

Parametric Polymorphism — Universally^{*}

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Abstract. In the 1980s, John Reynolds postulated that a parametrically polymorphic function is an ad-hoc polymorphic function satisfying a uniformity principle. This allowed him to prove that his set-theoretic semantics has a relational lifting which satisfies the *Identity Extension Lemma* and the *Abstraction Theorem*. However, his definition (and subsequent variants) have only been given for specific models. In contrast, we give a model-independent axiomatic treatment by characterising Reynolds’ definition via a universal property, and show that the above results follow from this universal property in the axiomatic setting.

1 Introduction

A polymorphic function is *parametric* if its behaviour is uniform across all of its type instantiations [18]. Reynolds [16] made this mathematically precise by formulating the notion of *relational parametricity*, and gave a set-theoretic model, where polymorphic programs are required to preserve all relations between instantiated types. Relational parametricity has proven to be one of the key techniques for formally establishing properties of software systems, such as representation independence [2,6], equivalences between programs [11], or deriving useful theorems about programs from their type alone [20].

In Reynolds’ original model of parametricity, every type constructor T of System F with n free type variables is represented not just by a functor $\llbracket T \rrbracket_0 : |\mathbf{Set}|^n \rightarrow \mathbf{Set}$, but also by a functor $\llbracket T \rrbracket_1 : |\mathbf{Rel}|^n \rightarrow \mathbf{Rel}$.¹ Notice how both of these functors have as domain discrete categories; this ensures that i) contravariant type expressions can be interpreted functorially; and ii) that the functorial interpretation of function types can be defined pointwise. The interpretation is given by induction on the structure of the type T . When T is a function type, say $T = U \rightarrow V$, we have

$$\begin{aligned} \llbracket U \rightarrow V \rrbracket_0 \vec{A} &= \llbracket U \rrbracket_0 \vec{A} \rightarrow \llbracket V \rrbracket_0 \vec{A} \\ (f, g) \in \llbracket U \rightarrow V \rrbracket_1 \vec{R} &\text{ iff } (a, b) \in \llbracket U \rrbracket_1 \vec{R} \Rightarrow (fa, gb) \in \llbracket V \rrbracket_1 \vec{R} \end{aligned} \tag{1}$$

Not only are the above definitions empirically natural, but they are also supported by universal properties. Indeed, $\llbracket U \rightarrow V \rrbracket_0$ and $\llbracket U \rightarrow V \rrbracket_1$ are in fact exponential

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¹ The category \mathbf{Rel} has as objects relations and as morphisms functions which preserve relatedness. This category will be introduced in detail in Section 2.

objects in their respective functor categories. The situation is less clear for \forall -types. If we denote the equality relation on the set X by EqX , and lift that notation to tuples of types, then Reynolds interpretation of \forall -types is as follows:

$$\begin{aligned} \llbracket \forall X.T \rrbracket_0 \vec{A} &= \{f : \prod_{X:\text{Set}} \llbracket T \rrbracket_0(\vec{A}, X) \mid \forall R \in \text{Rel}(A, B). (fA, fB) \in \llbracket T \rrbracket_1(Eq \vec{A}, R)\} \\ (f, g) \in \llbracket \forall X.T \rrbracket_1 \vec{R} &\text{ iff } \forall R \in \text{Rel}(A, B). (fA, gB) \in \llbracket T \rrbracket_1(\vec{R}, R) \end{aligned} \quad (2)$$

While these definitions are empirically natural, conforming to the intuition that related inputs are mapped to related outputs, and work in the sense that key theorems such as the Identity Extension Lemma and the Abstraction Theorem can be proved from them, they lack a theoretical justification as to why they are the way they are. That is,

Are there universal properties underpinning the definition of $\llbracket \forall X.T \rrbracket_0 \vec{A}$ and $\llbracket \forall X.T \rrbracket_1 \vec{R}$? Can these universal properties be used to prove the Identity Extension Lemma and Abstraction Theorem in an axiomatic manner that is independent of specific models?

This paper answers the above questions positively, for a class of models axiomatically built from subobject fibrations. We comment on an extension to a more general class of fibrations in the conclusion. We believe this is of interest because the notion of a universal property is a fundamental mechanism used to give categorical characterisations of key objects in mathematics, logic and computer science. Universal properties extract the core essence of structure. We believe Reynolds' definition of parametrically polymorphic functions is important enough to have its core essence uncovered.

Related Work: There is a significant body of work on the foundations of parametricity and, like us, many take a fibrational perspective. This can be traced back to the work of Hermida in his highly influential thesis [9] and subsequent work [10]. Other important work includes that of Reynolds and Ma [14], who gave the first categorical framework for *parametric* polymorphism, Dunphy and Reddy [7], who mixed fibrations with reflexive graphs, and Birkedal and Møgelberg [4], who gave detailed and sophisticated models of not just parametricity but also its logical structure. However, none of these papers tackles the question we tackle in this paper. Indeed, many follow the modern trend to *bake in* Identity Extension into their framework. In contrast, we dig deeper and prove the identity extension property from more primitive assumptions. Our own paper on parametric models [13] follows in the fibrational tradition, but distinguishes itself by using *bifibrations*. Since our work here requires bifibrations, this paper builds on the model presented there, and therefore further validates it.

