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Articulating ‘understanding’: Deploying mathematical cognition

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ABSTRACT

Scotland's Curriculum for Excellence (CfE), constitutes the demands that teachers are supposed to meet. Its intentions for the mathematics curriculum are similar to those in many countries: that learners be sufficiently mathematically literate to use mathematics in the personal, professional and societal dimensions of their lives. But like many attempts to reform mathematical curricula elsewhere, CfE does not address the fundamental issue of mathematical understanding. Of course, learner understanding transcends the mathematical curriculum per se, but the significance of mathematical understanding for competent functioning exemplifies the imperative of teaching for understanding or 'deep learning'. The article begins with some conceptual 'ground clearing' to establish mathematical understanding as a neglected issue. It then considers what mathematical understanding is, why it is important, the increasingly important role of metacognition, and the very necessary role of teachers. At each of these points, implications for pedagogical practice are raised.

MATHEMATICAL UNDERSTANDING AS A NEGLECTED ISSUE

This article builds on Henderson's (2012) excellent analysis of why improvements in mathematical achievement in Scotland are unlikely to be instantiated solely through the efforts of its curricular reform Curriculum for Excellence (CfE). Henderson argues that current guidance for mathematics is more opaque than that in previous curricular incarnations, thereby creating a pressing need for mathematics subject knowledge to be explicit in teacher education. This paper has no argument with the need for teachers to have mathematical knowledge but seeks to emphasise that such knowledge must reflect understanding. It will argue that meaningful mathematics teaching is rooted in routine engagement in mathematical metacognition. Opportunities to share and question one's own and others' thinking is central to developments in teaching and learning, because if teacher-educators, student-teachers and learners do not attend to their own and others' understanding, the outcomes of the teaching endeavour remain "tacit, elusive and difficult to define" (Loughran, 2010).

The learning goals for the mathematics curriculum in Scotland (see Table 1) make plain that understanding is a critical component in which the ability to reason with, and make sense of, what is learned is central to learners' experiences. However, teachers continue to struggle with ways to teach for mathematical understanding (Hiebert, 2013) and are bewildered by what they see as inadequate curricular guidance, arguing that if it is not evident what mathematical learning should take place, it cannot be surprising that it may not take place. In spite of the evidence that the documentation of curricular implementation *is* important (Thompson and Senk, 2008) the Scottish Government's response to requests for greater delineation emphasises that teachers "need to explore, experiment, and exchange ideas about how to make *Curriculum for Excellence* work for them. No amount of well-meaning advice by experts can replace this" (Wiliam, 2011). But curriculum policy documents of themselves do not explicate the meaning of the desired reform (Boesen et al., 2014). Nor is this gap between policy documentation and classroom practice filled by the provision of curricular activities and materials. While there is no shortage of commercially produced work/books, assessment checks or web resources all purportedly aligned to CfE guidance on mathematics; what all of these resources focus on is performance evidence of pupil procedural skill. Nowhere in Education Scotland’s CfE website at http://www.educationscotland.gov.uk/thecurriculum/index.asp, is there any substantive discussion of the meaning of understanding; of how teachers' accommodate to learners' cognitive affordances and constraints; or of the role of learners' cognitive mediations in making any sense of the mathematics curriculum.

This lack of explication of mathematical understanding is not peculiar to the Scottish context: the emphasis on procedural mastery of algorithmic skill is endemic worldwide (Thompson, 2013, Goldenberg, 2014, Tzur, 2010, Gill and Boote, 2012, Venkat et al., 2009). It does, however, pose a barrier to progress in mathematical pedagogy because subject-matter can neither be imposed upon nor inserted into learners, without their cognitive mediations (Dewey, 1902). So how particular learners construe (or misconstrue) the content to which they are exposed is a dynamic and central part of the relationship between teaching and learning; and is essentially psychological knowledge of profound importance to the teacher (Ausubel, 2000). Put simply, teachers cannot fully address their pedagogic task without knowing how and what their learners are actually learning.

