\, \phi_0) \in L^2(\Omega) \times H^1(\Omega)$ satisfying (3.1), if $(u(t),\, \phi(t)) \in \mathcal{C}(R^+;\, L^2(\Omega) \times H^1(\Omega))$ is a solution to (PFM′) satisfying these initial conditions and $||u_0 - u_s||_{L^2(\Omega)} + ||\phi_0 - \phi_s||_{H^1(\Omega)}$ is sufficiently small, there exists a constant D such that

$$\Big\{||u(t) - u_s||_{L^2(\Omega)} + ||\phi(t) - \phi_s||_{H^1(\Omega)}\Big\} \leq De^{-\beta t}\Big\{||u(0) - u_s||_{L^2(\Omega)} + ||\phi(0) - \phi_s||_{H^1(\Omega)}\Big\}, \tag{5.14}$$

where D is independent of the initial conditions.

We remark that conditions (A1)-(A3) guarantee that the (LPFM′) has a resolvent which is analytic in the sense of Definition 5.5. Condition (A4), which guarantees still further regularity, appears in Pruss [34] as (2.15), although there α is simply required to be positive. The upper bound on α has been introduced here to ensure asymptotic stability in the spaces indicated. Note that the kernel $a(t) = dt^{-1/2}e^{-\lambda_1 t}$ satisfies conditions (A1)-(A4) if d is taken to be a sufficiently small positive constant.

Proof. We proceed by expressing the problem in terms of the variables introduced in §3, verifying a variation of parameters formula for our problem based on a resolvent operator [34], and generalizing the estimates used in proving stability for semi-linear parabolic problems [21]. Note that while in [21] exponential stability of the form $e^{-\beta t}$ is obtained for any $0 < \beta < \lambda_1$, in the present context the exponent β is dictated by (A4).

Set

$$(\tilde{u}(t),\, \tilde{\phi}(t)) = (u(t),\, \phi(t)) - (u_s(t),\, \phi_s(t)),$$

where $(u(t),\, \phi(t))$ and $(u_s,\, \phi_s)$ refer respectively to a solution and to a steady state satisfying the conditions given in the statement of the theorem. In terms of the variables from §3 we may express the steady state as $\Phi_s = (\phi_s,\, \psi_s)$, the solution as $\Phi(t) = (\phi(t),\, \psi(t))$, and the perturbed solution as $\widetilde{\Phi}(t) = (\tilde{\phi}(t),\, \tilde{\psi}(t))$, where

$$\Phi(t) = \Phi_s + \widetilde{\Phi}(t), \quad \widetilde{\Phi}(t) \in V,$$

where $V := H^1(\Omega) \times H_0^1(\Omega)$ and $H_0^1(\Omega) = H^1(\Omega) \cap L_0^2(\Omega)$.

In terms of these variables, (PFM′) may be written as

$$\widetilde{\Phi}_t = -a * \mathcal{L}\widetilde{\Phi} + a * \mathcal{G}(\widetilde{\Phi};\, \Phi_s), \tag{5.15}$$

where

$$\mathcal{G}(\widetilde{\Phi};\,\Phi_s) = -\frac{1}{\alpha\eta}(3\phi_s\tilde{\phi}^2 + \tilde{\phi}^3)\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.16}$$

Integrating, (5.15) yields

$$\widetilde{\Phi}(t) = -1 * a * \mathcal{L}\widetilde{\Phi} + f(t), \tag{5.17}$$

with

$$f(t) = \widetilde{\Phi}(0) + 1 * a * \mathcal{G}(\widetilde{\Phi};\,\Phi_s). \tag{5.18}$$