Structure of paper: We first review Reynolds' model, and recast his definitions in a form suitable for generalisation in Section 2. We assume familiarity with category theory, but give a brief introduction to fibrations in Section 3, as well as our framework for models of System F. In Section 4, we instantiate it to the subobject fibration, and show that the expected properties hold. Finally we conclude in Section 5.

2 Reynolds' Parametrically Polymorphic Functions

We assume the reader is familiar with the syntax of System F and recall only those parts we need for our development — see e.g. [8] for more details. In particular, the type judgements of System F are generated as follows

$$\frac{X_i \in \Gamma}{\Gamma \vdash X_i \text{ Type}} \quad \frac{\Gamma \vdash T \text{ Type} \quad \Gamma \vdash U \text{ Type}}{\Gamma \vdash T \rightarrow U \text{ Type}} \quad \frac{\Gamma, X \vdash T \text{ Type}}{\Gamma \vdash \forall X. T \text{ Type}}$$

where Γ is a set of type variables. The term judgements of System F are of the form $\Gamma; \Delta \vdash t : T$ where T is a System F type definable in the context Γ and Δ is a term context associating distinct variables to a collection of types, each of which is also definable in Γ .

We write **Set** for the category of sets. Some care is needed here: in a metatheory using classical logic, there are no non-trivial set-theoretic parametric models of System F [17]. Instead, we should understand the category of sets e.g. internally to the Calculus of Constructions [5] with impredicative **Set** (see also Pitts [15] for other options). We further write **Rel** for the category whose objects are relations, i.e. subsets $R \subseteq A \times B$, and whose morphisms $(R \subseteq A \times B) \rightarrow (R' \subseteq A' \times B')$ consist of functions $(f : A \rightarrow A', g : B \rightarrow B')$ such that if $(a, b) \in R$, then $(fa, gb) \in R'$. In this case we say that the morphism in **Rel** is over the pair (f, g) . We write $U : \mathbf{Rel} \rightarrow \mathbf{Set} \times \mathbf{Set}$ for the functor defined by $U(R \subseteq A \times B) = (A, B)$, which we note is faithful. If $F, G : |\mathbf{Set}| \rightarrow \mathbf{Set}$ and $H : |\mathbf{Rel}| \rightarrow \mathbf{Rel}$ are functors such that $U \circ H = (F \times G) \circ U$, then we say that H is over (F, G) . We extend the notion of being over to natural transformations.

Using formulas (1) and (2) from the introduction, Reynolds gives a two level semantics for System F where, if $\Gamma \vdash T \text{ Type}$ and $|\Gamma| = n$, then $\llbracket T \rrbracket_0 : |\mathbf{Set}|^n \rightarrow \mathbf{Set}$ and $\llbracket T \rrbracket_1 : |\mathbf{Rel}|^n \rightarrow \mathbf{Rel}$ with $\llbracket T \rrbracket_1$ over $\llbracket T \rrbracket_0 \times \llbracket T \rrbracket_0$, i.e. if $\vec{R} : \mathbf{Rel}^n(\vec{A}, \vec{B})$, then $\llbracket T \rrbracket_1 \vec{R} : \mathbf{Rel}(\llbracket T \rrbracket_0 \vec{A}, \llbracket T \rrbracket_0 \vec{B})$. Reynolds also interprets terms $\llbracket t \rrbracket_0$, and then proves the following theorems which underpin most of the uses of parametricity.

Theorem 1 (Identity Extension Lemma). *If $\Gamma \vdash T$ with $|\Gamma| = n$, then $\llbracket T \rrbracket_1 \circ Eq^n = Eq \circ \llbracket T \rrbracket_0$.* \square

Theorem 2 (Abstraction Theorem). *If $\Gamma, \Delta \vdash t : T$ with $|\Gamma| = n$, then for every $\vec{R} : \mathbf{Rel}^n(\vec{A}, \vec{B})$, if $(u, v) \in \llbracket \Delta \rrbracket_1 \vec{R}$ then $(\llbracket t \rrbracket_0 \vec{A} u, \llbracket t \rrbracket_0 \vec{B} v) \in \llbracket T \rrbracket_1 \vec{R}$.* \square

So what makes Reynolds' definitions work? They are certainly fundamental, as can be seen by their numerous uses within the programming literature (see e.g. [2,6,11,19,1]). While valuable, this only provides a partial answer, which ought to be complemented by a deeper and more fundamental understanding. For us, that takes the form of showing that the above definitions satisfy axiomatic universal properties, and that those universal properties are strong enough to prove key theorems such as Theorems 1 and 2 in that axiomatic setting.

