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Abstract. In this paper, we study the distribution of the number occurrences of the simplest frame pattern, called the $\mu$ pattern, in $n$-cycles. Given an $n$-cycle $C$, we say that a pair $\langle i, j \rangle$ matches the $\mu$ pattern if $i < j$ and as we traverse around $C$ in a clockwise direction starting at $i$ and ending at $j$, we never encounter a $k$ with $i < k < j$. We say that $\langle i, j \rangle$ is a nontrivial $\mu$-match if $i + 1 < j$. We say that an $n$-cycle $C$ is incontractible if there is no $i$ such that $i + 1$ immediately follows $i$ in $C$. We show that number of incontractible $n$-cycles in the symmetric group $S_n$ is $D_n - 1$ where $D_n$ is the number of derangements in $S_n$. We show that number of $n$-cycles in $S_n$ with exactly $k$ $\mu$-matches can be expressed as a linear combination of binomial coefficients of the form $\binom{n-1}{i}$ where $i \leq 2k + 1$. We also show that the generating function $NTI_{n,\mu}(q)$ of $q$ raised to the number of nontrivial $\mu$-matches in $C$ over all incontractible $n$-cycles in $S_n$ is a new $q$-analogue of $D_n - 1$ which is different from the $q$-analogues of the derangement numbers that have been studied by Garsia and Remmel and by Wachs. We will show that there is a rather surprising connection between the charge statistic on permutations due to Lascoux and Schützenberger and our polynomials in that the coefficient of the smallest power of $q$ in $NTI_{2k+1,\mu}(q)$ is the number of permutations in $S_{2k+1}$ whose charge path is a Dyck path. Finally, we show that $NTI_{n,\mu}(q)$ and $NT_{n,\mu}(q)$ are the number of partitions of $k$ for sufficiently large $n$.

1. INTRODUCTION

Mesh patterns were introduced in [2] by Brändén and Claesson, and they were studied in a series of papers (e.g. see [6] by Kitaev and Liese, and references therein). A particular class of mesh patterns is boxed patterns introduced in [1] by Avgustinovich et al., who later suggested to call this type of patterns frame patterns. The simplest frame pattern which is called the $\mu$ pattern is defined as follows. Let $S_n$ denote the set of all permutations of $\{1, \ldots, n\}$. Given $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n \in S_n$, we say that a pair $\langle \sigma_i, \sigma_j \rangle$ is an occurrence of the $\mu$ pattern in $\sigma$ if $i < j$, $\sigma_i < \sigma_j$, and there is no $i < k < j$ such that $\sigma_i < \sigma_k < \sigma_j$ (the last condition is indicated by the shaded area in Figure 2). The $\mu$ pattern is shown in Figure 1 using the notation in [2].
Similarly, we say that the pair $\langle \sigma_i, \sigma_j \rangle$ is an occurrence of the $\mu'$ pattern in $\sigma$ if $i < j$, $\sigma_i > \sigma_j$, and there is no $i < k < j$ such that $\sigma_i > \sigma_k > \sigma_j$. For example, if $\sigma = 4 \, 8 \, 2 \, 6 \, 1 \, 3 \, 5 \, 7$, then the occurrences of $\mu$ in $\sigma$ are

$$\langle 4, 8 \rangle, \langle 4, 6 \rangle, \langle 4, 5 \rangle, \langle 2, 6 \rangle, \langle 2, 3 \rangle, \langle 6, 7 \rangle, \langle 1, 3 \rangle, \langle 3, 5 \rangle, \langle 5, 7 \rangle$$

and the occurrences of $\mu'$ in $\sigma$ are

$$\langle 4, 2 \rangle, \langle 4, 3 \rangle, \langle 8, 2 \rangle, \langle 8, 6 \rangle, \langle 8, 7 \rangle, \langle 2, 1 \rangle, \langle 6, 1 \rangle, \langle 6, 3 \rangle, \langle 6, 5 \rangle.$$  

We let $N_\mu(\sigma)$ (resp., $N_{\mu'}(\sigma)$) denote the number of occurrences of the $\mu$ (resp., $\mu'$) in $\sigma$. The reverse of $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, $\sigma^r$, is the permutation $\sigma_n \sigma_{n-1} \ldots \sigma_1$, and the complement of $\sigma$, $\sigma^c$, is the permutation $(n+1 - \sigma_1)(n+1 - \sigma_2)\ldots(n+1 - \sigma_n)$. It is easy to see that $N_\mu(\sigma) = N_{\mu'}(\sigma^r) = N_{\mu'}(\sigma^c)$ and thus, since the reverse and complement are trivial bijections from $S_n$ to itself, studying the distribution of $\mu$-matches in $S_n$ is equivalent to studying the distribution of $\mu'$-matches in $S_n$.

If we graph a given permutation $\sigma$ as dots on a grid as in Figure 2, then one can see that each occurrence of $\mu$ is a pair of increasing dots such that there are no dots within the rectangle created by the original dots. The occurrence $\langle 4, 6 \rangle$ is highlighted in Figure 2; in particular, there are no dots within the shaded rectangle.

Jones and Remmel studied the distribution of cycle-occurrences of classical consecutive patterns in [4]. See [5] for a comprehensive introduction to the theory of permutation patterns. In this paper, we shall study the distribution of the cycle-occurrences of $\mu$ in the cycle structure of permutations in the symmetric group $S_n$. That is, suppose that we are given a k-cycle $C$ in a permutation $\sigma$ in the symmetric group $S_n$. Then we will always write $C = (c_0, \ldots, c_{k-1})$ where $c_0$ is the smallest element of the cycle. We will always draw such a cycle with $c_0$ at the top and assume that we traverse around the cycle in a clockwise order.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{mu.png}
\caption{The mesh pattern $\mu$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{The graph of the permutation 48261357 with the occurrence $\langle 4, 6 \rangle$ highlighted.}
\end{figure}
direction. For example, if \( C = (1, 4, 6, 2, 7, 5, 8, 3) \), then we would picture \( C \) as in Figure 3. We say that \( c_i \) is cyclically between \( c_i \) and \( c_j \) in \( C \), if starting at \( c_i \), we encounter \( c_j \) before we encounter \( c_i \) as we traverse around the cycle in a clockwise direction. Alternatively, we can identify \( C \) with the permutation \( c_0c_1 \ldots c_{k-1} \). In this notation, \( c_s \) is cyclically between \( c_i \) and \( c_j \) if either \( i < s < j \), or \( j < i < s \), or \( s < j < i \). For example, in the cycle \( C = (1, 4, 6, 2, 7, 5, 8, 3) \), \( 8 \) is cyclically between \( 2 \) and \( 3 \). Then we say that the pair \( \langle c_i, c_j \rangle \) is a cycle-occurrence of \( \mu \) in \( C \) if \( c_i \) and \( c_j \) are either \( s < j < i \) or \( s < i < j \). Similarly, we say that the pair \( \langle c_i, c_j \rangle \) is a cycle-occurrence of \( \mu' \) in \( C \), if \( c_i > c_j \) and there is no \( c_s \) such that \( c_i > c_s > c_j \) and \( c_s \) is cyclically between \( c_i \) and \( c_j \). As is the case with permutations, the study of the number of cycle-occurrences of \( \mu \) in the cycle structures of permutations is equivalent to the study of the number of cycle-occurrences of \( \mu' \) in the cycle structures of permutations. That is, given the cycle structure \( C_1, \ldots, C_k \) of a permutation \( \sigma \in S_n \), the cycle complement of \( \sigma \), \( \sigma^{\text{cyc-c}} \), is the permutation whose cycle structure arises from the cycle structure of \( \sigma \) by replacing each number \( i \) by \( n + 1 - i \). For example, if \( \sigma \) consists of the cycles \( (1, 4, 2, 6), (3, 8), (5, 9, 7) \), then \( \sigma^{\text{cyc-c}} \) consists of the cycles \( (9, 6, 8, 4), (7, 2), (5, 1, 3) \). It is then easy to see that for all \( \sigma \in S_n \), \( \sigma \) has \( k \) cycle-occurrences of \( \mu \) if and only if \( \sigma^{\text{cyc-c}} \) has \( k \) cycle-occurrences of \( \mu' \).

Let \( \mathcal{C}_n \) be the set of \( n \)-cycles in \( S_n \). If \( C = (1, c_1, \ldots, c_{n-1}) \in \mathcal{C}_n \), then it is clear that \( \langle i, i+1 \rangle \) is always a cycle-occurrence of \( \mu \) in \( C \). We shall call such cycle-occurrences trivial occurrences of \( \mu \) or trivial \( \mu \)-matches in \( C \) and all other cycle-occurrences of \( \mu \) in \( C \) will be called nontrivial occurrences of \( \mu \) or nontrivial \( \mu \)-matches in \( C \). We let \( N_\mu(C) \) denote the number of occurrences of \( \mu \) in \( C \) and \( NT_\mu(C) \) denote the number of nontrivial occurrences of \( \mu \) in \( C \). For example, if \( C = (1, 4, 6, 2, 7, 5, 8, 3) \), the nontrivial occurrences of \( \mu \) in \( C \) are the pairs \( (1, 4), (2, 7), (2, 5), (4, 6), \) and \( (5, 8) \), so that \( NT_\mu(C) = 5 \). Clearly, if \( C = (1, c_1, \ldots, c_{n-1}) \) is an \( n \)-cycle in \( \mathcal{C}_n \), then \( N_\mu(C) = (n-1) + NT_\mu(C) \), since for all \( 1 \leq i \leq n-1 \), \( \langle i, i+1 \rangle \) will be a trivial occurrence of \( \mu \) in \( C \). If \( C \) is a 1-cycle, then \( NT_\mu(C) = N_\mu(C) = 0 \).

![Figure 3](image.png)

Figure 3. The cycle \( C = (1, 4, 6, 2, 7, 5, 8, 3) \).

If \( \sigma_1 \ldots \sigma_n \) is any sequence of distinct integers, we let \( \text{red}(\sigma) \) denote the permutation of \( S_n \) that is obtained by replacing the \( i \)th smallest element of \( \{\sigma_1, \ldots, \sigma_n\} \) by \( i \) for \( i = 1, \ldots, n \). For example, \( \text{red}(3\ 7\ 10\ 5\ 2) = 2\ 4\ 5\ 3\ 1 \). Similarly, if \( C = (c_0, \ldots, c_{k-1}) \) is a \( k \)-cycle in some permutation \( \sigma \in S_n \), we let \( \text{red}(C) \) be the \( k \)-cycle in \( S_k \) which is obtained
by replacing the \( i \)th largest element of \( C \) by \( i \). For example, if \( C = (2, 6, 7, 3, 9) \), then \( \text{red}(C) = (1, 3, 4, 2, 5) \). In such a situation, we let \( NT_{\mu}(C) = NT_{\mu}(\text{red}(C)) \). Finally, if \( \sigma \) consists of cycles \( C^{(1)}, \ldots, C^{(\ell)} \), then we let \( CNT_{\mu}(\sigma) = \sum_{i=1}^{\ell} NT_{\mu}(C^{(i)}) \).

We note that if one wishes to study the generating function

\[
CNT_{\mu}(q, x, t) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} q^{CNT_{\mu}(\sigma)} x^{\text{cyc}(\sigma)},
\]

where \( \text{cyc}(\sigma) \) denotes the number of cycles in \( \sigma \), then it is enough to study the generating function

\[
NT_{\mu}(q, t) = \sum_{n \geq 1} \frac{t^n}{n!} NT_{n, \mu}(q)
\]

where \( NT_{n, \mu}(q) = \sum_{C \in C_n} q^{NT_{\mu}(C)} \). That is, it easily follows from the exponential formula that if \( w \) is a weight function on \( n \)-cycles \( C \in C_n \) and for any permutation \( \sigma \) whose cycle structure is \( C_1, \ldots, C_k \), we define \( w(\sigma) = \prod_{i=1}^{k} w(\text{red}(C_i)) \), then

\[
1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} w(\sigma)x^{\text{cyc}(\sigma)} = e^x \sum_{n \geq 1} \frac{t^n}{n!} \sum_{C \in C_n} w(C).
\]

Hence

\[
CNT_{\mu}(q, x, t) = e^{x NT_{\mu}(q, t)}.
\]

The main focus of this paper is to study the polynomials \( NT_{n, \mu}(q) \).