Since by assumption $\widetilde{\Phi}(0) \in V$, and since $\Phi_s \in V$, it follows from the existence results quoted in §1 that there exists a solution $\widetilde{\Phi} \in \mathcal{C}(R^+;\,V)$ to (5.15). Therefore since we have assumed that $a \in L^1(R^+)$, it follows from (5.18) and Young's inequality that $f \in W^{1,\,1}_{loc}(R^+;\,X)$. If we can verify that (5.17) is *well posed* in the sense of Definition 1.2 in [34], then we may implement results from [34] to express our solution in a convenient form. Setting $X := L^2(\Omega) \times L^2_0(\Omega)$, $W := H^2(\Omega) \times H^2_0(\Omega)$ where $H^2_0(\Omega) = H^2(\Omega) \cap L^2_0(\Omega)$, and $\nu = 1 * a$, and paraphrasing [34, Definition 1.2]:

Definition 5.4. *Equation (5.17) is well posed if for each $x \in \mathcal{D}(\mathcal{L})\,(= W)$, there exists a unique solution $\widetilde{\Phi} \in \mathcal{C}(J;\,X_{\mathcal{L}})$ to*

$$\widetilde{\Phi} = x - \nu * \mathcal{L}\widetilde{\Phi}, \tag{5.19}$$

where $J = [0, T]$, $0 < T < \infty$, and $X_{\mathcal{L}} := \mathcal{D}(\mathcal{L})$ with the graph norm $||x||_{\mathcal{L}} := ||x||_X + ||\mathcal{L}x||_X$, and $\{x_n\} \subset \mathcal{D}(\mathcal{L})$, $x_n \to 0$ implies that $\widetilde{\Phi}(t;\,x_n) \to 0$ in X uniformly on compact intervals, where $\tilde{\Phi}(t;\,x_n)$ denotes the solution to (5.19) satisfying $\widetilde{\Phi}(0;\,x_n) = x_n$.

In order to demonstrate well-posedness, let us write (5.19) as

$$\widetilde{\Phi}_t = -a * \mathcal{L}\widetilde{\Phi}, \quad \widetilde{\Phi}(0) = x. \tag{5.20}$$

Existence of a solution $\widetilde{\Phi} \in \mathcal{C}(J;\,V)$ may be proven as in [32], and uniqueness of this solution may be demonstrated as in §3. The required additional regularity may be achieved by multiplying (5.20) by $\mathcal{L}^2\widetilde{\Phi}$ and using the self-adjointness of $\mathcal{L}$ and the assumption that a is of positive type to obtain the estimate

$$||\mathcal{L}\widetilde{\Phi}(t)||_X \le ||\mathcal{L}\widetilde{\Phi}(0)||_X.$$

Integrating (5.20), well-posedness of (5.17) now follows easily.

Since we have also seen that $f \in W^{1,1}_{loc}(R^+;\,X)$, we may now use the results of Propositions 1.1 and 1.2 in [34] to obtain that the solution to (5.17)-(5.18) may be written in terms of a resolvent $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ as

$$\widetilde{\Phi}(t) = S(t)\widetilde{\Phi}(0) + S * \{a * \mathcal{G}(\widetilde{\Phi};\,\Phi_s)\}, \tag{5.21}$$

where $S(t)$, the resolvent or solution operator for (5.17), satisfies

(S1) $S(t)$ is strongly continuous on R^+ and $S(0) = I$,

(S2) $S(t)\mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{L})$ and $\mathcal{L}S(t)x = S(t)\mathcal{L}x, \quad \forall x \in \mathcal{D}(\mathcal{L}),\ t \ge 0,$

(S3) $S(t)$ satisfies $S(t)x = x - a * \mathcal{L}S, \quad \forall x \in \mathcal{D}(\mathcal{L}),\ t \ge 0.$

(S4) $a * S(t)X \subset \mathcal{D}(\mathcal{L})$, $t \ge 0$,

(S5) $S(t)x = x - \mathcal{L}(a * Sx), \quad \forall x \in X,\ t \ge 0.$

In particular $\mathcal{L}(a * S)$ is strongly continuous on X. We remark that it is also readily proven that

(S2′) $\mathcal{L}^{1/2}S(t)x = S(t)\mathcal{L}^{1/2}x, \quad \forall x \in \mathcal{D}(\mathcal{L}^{1/2}),\ t \geq 0.$
(S2′) can be proven by first demonstrating, as in the proof of (S2), that the equality holds for all $x \in \mathcal{D}(\mathcal{L})$, and then using an approximation argument.