For function spaces, the answer is simply that $\llbracket U \rightarrow V \rrbracket_0 \vec{A}$ is the exponential of the functors $\llbracket U \rrbracket_0 \vec{A}$ and $\llbracket V \rrbracket_0 \vec{A}$; and that $\llbracket U \rightarrow V \rrbracket_1 \vec{R}$ is the exponential of the functors $\llbracket U \rrbracket_1 \vec{R}$ and $\llbracket V \rrbracket_0 \vec{R}$. These results in turn follow because **Rel** and **Set** are

cartesian closed categories and U preserves this cartesian closed structure. Our goal is to provide such a succinct and compelling equivalent explanation for the definitions of $\llbracket \forall X.T \rrbracket_0 \vec{A}$ and $\llbracket \forall X.T \rrbracket_1 \vec{R}$. To begin with, note that if we were only to consider ad-hoc polymorphic functions, i.e. the collection

$$\prod_{X:\mathbf{Set}} \llbracket T \rrbracket_0(\vec{A}, X)$$

then we could characterise this collection as the product of the functor $\llbracket T \rrbracket_0(\vec{A}, -) : \mathbf{Set} \rightarrow \mathbf{Set}$ (naïvely assuming the product exists), that is, as the terminal $\llbracket T \rrbracket_0(\vec{A}, -)$ -cone. Including Reynolds' condition that a parametrically polymorphic function $f : \prod_{S:\mathbf{Set}} \llbracket T \rrbracket_0(\vec{A}, S)$ is one where for every relation $R : \mathbf{Rel}(X, Y)$ we have that $(fX, fY) \in \llbracket T \rrbracket_1(\text{Eq}\vec{A}, R)$ cuts down the the number of ad-hoc polymorphic functions. Now the key bit. Define $\nu_X : \llbracket \forall X.T \rrbracket_0 \vec{A} \rightarrow \llbracket T \rrbracket_0(\vec{A}, X)$ to be type application, i.e. $\nu_X f = fX$. Then Reynolds' parametricity condition that for all $R : \mathbf{Rel}(A, B)$, if $f : \llbracket \forall X.T \rrbracket_0 \vec{A}$, then $(fA, fB) \in \llbracket T \rrbracket_1(\text{Eq}\vec{A}, R)$ is equivalent to a morphism $\text{Eq}(\llbracket \forall X.T \rrbracket_0 \vec{A}) \rightarrow \llbracket T \rrbracket_1(\text{Eq}\vec{A}, R)$ over ν_A and ν_B . Generalising, we have:

Definition 3. *Let $F = (F_0, F_1)$ be a pair of functors with $F_0 : |\mathbf{Set}| \rightarrow \mathbf{Set}$ and $F_1 : |\mathbf{Rel}| \rightarrow \mathbf{Rel}$ such that F_1 is over $F_0 \times F_0$. An F -eqcone is an F_0 -cone (A, ν) such that there is a (necessarily unique since U is faithful) F_1 -cone with vertex $\text{Eq}A$ over (ν, ν) . The category of such cones is the full subcategory of F_0 -cones whose objects are F -eqcones.*

Our axiomatic definition is linked to Reynolds' definition in the following way:

Theorem 4. *Assume $\Gamma, X \vdash T$ Type. For every tuple \vec{A} , Reynolds' set of parametrically polymorphic functions $\llbracket \forall X.T \rrbracket_0 \vec{A}$ from (2) is the terminal F -eqcone for the pair of functors $F = (\llbracket T \rrbracket_0(\vec{A}, -), \llbracket T \rrbracket_1(\text{Eq}\vec{A}, -))$.*

Proof. Application at X , defined by $\nu_X f = fX$, makes $\llbracket \forall X.T \rrbracket_0 \vec{A}$ a vertex of a $\llbracket T \rrbracket_0(\vec{A}, -)$ -cone. The uniformity condition on elements of $\llbracket \forall X.T \rrbracket_0 \vec{A}$ ensures this cone is an F -eqcone. To see that this is the terminal such, consider any other F -eqcone (A, η) . As this is a $\llbracket T \rrbracket_0(\vec{A}, -)$ -cone, there is a unique map $\bar{\eta}$ of such cones into $\prod_{X:\mathbf{Set}} \llbracket T \rrbracket_0(\vec{A}, X)$. However, the fact that (A, η) is an F -eqcone means the image of this mediating map lies within $\llbracket \forall X.T \rrbracket_0 \vec{A}$. Hence we have a morphism of F -eqcones $A \rightarrow \llbracket \forall X.T \rrbracket_0 \vec{A}$. The uniqueness of this mediating morphism follows from the uniqueness of $\bar{\eta}$. \square

We can also give a universal property to characterise $\llbracket \forall X.T \rrbracket_1 \vec{R}$.

Definition 5. *Let $F = (F_0, F_1)$ and $G = (G_0, G_1)$ be pairs of functors $|\mathbf{Set}| \rightarrow \mathbf{Set}$ and $|\mathbf{Rel}| \rightarrow \mathbf{Rel}$ with F_1 over $F_0 \times F_0$, G_1 over $G_0 \times G_0$, and let $H : |\mathbf{Rel}| \rightarrow \mathbf{Rel}$ with H over $F_0 \times G_0$. A fibred (F, G, H) -eqcone consists of an F -eqcone (A, ν) , a G -eqcone (B, μ) and a H -cone (Q, γ) over (ν, μ) . The category of such cones has as morphisms triples (f, g, h) , where f is a morphism between the underlying F -eqcones, g is a morphism between the underlying G -eqcones and h is a (again necessarily unique) morphism of H -cones above (f, g) .*