In Table 1 of the appendix, we give a table of \( NT_{n, \mu}(q) \) for \( n \leq 10 \) which was calculated using Mathematica. Given a polynomial \( f(x) = \sum_{i=0}^{n} c_i x^i \), we let \( f(x)\big|_{x^i} = c_i \) denote the coefficient of \( x^i \) in \( f(x) \). We shall show that for any fixed \( k \geq 0 \), there are constants \( c_0, c_1, \ldots, c_{2k+1} \) such that \( NT_{n, \mu}(q)\big|_{q^k} = \sum_{i=1}^{2k+1} c_i \left( \begin{array}{c} n-1 \\ i \end{array} \right) \) for all \( n \geq 2k + 2 \). To prove this result, we need to study what we call the contraction of a cycle. Define a bond of an \( n \)-cycle \( C \in C_n \) to be a pair of consecutive integers \((a, a+1)\) such that \( a \) is immediately followed by \( a+1 \) in the cycle \( C \). For example, the bonds of the cycle \((1, 4, 5, 6, 2, 7, 8, 3)\) are the pairs \((4, 5), (5, 6), \text{ and } (7, 8)\). If \( C = (c_1, c_2, \ldots, c_n) \) is an \( n \)-cycle where \( c_1 = 1 \), define the sets \( R_1, R_2, \ldots, R_k \) recursively as follows. First, let \( 1 = c_1 \in R_1 \). Then inductively, if \( c_i \in R_j \), then \( c_{i+1} \in R_j \) if \( c_{i+1} = c_i + 1 \), and \( c_{i+1} \in R_{j+1} \) otherwise. For example, for the cycle \( C = (1, 2, 4, 6, 7, 8, 3, 5) \) these sets are \( R_1 = \{1, 2\}, R_2 = \{4\}, R_3 = \{6, 7, 8\}, R_4 = \{3\}, R_5 = \{5\} \). We will call these sets the consecutive runs of \( C \). Define the contraction of a cycle \( C \) with consecutive runs \( R_1, R_2, \ldots, R_k \) to be

\[
\text{cont}(C) = \text{red}(\text{max}(R_1), \text{max}(R_2), \ldots, \text{max}(R_k)),
\]

where \( \text{max}(R_i) \) denotes the maximum element in a run \( R_i \). For example, the contraction of \( C = (1, 2, 4, 6, 7, 8, 3, 5) \) is

\[
\text{cont}(1, 2, 4, 6, 7, 8, 3, 5) = \text{red}(2, 4, 8, 3, 5) = (1, 3, 5, 2, 4).
\]

We say that \( C \) contracts to \( A \) if \( \text{cont}(C) = A \). We say that a cycle \( C \) is incontractible if \( \text{cont}(C) = C \). Thus an incontractible \( n \)-cycle \( C \) is an \( n \)-cycle such that there are no integers
i such that i + 1 immediately follows i in C. We let IC denote the set of incontractible n-cycles in C_n. We will show that |IC_n| is D_n−1 where D_n is the number of derangements of S_n, i.e. the number of σ ∈ S_n such that σ has no fixed points. We will also study the polynomials

\[ NTI_{n,\mu}(q) = \sum_{C \in IC_n} q^{NT_C(C)}. \]

Table 2 of the appendix gives the polynomials NTI_{n,\mu}(q) for n ≤ 10.

We shall show that for any n-cycle C, NTI_{n}\mu(C) = NTI_{n}(cont(C)) and that for any incontractible k-cycle A, there are precisely \( \binom{n-1}{k-1} \) n-cycles C such that cont(C) = A. It follows that

\[ NTI_{n,\mu}(q) \mid_{q^k} = \sum_{s=1}^{n} \sum_{\substack{A \in IC_s, \\ NTI_{n,\mu}(A) = k}} \binom{n-1}{s-1}. \]

Then we shall show that the lowest power of q that appears in either NTI_{2n,\mu}(q) or NTI_{2n+1,\mu}(q) is q^n. It will then follow that

\[ NTI_{n,\mu}(q) \mid_{q^k} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in IC_s, \\ NTI_{n,\mu}(A) = k}} \binom{n-1}{s-1}. \]

For example, it will then follow from our tables of the NTI_{n,\mu}(q)s that NTI_{n,\mu}(q) \mid_{q^2} = \binom{n-1}{2} for n ≥ 3 and NTI_{n,\mu}(q) \mid_{q^k} = \binom{n-1}{2} + 2\binom{n-1}{4} for n ≥ 5. Clearly, for n ≥ 1, NTI_{n,\mu}(q) \mid_{q^0} = 1 because (1, 2, ..., n) is the only n-cycle that has no nontrivial occurrences of \mu and NTI_{n,\mu}(q) \mid_{q^{n-1}} = 1 since the only n-cycle with \( \binom{n-1}{2} \) nontrivial occurrences of \mu is (1, n−1, n−2, ..., 2).

We computed that the sequence (NTI_{2k+1,\mu}(q) \mid_{q^k})_{k≥3} starts out 1, 2, 5, 14, ... which suggests that NTI_{2k+1,\mu}(q) \mid_{q^k} = C_k = \frac{1}{k+1} \binom{2n}{n} where C_k is the kth Catalan number.

Our proof of this fact lead us to a surprising connection between the charge statistic on permutations as defined by Lascoux and Schützenberger [7] and our problem. That is, given a permutation σ = σ_1...σ_n, one defines the index, ind_σ(σ_i), of σ_i in σ as follows. First, ind_σ(1) = 0. Then, inductively, ind_σ(i + 1) = ind_σ(i) if i + 1 appears to the right of i in σ and ind_σ(i + 1) = 1 + ind_σ(i) if i + 1 appears to the left i in σ. For example, if σ = 1 4 8 5 9 6 2 7 3 and we use subscripts to indicate the index of i in σ, then we see that the indices associated with σ are

\[ σ = 10 \ 41 \ 82 \ 51 \ 92 \ 61 \ 20 \ 71 \ 30. \]

The charge of σ, ch(σ), is equal to \( \sum_{i=1}^{n} \) ind_σ(i). Thus, for example, if σ = 1 4 8 5 9 6 2 7 3, then ch(σ) = 8. We will associate a path which we call the charge path of σ whose vertices are elements \( (i, \text{ind}_σ(σ_i)) \) and whose edges are \{\( (i, \text{ind}_σ(σ_i)), (i + 1, \text{ind}_σ(σ_{i+1})) \}\} for \( i = 1, ..., n \). For example, if σ = 1 4 8 5 9 6 2 7 3 as above, then charge path of σ, which we denote by cpath(σ), is pictured in Figure 4. In our particular example, the charge graph of σ is a Dyck path (a lattice path with steps (1,1) and (1,-1) from the origin.
(0,0) to (2n,0) that never goes below the x-axis) and if \( C = (1,4,8,5,9,6,2,7,3) \) is the \( n \)-cycle induced by \( \sigma \), then \( NT_\mu(C) = 4 \). This is no accident. That is, we shall show that if \( C = (1,\sigma_2,\ldots,\sigma_{2n+1}) \) is \( 2n+1 \)-cycle in \( I_C_{2n+1} \), then \( NT_\mu(C) = n \) if and only if the charge path of \( \sigma = 1\sigma_2\ldots\sigma_{2n+1} \) is a Dyck path of length \( 2n \).

We also observed that \( NT_{I_n,\mu}(q)|_{q^{(n-1)/2-k}} \) and \( NT_{I_n,\mu}(q)|_{q^{(n-1)/2-k}} \) is the number of partitions of \( k \) for sufficiently large \( n \). Here a sequence of positive integers \( \lambda = (\lambda_1,\ldots,\lambda_k) \) is a partition of \( n \) if \( \lambda_1 \geq \cdots \geq \lambda_k \) and \( \sum_{i=1}^{k} \lambda_i = n \). The Ferrers diagram \( F_\lambda \) of a partition \( \lambda = (\lambda_1,\ldots,\lambda_k) \) is the set of left justified rows of squares with \( \lambda_i \) squares in the \( i \) row from above. For example, \( F_{(3,2,2,1)} \) is pictured in Figure 5.

In fact, we will show that for \( k \leq n-2 \), \( NT_{I_n,\mu}(q)|_{q^{(n-1)/2-k}} \) equals the number of partitions of \( k \). We show this by plotting the non-\( \mu \)-matches of an \( n \)-cycle \( C \in C_n \), i.e. the pairs of integers \( \langle i,j \rangle \) with \( i < j \) which are not occurrences of the \( \mu \) pattern in \( C \), on an \( n \times n \) grid and shading a cell in the \( j \)th row and \( i \)th column if and only if \( \langle i,j \rangle \) is a non-\( \mu \)-match of \( C \). For example, the non-\( \mu \)-matches of the cycle \( C = (1,8,7,6,5,11,4,3,10,2,9) \) are \( \langle 1,9 \rangle, \langle 1,10 \rangle, \langle 1,11 \rangle, \langle 2,10 \rangle, \langle 2,11 \rangle, \langle 3,11 \rangle \) and \( \langle 4,11 \rangle \). These pairs can be plotted on the \( 11 \times 11 \) grid as shown in Figure 6. The shaded plot in Figure 6 is of the form of the Ferrers diagram of the integer partition \( \lambda = (4,2,1) \). We will show that if \( C \in C_n \) and there are
fewer than \(n - 2\) non-matches in \(C\), then shaded squares in the plot of the non-\(\mu\)-matches in \(C\) will be of the form of a Ferrers diagram of an integer partition. Moreover, we shall show that if \(C \in \mathcal{C}_n\) has fewer than \(n - 2\) non-matches, then \(C\) must be incontractible. Hence, it follows that \(NTI_{n,\mu}(q)\binom{n-1}{2-k}\) and \(NT_{n,\mu}(q)\binom{n-1}{2-k}\) are the number of partitions of \(k\) for sufficiently large \(n\).

The outline of this paper is as follows. In Section 2, we shall study the properties of contractions of \(n\)-cycles and show that for all \(n \geq 1\), there are positive integers \(c_1,k,\ldots,c_{2k+1,k}\) such that \(NT_{n,\mu}(q)\binom{n-1}{2-k}\) and \(NT_{n,\mu}(q)\binom{n-1}{2-k}\) are the number of partitions of \(k\) for sufficiently large \(n\).

In this section, we establish some basic properties about the contraction of \(n\)-cycles.

**Proposition 1.** For any \(n\)-cycle \(C \in \mathcal{C}_n\), \(NT_\mu(C) = NT_\mu(\text{cont}(C))\).

**Proof.** We proceed by induction on \(n\). The theorem is clearly true for \(n = 1\). Now, suppose that we are given an \(n\)-cycle \(C \in \mathcal{C}_n\) and \(R_1,\ldots,R_k\) are the consecutive runs. If \(k = n\), then, clearly, \(C\) is incontractible. Otherwise, suppose that \(s\) is the least \(i\) such that \(|R_i| \geq 2\). Let \(R_s = \{i, i+1, \ldots, j\}\). Then it is easy to see that in \(C\), there are no pairs \((a, t)\) which are nontrivial occurrences of \(\mu\) in \(C\) for \(i \leq a \leq j - 1\) since \(a + 1\) immediately follows \(a\) in \(C\). Similarly, there are no pairs \((s, b)\) which are nontrivial occurrences of \(\mu\) in \(C\) for \(i + 1 \leq b \leq j\) since \(b - 1\) immediately precedes \(b\) in \(C\). Now, suppose that \(C'\) arises from \(C\) by removing \(i + 1, \ldots, j\) and replacing each \(s > j\) by \(s - (j - i)\). Then it is easy to see that
(1) if \( s < t \leq i \), \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( C \) if and only if \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( C' \),

(2) if \( s \leq i < j < t \), then \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( C \) if and only if \( \langle s, t - (j - i) \rangle \) is a nontrivial occurrence of \( \mu \) in \( C' \), and

(3) if \( j \leq s < t \), then \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( C \) if and only if \( \langle s - (j - i), t - (j - i) \rangle \) is a nontrivial occurrence of \( \mu \) in \( C' \).

It follows that \( NT_\mu(C) = NT_\mu(C') \). Moreover, it is easy to see that \( \text{cont}(C) = \text{cont}(C') \).

By induction \( NT_\mu(C') = NT_\mu(\text{cont}(C')) \). Hence, \( NT_\mu(C) = NT_\mu(C') = NT_\mu(\text{cont}(C')) = NT_\mu(\text{cont}(C)) \). \( \square \)

It is easy to count the number of \( n \)-cycles \( C \in \mathcal{C}_n \) for which \( \text{cont}(C) = A \). That is, we have the following proposition.

**Proposition 2.** Let \( A \in \mathcal{IC}_\ell \) be an incontractible cycle of length \( \ell \). The number of \( n \)-cycles \( C \in \mathcal{C}_n \) such that \( \text{cont}(C) = A \) is \( \binom{n-1}{\ell-1} \).

**Proof.** If \( \text{cont}(C) = A \), then there are \( \ell \) consecutive runs of \( C \), namely \( R_1, \ldots, R_\ell \) such that \( \text{red}(\max(R_1), \ldots, \max(R_\ell)) = A \). Since the maximum of one of the consecutive runs must be \( n \), there are \( \binom{n-1}{\ell-1} \) ways to choose the rest of the maxima. This will determine the consecutive runs and then we order them so that \( \text{red}(\max(R_1), \ldots, \max(R_\ell)) = A \).