The resolvent which is obtained here can be considered as a generalization of the resolvent obtained in the parabolic case, (CPF), when $a(t) = \delta(t)$. In the parabolic case, one has that $S(t) = e^{-\mathcal{L}t}$ is an analytic semigroup, which possesses certain regularity properties; i.e., it can be shown that for any $\beta < \lambda_1$ there exists a constant $C \geq 1$ such that for any $x \in V$ and all $t > 0$,

$$(a)\ ||S(t)x||_V \leq Ce^{-\beta t}||x||_V, \quad \forall t > 0, (b)\ ||S(t)x||_V \leq Ct^{-1/2}e^{-\beta t}||x||_X. \quad (5.22)$$

See e.g. [21, Chapter 4] for a discussion of the semi-linear parabolic setting. In the Volterra context, if we simply require that $a \in L^1(R^+)$, $a \not\equiv 0$, and that a is of positive type, then as noted earlier $S(t)$ cannot be expected to have a semigroup structure. Thus, in particular, $S(t)$ cannot be expected to generate an analytic semigroup and estimates such as (5.22) cannot be concluded to hold. However, by requiring that $a(t)$ satisfy the conditions (A1)-(A3) from Theorem 5.3, we get that (5.17) admits an *analytic resolvent* and that an estimate such as (5.22a) does hold.

Definition 5.5. *A resolvent for (5.17) is said to be analytic if* $S(\cdot)\colon R^+ \to \mathcal{B}(X)$ *admits analytic extension to a sector* $\sum(0,\, \theta)$ *for some* $0 < \theta \leq \frac{\pi}{2}$. *An analytic resolvent* $S(t)$ *is said to be of analyticity type* $(\omega,\, \theta)$ *if for each* $0 < \theta_1 < \theta$ *and* $\omega_1 > \omega$ *there is a* $C_1 = C_1(\omega_1,\, \theta_1)$ *such that*

$$|S(z)| \leq C_1 e^{\omega_1 Re z}, \quad z \in \sum(0,\, \theta_1). \quad (5.23)$$

In the present context, we may rely on the following theorem to conclude that $S(t)$ is an analytic resolvent and of analyticity type $(-\lambda_1,\, \pi/2)$.

Theorem 5.6. [34, Theorem 2.1] *Let* $\mathcal{L}$ *be a closed linear unbounded operator on* X *with dense domain* $\mathcal{D}(\mathcal{L})$ *and let* $a \in L^1_{loc}(R^+)$ *satisfy* $\int_0^\infty |a(t)|e^{-\omega_a t}\, dt < \infty$ *for some* $\omega_a \in R$. *Then (5.17) admits an analytic resolvent* $S(t)$ *of analyticity type* $(-\lambda_1,\, \frac{\pi}{2})$ *iff conditions (A1)-(A3) from Theorem 5.3 hold.*

Since $\mathcal{L}^{1/2}\widetilde{\Phi} \in X$ if $\widetilde{\Phi} \in V$, it follows from (S2′) and Theorem 5.6 that for any $0 < \beta < \lambda_1$, $0 < \theta < \frac{\pi}{2}$,

$$||\mathcal{L}^{1/2}S(t)\widetilde{\Phi}||_X = ||S(t)\mathcal{L}^{1/2}\widetilde{\Phi}||_X \leq C_1 e^{-\beta t}||\mathcal{L}^{1/2}\widetilde{\Phi}||_X, \quad t > 0. \quad (5.24)$$

Moreover it is easy to demonstrate that there exist constants D_1 and D_2, which may depend on Ω and the parameters appearing in the definition of $\mathcal{L}$, such that for any $\widetilde{\Phi} \in V$,

$$D_1||\widetilde{\Phi}||_V - D_2||\widetilde{\Phi}||_X \leq ||\mathcal{L}^{1/2}\widetilde{\Phi}||_X \leq D_1||\widetilde{\Phi}||_V. \quad (5.25)$$

From (5.23)-(5.25) and the continuous embedding of V in X, it readily follows that for any $\widetilde{\Phi} \in V$

$$||S(t)\widetilde{\Phi}||_V \leq \widetilde{C}_1 e^{-\beta t}||\widetilde{\Phi}||_V, \quad t \geq 0, \quad (5.26)$$

where $\widetilde{C}_1$ may depend on β, θ as well as on Ω and the parameters in $\mathcal{L}$. Note that (5.26) parallels (5.22a).