The above definition can be understood as follows. For every relation $R : \text{Rel}(X, Y)$ we need two things to be related, which is forced by γ . That the related things are instances of polymorphic functions is reflected by the fact that γ_R is over (ν_X, μ_Y) . This intuition can be formalised via the following theorem:

Theorem 6. *Assume $\Gamma, X \vdash T$ Type. For every $\vec{R} : \text{Rel}(\vec{A}, \vec{B})$, the relation $\llbracket \forall X.T \rrbracket_1 \vec{R}$ from (2) is the terminal fibred (F, G, H) -eqcone for the functors $F = (\llbracket T \rrbracket_0(\vec{A}, -), \llbracket T \rrbracket_1(\text{Eq}\vec{A}, -))$, $G = (\llbracket T \rrbracket_0(\vec{B}, -), \llbracket T \rrbracket_1(\text{Eq}\vec{B}, -))$ and $H = \llbracket T \rrbracket_1(\vec{R}, -)$.*

Proof. Straightforward calculation, similar to the proof of Theorem 4. \square

We postpone the proof that the Identity Extension Lemma and Abstraction Theorem follow from the universal properties in Reynolds' concrete model, as such proofs would simply be instantiations of the proofs of Theorems 15 and 18 in Section 4. Instead, we turn to our axiomatic setting for the study of parametricity.

3 Fibrational Tools

The previous section only covered a specific model, but what we really want is an axiomatic approach which can then be instantiated. There are a number of axiomatic approaches to parametricity, e.g. Ma and Reynolds [14], Dunphy and Reddy [7], Birkedal and Møgelberg [4], and Hermida [10]. As we shall see, our axiomatisation requires a bifibration, and for this reason, we build upon our own treatment [13], whose distinguishing feature is exactly bifibrational structure. We give a brief introduction to fibrations; for more details see Jacobs [12].

Definition 7. *Let $U : \mathcal{E} \rightarrow \mathcal{B}$ be a functor. A morphism $g : Q \rightarrow P$ in \mathcal{E} is cartesian over $f : X \rightarrow Y$ in \mathcal{B} if $Ug = f$ and, for every $g' : Q' \rightarrow P$ in \mathcal{E} with $Ug' = f \circ v$ for some $v : UQ' \rightarrow X$, there exists a unique $h : Q' \rightarrow Q$ with $Uh = v$ and $g' = g \circ h$. Dually, a morphism $g : P \rightarrow Q$ in \mathcal{E} is opcartesian over $f : X \rightarrow Y$ in \mathcal{B} if $Ug = f$ and, for every $g' : P \rightarrow Q'$ in \mathcal{E} with $Ug' = v \circ f$ for some $v : Y \rightarrow UQ'$, there exists a unique $h : Q \rightarrow Q'$ with $Uh = v$ and $g' = h \circ g$.*

We write f_P^{\S} for the cartesian morphism over f with codomain P and f_{\S}^P for the opcartesian morphism over f with domain P . These are unique up to isomorphism. If P is an object of \mathcal{E} then we write f^*P for the domain of f_P^{\S} and $\Sigma_f P$ for the codomain of f_{\S}^P .

Definition 8. *A functor $U : \mathcal{E} \rightarrow \mathcal{B}$ is a fibration if for every object P of \mathcal{E} and every morphism $f : X \rightarrow UP$ in \mathcal{B} , there is a cartesian morphism $f_P^{\S} : Q \rightarrow P$ in \mathcal{E} over f . Similarly, U is an opfibration if for every object P of \mathcal{E} and every morphism $f : UP \rightarrow Y$ in \mathcal{B} , there is an opcartesian morphism $f_{\S}^P : P \rightarrow Q$ in \mathcal{E} over f . A functor U is a bifibration if it is both a fibration and an opfibration.*

Example 9. Consider a category \mathcal{B} with pullbacks, and let $\text{Sub}_{\mathcal{B}}(A)$ be the category of subobjects of $A \in \mathcal{B}$ (i.e. equivalence classes of monos $m : X \hookrightarrow A$). Let $\text{Sub}(\mathcal{B})$ be the category with objects pairs (A, m) where m is in $\text{Sub}_{\mathcal{B}}(A)$.

A morphism $(f, \alpha) : (A, m : X \hookrightarrow A) \rightarrow (B, n : Y \hookrightarrow B)$ consists of morphisms $f : A \rightarrow B$ and $\alpha : X \rightarrow Y$ in \mathcal{B} such that $f \circ m = n \circ \alpha$. The functor $U : \text{Sub}(\mathcal{B}) \rightarrow \mathcal{B}$ defined by $U(A, m) = A$ is then a fibration (with reindexing given by pullback), and further a bifibration if \mathcal{B} has image factorisations. For $\mathcal{B} = \text{Set}$, subobjects of A can be identified with subsets of A .