For example, suppose \( A = (1, 3, 5, 2, 4) \) and \( n = 8 \). The 8-cycle that contracts to \( A \) corresponding to the choice \( \{1, 3, 4, 7\} \) out of the \( \binom{7}{4} \) choices is obtained by letting the maxima of the consecutive runs be \( \{1, 3, 4, 7, 8\} \). Therefore, the consecutive runs are \( \{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}, \{8\} \). Then we arrange them so that \( \text{red}(\max(R_1), \ldots, \max(R_\ell)) = A \) and you get \( \{1\}, \{4\}, \{8\}, \{2, 3\}, \{5, 6, 7\} \), and so the cycle is \( (1, 4, 8, 2, 3, 5, 6, 7) \). \( \square \)

By partitioning the \( n \)-cycles \( C \in \mathcal{C}_n \) such that \( NT_\mu(C) = k \) by their contractions and applying Propositions 1 and 2, we see that for any \( k \geq 0 \) and \( n \geq 1 \),

\[
NT_{n, \mu}(q)|_{q^k} = \sum_{s=1}^{n} \sum_{\substack{A \in \mathcal{IC}_s, \\
NT_\mu(A) = k}} \binom{n-1}{s-1}.
\]

We shall prove in the next section that the smallest power of \( q \) that occurs in either \( NT_{2n, \mu}(q) \) or \( NT_{2n+1, \mu}(q) \) is \( q^n \). Hence it follows that

\[
\sum_{s=1}^{n} \sum_{\substack{A \in \mathcal{IC}_s, \\
NT_\mu(A) = k}} \binom{n-1}{s-1} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\
NT_\mu(A) = k}} \binom{n-1}{s-1}.
\]

We observed in the introduction the maximum number of nontrivial occurrences of \( \mu \) that one can have for \( s \)-cycle is \( \binom{s-1}{2} \). This means that for a cycle of length \( s \) to have \( k \) nontrivial occurrences, we must have that \( \frac{(s-1)(s-2)}{2} \geq k \) or, equivalently, \( s^2 - 3s + 2 - 2k \geq 0 \). It
follows that it must be the case that $s \geq \frac{3+\sqrt{1+8k}}{2}$. Hence,

$$(5) \quad \sum_{s=1}^{2k+1} \sum_{A \in IC_s, NT_s(A) = k} \binom{n-1}{s-1} = \sum_{s=\left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor}^{2k+1} \sum_{A \in IC_s, NT_s(A) = k} \binom{n-1}{s-1}.$$ 

Thus we have the following theorem.

**Theorem 3.** For any $k \geq 0$ and $n \geq 1$,

$$(6) \quad NT_{n,\mu}(q)^k = \sum_{s=\left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor}^{2k+1} c_{s,k} \binom{n-1}{s-1},$$

where $c_{s,k} = NTI_{s,\mu}(q)|_{q^k}$.

It follows from our tables for $NTI_{n,\mu}(q)$ and the fact that $NTI_{11,\mu}(q)|_{q^5} = C_5 = 42$ which we will prove in Corollary 7 that for all $n \geq 1$

\begin{align*}
NT_{n,\mu}(q) |_q & = \binom{n-1}{2}, \\
NT_{n,\mu}(q) |_{q^2} & = \binom{n-1}{3} + 2 \binom{n-1}{4}, \\
NT_{n,\mu}(q) |_{q^3} & = \binom{n-1}{3} + 3 \binom{n-1}{4} + 6 \binom{n-1}{5} + 5 \binom{n-1}{6}, \\
NT_{n,\mu}(q) |_{q^4} & = 2 \binom{n-1}{4} + 13 \binom{n-1}{5} + 27 \binom{n-1}{6} + 29 \binom{n-1}{7} + 14 \binom{n-1}{8}, \\
NT_{n,\mu}(q) |_{q^5} & = \binom{n-1}{4} + 10 \binom{n-1}{5} + 51 \binom{n-1}{6} + 134 \binom{n-1}{7} + \\
& \quad 181 \binom{n-1}{8} + 130 \binom{n-1}{9} + 42 \binom{n-1}{10}. \\
\end{align*}

3. **Incontractible $n$-cycles**

Let $IC_n$ denote the number of incontractible $n$-cycles in $C_n$. Clearly, there is only 1 incontractible 3-cycle, namely $(1, 3, 2)$, and there are only 2 incontractible 4-cycles, namely $(1, 4, 3, 2)$ and $(1, 3, 2, 4)$. Let $D_n$ denote the number of permutations of length $n$ with no fixed points. For example, $D_1 = 0$, $D_2 = 1$, and $D_3 = 2$. We call $D_n$ the $n$-derangement number. It is well known that $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$ for $n \geq 3$. We will show that $IC_n = (n-2)IC_{n-1} + (n-2)IC_{n-2}$ for $n \geq 5$. Our proof of this recursion will allows us to prove the following theorem.

**Theorem 4.**

1. For all $n \geq 3$, $IC_n = D_{n-1}$.
2. For all $n \geq 2$, the lowest power of $q$ that appears in $NTI_{2n}(q)$ is $q^n$ and the lowest power of $q$ that occurs in $NTI_{2n+1}(q)$ is $q^n$. 

Proof. For part (1), we have computed that \( IC_n = D_{n-1} \) for \( n \in \{3, 4\} \). Thus we need only prove that for \( n \geq 5 \),

\[
(7) \quad IC_n = (n - 2)IC_{n-1} + (n - 2)IC_{n-2}.
\]

Let \([n] = \{1, \ldots, n\}\). Suppose that \( n \geq 5 \) and \( C = (1, c_2, \ldots, c_n) \) is an \( n \)-cycle in \( \mathcal{IC}_n \). Let \( C \upharpoonright_{[n-1]} = (1, c'_2, \ldots, c'_{n-1}) \) be the cycle obtained from \( C \) by removing \( n \) from \( C \). For example if \( C = (1, 3, 6, 2, 4, 5) \) then \( C \upharpoonright_{[5]} = (1, 3, 2, 4, 5) \). We then have two cases depending on whether \( C \upharpoonright_{[n-1]} \) is incontractible or not.

**Case 1.** \( C \upharpoonright_{[n-1]} \in \mathcal{IC}_{n-1} \).

In this situation, it is easy to see that if \( D = C \upharpoonright_{[n-1]} \), there are exactly \( n - 2 \) cycles \( C' \in \mathcal{IC}_n \) such that \( D = C' \upharpoonright_{[n-1]} \). These cycles are the result of inserting \( n \) immediately after \( i \) in \( D \) for \( i = 1, \ldots, n-2 \). That is, let \( D^{(i)} \) be the result of inserting \( n \) immediately after \( i \) in the cycle structure of \( D \) where \( 1 \leq i \leq n-2 \). Then it is easy to see that \( D^{(i)} \in \mathcal{IC}_n \) and \( D = D^{(i)} \upharpoonright_{[n-1]} \). For example, if \( D = (1, 3, 5, 4, 2) \) is the 5-cycle pictured at the top of Figure 7, then \( D^{(1)}, \ldots, D^{(4)} \) are pictured on the second row of Figure 7, reading from left to right. Thus, there are \( (n - 2) IC_{n-1} \) \( n \)-cycles \( C \in \mathcal{IC}_n \) such that \( C \upharpoonright_{[n-1]} \) is an element of \( \mathcal{IC}_{n-1} \). (Note that \( D^{(n-1)} \) is not incontractible because it is obtained by inserting \( n \) directly after \( n - 1 \).)

![Figure 7. The four elements of \( \mathcal{IC}_6 \) that arise by inserting 6 into \( C = (1, 3, 5, 4, 2) \).](image)

We note that if \( D = (1 = d_1, \ldots, d_{n-1}) \) is an \( (n - 1) \)-cycle in \( \mathcal{IC}_{n-1} \), then

\[
NT_{\mu}(D^{(i)}) \geq 1 + NT_{\mu}(D) \quad \text{for} \quad i = 1, \ldots, n-2.
\]

That is, if \( \langle d_s, d_t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D \), then the insertion of \( n \) does not effect whether \( \langle d_s, d_t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D^{(i)} \). Moreover, we are always guaranteed that \( \langle i, n \rangle \) is a nontrivial occurrence of \( \mu \) in \( D^{(i)} \) since \( i \leq n - 2 \). It is possible that \( NT_{\mu}(D^{(i)}) - NT_{\mu}(D) \) is greater than or equal to 1. For example, in Figure 7, it is easy to see that if \( D = (1, 3, 5, 4, 2) \), then there is only two pairs which are nontrivial occurrences of \( \mu \) in \( C \), namely \( \langle 1, 3 \rangle \) and \( \langle 3, 5 \rangle \), while in \( D^{(1)} \), the insertion of 6 after 1 created three new nontrivial occurrences of \( \mu \) in \( D^{(1)} \) namely, the pairs \( \langle 1, 6 \rangle, \langle 2, 6 \rangle, \) and \( \langle 4, 6 \rangle \). Thus,
$NT_\mu(D^{(1)}) - NT_\mu(D) = 5 - 2 = 3$. On the other hand, it is easy to see that if $D \in IC_{n-1}$, then

$$NT_\mu(D^{(n-2)}) = 1 + NT_\mu(D).$$

That is, if we insert $n$ immediately after $n - 2$ in $D$, then for $i = 1, \ldots, n - 3$, $\langle i, n \rangle$ cannot be a nontrivial occurrence of $\mu$ in $D^{(n-2)}$ because $n - 2$ will be between $i$ and $n$ in $D$. Thus, we will create exactly one more pair which is a nontrivial occurrence of $\mu$ in $D^{(n-2)}$ that was not in $D$, namely, $\langle n-2, n \rangle$.

**Case 2 $C \mid_{[n-1]} \not\in IC_n$.**

In this case, it must be that in $C$, $n$ is the only element between $i$ and $i + 1$ in $C$ for some $i$ so that in $C \mid_{[n-1]}$, $i + 1$ immediately follows $i$. Clearly, $i$ is the only $j$ in $C \mid_{[n-1]}$ such that $j + 1$ immediately follows $j$. Hence, we can construct an element of $D \in IC_{n-2}$ from $C \mid_{[n-1]}$ by removing $i + 1$ and then replacing $j$ by $j - 1$ for $i + 1 < j \leq n - 1$. Vice versa, given $D \in IC_{n-2}$ and $1 \leq i \leq n - 2$, let $D[i]$ be the cycle that results from $D$ by first replacing elements $j \geq i + 1$, by $j + 1$, then replacing $i$ by a pair $i$ immediately followed by $i + 1$, and finally inserting $n$ between $i$ and $i + 1$. This process is pictured in Figure 8 for the cycle $D = (1, 3, 2, 4)$. In such a situation, we shall say that $D[i]$ arises by expanding $D$ at $i$.

![Figure 8. The four elements of IC6 that arise by expanding D = (1, 3, 2, 4).](image-url)
there will be no pair \( \langle i, t \rangle \) which is a nontrivial occurrence of \( \mu \) in \( D_i \) since \( i \) is immediately followed by \( i + 1 \) in \( D_i \). Moreover, it is easy to check that

1. if \( s < t \leq i \), then \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \) if and only if \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \).

2. if \( s < i < t \), then \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \) if and only if \( \langle s, t + 1 \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \).

3. \( \langle i, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \) if and only if \( \langle i + 1, t + 1 \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \), and

4. if \( i < s < t \), then \( \langle s, t \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \) if and only if \( \langle s + 1, t + 1 \rangle \) is a nontrivial occurrence of \( \mu \) in \( D_i \).

Then for every pair \( \langle r, s \rangle \) which is a nontrivial occurrence of \( \mu \) in \( D_i \), there exists a nontrivial occurrence of \( \mu \) in \( D_i \) after inserting \( n \) between \( i \) and \( i + 1 \). Finally we will create at least one new nontrivial occurrence of \( \mu \) in \( D \), namely \( \langle i, n \rangle \).

Again it is possible that \( NT_\mu(D_i) - NT_\mu(D) \) is greater than or equal to 1. For example, in Figure 8, it is easy to see that if \( D = (1, 3, 5, 4, 2) \), there are only two pairs that are nontrivial occurrences of \( \mu \) in \( D \), namely \( \langle 1, 3 \rangle \) and \( \langle 2, 4 \rangle \), which correspond to the pairs \( (1, 4) \) and \( (3, 5) \) that are nontrivial occurrences of \( \mu \) in \( D^{[2]} \). However, the insertion of 6 between 2 and 3 in \( D_2 \) created two new nontrivial occurrences of \( \mu \) in \( D^{[2]} \), namely, \( \langle 2, 6 \rangle \) and \( \langle 4, 6 \rangle \). Thus, \( NT_\mu(D^{[2]}) - NT_\mu(D) = 4 - 2 = 2 \). On the other hand, it is easy to see that if \( D \in IC_{n-2} \), then

\[
NT_\mu(D^{[n-2]}) = 1 + NT_\mu(D).
\]

That is, if we insert \( n \) immediately after \( n - 2 \) in \( D_{n-2} \), then for \( i = 1, \ldots, n - 3 \), \( \langle i, n \rangle \) cannot be a nontrivial occurrence of \( \mu \) in \( D^{[n-2]} \) because \( n - 2 \) will be between \( i \) and \( n \) in \( D^{[n-2]} \). Thus, we will create exactly one occurrence of \( \mu \) in \( D^{[n-2]} \) that was not in \( D_{n-2} \), namely, \( \langle n - 2, n \rangle \).