To obtain an estimate similar to (5.22b), we rely on the assumption that a satisfies (A4) and on the theorem below which generalizes Theorem 2.2 in [34].

Theorem 5.7. *Suppose that $S(t)$ is an analytic resolvent for (5.17) of type $(-\lambda_1,\, \frac{\pi}{2})$, and suppose that there are constants $0 < \beta < \lambda_1$, $0 < \theta < \pi/2$, $c > 0$, $\alpha > 0$ such that*

$$|\hat{a}(\lambda)| \geq c|\lambda|(|\lambda + \omega|^{\alpha} + 1)^{-1}, \quad \forall \lambda \in \sum(-\beta,\, \theta + \frac{\pi}{2}), \tag{5.27}$$

then for any $\gamma \in [0,\, 1]$ there is a constant $C_2 = C_2(\beta,\, \theta,\, \gamma)$ such that

$$|\mathcal{L}^{\gamma} S(t)| \leq C_2 e^{-\beta t}(1 + t^{-\alpha\gamma}), \quad t > 0, \tag{5.28}$$

and $S(t)X \subset \mathcal{D}(\mathcal{L})$.

Proof. The proof of (5.28) for $\gamma = 1$ can be found in [34, Theorem 2.2], and the statement that $S(t)X \subset \mathcal{D}(\mathcal{L})$ then follows. A very similar proof can be given for $\gamma \in [0,\, 1)$. Both are based on the formula

$$\mathcal{L}^{\gamma} S(z) = (2\pi i)^{-1} \int_{\Gamma_R} e^{\lambda z} \mathcal{L}^{\gamma} H(\lambda)\, d\lambda, \quad z \in \sum(0,\, \theta), \tag{5.29}$$

for $\gamma \in [0,\, 1]$ where Γ_R denotes the contour in the complex plane consisting of the two rays $\omega + ire^{i\theta'}$ and $\omega - ire^{i\theta'}$ with $r \geq R$, $0 < \theta' < \theta$, and the larger part of the circle $|\lambda - \omega| = R$ connecting these rays. The estimate (5.28) may be obtained by noting that $\mathcal{L}H = [H(\lambda) - 1/\lambda]\frac{\lambda}{\hat{a}(\lambda)}$, then writing $\mathcal{L}^{\gamma} H(\lambda)$ as

$$\mathcal{L}^{\gamma} H(\lambda) = \mathcal{L}^{\gamma} H^{\gamma}(\lambda) H^{1-\gamma}(\lambda),$$

and using the estimates in (A3) and (5.27) to obtain bounds and evaluate (5.29). $\square$

From (5.28) and (A4), it follows that for some $0 < \beta < \lambda_1$, $0 < \theta_0 < \frac{\pi}{2}$, $0 < \alpha < 2$,

$$||\mathcal{L}^{1/2} S\widetilde{\Phi}||_X \leq C_2 e^{-\beta t}(1 + t^{-\alpha/2})||\widetilde{\Phi}||_X, \quad t > 0.$$

Hence for some $0 < \beta < \lambda_1$, $0 < \theta_0 < \frac{\pi}{2}$, $0 < \alpha < 2$, we find using (5.23),(5.25) that

$$||S\widetilde{\Phi}||_V \leq \widetilde{C}_2 e^{-\beta t}(1 + t^{-\alpha/2})||\widetilde{\Phi}||_X, \quad t > 0, \tag{5.30}$$

which generalizes (5.22b). Here $\widetilde{C}_2$ depends on β and θ_0, as well as on the parameters D_1, D_2 from (5.25).