If $U : \mathcal{E} \rightarrow \mathcal{B}$ is a fibration, opfibration, or bifibration, then \mathcal{E} is its *total category* and \mathcal{B} is its *base category*. An object P in \mathcal{E} is *over* its image UP and similarly for morphisms. A morphism is *vertical* if it is over id . We write \mathcal{E}_X for the *fibre over* an object X in \mathcal{B} , i.e., the subcategory of \mathcal{E} of objects over X and vertical morphisms. For $f : X \rightarrow Y$ in \mathcal{B} , the function mapping each object P of \mathcal{E} to f^*P extends to a *reindexing functor* $f^* : \mathcal{E}_Y \rightarrow \mathcal{E}_X$. Similarly for opfibrations, the function mapping each object P of \mathcal{E}_X to $\Sigma_f P$ extends to the *opreindexing functor* $\Sigma_f : \mathcal{E}_X \rightarrow \mathcal{E}_Y$. We write $|\mathcal{E}|$ for the discrete category of \mathcal{E} . If $U : \mathcal{E} \rightarrow \mathcal{B}$ is a functor, then the *discrete functor* $|U| : |\mathcal{E}| \rightarrow |\mathcal{B}|$ is induced by the restriction of U to $|\mathcal{E}|$, and is always a bifibration. If $n \in \mathbb{N}$, then \mathcal{E}^n denotes the n -fold product of \mathcal{E} in Cat . The *n -fold product of U* , denoted $U^n : \mathcal{E}^n \rightarrow \mathcal{B}^n$, is the functor defined by $U^n(X_1, \dots, X_n) = (UX_1, \dots, UX_n)$. If U is a bifibration, then so is U^n . Since parametricity is about relations, we describe relations in a fibrational setting. If U is a fibration whose base has products, then the associated fibration of relations $\text{Rel}(U)$ is obtained by change of base, i.e. the following pullback:

$$\begin{array}{ccc} \text{Rel}(\mathcal{E}) & \longrightarrow & \mathcal{E} \\ \text{Rel}(U) \downarrow & \lrcorner & \downarrow U \\ \mathcal{B} \times \mathcal{B} & \xrightarrow{- \times -} & \mathcal{B} \end{array}$$

If U is a bifibration, then so is $\text{Rel}(U)$. The bifibration $\text{Rel} \rightarrow \text{Set} \times \text{Set}$ from Section 2 arises as the relations fibration associated to the subobject fibration $\text{Sub}(\text{Set}) \rightarrow \text{Set}$ from Example 9.

To treat equality in this axiomatic framework, we first need the notion of truth. Let $U : \mathcal{E} \rightarrow \mathcal{B}$ be a fibration with fibred terminal objects, i.e. each fibre \mathcal{E}_X has a terminal object KX , and reindexing preserves it. Then the assignment $X \mapsto KX$ extends to the functor $K : \mathcal{B} \rightarrow \mathcal{E}$, called the *truth functor*. This functor is right adjoint to the fibration U . Equality arises axiomatically as follows:

Lemma 10. *Let $U : \mathcal{E} \rightarrow \mathcal{B}$ be a bifibration with fibred terminal objects. If \mathcal{B} has products, then the map $A \mapsto \Sigma_{\langle \text{id}_A, \text{id}_A \rangle} KA$ extends to a functor $Eq : \mathcal{B} \rightarrow \text{Rel}(\mathcal{E})$, called the equality functor. \square*

Example 11. The subobject fibration from Example 9 has fibred terminal objects given by $KA = \text{id} : A \hookrightarrow A$. Opreindexing by a mono is by composition in the subobject fibration, hence equality is given by $EqA = \langle \text{id}_A, \text{id}_A \rangle : A \hookrightarrow A \times A$.

Let $U : \mathcal{E} \rightarrow \mathcal{B}$ and $U' : \mathcal{E}' \rightarrow \mathcal{B}'$ be fibrations. A *fibred functor* $T : U \rightarrow U'$ comprises two functors $T_0 : \mathcal{B} \rightarrow \mathcal{B}'$ and $T_1 : \mathcal{E} \rightarrow \mathcal{E}'$ such that T_1 is over T_0 , i.e. $U' \circ T_1 = T_0 \circ U$, and T_1 preserves cartesian morphisms. If $T' : U \rightarrow U'$ is another

fibred functor, then a *fibred natural transformation* $\nu : T \rightarrow T'$ comprises two natural transformations $\nu_0 : T_0 \rightarrow T'_0$ and $\nu_1 : T_1 \rightarrow T'_1$ such that $U' \nu_1 = \nu_0 U$. Note that in the case of fibred functors $|\text{Rel}(U)| \rightarrow \text{Rel}(U)$, the requirement that cartesian morphisms are preserved is vacuous. Nevertheless, we will avoid introducing more terminology and stick with ‘fibred’ also in this case.

Armed with these definitions, we can introduce the axiomatic framework for parametricity within which we can generalise the universal properties of Section 2. Given a bifibration $U : \mathcal{E} \rightarrow \mathcal{B}$ with appropriate structure, we interpret types $\Gamma \vdash T$ as fibred functors $\llbracket T \rrbracket = (\llbracket T \rrbracket_0 \times \llbracket T \rrbracket_0, \llbracket T \rrbracket_1) : |\text{Rel}(\mathcal{E})|^{|\Gamma|} \rightarrow \text{Rel}(\mathcal{E})$, and terms $\Delta \vdash t : T$ as fibred natural transformations $(\llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0, \llbracket t \rrbracket_1) : \llbracket \Delta \rrbracket \rightarrow \llbracket T \rrbracket$. Thus a type T has a “standard” semantics $\llbracket T \rrbracket_0$, as well as a relational semantics $\llbracket T \rrbracket_1$. The interpretation can be summed up as follows:

$$\begin{array}{ccc}
|\text{Rel}(\mathcal{E})|^{|\Gamma|} & \begin{array}{c} \xrightarrow{\llbracket \Delta \rrbracket_1} \\ \Downarrow \llbracket t \rrbracket_1 \\ \xrightarrow{\llbracket T \rrbracket_1} \end{array} & \text{Rel}(\mathcal{E}) \\
\downarrow |\text{Rel}(U)|^{|\Gamma|} & & \downarrow \text{Rel}(U) \\
|\mathcal{B}|^{|\Gamma|} \times |\mathcal{B}|^{|\Gamma|} & \begin{array}{c} \xrightarrow{\llbracket \Delta \rrbracket_0 \times \llbracket \Delta \rrbracket_0} \\ \Downarrow \llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0 \\ \xrightarrow{\llbracket T \rrbracket_0 \times \llbracket T \rrbracket_0} \end{array} & \mathcal{B} \times \mathcal{B}
\end{array} \tag{3}$$

In Reynolds model, the Abstraction Theorem states that if $\vec{R} : \text{Rel}(\mathcal{E})^{|\Gamma|}(\vec{A}, \vec{B})$, and $(u, v) \in \llbracket \Delta \rrbracket_1 \vec{R}$, then $(\llbracket t \rrbracket_0 \vec{A} u, \llbracket t \rrbracket_0 \vec{B} v) \in \llbracket T \rrbracket_1 \vec{R}$. This is equivalent to a natural transformation $\llbracket t \rrbracket_1$ over $\llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0$. Thus the existence of $\llbracket t \rrbracket_1$ is the fibrational analogue of the Abstraction Theorem. See [13] for more details.

4 Parametrically Polymorphic Functions, Axiomatically

We now turn to the universal property we will use to define the object of parametrically polymorphic functions in our axiomatic framework. We carefully formulated the definitions of Section 2 so that they seamlessly generalise once we have axiomatic notions of relations and equality, which we developed in Section 3.

Definition 12. *Let $F = (F_0, F_1) : |\text{Rel}(U)| \rightarrow \text{Rel}(U)$ be a fibred functor. An F -eqcone is an F_0 -cone (A, ν) such that there is a (necessarily unique since U is faithful) F_1 -cone with vertex $\text{Eq}A$ over (ν, ν) . The category of such cones is the full subcategory of F_0 -cones whose objects are F -eqcones. We denote the terminal object of this category $\forall_0 F$, if it exists.*

The universal property defining the relational interpretation of parametrically polymorphic functions also smoothly generalises to the fibrational setting:

Definition 13. *Let $F = (F_0, F_1)$ and $G = (G_0, G_1)$ be fibred functors $|\text{Rel}(U)| \rightarrow \text{Rel}(U)$ and let $H : |\text{Rel}(\mathcal{B})| \rightarrow \text{Rel}(\mathcal{B})$ be over $F_0 \times G_0$. A fibred (F, G, H) -eqcone*

consists of an F -eqcone (A, ν) , a G -eqcone (B, μ) and an H -cone (Q, γ) such that Q is over $A \times B$, and γ is over (ν, μ) . A morphism $(A, \nu, B, \mu, Q, \gamma) \rightarrow (A', \nu', B', \mu', Q', \gamma')$ in the category of such cones consists of triples (f, g, h) where $f : (A, \nu) \rightarrow (A', \nu')$, $g : (B, \mu) \rightarrow (B', \mu')$, and h is a (again necessarily unique) morphism of H -cones above (f, g) . We denote the terminal object of this category $\forall_1(F, G, H)$, if it exists.

For the rest of this section, let \mathcal{B} be a category with pullbacks and image factorisations [12]. We instantiate our framework to subobject fibrations $\mathbf{Sub}(\mathcal{B}) \rightarrow \mathcal{B}$; the results of Section 2 arise from the subobject fibration over \mathbf{Set} . We show the utility of our axiomatic definition by showing that it supports:

- (i) *A Fibred Semantics:* Our axiomatic definitions do not by definition guarantee that if $\vec{R} : \mathbf{Rel}(U)^n(\vec{A}, \vec{B})$, then $\llbracket \forall X.T \rrbracket_1 \vec{R}$ is a relation between $\llbracket \forall X.T \rrbracket_0 \vec{A}$ and $\llbracket \forall X.T \rrbracket_0 \vec{B}$, so we prove this axiomatically.
- (ii) *The Identity Extension Lemma:* Since we do not “bake-in” the Identity Extension Lemma using for example reflexive graphs, we need to prove it.
- (iii) *The Abstraction Theorem:* We prove the Abstraction Theorem in the axiomatic setting. Most of the work in doing so involves the construction of a model of System F in the form of a $\lambda 2$ -fibration.

A Fibred Semantics: Our proof that if $\vec{R} : \mathbf{Rel}^n(\vec{A}, \vec{B})$, then $\llbracket \forall X.T \rrbracket_1 \vec{R}$ is a relation between $\llbracket \forall X.T \rrbracket_0 \vec{A}$ and $\llbracket \forall X.T \rrbracket_0 \vec{B}$ crucially requires opfibrational structure, which plays a distinguishing role in our framework [13].