For part (2), note that we have shown by direct calculation that the lowest power of \( q \) appearing in \( NTI_{4, \mu}(q) \) and \( NTI_{5, \mu}(q) \) is \( q^2 \). Thus, the part (2) holds for \( n = 2 \). Now, assume part (2) holds for \( n \geq 2 \). Then we shall show the corollary holds for \( n + 1 \). We have shown that each \( (2n + 2) \)-cycle \( C \) in \( IC_{2n+2} \) is either of the form \( D^{(i)} \) for some \( D \in IC_{2n+1} \) and \( 1 \leq i \leq 2n \) in which case \( NT_\mu(D^{(i)}) \geq 1 + NT_\mu(D) \), or of the form \( E^{[i]} \) for some \( E \in IC_{2n} \) and \( 1 \leq i \leq 2n \) in which case \( NT_\mu(E^{[i]}) \geq 1 + NT_\mu(E) \). But by induction, we know that \( NT_\mu(D) \geq n \) and \( NT_\mu(E) \geq n \) so that \( NT_\mu(C) \geq n + 1 \). Thus, the smallest possible power of \( q \) that can appear in \( NTI_{2n+2, \mu}(q) \) is \( n + 1 \). On the other hand, we can assume by induction that there is a \( D \in IC_{2n+1} \) such that \( NT_\mu(D) = n \) in which case we know that \( NT_\mu(D^{(n-2)}) = 1 + NT_\mu(D) = n + 1 \). Hence, the coefficient of \( q^{n+1} \) in \( NTI_{2n+2, \mu}(q) \) is non-zero.

We have shown that each \( (2n + 3) \)-cycle \( C \) in \( IC_{2n+3} \) is either of the form \( D^{(i)} \) for some \( D \in IC_{2n+2} \) and \( 1 \leq i \leq 2n + 1 \) in which case \( NT_\mu(D^{(i)}) \geq 1 + NT_\mu(D) \), or of the form \( E^{[i]} \) for some \( E \in IC_{2n+1} \) and \( 1 \leq i \leq 2n + 1 \) in which case \( NT_\mu(E^{[i]}) \geq 1 + NT_\mu(E) \). But by induction, we know that \( NT_\mu(D) \geq n + 1 \) and \( NT_\mu(E) \geq n \) so that \( NT_\mu(C) \geq n + 1 \).
Thus, the smallest possible power of \( q \) that can appear in \( NTI_{2n+3,\mu}(q) \) is \( n+1 \). On the other hand, we can assume by induction that there is a \( D \in \mathcal{I}C_{2n+1} \) such that \( NT_\mu(D) = n \) in which case we know that \( NT_\mu(D^{[n-2]}) = 1 + NT_\mu(D) = n+1 \). Hence, the coefficient of \( q^{n+1} \) in \( NTI_{2n+3,\mu}(q) \) is non-zero.

We have not been able to find a recursion for the polynomials \( NTI_{n,\mu}(q) \). The problem with the recursion implicit in the proof of Theorem 4 is that for \( n \)-cycles \( D \in \mathcal{I}C_n \), the contributions of \( \sum_{i=1}^{\mu(D)} q^{NT_\mu(D[i])} \) are not uniform. For example, if \( D = (1,3,5,4,2) \), then \( NT_\mu(D) = 2 \), \( NT_\mu(D[1]) = 5 \), \( NT_\mu(D[2]) = 4 \), \( NT_\mu(D[3]) = 4 \), and \( NT_\mu(D[4]) = 3 \) so that \( \sum_{i=1}^{\mu(D)} q^{NT_\mu(D[i])} = (q + 2q + q^3)q^{NT_\mu(D)} \). However, \( D = (1,5,4,3,2) \), then \( NT_\mu(D) = 6 \), \( NT_\mu(D[1]) = 10 \), \( NT_\mu(D[2]) = 9 \), \( NT_\mu(D[3]) = 8 \), and \( NT_\mu(D[4]) = 7 \) so that \( \sum_{i=1}^{\mu(D)} q^{NT_\mu(D[i])} = (q + q^2 + q^3 + q^4)q^{NT_\mu(D)} \). A similar phenomenon occurs for \( \sum_{i=1}^{\mu(D)} q^{NT_\mu(D[i])} \). For example, it is easy to see from Figure 6 that if \( D = (1,3,2,4) \), then \( NT_\mu(D) = 2 \), \( NT_\mu(D[1]) = 3 \), \( NT_\mu(D[2]) = 4 \), \( NT_\mu(D[3]) = 3 \), and \( NT_\mu(D[4]) = 3 \) so that \( \sum_{i=1}^{\mu(D)} q^{NT_\mu(D[i])} = (3q + q^2)q^{NT_\mu(D)} \). However, as one can see from Figure 9 below that if \( D = (1,4,3,2) \), then \( NT_\mu(D) = 3 \), \( NT_\mu(D[1]) = 6 \), \( NT_\mu(D[2]) = 5 \), \( NT_\mu(D[3]) = 4 \), and \( NT_\mu(D[4]) = 4 \) so that \( \sum_{i=1}^{\mu(D)} q^{NT_\mu(D[i])} = (2q + q^2 + q^3)q^{NT_\mu(D)} \).

**Figure 9.** The four elements of \( \mathcal{I}C_6 \) that arise by expanding \( D = (1,4,3,2) \).

Our next goal is to show that \( NTI_{2n+1,\mu}(q) \mid_{q^n} = C_n \) where \( C_n \) is the \( n \)th Catalan number.

**Theorem 5.** Suppose that \( \sigma = \sigma_1 \ldots \sigma_{2n+1} \in S_{2n+1} \) and chpath(\( \sigma \)) = \( P \) is a Dyck path of length \( 2n \). Then the \( n \)-cycle \( C_\sigma = (\sigma_1, \ldots, \sigma_{2n+1}) \) is incontractible and \( NT_\mu(C) = n \).

**Proof.** Since the path \( P \) starts at \((0,0)\), it follows that \( \text{ind}_\sigma(\sigma_1) = 0 \). This can only happen if \( \sigma_1 = 1 \). Thus, \( C_\sigma = (1,\sigma_2, \ldots, \sigma_{2n+1}) \) has the standard form of an \( n \)-cycle. Note that the
only way $i + 1$ immediately follows $i$ in $C_\sigma$ is if there is a $j$ such that $\sigma_j = i$ and $\sigma_{j+1} = i + 1$. But then $\text{ind}_\sigma(\sigma_j) = \text{ind}_\sigma(\sigma_{j+1})$ which would imply that the charge path of $\sigma$ has a level which means that there is an edge drawn horizontally. Since we are assuming that the charge path of $C_\sigma$ is a Dyck path, there can be no such $i$ and hence, $C_\sigma$ is intractable.

Next we claim that if the $i$th step of $P$ is an up-step, then $\langle \sigma_i, \sigma_{i+1} \rangle$ is a nontrivial occurrence of $\mu$ in $C_\sigma$. That is, if the $i$th step of $P$ is an up-step, then $\text{ind}_\sigma(\sigma_i) = k$ and $\text{ind}_\sigma(\sigma_{i+1}) = k + 1$ for some $k \geq 0$. Since $P$ is a Dyck path, there must be some $j > i + 1$ such that $\text{ind}_\sigma(\sigma_j) = k$ because we must pass through level $k$ from the point $(i + 1, k + 1)$ in $\text{clpath}(\sigma)$ to get back to the point $(2n, 0)$ which is the end point of the path. Let $i_j$ denote the smallest $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$. It follows from the definition of the function $\text{ind}_\sigma$ that it must be the case that $\sigma_i = \ell$ and $\sigma_{i_j} = \ell + 1$. Moreover, it must be the case that $\sigma_{i+1} > \ell + 1$ so that $\langle \sigma_i, \sigma_{i+1} \rangle$ is a nontrivial occurrence of $\mu$ in $C_\sigma$. We claim that there can be no $t \neq i_{i+1}$ such that $\langle \sigma_i, t \rangle$ is a nontrivial occurrence of $\mu$ in $C_\sigma$. That is, since $P$ is a Dyck path, we know that all the vertices on $P$ between $(i, k)$ and $(i_j, k)$ lie on levels strictly greater than $k$. It is easy to see from our inductive definition of $\text{ind}_\sigma$ that it must be the case that $\sigma_{i+1}$ is the least element of $\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{i_j} - 1$ and hence the fact that $\sigma_{i+1}$ immediately follows $\sigma_i$ in $C_\sigma$ implies that none of the pairs $\langle \sigma_i, \sigma_s \rangle$ is a nontrivial occurrence of $\mu$ in $C_\sigma$ for $i + 1 < s < i_j$. But then as we traverse around the cycle $C_\sigma$, the fact that $\sigma_i = \ell$ and $\sigma_{i_j} = \ell + 1$ implies that none of the pairs $\langle \sigma_i, \sigma_s \rangle$ where either $i_j < s \leq 2n + 1$ or $1 \leq s < i$ are nontrivial occurrences of $\mu$ in $C_\sigma$. Thus, if the $i$th step of $P$ is an up-step, then $\langle \sigma_i, t \rangle$ is an occurrence of $\mu$ in $C_\sigma$ for exactly one $t$, namely $t = \sigma_{i+1}$.

Next suppose that the $i$th step of $P$ is a down-step. Then, we claim that there is no $t$ such that $\langle \sigma_i, t \rangle$ is a nontrivial occurrence of $\mu$ in $C_\sigma$. We then have two cases.

**Case 1.** $\text{ind}_\sigma(\sigma_i) = k$ and there is a $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$.

In this case, let $i_j$ be the least $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$. It then follows that if $\sigma_i = \ell$, then $\sigma_{i_j} = \ell + 1$. Because $P$ is a Dyck path, it must be the case that $\text{ind}_\sigma(\sigma_s) < k$ for all $i < s < i_j$ so that $\sigma_s < \ell$ for all $i < s < i_j$. But this means that as we traverse around the cycle $C_\sigma$, the first element that we encounter after $\sigma_i = \ell$ which is bigger than $\ell$ is $\sigma_{i_j} = \ell + 1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_i, t \rangle$ is a nontrivial occurrence of $\mu$ in $C_\sigma$.

**Case 2.** $\text{ind}_\sigma(\sigma_i) = k$ and there is no $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$.

In this case since $P$ is a Dyck path, all the elements $\sigma_s$ such that $s > i$ must have $\text{ind}_\sigma(\sigma_s) < k$ and hence $\sigma_s < \sigma_i$ since, in a charge graph, all the elements whose index in $\sigma$ is less than $k$ are smaller than all the elements whose index is equal to $k$ in $\sigma$. We now have two subcases.

**Subcase 2.1.** There is an $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

Let $s_j$ be the smallest $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$. Since $\sigma_i$ was the rightmost element whose index relative to $\sigma$ equals $k$, it follows that if $\sigma_i = \ell$, then $\sigma_{s_j} = \ell + 1$ since it is the leftmost element whose index relative to $\sigma$ equals $k + 1$. Moreover, for all $t < s_j$, either...
\(\sigma_t\) has index less than \(k\) in \(\sigma\) or \(\sigma_t\) has index \(k\) in \(\sigma\). In either case, our definition of \(\text{ind}_\sigma\) ensures that \(\sigma_t < \ell\). But this means that as we traverse around the cycle \(C_\sigma\), the first element that we encounter after \(\sigma_t = \ell\) which is bigger than \(\ell\) is \(\sigma_{s_j} = \ell + 1\) and hence there is no \(t > \sigma_t\) such that \(\langle \sigma_t, t \rangle\) is a nontrivial occurrence of \(\mu\) in \(C_\sigma\).

**Subcase 2.2.** There is no \(s < i\) such that \(\text{ind}_\sigma(\sigma_s) = k + 1\).

In this case, \(\sigma_i\) has the highest possible index in \(\sigma\) and it is the rightmost element whose index in \(\sigma\) is \(k\) which means that \(\sigma_i = 2n + 1\). Hence, there is no \(t > \sigma_i\) such that \(\langle \sigma_i, t \rangle\) is an occurrence of \(\mu\) in \(C_\sigma\).

The only other possibility is that we have not considered is that \(i = 2n + 1\). Since \(P\) is a Dyck path, we know that \(\text{ind}_\sigma(\sigma_{2n+1}) = 0\), \(\text{ind}_\sigma(\sigma_1) = 0\), and \(\text{ind}_\sigma(\sigma_2) = 1\). But then it follows that if \(\sigma_{2n+1} = \ell\), then \(\sigma_1 < \ell\) and \(\sigma_2 = \ell + 1\). But this means that as we traverse around the cycle \(C_\sigma\), the first element that we encounter after \(\sigma_{2n+1} = \ell\) which is bigger than \(\ell\) is \(\sigma_2 = \ell + 1\) and hence there is no \(t > \sigma_i\) such that \(\langle \sigma_{2n+1}, t \rangle\) is a nontrivial occurrence of \(\mu\) in \(C_\sigma\).

Thus, we have shown that the nontrivial occurrences of \(\mu\) in \(C_\sigma\) correspond to the up-steps of 
\[\text{chpath}(\sigma)\] and hence \(NT_\mu(C_\sigma) = n\). \(\square\)

Our next theorem will show that if \(C = (1, \sigma_2, \ldots, \sigma_{2n+1})\) is a \((2n + 1)\)-cycle in \(IC_{2n+1}\) where \(NT_\mu(C) = n\), then the charge path of \(\sigma_C = 1\sigma_2 \ldots \sigma_{2n+1}\) must be a Dyck path.