With estimates (5.26) and (5.30) in hand, we may now proceed roughly as in the stability proof given in [21, Theorem 5.1.1], making certain adjustments for the appearance of the memory kernel in the equation. Let $0 < \beta' < \beta < \lambda_1$ and $0 < \theta_0 < \frac{\pi}{2}$, where β and θ_0 have been chosen in accordance with (A4) and (5.30). Note that (A1) and Hypothesis I imply that

$$||a||_{L^1(R^+)} \leq \int_0^{\infty} a(q) e^{\beta q}\, dq = \hat{a}(-\beta) < \infty.$$

Because $\alpha < 2$, we may choose $\sigma > 0$ so small that

$$M\,\Gamma\,\sigma \int_0^{\infty} (1 + s^{-\alpha/2}) e^{-(\beta - \beta')s}\, ds < \frac{1}{2}, \tag{5.31}$$

where $M := \max\{\widetilde{C}_1(\beta, \theta_0),\, \widetilde{C}_2(\beta, \theta_0),\, 1\}$ and $\Gamma := ||a||_{L^1(R^+)}$. Since it is readily seen that there exist constants $C_3 = C_3(\Phi_s)$ and $\delta' > 0$ such that

$$||\mathcal{G}(y;\, \Phi_s)||_X \leq C_3 ||y||_V^{1+\delta'}, \quad \forall y \in V,$$

we may choose $\delta > 0$ so small that

$$||\mathcal{G}(y;\, \Phi_s)||_X \leq \sigma ||y||_V, \tag{5.32}$$

for any $y \in V$ such that $||y||_V \leq \delta$.

We first prove that if $||\widetilde{\Phi}(0)||_V \leq \delta/(2M)$, then the solution stays in the ball $||\widetilde{\Phi}(t)||_V \leq \delta$ for all $t \geq 0$. This is accomplished by noting that if $||\widetilde{\Phi}(0)||_V \leq \delta/(2M)$, then since $\widetilde{\Phi} \in \mathcal{C}(R^+; V)$, the solution satisfies $||\widetilde{\Phi}(t)||_V < \delta$ on some finite positive time interval, $t \in [0, T)$. Let $T_{\max} := \sup\{T \,|\, ||\widetilde{\Phi}(t)||_V < \delta,\ 0 < t < T\}$, and suppose that $T_{\max} < \infty$. Then for $t \in [0, T_{\max}]$, we have from (5.21) using (5.26), (5.30), and (5.32) that

$$\begin{aligned}
||\widetilde{\Phi}(t)||_V &= ||S(t)\widetilde{\Phi}(0) + S * a * \mathcal{G}(\widetilde{\Phi};\, \Phi_s)||_V \\
&\leq ||S(t)\widetilde{\Phi}(0)||_V + ||S * a * \mathcal{G}||_V \\
&\leq Me^{-\beta t}||\widetilde{\Phi}(0)||_V + M\int_0^t (1 + (t-s)^{-\alpha/2})e^{-\beta(t-s)}||a * \mathcal{G}(s)||_X\, ds \\
&\leq Me^{-\beta t}||\widetilde{\Phi}(0)||_V \\
&\quad + M\int_0^t \left(1 + (t-s)^{-\alpha/2}\right) e^{-\beta(t-s)}||a||_{L^1(R^+)} \sup_{(0,\,t)} ||\mathcal{G}||_X\, ds \\
&\leq Me^{-\beta t}||\widetilde{\Phi}(0)||_V + M||a||_{L^1(R^+)}\sigma\delta \int_0^t (1 + (t-s)^{-\alpha/2})e^{-\beta(t-s)}\, ds,
\end{aligned}$$

and thus by (5.31) we obtain in particular that

$$||\widetilde{\Phi}(T_{\max})||_V \leq \frac{\delta}{2}e^{-\beta T_{\max}} + \frac{\delta}{2} < \delta. \tag{5.33}$$

Since $\widetilde{\Phi} \in \mathcal{C}(R^+; V)$, (5.33) contradicts the maximality of $T_{\max}$, and hence $T_{\max} = \infty$.