TABLE 1: NUMERACY AND MATHEMATICS EXPERIENCES AND OUTCOMES

Develop a secure understanding of the concepts, principles and processes of mathematics and apply these in different contexts, including the world of work;
Engage with more abstract mathematical concepts and develop important new kinds of thinking;
Understand the application of mathematics, its impact on our society past and present, and its potential for the future;
Develop essential numeracy skills which will allow me to participate fully in society
Establish firm foundations for further specialist learning;
Understand that successful independent living requires financial awareness, effective money management, using schedules and other related skills;
Interpret numerical information appropriately and use it to draw conclusions, assess risk, and make reasoned evaluations and informed decisions;
Apply skills and understanding creatively and logically to solve problems, within a variety of contexts;
Appreciate how the imaginative and effective use of technologies can enhance the development of skills and concepts.

CONSTITUENT ASPECTS OF MATHEMATICAL UNDERSTANDING

There is considerable agreement that understanding means being able to explain why a particular skill or procedure allows a particular outcome, and being able to modify/adapt/invent skills or procedures appropriately in the face of varying contextual constraints (Hatano and Inagaki, 1986, Ma, 1999, Skemp, 1976). So characterised, any particular piece of understanding is like a jigsaw piece – connecting to other jigsaw pieces and supported by a grasp of the larger network of knowledge (or entire jigsaw). But understanding is a slippery concept with different nuances of interpretation. It is possible for people to behave effectively with procedural knowledge: algorithmic skills and the decision rules for how and when to apply the skills (Hiebert and Lefevre, 1986). Procedural knowledge is important for the functioning of society and can be acquired through direct observation, verbal instruction, corrective feedback, and/or supervision. For some learners and teachers, procedural knowledge is tantamount to understanding when it results in the 'right answer' and speedier solution processes (Federici and Skaalvik, 2014, Skemp, 1989). Procedural knowledge has been called instrumental understanding (Skemp, 1976) in that it is formulaic recall for achieving results with no back-up if the procedure does not solve the problem. A more profound (Ma, 1999) or relational (Skemp, 1976) understanding of mathematics reflects a capacity to reason about why a particular procedure works without recourse to the memorisation so necessary for instrumental understanding (Schoenfeld, 1988, Richland et al., 2012). Relational or profound understanding allows an individual to modify an already acquired procedure in the face of the demand for 'high-road transfer' (Salomon and Perkins, 1989). Distinctions between relational and instrumental understanding have been reported in terms of learner behaviour (Duffin and Simpson, 2000).

A relational learner:

- Can explain 'why'
- Contemplates before acting
- Will come up with an answer eventually
- Adapts to any task, using previous knowledge
- Attempts to understand, and asks, 'Why?'
- Independently creates mathematics new to them
- Enjoys doing mathematics for its own sake

An instrumental learner:

- Cannot explain 'why'
- Can give immediate answers to particular questions
- Can sometimes make no progress and be 'stuck'
- Is inflexible in method
- Attempts to memorise
- Is dependent on examples from the teacher
- Is not likely to find mathematics enjoyable (Reason, 2003)

The implications of emphasising relational understanding could be quite considerable for teachers' practice. The learner's stage of development and pace of learning would take precedence over pre-specified curricular coverage. Allowing learners the time to explore the mathematical idea under consideration and supporting them in their attempts to reflect on their own ideas might create considerable change in how the maths class was conducted. Such change might be resisted by parents and policy-makers who believed primarily in documenting algorithmic and procedural progress.

THE IMPORTANCE OF RELATIONAL/PROFOUND UNDERSTANDING

Learning mathematics with understanding requires learners to make connections between

- New information and current knowledge
- Different mathematical ideas and representations (such as drawings and physical objects).
- School mathematics and the mathematical aspects of other everyday contexts (Mousley, 2004)

Making these connections is an iterative and time-demanding process and learners must make these connections for themselves as no amount of clear explanation and demonstration by the teacher leads directly to relational understanding (Hiebert et al., 1997). Rather than being the outcome of a declaratively didactic approach to topics such as place value or mutlidigit multiplication or subtraction with regrouping or whatever, learners' relational understanding of such topics emerges from experience of carefully selected mathematical problems (Hiebert et al., 1997). This approach to teaching, in which problem solving is the organisational focus of any lesson, is well documented in America (Carpenter et al., 2004) and is in nascent form here in Scotland (Moscardini, 2010, Moscardini, 2013, Moscardini, 2014). The integrity of relational learning remains intact in the problem-solving focus of Cognitively