**Theorem 6.** Let \(C = (1, \sigma_2, \ldots, \sigma_{2n+1}) \in IC_{2n+1}\) and \(\sigma_C = 1\sigma_2 \ldots \sigma_{2n+1}\). Then, if \(NT_\mu(C) = n\), the charge path of \(\sigma_C\) must be a Dyck path.

**Proof.** Our proof proceeds by induction on \(n\). For \(n = 1\), the only element of \(IC_3\) is \(C = (1, 3, 2)\) and it is easy to see that 
\[\text{chpath}(132)\] is a Dyck path.

Now assume by induction that if \(D = (1, \sigma_2, \ldots, \sigma_{2n+1})\) is a \((2n + 1)\)-cycle in \(IC_{2n+1}\) such that \(NT_\mu(D) = n\), then the charge path of \(\sigma_D = 1\sigma_2 \ldots \sigma_{2n+1}\) is a Dyck path. Now suppose that \(C = (1, \sigma_2, \ldots, \sigma_{2n+3})\) is a \((2n + 3)\)-cycle in \(IC_{2n+3}\) such that \(NT_\mu(C) = n + 1\). Let \(\sigma_C = \sigma = 1\sigma_2 \ldots \sigma_{2n+3}\). Then, we know by our proof of Theorem 4 that either \(C = D^{[j]}\) for some \(D \in IC_{2n+2}\) or \(C = D^{[j]}\) for some \(D \in IC_{2n+1}\). We claim that \(C\) cannot be of the form \(D^{[j]}\) for some \(D \in IC_{2n+2}\). That is, we know that for all \(D \in IC_{2n+2}\) we have \(NT_\mu(D) \geq n + 1\) and, hence, \(NT_\mu(D^{[j]}) \geq 1 + NT_\mu(D) \geq n + 2\). Thus, there must be some \(D \in IC_{2n+1}\) such that \(C = D^{[j]}\) for some \(j\). But then we know that \(P_D = \text{chpath}(D)\) is a Dyck path. Let \(D = (1, \tau_2, \ldots, \tau_{2n+1})\) and \(\sigma_D = \tau_1 \ldots \tau_{2n+1}\). Note that it follows from our arguments in Theorem 4 that \(NT_\mu(D^{[j]}) - NT_\mu(D)\) is equal to the number of pairs \(\langle s, 2n + 3 \rangle\) which are nontrivial occurrences of \(\mu\) in \(D^{[j]}\). It is also the case that the charge path of \(\sigma\) is easily constructed from the charge path of \(\tau\). That is, suppose that \(\tau_i = j\). Then, \(1, \ldots, j\) are in the same order in \(\sigma\) and \(\tau\) so that \(\text{ind}_\sigma(s) = \text{ind}_\tau(s)\) for \(1 \leq s \leq j\). In going from \(\tau\) to \(\sigma\), we first increased the value of any \(t > j\) by one and then inserted \(j + 1\) immediately after \(j\) to get a permutation \(\alpha\). Thus \(\text{ind}_\alpha(j) = \text{ind}_\alpha(j + 1)\). Moreover, \(j + 1, \ldots, 2n + 2\) are in the same relative order in \(\alpha\) as \(j, \ldots, 2n + 1\) in \(\tau\) so that for \(j + 1 \leq t \leq 2n + 1\), \(\text{ind}_\tau(t) = \text{ind}_\alpha(t + 1)\). Finally, inserting \(2n + 3\) in \(\alpha\) between \(j\) and \(j + 1\)
has no effect on the indices assigned to 1, \ldots, 2n + 2. Thus for 1 \leq s \leq j, \text{ind}_\sigma(j) = \text{ind}_\tau(j) and for j + 1 \leq t \leq 2n + 1, \text{ind}_\sigma(t + 1) = \text{ind}_\tau(t). This means that one can construct the charge path of \sigma by essentially starting with the charge graph of \tau, then replacing the vertex \((i, \text{ind}_\tau(\tau_i))\) by a horizontal edge, and finally replacing that horizontal edge by a pair of edges \{(i, \text{ind}_\sigma(\tau_i)), (i + 1, \text{ind}_\sigma(2n + 3))\} and \{(i + 1, \text{ind}_\sigma(2n + 3)), ((i + 2), \text{ind}_\sigma(\tau_{i+2}))\}. In particular, this means the charge paths from 1 up to i are identical for both \sigma and \tau and the charge path from \(i + 2\) to \(2n + 3\) in \sigma is identical to the charge path from \(i\) to \(2n + 1\) in \tau.

We now consider several cases depending on which \(i\) is such that \(\tau_i = j\) and the value \(\text{ind}_\tau(\tau_i)\) in the Dyck path \(P_D\).

**Case 1.** \(j = \tau_i\) where \(\text{ind}_\tau(\tau_i) < \text{ind}_\tau(\tau_{i+1})\).

In this case, we will have \(\sigma_i = \tau_i, \sigma_{i+1} = 2n + 3, \sigma_{i+2} = 1 + \tau_i, \text{ and } \sigma_{i+3} = 1 + \tau_i + 1\). Since \(P_D\) is a Dyck path, there will be some \(k\) such that \(\text{ind}_\tau(\tau_i) = k, \text{ and } \text{ind}_\tau(\tau_{i+1}) = k + 1\) so that we will be in the situation pictured in Figure 10. Because \(\text{ind}_\sigma(\sigma_{i+1}) = k + 1\), it must be the case that \(\text{ind}_\sigma(2n + 3) > k + 1\). It is easy to see that \(\langle \sigma_i, 2n + 3 \rangle\) is an occurrence of \(\mu\) in \(C\). We then have two subcases.

**Subcase 1A.** There is an \(s < i\) such that \(\text{ind}_\sigma(\sigma_s) = k + 1\).

In this case, let \(s_i\) be the largest \(s\) such that \(s < i\) and \(\text{ind}_\sigma(\sigma_s) = k + 1\). Because \(P_D\) is a Dyck path, it must be the case that for all \(s_i < t < i\), \(\text{ind}_\sigma(\sigma_s) \leq k\) because on the path from \((s, k + 1)\) to \((i, k)\), we cannot have an \(s < t < i\) such that \(\text{ind}_\sigma(\sigma_t) > k + 1\) without having some \(t < u < i\) such that \(\text{ind}_\sigma(\sigma_u) = k + 1\) which would violate our choice of \(s_i\). But this means that \(\sigma_t < \sigma_i\) for all \(s_i < t < \sigma_i\) and hence, \(\langle \sigma_{s_i}, 2n + 3 \rangle\) is a nontrivial occurrence of \(\mu\) in \(C\). Thus, \(NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2\).

**Subcase 1B.** There is no \(s < i\) such that \(\text{ind}_\sigma(\sigma_s) = k + 1\).

In this case, let \(s_i\) be the largest \(s\) such that \(i < s\) and \(\text{ind}_\sigma(\sigma_s) = k\). Because \(P_D\) is a Dyck path such an \(s\) must exist since \(\text{ind}_\sigma(\sigma_{i+3}) = k + 1\) and a Dyck path that reaches level \(k + 1\) must subsequently descend to level \(k\). But this means that for all \(t < i\), \(\text{ind}_\sigma(\sigma_t) \leq k\) and hence, \(\sigma_t \leq \sigma_i\). Moreover, for all \(r > s_i\), \(\text{ind}_\sigma(\sigma_r) < k\) and hence, \(\sigma_r \leq \sigma_i\). But it then follows that since \(\sigma_{s_i} > \sigma_t, \langle \sigma_{s_i}, 2n + 3 \rangle\) is a nontrivial occurrence of \(\mu\) in \(C\). Thus, \(NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2\).

**Case 2.** \(j = \tau_{2n+1}\).

In this case, we will have \(\sigma_{2n+1} = \tau_{2n+1}, \sigma_{2n+2} = 2n + 3\) and \(\sigma_{2n+3} = \tau_{2n+1} + 1\). Thus, the charge path of \(D^{[\tau_{2n+1}]}\) looks like one of the two situations pictured in Figure 11. That is, we have two subcases.

**Subcase 2A.** \(\text{ind}_\sigma(\sigma_{2n+2}) = 1\).

In this case, \(\text{chpath}(\sigma)\) will be a Dyck path. This case can only happen if \(\text{ind}_\tau(\tau_i) \leq 1\) for all \(1 \leq i \leq 2n + 1\).

**Subcase 2B.** \(\text{ind}_\sigma(\sigma_{2n+2}) \geq 2\).
In this case, chpath(\(\sigma\)) will not be a Dyck path. This case can only happen if there is an \(s\) such that \(\text{ind}_r(\tau_s) = \text{ind}_\sigma(\tau_i + s) \geq 2\). But then since \(\text{ind}_\sigma(\sigma_{2n}) = 1\), it must be the case that \(\sigma_{2n} < \tau_s + 1 < 2n + 3\). Hence both \(\langle \sigma_{2n+1}, 2n + 3 \rangle\) and \(\langle \sigma_{2n}, 2n + 3 \rangle\) are nontrivial occurrences of \(\mu\) in \(C\) so that \(NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2\).

**Figure 11.** The charge path of \(D^{|\tau_{2n+1}|}\) in Case 2.

**Case 3.** \(j = \tau_i\) where \(\text{ind}_r(\tau_{i-1}) > \text{ind}_r(\tau_i) > \text{ind}_r(\tau_{i+1})\).

In this case, we will have \(\sigma_{i-1} = 1 + \tau_{i-1}, \sigma_i = \tau_i, \sigma_{i+1} = 2n + 3, \sigma_{i+2} = 1 + \tau_i,\) and \(\sigma_{i+3} = \tau_{i+1}\). There are now two cases.

**Case 3A.** \(\tau_{i-1} < 2n + 1\).

In this case, we will have the situation pictured in Figure 12. That is, since \(\sigma_{i-1} = 1 + \tau_{i-1} < 2n + 2\), it must be the case the either \(2n + 2\) is to the left of \(\sigma_{i-1}\) in which case
$\text{ind}_\sigma(2n + 2) > k + 1$ or $2n + 2$ is the right of $\sigma_{i+3}$ in which case $\text{ind}_\sigma(2n + 2) \geq k + 1$ and $2n + 3$ is to the left of $2n + 2$ in $\sigma$. In either case, it must be that $\text{ind}_\sigma(2n + 3) > k + 1$. It is then easy to see that both $\langle \sigma_{i-1}, 2n + 3 \rangle$ and $\langle \sigma_{i}, 2n + 3 \rangle$ are nontrivial occurrences of $\mu$ in $C$ so that $\text{NT}_\mu(C) \geq \text{NT}_\mu(D) + 2 = n + 2$.

Case 3B. $\tau_{i-1} = 2n + 1$.
In this case, we will have the situation pictured in Figure 13. In this situation, the charge path of $D[\tau_i]$ will be a Dyck path.
Case 4. \( j = \tau_i \) where \( \text{ind}_\tau(\tau_{i-1}) < \text{ind}_\tau(\tau_i) > \text{ind}_\tau(\tau_{i+1}) \).

In this case, we will have \( \sigma_{i-1} = \tau_{i-1}, \sigma_i = \tau_i, \sigma_{i+1} = 2n+3, \sigma_{i+2} = 1+\tau_i, \) and \( \sigma_{i+3} = \tau_{i+1} \). There are now two subcases.

Subcase 4A. \( \text{ind}_\sigma(2n+3) > k+1 \).

In this case, we have the situation pictured in Figure 14. It will always be the case that \( \langle \sigma_i, 2n+3 \rangle \) is a nontrivial occurrence of \( \mu \) in \( C \). We then have two more subcases.

Subcase 4A1. There is an \( s < i \) such that \( \text{ind}_\sigma(\sigma_s) = k+1 \).

In this case, let \( s_i \) be the largest \( s \) such that \( s < i \) and \( \text{ind}_\sigma(\sigma_s) = k+1 \). Because \( P_D \) is a Dyck path, we can argue as in Case 1A that it must be the case that for all \( s_i < t < i \), \( \text{ind}_\sigma(\sigma_s) \leq k \). But this means that \( \sigma_t < \sigma_i \) for all \( s_i < t < \sigma_i \) and hence, \( \langle \sigma_s, 2n+3 \rangle \) is also a nontrivial occurrence of \( \mu \) in \( C \). Thus, \( NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2 \).

Subcase 4A2. There is no \( s < i \) such that \( \text{ind}_\sigma(\sigma_s) = k+1 \).

In this case, let \( s_i \) be the largest \( s \) such that \( i < s \) and \( \text{ind}_\sigma(\sigma_s) = k \). Note that \( s_i \) exists since \( \text{ind}_\sigma(\sigma_{i+2}) = k \). But this means that for all \( t < i \), \( \text{ind}_\sigma(\sigma_t) \leq k \) and, hence, \( \sigma_t < \sigma_i \). Moreover, for all \( r > s_i \), \( \text{ind}_\sigma(\sigma_r) < k - 1 \) and hence, \( \sigma_r < \sigma_i \). But it then follows that since \( \sigma_s > \sigma_i \), \( \langle \sigma_s, 2n + 3 \rangle \) is also a nontrivial occurrence of \( \mu \) in \( C \). Thus, \( NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2 \).