Next, defining $u(t) := \sup\{||\widetilde{\Phi}(s)||_V\, e^{\beta' s},\ 0 \leq s \leq t\}$, we get that

$$\begin{aligned}
||\widetilde{\Phi}(t)||_V &\leq Me^{-\beta t}||\widetilde{\Phi}(0)||_V \\
&\quad + M\,\sigma u(t)\int_0^t (1 + (t-s)^{-\alpha/2})e^{-\beta(t-s)}\int_0^s a(s-\tau)e^{-\beta'\tau}\, d\tau\, ds.
\end{aligned}$$

Hence

$$u(t) \leq Me^{-(\beta-\beta')t}||\widetilde{\Phi}(0)||_V + M\,\Gamma\,\sigma u(t)\int_0^t (1 + (t-s)^{-\alpha/2})e^{-(\beta-\beta')(t-s)}\, ds.$$

Therefore using (5.31) we obtain that

$$\frac{1}{2}u(t) \leq Me^{-(\beta-\beta')t}||\widetilde{\Phi}(0)||_V,$$

from which (5.14) readily follows. □

6. **Appendix A. Proof of Theorem 5.1:**

Integrating (5.3) we obtain that

$$\mathcal{F}(t) + \alpha^{-1}\int_0^t \langle g(u,\,\phi),\, a * g(u,\,\phi)\rangle\, d\tau + 2l^{-1}\int_0^t \langle \nabla u,\, a * \nabla u\rangle\, d\tau = \mathcal{F}(0),$$

where

$$\mathcal{F}(t) = \int_\Omega \Big\{\frac{\xi^2}{2}|\nabla\phi|^2 + \frac{1}{4\eta}(\phi^2 - 1)^2 + \frac{1}{l}u^2\Big\}\, dx, \quad g(u,\,\phi) = \xi^2\triangle\phi + \frac{1}{\eta}(\phi - \phi^3) + u.$$

Since by assumption $(u_0,\, \phi_0) \in L^2(\Omega) \times H^1(\Omega)$ and a is a kernel of positive type, we obtain that

$$\mathcal{F}(t) \leq C_0,\ \int_0^t \langle g(u,\,\phi),\, a * g(u,\,\phi)\rangle\, d\tau \leq C_0,\ \int_0^t \langle \nabla u,\, a * \nabla u\rangle\, d\tau \leq C_0, \tag{A.1}$$

where C_0 denotes a generic constant whose value can change from line to line, but which depends only on the initial conditions, on the parameters of the problem, and possibly on the domain, Ω.

From (A.1) and (1.1) (see condition $(*)$ just beneath (1.1)), it follows that

$$||u||_{L^\infty(R^+;\,L^2(\Omega))} \le C_0, \quad ||\phi||_{L^\infty(R^+;\,H^1(\Omega))} \le C_0, \tag{A.2}$$

$$||a * \nabla u||_{L^2(R^+;\,L^2(\Omega))} \le C_0, \quad ||e^{-t} * \nabla u||_{L^2(R^+;\,L^2(\Omega))} \le C_0, \tag{A.3}$$

$$||a * g(u,\,\phi)||_{L^2(R^+;\,L^2(\Omega))} \le C_0, \quad ||e^{-t} * g(u,\,\phi)||_{L^2(R^+;\,L^2(\Omega))} \le C_0. \tag{A.4}$$

From (PFM$'$), we see that (A.3), (A.4) imply that

$$||u_t||_{L^2(R^+;\,H^{-1}(\Omega))} \le C_0, \quad ||\phi_t||_{L^2(R^+;\,L^2(\Omega))} \le C_0. \tag{A.5}$$

From (A.2) and (A.5) it follows that $(u,\,\phi)$ is sequentially weakly precompact in $L^2(\Omega) \times H^1(\Omega)$, as well as sequentially precompact in $H^{-1}(\Omega) \times L^2(\Omega)$, where H^{-1} denotes the dual space of $H^1(\Omega)$. Let $(u_\infty,\,\phi_\infty) \in L^2(\Omega) \times H^1(\Omega)$ denote a limit point, and let t_N, $N = 1,\,2,\,\ldots,$ denote an increasing sequence such that

$$u(x,\,t_N) \to u_\infty, \quad \phi(x,\,t_N) \to \phi_\infty, \qquad \text{as } N \to \infty, \tag{A.6}$$

strongly in $H^{-1}(\Omega)$ and $L^2(\Omega)$ respectively. Our goal is to demonstrate that $(u_\infty,\,\phi_\infty)$ constitutes a steady state of (PFM$'$).