Lemma 14. *Let $F = (F_0, F_1)$ and $G = (G_0, G_1)$ be fibred functors $|\mathbf{Rel}(U)| \rightarrow \mathbf{Rel}(U)$ and assume H is over $F_0 \times G_0$. Then $\forall_1(F, G, H)$ is over $\forall_0 F \times \forall_0 G$.*

Proof. The forgetful functor which maps a fibred (F, G, H) -eqcone to its pair of underlying F -eqcones and G -eqcones is an opfibration, since it inherits the opfibrational structure of $\mathbf{Rel}(U) : \mathbf{Rel}(\mathcal{E}) \rightarrow \mathcal{B} \times \mathcal{B}$. Any opfibration which has terminal objects in the base and in the total category, also has a terminal object in the total category over that in the base. Since terminal objects are defined up to isomorphism, we can take $\forall_1(F, G, H)$ to be over $\forall_0 F \times \forall_0 G$. \square

This lemma, when taken with the usual treatment of function spaces, ensures that we have replicated Reynolds’ original fibred semantics within our axiomatic framework. That is, for all judgments $\Gamma \vdash T \text{ Type}$, $(\llbracket T \rrbracket_0 \times \llbracket T \rrbracket_0, \llbracket T \rrbracket_1)$ forms a fibred functor $|\mathbf{Rel}(U)|^n \rightarrow \mathbf{Rel}(U)$.

The Identity Extension Lemma: In a subobject fibration, $Eq : \mathcal{B} \rightarrow \mathbf{Rel}(\mathcal{E})$ maps an object X to the mono $\langle \text{id}_X, \text{id}_X \rangle : X \hookrightarrow X \times X$. Thus we need to show:

Lemma 15. *Let U be a subobject bifibration and $F = (F_0 \times F_0, F_1) : |\mathbf{Rel}(U)| \rightarrow \mathbf{Rel}(U)$ be a fibred functor which is equality preserving. Then the subobject $\langle v_1, v_2 \rangle : \forall_1(F, F, F_1) \hookrightarrow \forall_0 F \times \forall_0 F$ is $Eq(\forall_0 F) = \langle \text{id}, \text{id} \rangle : \forall_0 F \hookrightarrow \forall_0 F \times \forall_0 F$.*

Proof. The heart of the proof is to show that $v_1 = v_2$. To see this, let $\pi_X : \forall_0 F \rightarrow F_0 X$ be the projection maps associated with $\forall_0 F$, and let $\gamma_R : \forall_1(F, F, F_1) \rightarrow F_1 R$ be the projection maps associated with $\forall_1(F, F, F_1)$. By Lemma 14, for every X , $\gamma_{EqX} : \forall_1(F, F, F_1) \rightarrow F_1(EqX) = Eq(F_0 X) = F_0 X$ is over $(\pi_X \times \pi_X)$. By the definition of the equality functor in a subobject fibration, we have

$$\begin{array}{ccc} \forall_1(F, F, F_1) & \xrightarrow{\gamma_{EqX}} & F_1(EqX) = Eq(F_0 X) = F_0 X \\ \langle v_1, v_2 \rangle \downarrow & & \downarrow \langle \text{id}, \text{id} \rangle \\ \forall_0 F \times \forall_0 F & \xrightarrow{\pi_X \times \pi_X} & F_0 X \times F_0 X \end{array}$$

Thus $\pi_X v_1 = \gamma_{Eq(X)} = \pi_X v_2$ and $(\forall_1(F, F, F_1), \gamma_{Eq(-)})$ is a F -eqcone with F_1 -cone given by γ_R . Hence both v_1 and v_2 are mediating morphisms into the terminal F -eqcone, and thus $v_1 = v_2$. Furthermore, they are vertical since $\langle \text{id}, \text{id} \rangle \circ v_i = \langle v_1, v_2 \rangle$. We can now show that $Eq(\forall_0 F)$ is isomorphic to $\forall_1(F, F, F_1)$. In one direction, $Eq(\forall_0 F)$ is easily seen to be a fibred (F, F, F_1) -eqcone and hence there is a map of subobjects $Eq(\forall_0 F) \rightarrow \forall_1(F, F, F_1)$. In the other direction, v_1 is a map of subobjects since $v_1 = v_2$. These maps are mutually inverse, as they are both vertical and the fibration is faithful. \square

The Identity Extension Lemma for fibred functors $(T_0 \times T_0, T_1) : \text{Rel}(U)^{|n+1} \rightarrow \text{Rel}(U)$ immediately follows by instantiating $F_0 = T_0(\vec{A}, -)$ and $F_1 = T_1(Eq\vec{A}, -)$. When taken with an appropriate treatment of arrow types, this lemma shows that in our axiomatic setting, all type expressions are interpreted not just as fibred functors $\text{Rel}(U)^{|n} \rightarrow \text{Rel}(U)$, but as *equality preserving* fibred functors. This forms the interpretation of types into a model which we now turn to, and from which we derive our axiomatic derivation of the Abstraction Theorem.