Subcase 4B. \( \text{ind}_\sigma(2n+3) = k+1 \).

In this case, we have the situation pictured in Figure 15 and, hence, \( \text{chpath}(\sigma) \) will be a Dyck path.

![Figure 14](image-url). The charge path of \( D^{[\tau_i]} \) in Case 4A.
Thus, we have shown that if $D \in IC_{2n+1}$ is such that $NT_\mu(D) = n$ and $D[i] = C$, then either chpath($\sigma$) is a Dyck path or $NT_\mu(D[i]) \geq n + 2$. Hence, if $NT_\mu(C) = n + 1$, then chpath($\sigma$) is a Dyck path. \hfill \Box

Theorems 5 and 6 yield the following corollary.

**Corollary 7.** $NTI_{2n+1,\mu}(q^n) = C_n$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$th Catalan number.

### 4. Plots and Integer Partitions

In this section we will show that $NTI_{n,\mu}(q)_{\mu} | q^{n-1} - k$ and $NTn,\mu(q)_{\mu} | q^{n-1} - k$ are equal to the number of partitions of $k$ for sufficiently large $n$ by plotting the non-$\mu$-matches on a grid and mapping them to Ferrers diagrams of partitions.

Given an $n$-cycle $C \in C_n$, we say that a pair $\langle i, j \rangle$ with $i < j$ is a non-$\mu$-match of $C$ if $\langle i, j \rangle$ is not an occurrence of $\mu$ in $C$. In other words, $\langle i, j \rangle$ is a non-$\mu$-match if there exists an integer $x$ such that $i < x < j$ and $x$ is cyclically between $i$ and $j$ in $C$. Furthermore, let $N_M_{\mu}(C)$ be the set of non-$\mu$-matches in $C$ and let $NM_{\mu}(C) = |N_M_{\mu}(C)|$. Let

$$NM_{n,\mu}(q) = \sum_{C \in C_n} q^{NM_{\mu}(C)}.$$  

Note that for any $C \in C_n$, $NM_{\mu}(C) + NT_\mu(C) = \binom{n-1}{2}$ so that

$$NM_{n,\mu}(q)_{\mu} | q^{n-1} - k.$$  

Table 3 of the appendix shows the polynomials $NM_{n,\mu}(q)$ for $1 \leq n \leq 10$. Our data suggests the following theorem.

**Theorem 8.** For $k < n - 2$, $NM_{n,\mu}(q)_{\mu} = a(k)$ where $a(k)$ is the number of integer partitions of $k$. 

![Figure 15. The charge path of $D[\tau]$ in Case 4B.](image)
To prove Theorem 8, we need to define what we call the plot of non-$\mu$-matches in $C \in \mathcal{C}_n$. We consider an $n \times n$ grid where the rows are labeled with $1, 2, \ldots, n$, reading from bottom to top, and the columns are labeled with $1, 2, \ldots, n$, reading from left to right. The cell $(i, j)$ is the cell which lies in the $i$th row and $j$th column. Then, given a set $S$ of ordered pairs $(i, j)$ with $1 \leq i \leq n$, we let $\text{plot}_n(S)$ denote the diagram that arises by shading a cell $(j, i)$ on the $n \times n$ grid if and only if $(i, j) \in S$. Given an $n$-cycle $C \in \mathcal{C}_n$, we let $\text{NM}_\mu(C) = \text{plot}_n(\mathcal{N}\mathcal{M}_\mu(C))$. For example, if $C = (1, 4, 5, 3, 8, 7, 2, 6)$, then

$$\mathcal{N}\mathcal{M}_\mu(C) = \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 7), (2, 8), (3, 5), (4, 6), (4, 7), (4, 8)\}$$

and $\text{NM}_\mu(C)$ is pictured in Figure 16.

![Figure 16. $\text{NM}_\mu((1, 4, 5, 3, 8, 7, 2, 6)$).](image)

First, we observe that if $C \in \mathcal{C}_n$, then $\mathcal{N}\mathcal{M}_\mu(C)$ completely determines $C$. That is, we have the following theorem.

**Theorem 9.** If $S$ is a set of ordered pairs such that $\mathcal{N}\mathcal{M}_\mu(C) = S$ for some $n$-cycle $C \in \mathcal{C}_n$, then $C$ is the only $n$-cycle such that $\mathcal{N}\mathcal{M}_\mu(C) = S$.

**Proof.** Our proof proceeds by induction on $n$. Clearly, the theorem holds for $n = 1$ and $n = 2$. Now, assume that the theorem holds for $n$. Let $C' \in \mathcal{C}_{n+1}$ and $S = \mathcal{N}\mathcal{M}_\mu(C')$. Let $C = (1, c_2, \ldots, c_n)$ be the $n$-cycle that is obtained from $C$ by removing $n+1$. Then it is easy to see that $\mathcal{N}\mathcal{M}_\mu(C) = S - \{(i, n+1) : (i, n+1) \in S\}$. Hence, $C$ is the unique $n$-cycle in $\mathcal{C}_n$ such that $\mathcal{N}\mathcal{M}_\mu(C) = S - \{(i, n+1) : (i, n+1) \in S\}$. Let $C^{(i)}$ denote the cycle that results by inserting $n+1$ immediately after $i$ in $C$. Since $C$ is unique, it follows that $C'$ must equal $C^{(i)}$ for some $1 \leq i \leq n$. However, it is easy to see that if $1 \leq i < n$, then $(j, n+1) \in \mathcal{N}\mathcal{M}_\mu(C^{(i)})$ for $1 \leq j < i$ and $(i, n+1) \notin \mathcal{N}\mathcal{M}_\mu(C^{(i)})$. If $i = n$, then $(j, n+1) \in \mathcal{N}\mathcal{M}_\mu(C^{(i)})$ for $1 \leq j < n$. Thus, it follows that $\mathcal{N}\mathcal{M}_\mu(C^{(1)}), \ldots, \mathcal{N}\mathcal{M}_\mu(C^{(n)})$ are pairwise distinct. Hence, there is exactly one cycle $C'$ such that $\mathcal{N}\mathcal{M}_\mu(C') = S$. □

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of $n$, the Ferrers diagram of a partition $\lambda$ on the $m \times m$ grid, denoted $FD_m(\lambda)$, is the diagram that results by shading the squares $(1, m - i + 1), (2, m - i + 1), \ldots, (\lambda_i, m - i + 1)$ in the $i$th row from the top for $i = 1, \ldots, \ell$. For example, the Ferrers diagram of $\lambda = (5, 4, 3, 1, 1)$ on a $8 \times 8$ grid is pictured in Figure 17.
It is easy to see that the following three properties characterize the Ferrers diagrams of partitions $\lambda$ in an $n \times n$ grid.

(1) If $\lambda$ is not empty, then the square $(n, 1)$ is shaded in $FD_m(\lambda)$.
(2) If $x \neq 1$ and the square $(y, x)$ is shaded in $FD_m(\lambda)$, then the square $(y, x - 1)$ is shaded in $FD_m(\lambda)$.
(3) If $y \neq n$ and the square $(y, x)$ is shaded in $FD_m(\lambda)$, then the square $(y + 1, x)$ is shaded in $FD_m(\lambda)$.

This means that if $S$ is a set of pairs $(i, j)$ such that $\text{plot}_n(S)$ is a Ferrers diagram, then it must be the case that

(1) if $S$ is not empty, then the square $(1, n) \in S$,
(2) if $x \neq 1$ and the square $(x, y) \in S$, then $(x - 1, y) \in S$, and
(3) if $y \neq n$ and the square $(x, y) \in S$, then $(x, y + 1) \in S$.

Our next lemma and corollary will show that if $C \in C_n$ and $N\mathcal{M}_\mu(C) < n - 2$, then $N\mathcal{M}_\mu(C)$ has the same three properties.

**Lemma 10.** Assume that $C \in C_n$.

(1) If $(1, n) \notin N\mathcal{M}_\mu(C)$ and $N\mathcal{M}_\mu(C) \neq 0$, then $N\mathcal{M}_\mu(C) \geq n - 2$.
(2) If $x \neq 1$, $(x, y) \in N\mathcal{M}_\mu(C)$, and $(x - 1, y) \notin N\mathcal{M}_\mu(C)$, then $N\mathcal{M}_\mu(C) \geq n - 2$.
(3) If $y \neq n$, $(x, y) \in N\mathcal{M}_\mu(C)$, and $(x, y + 1) \notin N\mathcal{M}_\mu(C)$, then $N\mathcal{M}_\mu(C) \geq n - 2$.

**Proof.** Let $C = (1, c_1, \ldots, c_{n-1}) \in C_n$ be an $n$-cycle where $n \geq 4$.

For part (1), suppose that $(1, n) \notin N\mathcal{M}_\mu(C)$ and $N\mathcal{M}_\mu(C) \neq 0$. Then, there are no integers that are cyclically in between 1 and $n$ in $C$ so that $c_1 = n$. Since $N\mathcal{M}_\mu(C) \neq 0$, $C \neq (1, n, n-1, \ldots, 3, 2)$. Thus, $C$ must be of the form

$$C = (1, n, n-1, \ldots, n-a, b, \ldots, b+1, \ldots)$$

where $b + 1 < n - a$. That is, the sequence $c_1 > c_2 > \cdots > c_{a+1}$ consists of the decreasing interval from $n$ to $n - a$ and then $b \leq n - a - 2$. It follows that the pairs $(1, b+1), (b, n), \ldots, (b, n-a)$ are all non-$\mu$-matches in $C$ which accounts for $a + 2$ non-$\mu$-matches in $C$. Let $A$ be the set of integers cyclically between $b$ and $b + 1$ in $C$ and let $B$ be the set of integers cyclically between $b + 1$ and 1 in $C$. Note that

**Figure 17.** $FD_8 (5, 4, 3, 1, 1)$
(1) if \(x \in A\) and \(x < b\), then \(\langle x, n \rangle \in \mathcal{NM}_\mu(C)\),
(2) if \(x \in A\) and \(x > b\), then \(\langle 1, x \rangle \in \mathcal{NM}_\mu(C)\),
(3) if \(x \in B\) and \(x < b\), then \(\langle x, b + 1 \rangle \in \mathcal{NM}_\mu(C)\), and
(4) if \(x \in B\) and \(x > b\), then \(\langle 1, x \rangle \in \mathcal{NM}_\mu(C)\).

It follows that each element in \(A\) and \(B\) is part of at least one non-\(\mu\)-match in \(C\) so that we know that \(\mathcal{NM}_\mu(C) \geq a + 2 + |A| + |B|\). Thus, since \(|A| + |B| = n - a - 4\), we have \(\mathcal{NM}_\mu(C) \geq n - 2\).

For part (2), suppose that \(x \neq 1\), \(\langle x, y \rangle \in \mathcal{NM}_\mu(C)\), and \(\langle x - 1, y \rangle \notin \mathcal{NM}_\mu(C)\). Then \(x\) cannot be cyclically between \(x - 1\) and \(y\) in \(C\). Also, there exists an integer \(z\) with \(x < z < y\) such that \(z\) is cyclically between \(x\) and \(y\) in \(C\), but \(z\) is not cyclically between \(x - 1\) and \(y\) in \(C\). It follows that \(C\) is of the form

\[C = (1, \ldots, x, \ldots, z, \ldots, x - 1, \ldots, y, \ldots)\]

where \(A_1\) is the set of integers cyclically between \(x\) and \(x - 1\) in \(C\) that are not equal to \(z\), \(A_2\) is the set of integers cyclically between \(x - 1\) and \(y\) in \(C\), and \(A_3\) is the set of integers between \(y\) and \(x\) in \(C\). Note that \(\langle x, y \rangle\) and \(\langle x - 1, z \rangle\) are in \(\mathcal{NM}_\mu(C)\). Moreover,

(1) if \(a \in A_1\) and \(a < x - 1\), then \(\langle a, y \rangle \in \mathcal{NM}_\mu(C)\),
(2) if \(a \in A_1\) and \(a > x\), then \(\langle x - 1, a \rangle \in \mathcal{NM}_\mu(C)\),
(3) if \(a \in A_2\) and \(a < x - 1\), then \(\langle a, z \rangle \in \mathcal{NM}_\mu(C)\),
(4) there are no \(a \in A_2\) with \(x - 1 < a < y\) since \(\langle x - 1, y \rangle \notin \mathcal{NM}_\mu(C)\),
(5) if \(a \in A_2\) and \(a > y\), then \(\langle 1, a \rangle \in \mathcal{NM}_\mu(C)\),
(6) if \(a \in A_3\) and \(a < z\), then \(\langle a, y \rangle \in \mathcal{NM}_\mu(C)\), and
(7) if \(a \in A_3\) and \(a > z\), then \(\langle x, a \rangle \in \mathcal{NM}_\mu(C)\).