We now define the translates

$$u^N(x,\,t) = u(x,\,t_N + t), \quad \phi^N(x,\,t) = \phi(x,\,t_N + t), \quad t \ge 0,$$

as well as

$$\Psi^N(x,\,t) = \Psi(x,\,t_N + t), \quad t \ge 0, \tag{A.7}$$

where $\Psi(x,\,t)$ denotes an arbitrary function which is well defined on $\Omega \times R^+$.

Let us first focus on the implication of the estimates above for u. We proceed here roughly as in the proof of Theorem 3.3 in [1]. Defining $y = e^{-t} * (u - \bar{u})$ where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx$, we see that y satisfies

$$y_t + y = (u - \bar{u}), \quad t \ge 0, \quad y(0) = 0. \tag{A.8}$$

From (A.2)-(A.3) and Young's inequality, it follows that

$$y \in L^\infty(R^+;\,L^2(\Omega)) \cap L^2(R^+;\,H^1(\Omega)). \tag{A.9}$$

Multiplying (A.8) by y_t, integrating over space and time, and integrating the last term by parts, we obtain the estimate

$$\int_0^t \int_\Omega y_t^2\,dx\,d\tau \le ||(u - \bar{u})(t)||_{L^2(\Omega)}\,||y(t)||_{L^2(\Omega)} + \int_0^t ||u_t||_{H^{-1}(\Omega)}\,||y||_{H^1(\Omega)}\,d\tau.$$

Hence $y_t \in L^2(R^+;\,L^2(\Omega))$. Returning to (A.8) and noting (A.9), we see that $u - \bar{u} \in L^2(R^+;\,L^2(\Omega))$. Together with (A.2) and (A.5), we see that $u^N - \bar{u}^N \to 0$ in $\mathcal{C}([0,\,T];\,H^{-1}(\Omega))$, for any $0 < T < \infty$. Noting that by (A.5), $\bar{u}^N(t) - \bar{u}^N(0) \to 0$ uniformly as $N \to \infty$ for $t \in [0,\,T]$, and by (A.6), $\bar{u}^N(0) \to \bar{u}_\infty$, we obtain that

$$u^N(x,\,t) \to u_\infty = \bar{u}_\infty \quad \text{as } N \to \infty \text{ in } \mathcal{C}([0,\,T];\,H^{-1}(\Omega)),\ \forall\ 0 < T < \infty. \tag{A.10}$$

We now consider the long time behavior of ϕ. We claim that

$$||g(u,\,\phi)||_{L^\infty(R^+;H^{-1}(\Omega))} \le C_0, \quad ||g_t||_{L^2(R^+;\,W^{-2,\,1}(\Omega))} \le C_0. \tag{A.11}$$

The first estimate follows easily from (A.1) and the latter estimate follows from (A.2) and (A.5) by noting that for any $t > 0$

$$||(\phi^3)_t||^2_{L^2(0,\,t;\,L^1(\Omega))} \le 9 \int_0^t ||\phi_t||^2_{L^2(\Omega)}\, ||\phi||^4_{L^4(\Omega)}\, d\tau \le C_0 ||\phi_t||^2_{L^2(R^+;\,L^2(\Omega))}.$$

Defining $g^N(u,\,\phi)$ in accordance with (A.7), from (A.11) we obtain for any $0 < T < \infty$ that along subsequences

$$g^{N'} \to g_\infty \quad \text{as} \quad N' \to \infty \quad \text{in} \quad \mathcal{C}([0,\,T];\,W^{-2,1}(\Omega)), \tag{A.12}$$

where g_∞ is independent of time.