Guided Instruction (CGI) because concepts and procedures develop in concert (Carpenter et al., 1999, Empson and Levi, 2011). As Moscardini's work demonstrates, those teachers who have themselves experienced the transformation in their own practice as a result of high-quality CGI input, are astonished by learners' enjoyment of, engagement in and meaningful exploration of mathematical concepts. Unfortunately, however, perhaps because of the pressure of accountability (Vogler and Burton, 2010) or because of reverting to ways in which they themselves were taught mathematics (Seaman and Szydlik, 2007); teachers may be resistant to privileging relational learning (Beswick, 2005), viewing it as an unrealistic expectation over which they believe themselves to have little or no control. To relegate relational understanding to laudable but 'optional' status misses the exquisite challenge of being able to adapt extant knowledge to address new problems. Relational understanding is fundamental to engaging in knowledge construction (Bereiter and Scardamalia, 2003). Using extant knowledge to leverage further (rather than merely rehearse existing) learning is the activity of 'high-road transfer'. In other words, transfer is not predominantly a matter of 'learn it here, apply it there' (Perkins and Salomon, 2012) but an opportunity to revise, reorganise and invent knowledge (Schwartz et al., 2012, Lobato, 2012).

Relational understanding is not a dichotomous state but part of an unending continuum. It can always be enhanced but never fully mastered. Further, relational understanding is implicated in the needs of learners (who need to develop more sophisticated conceptions of mathematics) and in the needs of teachers (who are charged with enabling learners' understanding). The embodiment of relational understanding in the mathematics classroom is problem-solving. Mathematical problem-solving involves multiple cognitive processes. Problem solving is <u>directed</u> <u>cognitive</u> <u>processing</u>: directed because it seeks to achieve a goal; cognitive because it occurs in the problem solver's mind (and must be inferred indirectly through the problem solver's behaviour); and processing because it involves the mental manipulation of knowledge representations in the problem solver's cognitive system (Mayer, 2014). While solving a mathematical problem means employing mathematical concepts, facts, procedures, and reasoning to generate a solution, the more fundamental part of problem solving is to construct a representation of the problem (Mayer and Hegarty, 1996). Representing the problem means translating a real-world scenario into a mental image of what is known/given, what needs to be found out, and the allowable operations for solution, without overlooking or omitting all relevant detail stated in the problem. Thus the learner has to translate linguistic and numerical information into verbal, graphic, symbolic, and quantitative representations which capture the relationship(s) in the problem information. This is the process of formulating a situation mathematically (OECD, 2013) or mathematising a real-life scenario (Van Oers, 2013, Boesen et al., 2014). This translation is both necessary (since without a productive problem representation, a solution path cannot be planned) and challenging because of the cognitive demands it makes (OECD, 2013).

Clearly if learners are to construct problem representations for themselves, teachers' support of this learning must come from triggering learners to use their own cognitive processing and not from giving 'straightforward' procedural guidance. Solving mathematical problems is inherently cognitive activity. Complex as it is for learners to negotiate problem representation, and subsequently to deploy relevant mathematical skills, learners must also monitor and evaluate their progress in executing problem-solving processes and use such knowledge to drive forward their own learning. In summary, enabling learners to achieve relational understanding requires that they take responsibility for their own learning: through systematic planning and execution; as well as monitoring, evaluating and reflecting on their learning; all of which demand their metacognitive skillfulness.

THE IMPORTANT ROLE OF METACOGNITION

The literature on the architecture of cognition emphasises the efforts that individuals make to monitor and control their thoughts and actions. For learners to fully internalise their thinking (their cognitions), they must exercise metacognition (Gourgey, 1998). Metacognition is essential for learners to regulate their own learning, since purely cognitive activity is not the sole determinant of mathematical achievement (Furinghetti and Morselli, 2009). Metacognition is knowledge of self-instructions for regulating task performance and it both feeds on, and helps advance, extant cognition (Veenman et al., 2006). Metacognition is being invoked when learners use their knowledge to determine what information is given and what is needed; when learners plan a solution path rather than depend on trial-and-error; when they avoid/repair errors while working on the solution; and when they compare the plausibility of their answer(s) with the problem statement (Van der Stel et al., 2010). For example, Iiskala et al. (2011) found that 10-year-olds engaged in collaborative mathematical word-problem solving shared their online thinking to develop solution processes. Notably, they intensified their verbal and non-verbal exchanges as the problems became more complex. By making their thinking explicit through a continuous oral commentary, the learners put their thinking 'on the table' for the comment and reflection of peers and teachers, as well as themselves (Desoete, 2007). Such commenting allows learners to express how they feel about the task and their own progress within it (Efklides et al., 1999, Usher, 2009). The teacher can then use learners' comments and views to facilitate powerful mathematical discussion so that learners not only gain insight into their own understanding but are supported to bring their ideas into alignment with canonical understandings of mathematics (Stein et al., 2008).