The Abstraction Theorem: As described in the concrete model in Section 2, and in the axiomatic framework of Section 4, Reynolds interprets System F types as equality preserving fibred functors. In order to interpret terms — and then establish the Abstraction Theorem — we must therefore discuss models. For our purposes, the notion of a $\lambda 2$ -fibration as a generic model of System F is most directly applicable. Recall that a (split) fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ has a generic object $\Omega \in \mathcal{B}$ if there is a collection of isomorphisms $\theta_I : \mathcal{B}(I, \Omega) \cong \mathcal{E}_I$, natural in I , and that p has simple Ω -products if each reindexing π^* along a projection $\pi : A \times \Omega \rightarrow \Omega$ has a right adjoint $\Pi_A : \mathcal{E}_{A \times \Omega} \rightarrow \mathcal{E}_A$ such that the *Beck-Chevalley condition* holds, i.e. $f^* \circ \Pi_B = \Pi_A \circ (f \times \text{id})^*$ for every $f : A \rightarrow B$.

Definition 16 ($\lambda 2$ -fibration). *A fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ is a $\lambda 2$ -fibration if it is fibred Cartesian closed, \mathcal{B} has finite products, p has a generic object $\Omega \in \mathcal{B}$, and p has simple Ω -products.*

We have already taken the first steps towards a model, by showing that types in n free variables can be modelled as equality preserving fibred functors $|\text{Rel}(U)|^n \rightarrow \text{Rel}(U)$. We now complete the construction, based upon our axiomatic definitions. We first define the fibres of the $\lambda 2$ -fibration, and then the base category:

Definition 17. For each natural number n , let the category $\mathcal{F}_n^{\text{Eq}}$ have as objects equality preserving fibred functors of the form $(F_0 \times F_0, F_1) : |\text{Rel}(U)|^n \rightarrow \text{Rel}(U)$. Morphisms are fibred natural transformation of the form $(\tau_0 \times \tau_0, \tau_1) : (F_0 \times F_0, F_1) \rightarrow (G_0 \times G_0, G_1)$. Let \mathcal{L} be the category with the natural numbers as objects, and where morphisms $n \rightarrow m$ are m -tuples of objects of $\mathcal{F}_n^{\text{Eq}}$.

This clearly defines a split fibration $\mathcal{F}^{\text{Eq}} \rightarrow \mathcal{L}$ with reindexing given by composition. The base category has finite products given by addition. In particular, projection $\pi : n + 1 \rightarrow n$ has as i th component the fibred functor $|\text{Rel}(U)|^{n+1} \rightarrow \text{Rel}(U)$ which selects the i th input. By construction, 1 is a generic object. Our previous paper [13] showed how the fibred cartesian closed structure arises from standard structure, e.g. that our original fibration $U : \mathcal{E} \rightarrow \mathcal{B}$ is cartesian closed, and that the functor Eq has a left adjoint satisfying Frobenius. In the case of subobject fibrations, it is enough to ask that the base \mathcal{B} is a regular LCCC and has coequalisers [12]. All that is left to prove is that we have simple 1-products.

Lemma 18. For each projection $\pi : n + 1 \rightarrow n$, the functor $\pi^* : \mathcal{F}_n^{\text{Eq}} \rightarrow \mathcal{F}_{n+1}^{\text{Eq}}$ has a right adjoint $\Pi = (\Pi_0, \Pi_1)$ with $(\Pi_0 F)\vec{A} = \forall_0(F_0(\vec{A}, -), F_1(Eq\vec{A}, -))$ and $(\Pi_1 F)\vec{R} = \forall_1((F_0(\vec{A}, -), F_1(Eq\vec{A}, -)), (F_0(\vec{B}, -), F_1(Eq\vec{B}, -)), F_1(\vec{R}, -))$, and the Beck-Chevalley condition holds. \square

To summarise, in this section we have proven:

Theorem 19. Let \mathcal{B} be a regular LCCC with coequalisers, and assume terminal fibred eqcones exist. The construction in Definition 17 gives rise to a $\lambda 2$ -fibration, where types are interpreted as fibred functors, and terms as fibred natural transformations. By construction, the Abstraction Theorem holds in the sense of (3). \square

From this theorem, together with Lemma 15, all the usual expected consequences of parametricity — e.g. the existence of initial algebras and final coalgebras, dinaturality — follow. See our other paper [13] for details and examples of models.

5 Conclusion

We have taken Reynolds definition of the set of parametric polymorphic functions in his relational model, and given an abstract characterisation of it as a universal property in an axiomatic fibrational framework. Further, we have shown the value of the axiomatisation by proving the two key theorems of parametricity from it, i.e. the Identity Extension Lemma and the Abstraction Theorem.

Throughout this paper, we worked with subobject fibrations. In unpublished work, we have relaxed this to require only a faithful fibration with comprehension. Recall that comprehension is defined to be right adjoint to truth and as such is a fundamental structure in categorical logic and type theory. Faithfulness is also reasonable, as it corresponds to proof-irrelevant relations, which is a standing assumption in the literature. We are in the process of lifting this restriction, and thereby tackling proof-relevant parametricity. This is a significant undertaking as it involves blending parametricity with higher dimensional cubical structure that, intriguingly, also arises in the semantics of Homotopy Type Theory [3].

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