Let us now define $Y = e^{-t} * g(u,\,\phi)$, and note that

$$Y_t + Y = g(u,\,\phi), \quad t \ge 0, \quad Y(0) = 0. \tag{A.13}$$

Since by assumption a is of strong positive type and since e^{-t} satisfies all the assumptions made on a, it follows from (A.1) that

$$||Y||_{L^2(R^+;\,L^2(\Omega))} \le C_0, \tag{A.14}$$

and from Young's inequality and (A.11), it follows that

$$||Y||_{L^\infty(R^+;\,H^{-1}(\Omega))} \le C_0.$$

Referring to (A.13) and (A.11), we obtain that

$$||Y_t||_{L^\infty(R^+;\,H^{-1}(\Omega))} \le C_0. \tag{A.15}$$

Upon defining Y^N and Y_t^N in accordance with (A.7), it follows from (A.14) and (A.15) that

$$Y^N(x,t) \to 0 \quad \text{as} \quad N \to \infty \quad \text{in} \quad \mathcal{C}([0,\,T];\,H^{-1}(\Omega)).$$

In particular

$$Y^N(x,t) \to 0 \quad \text{as} \quad N \to \infty \quad \text{in} \quad \mathcal{C}([0,\,T];\,W^{-2,1}(\Omega)). \tag{A.16}$$

We now show that $g_\infty = 0$. This is accomplished by noting that

$$\begin{aligned} e^{-t} * ||g_\infty||_{W^{-2,1}(\Omega)} &\le e^{-t} * ||g_\infty - g^{N'}||_{W^{-2,1}(\Omega)} \\ &\quad + ||e^{-t} * g^{N'} - (e^{-t} * g)^{N'}||_{W^{-2,1}(\Omega)} + ||(e^{-t} * g)^{N'}||_{W^{-2,1}(\Omega)}, \end{aligned}$$

and estimating the terms on the right hand side as in the proof of Lemma 19 (see also [6, 5]). Thus

$$\phi^N - (\phi^N)^3 + \xi^2 \triangle \phi^N + u^N \to 0 \quad \text{as} \quad N \to \infty \quad \text{in} \quad \mathcal{C}([0,\,T];\,W^{-2,1}(\Omega)).$$

Noting (A.2), (A.5), and (A.6), we obtain that

$$\phi^N + \xi^2 \triangle \phi^N + u^N \to \phi_\infty + \xi^2 \triangle \phi_\infty + u_\infty \quad \text{as} \quad N \to \infty \quad \text{in} \quad \mathcal{C}([0,\,T];\,H^{-2}(\Omega)).$$

Arguing, for example, as in [33], it follows that

$$(\phi^N)^3 \to \phi^3_\infty \quad \text{as} \quad N \to \infty \quad \text{in} \quad \mathcal{C}([0,\,T];\,W^{-2,1}(\Omega)).$$

Finally recalling from (A.10) that $u_\infty = \bar{u}_\infty$, we obtain that $(u_\infty,\,\phi_\infty)$ satisfies

$$\phi_\infty - \phi^3_\infty + \xi^2 \triangle \phi_\infty + \bar{u}_\infty = 0. \tag{A.17}$$

This completes the proof of Theorem 5.1.

Remark 6.1. *Since $u_\infty = \bar{u}_\infty$, and noting by (PFM′) that $\bar{u}(t) + \frac{l}{2}\bar{\phi}(t) = \bar{u}_0 + \frac{l}{2}\bar{\phi}_0$ for all $t \geq 0$, it follows from (A.6) that $\bar{u}_\infty + \frac{l}{2}\bar{\phi}_\infty = \bar{u}_0 + \frac{l}{2}\bar{\phi}_0$, and therefore (A.17) may be written as*

$$\phi_\infty - \phi_\infty^3 + \xi^2 \triangle \phi_\infty - \frac{l}{2}\bar{\phi}_\infty = -\bar{u}_0 - \frac{l}{2}\bar{\phi}_0. \tag{A.18}$$

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