Therefore, any integer \(a \in A_1 \cup A_2 \cup A_3\) is part of a distinct non-\(\mu\)-match in \(C\). Since \(|A_1 \cup A_2 \cup A_3| = n - 4\), it follows that \(\mathcal{NM}_\mu(C) \geq 2 + n - 4 = n - 2\).

For part (3), suppose that \(y \neq n\), \(\langle x, y \rangle \in \mathcal{NM}_\mu(C)\), and \(\langle x, y + 1 \rangle \notin \mathcal{NM}_\mu(C)\). Then \(y\) cannot be cyclically between \(x\) and \(y + 1\) in \(C\). Also, there exists an integer \(z\) with \(x < z < y\) such that \(z\) is cyclically between \(x\) and \(y\) in \(C\), but \(z\) is not cyclically between \(x\) and \(y + 1\) in \(C\). Thus, \(C\) must be of the form

\[C = (1, \ldots, x, \ldots, y + 1, \ldots, z, \ldots, y, \ldots)\]

where \(B_1\) is the set of integers cyclically between \(x\) and \(y + 1\) in \(C\), \(B_2\) is the set of integers cyclically between \(y + 1\) and \(y\) in \(C\) that are not equal to \(z\), and \(B_3\) is the set of integers cyclically between \(y\) and \(x\) in \(C\). Note that \(\langle x, y \rangle\) and \(\langle z, y + 1 \rangle\) are in \(\mathcal{NM}_\mu(C)\). Moreover,

(1) if \(b \in B_1\) and \(b < x\), then \(\langle b, y \rangle \in \mathcal{NM}_\mu(C)\),
(2) there is no \(b \in B_1\) with \(x < b < y + 1\) since \(\langle x, y + 1 \rangle \notin \mathcal{NM}_\mu(C)\),
(3) if \(b \in B_1\) and \(b > y + 1\), then \(\langle z, b \rangle \in \mathcal{NM}_\mu(C)\),
(4) if \(b \in B_2\) and \(b < y\), then \(\langle y, b + 1 \rangle \in \mathcal{NM}_\mu(C)\),
(5) if \(b \in B_2\) and \(b > y + 1\), then \(\langle x, b \rangle \in \mathcal{NM}_\mu(C)\),
(6) if \( b \in B_3 \) and \( b < z \), then \( \langle b, y \rangle \in \mathcal{NM}_\mu(C) \), and
(7) if \( b \in B_3 \) and \( b > z \) then \( \langle x, b \rangle \in \mathcal{NM}_\mu(C) \).

Therefore, any integer \( b \in B_1 \cup B_2 \cup B_3 \) is part of a distinct non-\( \mu \)-match in \( C \). Since \( |B_1 \cup B_2 \cup B_3| = n - 4 \), it follows that \( \text{NM}_\mu(C) \geq 2 + n - 4 = n - 2. \) □

**Corollary 11.** If \( \text{NM}_\mu(C) < n - 2 \), then \( \text{NM}_{\text{plot}}(C) \) is a Ferrers diagram.

**Proof.** This corollary follows directly from Lemma 10. That is, suppose that \( C \in \mathcal{C}_n \) and \( \text{NM}_\mu(C) < n - 2 \). First, if \( \text{NM}_\mu(C) = 0 \), then \( C = (1, n, n - 1, \ldots, 2) \) in which case \( FD_n(C) \) has no shaded squares which correspond to the empty partition. Thus, assume that \( 1 \leq \text{NM}_\mu(C) < n - 2 \). Then, it follows from part (1) of Lemma 10 that \( \langle 1, n \rangle \in \mathcal{NM}_\mu(C) \). Next it follows from part (2) of Lemma 10 that if \( x \neq 1 \) and \( \langle x, y \rangle \in \mathcal{NM}_\mu(C) \), then \( \langle x - 1, y \rangle \in \mathcal{NM}_\mu(C) \). Finally, it follows from part (3) of Lemma 10 that if \( y \neq n \) and \( \langle x, y \rangle \in \mathcal{NM}_\mu(C) \), then \( \langle x, y + 1 \rangle \in \mathcal{NM}_\mu(C) \). Hence, the shaded cells \( \text{NM}_{\text{plot}}(C) \) must be a Ferrers diagram of a partition \( \lambda \) of \( \text{NM}_\mu(C) \). □

If \( n \geq 3 \), we let \( T_n \) be the plot of the Ferrers diagram in the \( n \times n \) grid corresponding to partition \( \lambda = (n - 2, n - 3, \ldots, 1) \). Thus, for example, \( T_7 \) is pictured in Figure 18. One can see that the sets that \( T_n \) have the property that \( |\{ \lambda : \lambda \text{ is a partition and } FD_n(\lambda) \subseteq T_n \}| = C_n \) where \( C_n \) is the \( n \)th Catalan number since the lower boundaries of the plots of \( FD(\lambda) \) for such \( \lambda \) correspond to Dyck paths.

![Figure 18. The Ferrers diagram \( T_7 \) on the 7 x 7 grid.](image)

Our next theorem will show that for any \( n \geq 3 \) and any partition \( \lambda \subseteq T_n \), we can construct an \( n \)-cycle \( C \in \mathcal{C}_n \) such that \( \text{NM}_{\text{plot}}(C) = FD_n(\lambda) \).

**Theorem 12.** Suppose that \( n \geq 3 \) and \( \lambda \) is a partition such that \( \lambda \subseteq T_n \). Then, there is an \( n \)-cycle \( C \in \mathcal{C}_n \) such that \( \text{NM}_{\text{plot}}(C) = FD_n(\lambda) \).

**Proof.** We proceed by induction on \( n \). For \( n = 3 \), it is easy to see that if \( C^{(1)} = (1, 3, 2) \), the \( \text{NM}_{\text{plot}}(C^{(1)}) \) is empty and if \( C^{(2)} = (1, 2, 3) \), then \( \text{NM}_{\text{plot}}(C^{(2)}) = T_3 \). Note that \( C^{(1)} \) and \( C^{(2)} \) have the property that if the largest part of the corresponding partition is of size \( i \), then \( 3 \) immediately follows \( i + 1 \) in the cycle. Thus, our theorem holds for \( n = 3 \).
Now, suppose that \( n > 3 \) and \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition which is contained in \( T_n \). It follows that \( \lambda^- = (\lambda_2, \ldots, \lambda_k) \) is contained in \( T_{n-1} \). Assume by induction that there is an \((n - 1)\)-cycle \( C' \) such that \( N M plot(C') = F D_{n-1}(\lambda^-) \) and \( n - 1 \) immediately follows \( \lambda_2 + 1 \) in \( C' \). Thus, in \( C' \), the pairs \((\lambda_2 + 1, n - 1), (\lambda_2 + 2, n - 1), \ldots, (n - 3, n - 1)\) must match \( \mu \) in \( C' \). This means that if \( \lambda_2 + 1 \leq n - 3 \), then \( n - 3 \) must lie between \( n - 2 \) and \( n - 1 \) in \( C' \). Next if \( \lambda_2 + 1 \leq n - 4 \), then \( n - 4 \) must lie between \( n - 3 \) and \( n - 1 \) in \( C' \). In general, if \( \lambda_2 + 1 \leq n - k \), then \( n - k \) must lie between \( n - k + 1 \) and \( n - 1 \) in \( C' \). Thus, \( C' \) must be of the following form

\[
C' = (\ldots n - 2 \ldots n - 3 \ldots n - 4 \ldots (\lambda_2 + 2) \ldots (\lambda_2 + 1), (n - 1) \ldots),
\]

where \( A_i \) is the set of elements cyclically between \( n - i \) and \( n - i - 1 \) in \( C' \) for \( i = 1, \ldots, n - \lambda_2 - 2 \). Now let \( C \) be the cycle that results from \( C' \) by inserting \( n \) immediately after \( \lambda_1 + 1 \) in \( C' \). Inserting \( n \) into \( C' \) does not effect on whether pairs \((i, j)\) with \( 1 \leq i < j \leq n - 1 \) are \( \mu \)-matches in \( C \). That is, for such pairs \((i, j) \in N M_{\mu}(C)\) if and only if \((i, j) \in N M_{\mu}(C')\).

Thus, the diagram of \( N M plot(C') \) and \( N M plot(C) \) are the same up to row \( n - 1 \). Now, in row \( n \), we know that the cells \((n, 1), \ldots, (n, \lambda_1)\) are shaded since the fact that \( n \) immediately follows \( \lambda_1 + 1 \) in \( C \) means that \((i, n) \in N M_{\mu}(C)\) for \( i = 1, \ldots, \lambda_1 \). However, it is easy to see from the form of \( C' \) above that \((n - 2, n), (n - 3, n), \ldots, (\lambda_1 + 1, n)\) are \( \mu \)-matches in \( C \) since \( n - 2, n - 3, \ldots, \lambda_1 + 1 \) appear in decreasing order as we traverse clockwise around the cycle \( C \). Thus, \( N M plot(C) = F D_n(\lambda) \).

Note the proof of Theorem 12 gives a simple algorithm to construct an \( n \)-cycle \( C_\lambda \) such that \( N M plot(C_\lambda) = F D_n(\lambda) \) for any \( \lambda \subseteq T_n \). For example, suppose that \( n = 12 \) and \( \lambda = (3, 3, 2, 1) \) so that the Ferrers diagram in the \( 12 \times 12 \) grid is pictured in Figure 19.

![Figure 19. sh12(3, 3, 2, 1).](image-url)

Since there are no non-\( \mu \)-matches in the first eight rows, we must start with the cycle \( C_8 = (1, 8, 7, 6, 5, 4, 3, 2) \). Then, our proof of Theorem 12 tells us that we should build up the cycle structure by first creating a cycle \( C_9 \in C_8 \) by inserting 9 immediately after 2 in \( C_8 \), since the number of non-\( \mu \)-matches in row 9 is 1. Then we create a cycle \( C_{10} \in C_{10} \)
by inserting 10 immediately after 3 in $C_9$ since the number of non-$\mu$-matches in row 10 is 2. Then, we create a cycle $C_{11} \in C_{11}$ by inserting 11 immediately after 4 in $C_{10}$ since the number of non-$\mu$-matches in row 11 is 3. Finally, we create a cycle $C_{12} = C_\lambda \in C_{12}$ by inserting 12 immediately after 4 in $C_{11}$ since the number of non-$\mu$-matches in row 12 is 3. Thus,

$$C_9 = (1, 8, 7, 6, 5, 4, 3, 2, 9),$$
$$C_{10} = (1, 8, 7, 6, 5, 4, 3, 10, 2, 9),$$
$$C_{11} = (1, 8, 7, 6, 5, 4, 11, 3, 10, 2, 9),$$
$$C_{12} = (1, 8, 7, 6, 5, 4, 12, 11, 3, 10, 2, 9).$$

Then we have that

$$\mathcal{N}\mathcal{M}_\mu((1, 8, 7, 6, 5, 4, 12, 11, 3, 10, 2, 9)) = \langle 1, 12 \rangle, \langle 2, 12 \rangle, \langle 3, 12 \rangle,$$
$$\langle 1, 11 \rangle, \langle 2, 11 \rangle, \langle 3, 11 \rangle,$$
$$\langle 1, 10 \rangle, \langle 2, 10 \rangle,$$
$$\langle 1, 9 \rangle.$$ 

Now we can prove Theorem 8.

**Proof.** Suppose that $k \leq n - 2$. Let

$$FD_n(k) = \{ C \in \mathcal{C}_n : NM\text{plot}(C) = FD_n(\lambda) \text{ for some } \lambda \vdash k \}$$
$$NM\mathcal{P}_n(k) = \{ C \in \mathcal{C}_n : NM\mathcal{P}_n(C) = k \}.$$

Theorem 11 shows that $NM\mathcal{P}_n(k) \subseteq FD_n(k)$ and Theorem 12 shows that $FD_n(k) \subseteq NM\mathcal{P}_n(k)$. Thus, $NM\mathcal{P}_n(\mu(q)) = |NM\mathcal{P}_n(k)| = |FD_n(k)|$. Hence, $NM\mathcal{P}_n(\mu(q)) = a(k)$ equals the number of partitions of $k$. \hfill \Box

Now, we will show that for $k < n - 2$, $NTI_{n,\mu}(q) \mid q^{(n-1)}_{k} - k = NTI_{n,\mu}(q) \mid q^{(n-1)}_{k} = a(k)$, where $a(k)$ is the number of partitions of $k$. First we shall show that if $C \in \mathcal{C}_n$ has fewer than $n - 2$ non-matches, then $C$ must be incontractible.

**Lemma 13.** If $C \in \mathcal{C}_n$ has fewer than $n - 2$ non-matches, then $C$ must be incontractible.

**Proof.** If $i + 1$ immediately follows $i$ in an $n$-cycle $C \in \mathcal{C}_n$, then, clearly, $\langle j, i + 1 \rangle$ are non-$\mu$-matches for $1 \leq j < i$ and $\langle i, k \rangle$ is a non-$\mu$-match for $i + 2 \leq k \leq n$. This gives $n - 2$ non-$\mu$-matches. \hfill \Box

**Corollary 14.** For $k < n - 2$, $NTI_{n,\mu}(q) \mid q^{(n-1)}_{k} - k = NTI_{n,\mu}(q) \mid q^{(n-1)}_{k} = a(k)$ where $a(k)$ is the number of partitions of $k$.