So powerful is metacognition in learning mathematics that it can outweigh ability in predicting achievement when learners are operating at the boundaries of their knowledge (Prins et al., 2006, Schneider and Artelt, 2010). However, the circular relationship of metacognition and cognition is often not explicit in the mathematics lesson (Montague et al., 2014), although it is now clear that a metacognitively based pedagogical protocol for problem solving can generate strong learner understanding (Lee et al., 2014). Metacognition does not emerge spontaneously (Veenman et al., 2006, Leutwyler, 2009), but it can be invoked when pedagogical practices routinely surface learners' descriptions, explanations and justifications (Van der Stel et al., 2010) . One protocol for embedding metacognition suggests training in the use of four broad questions:

(a) Comprehension: What is the problem all about? (Learners have to describe the problem in their own words by focusing on the mathematical content.)

(b) Connection: In what way is the problem at hand similar to or different from problems you have solved in the past? (Learners have to explain their reasoning.)

(c) Strategies: Which strategies are appropriate for solving the problem, and why? (Learners have to describe, and justify, their choice of strategy)

(d) Reflection: Does the solution make sense? Am I stuck? Can I solve the problem differently? (Learners have to reflect on themselves as solvers, the solution processes, and the solution itself). (Mevarech et al., 2010).

The effective use of these questions presupposes that learners will possess necessary underpinning knowledge – otherwise known as 'offline metacognition' (Desoete et al., 2003) - and where this is shown to be in deficit, pedagogical support such as is advocated by Du Toit and Kotze (2009) is required. Nevertheless, such 'offline metacognition' must not be used as blunderbuss input without regard for content. It must be calibrated to extant curricular achievement (Huff and Nietfeld, 2009, Rosenzweig et al., 2011). The effective use of these questions also presupposes that teachers themselves are metacognitively sophisticated. Historically, the construct of metacognition has not been an explicit feature of teacher-education (Veenman et al., 2006); although there is now evidence that it is amenable to well thought-out professional development (Zohar and Peled, 2008, Zohar and David, 2009). As both a covert internal activity to promote cognition, and a socially situated activity to drive discussion illuminating individual and group thinking (Garrison and Akyol, 2013), metacognition is a valuable dimension of pedagogical practice (Magno, 2010, Wilson and Bai, 2010)

THE TEACHER'S METACOGNITIVE FOCUS

In promoting mathematical understanding (Ellis et al., 2014) teachers seek to move learners move from appreciating the properties of concrete objects to theoretical relations between objects and structures (Steinbring, 2007). In other words, we are concerned with learners' mental processing of mathematical ideas. To do this, we must focus on learners' representations - the tools with which we think, mathematically (Davis and Maher, 1997). Representations refer to both internal and external manifestations of ideas and are essential for the development of mathematical thought (Duval, 2006). Internal representations are individuals' hypothesised semantically-based images, propositions or precepts which maintain a relationship with an object or event in its absence (Olson and Campbell, 1993, Goldin, 1998); and are abstracted from experience (Von Glasersfeld, 1991). External representations are observable physically embodiments, taking the form of graphs, diagrams, tables, grids, formulae, symbols, words, gestures, software code, videos, manipulatives, concrete models, pictures, and sounds (Goldin and Kaput, 1996). External and internal representations (which have an interdependent relationship) can be known as the signifier and the signified, or the symbol and the symbolised (Mason, 1987). We cannot *think* about anything (such as dogs, holidays or happiness) without having an internal representation of the 'object', but representations are internally elaborated through the availability of external configurations (Von Glasersfeld, 1987, Greeno, 1991). In other words, we do not think about 'nothing' but must have some 'object' about which to think. Thus, mathematical thinking is the ongoing interactional (re)interpretation(s) of internal and external representations.

The distinction between internal and external representation is important for learning mathematics but extremely complex for the teacher to manage. External representations are, in principle, accessible to observation by anyone with suitable knowledge; but internal representations are not directly accessible to anyone other than the individuals (who *may* be enabled to share their own metacognitive awareness). And yet it is the internal representation -the signified or the symbolised - which teacher behaviour is attempting to modify (Marton, 1974, Richland et al., 2012). The relationship between the external sign and its internal interpretation by the learner is a function of the individual's existing cognitive structure; and it is this that teachers need to understand if they want learners to become more mathematically sophisticated (Duval, 2006), as the external representations are merely tools, albeit powerful ones (Schliemann, 2002). Internal representations, on the other hand, transcend concrete situations and are advantageous in solving complex problems (Ding and Li, 2014). Constructing internal representations takes time, and requires teachers' attention to learners' "concept images" (Tall and Vinner, 1981). For example an early part of the learner's concept image (or internal representation) may be that subtraction always *reduces* the answer. Such a view is compromised if negative numbers are invoked. Similarly a concept image of multiplication always *increasing* the answer is problematic when fractional factors are introduced. In attending to learners' internal representations, teachers need to surface all of the mental attributes that learners associate with a given concept, even if these are potentially conflicting (Mason and Spence, 1999). If learners can indicate their thinking, verbally and/or gesturally, teachers may infer the level of learner understanding and, assuming teachers' diagnostic competence in mathematical communication (Bräuning and Steinbring, 2011), can deploy instructional practices to encourage learners' engagement with their own and other's thinking (Webb et al., 2014). However, if teachers make inferences tacitly rather than explicitly, as they often do (Goldin and Kaput, 1996) and treat representations as pre-given subject matter without regard for individual learners' possible understandings, they are (perhaps unwittingly) assuming a transmission view of teaching to be superior.

FOCUSING ON REPRESENTATIONS

External representations have been categorised into five systems (Lesh et al., 1987):

1. Real world scripts – scenarios that serve as contexts for interpreting and solving numerical and mathematical problems (problem-solving)

2. Manipulative models - Cuisenaire Rods, Unifix Cubes, Fraction Bars, Arithmetic Blocks, Number Lines - which have little meaning in themselves but can be used to model aspects of reality
3. Static pictures - pictures or diagrams that can be internalised as images
4. Written symbols - numerical and mathematical notation and formulae
5. Spoken language – particular mathematical meaning of words in ordinary English and conventional mathematical language

No one representation describes fully a mathematical construct (Dreyfus and Eisenberg, 1996). Each has different advantages, so using multiple representations for the same mathematical situation seems a logical extension. Just as the traveller may have a better understanding of the route he is navigating if he has a map as well as a set of directions listing street names and turns to follow, so mathematical understanding is strengthened by accessing more than one representation. Translating between and among different external representations (Janvier, 1987) is highly correlated with mathematical competence (Gagatsis and Shiakalli, 2004). Translation essentially means:

- recognising an idea which is embedded in a variety of qualitatively different representational systems (as in the concept of 'threeness' being represented in the written word three, the numeral 3, the story of the Three Bears, part of the meaning of a triangle, the ordinal position of third, that number written as ⅓, the sum of '2 and 1' and so on)
- manipulating the idea flexibly within given representational systems (such as recognising and performing numerical equivalence tasks in terms of decimal, percentage, fractional or ratio as outlined for example at http://uk.ixl.com/standards/scotland/maths)
- translating the idea from one system to another accurately, as most commonly recognised in problem-solving (Lesh et al., 1987).

Flexibility and adaptability in using multiple external representations as tools is clearly desirable, but is difficult to instantiate (Acevedo Nistal et al., 2009, Heinze et al., 2009). It is also the very element which is recognised to be problematic in Scottish schools. Lest readers regard this claim to be overstated, they need only skim the results of reports on mathematical performance over the last 20 years or so to learn that:

- Children do not use mathematics enough in real contexts and for meaningful purposes.
- Their skills in using a range of strategies to solve mathematical problems are not well developed.
- Successive surveys of mathematical achievement report a dip in performance from the middle of primary school onwards; evidenced by lack of understanding of common and decimal fractions, of ratio and proportion, of percentages and of the relationships between these numerical representations: understandings which are persistently difficult to construct (Ngu et al., 2014)
- Average achievement scores have changed little.
(Her Majesty's Inspectorate for Education (HMIE), 2009, Her Majesty's Inspectorate for Education (HMIE), 2010, Scottish Government, 2010, Scottish Government, 2014).