**Proof.** By definition, $NTI_{n,\mu}(q) \mid q^{(n-1)}_{k} = NM\mathcal{P}_n(\mu(q)) \mid q^{k}$. Thus, by Theorem 8, for $k < n - 2$, $NTI_{n,\mu}(q) \mid q^{(n-1)}_{k} = a(k)$. 


By Lemma 13, we have that if a cycle $C$ has $k$ non-$\mu$-matches where $k < n - 2$, then $C$ is incontractible. It follows that if a cycle has $\binom{n-1}{2} - k$ non-trivial $\mu$-matches, then it is incontractible. Thus, for $k < n - 2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$. And so $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$ for $k < n - 2$. 

5. Conclusions and direction for further research

The main focus of this paper was to study the polynomials $NTI_{n,\mu}(q) = \sum_{C \in \mathcal{C}_n} q^{NTI(C)}$ and $NTI_{n,\mu}(q) = \sum_{C \in \mathcal{IC}_n} q^{NTI(C)}$. We showed that $NTI_{n,\mu}(1)$ is the number of derangements of $S_{n-1}$. Thus, the polynomial $NTI_{n,\mu}(q)$ is a $q$-analogue of the derangement number $D_{n-1}$. There are several $q$-analogues of the derangement numbers that have been studied in the literature, see the papers by Garsia and Remmel [3] and Wachs [9]. Our $q$-analogue of $D_{n-1}$ is different from either the Garsia-Remmel $q$-analogue or the Wachs $q$-analogue of the derangement numbers. Moreover, we proved that

$$NTI_{n,\mu}(q)|_{q^k} = \sum_{s=1}^{2k+1} \sum_{A \in \mathcal{IC}_s, \, NTI_{\mu}(A)=k} \binom{n-1}{s-1}$$

so that the coefficients of the polynomial $NTI_{n,\mu}(q)$ can be expressed in terms of the coefficients of the polynomials $NTI_{j,\mu}(q)$. We also showed that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ equals the number of partition of $k$ for $k < n - 2$.

The main open question is to find some sort of recursion or generating function that would allow us to compute $NTI_{n,\mu}(q)$. Several of the sequences $(NTI_{n,\mu}(q)|_{q^k})_{n \geq 2}$ appear in the OEIS [8]. For example, we showed that $NTI_{n,\mu}(q)|_{q^2} = \binom{n-1}{3} + 2\binom{n-1}{4}$ so that the sequence $(NTI_{n,\mu}(q)|_{q^2})_{n \geq 4}$ starts out 1, 6, 20, 50, 105, 196, 336, 540, 825, .... The $n$th term of this sequence has several combinatorial interpretations including being the number of $\sigma \in S_n$ which are 132-avoiding and have exactly two descents, the number of Dyck paths on length $2n+2$ with $n-1$ peaks, and the number of squares with corners on the $n \times n$ grid. It would be interesting to find bijections from such objects to our $(n+4)$-cycles $C \in \mathcal{C}_n$ such that $NTI_{\mu}(C) = 2$. We proved that $NTI_{n,\mu}(q)|_{q^3} = \binom{n-1}{3} + 3\binom{n-1}{4} + 6\binom{n-1}{5} + 5\binom{n-1}{6}$ so that the sequence $(NTI_{n,\mu}(q)|_{q^3})_{n \geq 4}$ starts out

$$1, 7, 31, 102, 273, 630, 1302, 2472, 4389 \ldots$$

This sequence does not appear in the OEIS. Similarly, the sequence $(NTI_{n,\mu}(q)|_{q^4})_{n \geq 5}$ starts out 2, 23, 135, 561, 1870, 5328, 13476, ... and it does not appear in the OEIS.

Finally, it would be interesting to characterize the charge graphs of those cycles $C \in \mathcal{IC}_{2n}$ for which $NM_{\mu}(C) = n$. One can see from our tables that sequence $(NTI_{2n,\mu}(q)|_{q^n})_{n \geq 2}$ starts out 1, 6, 29, 130, .... Moreover, we have computed the $NTI_{12,\mu}(q)|_{q^6} = 562$. This suggests that this sequence is sequence A008549 in the OEIS. If so, this would mean that $NTI_{2n,\mu}(q)|_{q^n} = \sum_{i=0}^{n-2} \binom{2n-1}{i}$ for $n \geq 2$. 


References


Appendix: Tables of $NT_{n,\mu}(q)$, $NTI_{n,\mu}(q)$, and $NM_{n,\mu}(q)$.

Table 1. Polynomials $NT_{n,\mu}(q)$.

<table>
<thead>
<tr>
<th>$NT_{1,\mu}(q)$</th>
<th>1</th>
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</thead>
<tbody>
<tr>
<td>$NT_{2,\mu}(q)$</td>
<td>1</td>
</tr>
<tr>
<td>$NT_{3,\mu}(q)$</td>
<td>$1 + q$</td>
</tr>
<tr>
<td>$NT_{4,\mu}(q)$</td>
<td>$1 + 3q + q^2 + q^3$</td>
</tr>
<tr>
<td>$NT_{5,\mu}(q)$</td>
<td>$1 + 6q + 6q^2 + 7q^3 + 2q^4 + q^5 + q^6$</td>
</tr>
<tr>
<td>$NT_{6,\mu}(q)$</td>
<td>$1 + 10q + 20q^2 + 31q^3 + 23q^4 + 15q^5 + 13q^6 + 3q^7 + 2q^8 + q^9 + q^{10}$</td>
</tr>
<tr>
<td>$NT_{7,\mu}(q)$</td>
<td>$1 + 15q + 50q^2 + 106q^3 + 135q^4 + 126q^5 + 119q^6 + 66q^7 + 46q^8 + 25q^9 + 19q^{10} + 5q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15}$</td>
</tr>
<tr>
<td>$NT_{8,\mu}(q)$</td>
<td>$1 + 21q + 105q^2 + 301q^3 + 561q^4 + 736q^5 + 850q^6 + 726q^7 + 603q^8 + 418q^9 + 299q^{10} + 174q^{11} + 101q^{12} + 65q^{13} + 33q^{14} + 27q^{15} + 7q^{16} + 5q^{17} + 3q^{18} + 2q^{19} + q^{20} + q^{21}$</td>
</tr>
<tr>
<td>$NT_{9,\mu}(q)$</td>
<td>$1 + 28q + 196q^2 + 742q^3 + 1870q^4 + 3311q^5 + 4820q^6 + 5541q^7 + 5675q^8 + 5007q^9 + 4055q^{10} + 3093q^{11} + 2116q^{12} + 1461q^{13} + 888q^{14} + 646q^{15} + 338q^{16} + 217q^{17} + 126q^{18} + 80q^{19} + 44q^{20} + 35q^{21} + 11q^{22} + 7q^{23} + 5q^{24} + 3q^{25} + 2q^{26} + q^{27} + q^{28}$</td>
</tr>
<tr>
<td>$NT_{10,\mu}(q)$</td>
<td>$1 + 36q + 336q^2 + 1638q^3 + 5328q^4 + 12253q^5 + 22392q^6 + 32864q^7 + 41488q^8 + 5433q^9 + 44119q^{10} + 40008q^{11} + 32781q^{12} + 25689q^{13} + 18551q^{14} + 13710q^{15} + 9137q^{16} + 6179q^{17} + 3971q^{18} + 2568q^{19} + 1640q^{20} + 1098q^{21} + 640q^{22} + 374q^{23} + 251q^{24} + 148q^{25} + 100q^{26} + 56q^{27} + 46q^{28} + 15q^{29} + 11q^{30} + 7q^{31} + 5q^{32} + 3q^{33} + 2q^{34} + q^{35} + q^{36}$</td>
</tr>
</tbody>
</table>
Table 2. Polynomials $NTI_{n,\mu}(q)$.

<table>
<thead>
<tr>
<th>$NTI_{1,\mu}(q)$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NTI_{2,\mu}(q)$</td>
<td>0</td>
</tr>
<tr>
<td>$NTI_{3,\mu}(q)$</td>
<td>$q$</td>
</tr>
<tr>
<td>$NTI_{4,\mu}(q)$</td>
<td>$q^2 + q^4$</td>
</tr>
<tr>
<td>$NTI_{5,\mu}(q)$</td>
<td>$2q^2 + 3q^4 + 2q^6$</td>
</tr>
<tr>
<td>$NTI_{6,\mu}(q)$</td>
<td>$6q^4 + 13q^4 + 10q^6 + 8q^6 + 3q^8 + 2q^9 + q^{10}$</td>
</tr>
<tr>
<td>$NTI_{7,\mu}(q)$</td>
<td>$5q^4 + 27q^6 + 51q^8 + 56q^{10} + 48q^{12} + 34q^{14} + 19q^{16} + 13q^{18}$ $+ 5q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15}$</td>
</tr>
<tr>
<td>$NTI_{8,\mu}(q)$</td>
<td>$29q^4 + 134q^6 + 255q^8 + 327q^{10} + 323q^{12} + 264q^{14} + 187q^{16} + 139q^{18} + 80q^{12} + 51q^{13} + 26q^{14} + 20q^{15} + 7q^{16} + 5q^{17} + 3q^{18} + 2q^{19} + q^{20} + q^{21}$</td>
</tr>
<tr>
<td>$NTI_{9,\mu}(q)$</td>
<td>$14q^4 + 181q^6 + 694q^8 + 1413q^{10} + 2027q^{12} + 2307q^{14} + 2139q^{16} + 1841q^{18} + 1392q^{12} + 997q^{14} + 652q^{16} + 458q^{18} + 282q^{20} + 177q^{22} + 102q^{24} + 64q^{26} + 36q^{20} + 27q^{22} + 11q^{24} + 7q^{26} + 5q^{28} + 3q^{30} + 2q^{32} + q^{34} + q^{36}$</td>
</tr>
<tr>
<td>$NTI_{10,\mu}(q)$</td>
<td>$130q^6 + 1128q^8 + 3965q^{10} + 8509q^{12} + 13444q^{14} + 16918q^{16} + 18015q^{18} + 17121q^{20} + 14712q^{12} + 11663q^{14} + 8784q^{16} + 6347q^{18} + 4406q^{20} + 2945q^{22} + 1920q^{24} + 1280q^{26} + 819q^{24} + 541q^{26} + 311q^{28} + 206q^{30} + 121q^{32} + 82q^{34} + 47q^{36} + 37q^{38} + 15q^{40} + 11q^{42} + 7q^{44} + 5q^{46} + 3q^{48} + 2q^{50} + q^{52} + q^{54}$</td>
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Table 3. Polynomials $NM_{n,\mu}(q)$

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<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$1 + q$</td>
</tr>
<tr>
<td>4</td>
<td>$1 + q + 3q^2 + q^3$</td>
</tr>
<tr>
<td>5</td>
<td>$1 + q + 2q^2 + 7q^3 + 6q^4 + 6q^5 + q^6$</td>
</tr>
<tr>
<td>6</td>
<td>$1 + q + 2q^2 + 3q^3 + 13q^4 + 15q^5 + 23q^6 + 31q^7 + 20q^8 + 10q^9 + q^{10}$</td>
</tr>
<tr>
<td>7</td>
<td>$1 + q + 2q^2 + 3q^3 + 5q^4 + 19q^5 + 25q^6 + 46q^7 + 66q^8 + 119q^9 + 126q^{10} + 135q^{11} + 106q^{12} + 50q^{13} + 15q^{14} + q^{15}$</td>
</tr>
<tr>
<td>8</td>
<td>$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 27q^6 + 33q^7 + 65q^8 + 101q^9 + 174q^{10} + 299q^{11} + 418q^{12} + 603q^{13} + 726q^{14} + 850q^{15} + 736q^{16} + 561q^{17} + 301q^{18} + 105q^{19} + 21q^{20} + q^{21}$</td>
</tr>
<tr>
<td>9</td>
<td>$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 35q^7 + 44q^8 + 80q^9 + 126q^{10} + 217q^{11} + 338q^{12} + 646q^{13} + 888q^{14} + 1461q^{15} + 2116q^{16} + 3093q^{17} + 4055q^{18} + 5007q^{19} + 5675q^{20} + 5541q^{21} + 4820q^{22} + 3311q^{23} + 1870q^{24} + 742q^{25} + 196q^{26} + 28q^{27} + q^{28}$</td>
</tr>
<tr>
<td>10</td>
<td>$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 46q^8 + 56q^9 + 100q^{10} + 148q^{11} + 251q^{12} + 374q^{13} + 640q^{14} + 1098q^{15} + 1640q^{16} + 2568q^{17} + 3971q^{18} + 6179q^{19} + 9137q^{20} + 13710q^{21} + 18551q^{22} + 25689q^{23} + 32781q^{24} + 40008q^{25} + 44119q^{26} + 45433q^{27} + 41488q^{28} + 32864q^{29} + 22392q^{30} + 12253q^{31} + 5328q^{32} + 1638q^{33} + 336q^{34} + 36q^{35} + q^{36}$</td>
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