Teaching for mathematical understanding would appear to demand more than what is on offer just now. Essentially, we are trying to enable learners to develop mathematical principles. These principles are by definition abstract and lack close relevance to learners' lives. How we enable learners to transcend the concrete-abstract divide is not yet well understood, though the studies reviewed here suggest promising directions. We must always keep at the front of our minds that we seek to enable learners to:

- understand why their procedures work
- modify known procedures
- invent new procedures
- respond flexibly to contextual variations
- cross boundary domains to find better solutions (Hatano and Oura, 2003)

THE VERY NECESSARY ROLE OF TEACHERS

If we actually believe understanding to be a genuine goal of mathematics education, we must be prepared to grapple with the role of thinking in the construction of such understanding. Understanding is the *product* of thinking (Thompson, 2013) and thinking is the individual's mental activity to make connections and links amongst facts and conditions which, of themselves, are "isolated, fragmentary, and discrepant" (Dewey, 1910). Understanding is not 'out there' but in the minds of the persons who construct and interpret it. It is therefore a necessary part of teaching to enable learners to make explicit their perspectives, reasoning and conceptualisations (Schliemann, 2002). To achieve this teachers need to focus not primarily on the 'curriculum to be covered' but on the learner's mathematical thinking (Steinberg et al., 2004). Providing cognitively complex tasks which require learners to think, reason and solve problems challenges learners to use their cognitive abilities and converse mathematically (Simon and Tzur, 2004, Simon et al., 2010, Clarke et al., 2009, Sullivan et al., 2013, Gravemeijer, 2014). This demands the attention of classroom teachers and academic researchers working interactively. Significant political clout is also needed to trigger such a task in the first instance, although recognising that these difficulties are not ameliorated by quick procedural fixes would be a useful start (Ainsworth, 2006, Lesser and Tchoshanov, 2005, Elia et al., 2007). Yes, of course it is possible to induct teachers in the delivery of all manner of 'programmes' claimed to enhance practice, but these programmes have the potential to fail unless there is significant change in curricular policy-making.

Curricular policy-making must position teachers to be key stakeholders in mathematics education research so that they are co-producers of professional and/or scientific knowledge. Involving teachers researching their own and/or their colleagues’ practice allows the dynamic duality of extending knowledge of the field of mathematical pedagogy as well as providing continuing professional development for the teachers involved (Kieran et al., 2013). The traditional divide between teachers and other stakeholders in education at best identifies teachers as being knowledge recipients rather than knowledge builders, and at worst stereotypes them as recalcitrant technocrats who need to be kept in line with ever-increasing accountability mechanisms. Neither portrayal promotes progress (Bevins and Price, 2014, White et al., 2013). Amongst the many stakeholders involved in the education enterprise, it is teachers who are pedagogically expert and their voice must be integral to every level of curricular decision-making; be this at executive, local authority, cluster group or individual school level. Creating mechanisms to permit full teacher-engagement in curricular policy decision making is not a trivial task (Daly et al., 2014) but a necessary one, to attenuate the tendency to use research evidence to confirm espoused positions rather than as part of a learning process to diagnose problems and uncover solutions (Finnigan and Daly, 2014).

CONCLUSION

Stimulated by the current curriculum reform in Scotland, this I essay has addressed a gap in the CfE documentation: that of a lack of explication of what mathematical understanding might mean. Deep reading of the pedagogical literature indicates that:

- Understanding can be at a relational or instrumental level with each having very different consequences for teaching and learning
- Relational understanding is much more profound that instrumental understanding but is also more difficult to construct – partly because it is an infinite activity
- Learners need to be supported to make their understandings plain, through pedagogical practices to stimulate metacognition
- Teachers need to be constantly concerned with the ways in which learners represent their understanding: an exquisitely demanding but nevertheless fundamental aspect of the teacher's role.

There is now sufficient evidence available to suggest that if Scotland (or indeed any developed country) wants to educate its young so that they can communicate coherently through using the resources of mathematics, there must be a concerted effort to teach mathematics for meaning rather than for measurement.

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