

The ion channel free-electron laser with varying betatron amplitude

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2014 New J. Phys. 16 093025

(<http://iopscience.iop.org/1367-2630/16/9/093025>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 130.159.68.171

This content was downloaded on 24/09/2014 at 17:13

Please note that [terms and conditions apply](#).

The ion channel free-electron laser with varying betatron amplitude

B Ersfeld¹, R Bonifacio¹, S Chen¹, M R Islam¹, P W Smorenburg² and D A Jaroszynski¹

¹Department of Physics, Scottish Universities Physics Alliance and University of Strathclyde, Glasgow G4 0NG, UK

²Eindhoven University of Technology, Coherence and Quantum Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

E-mail: bernhard.ersfeld@strath.ac.uk and d.a.jaroszynski@strath.ac.uk

Received 25 March 2014, revised 11 June 2014

Accepted for publication 21 July 2014

Published 22 September 2014

New Journal of Physics **16** (2014) 093025

doi:[10.1088/1367-2630/16/9/093025](https://doi.org/10.1088/1367-2630/16/9/093025)

Abstract

The ion-channel laser (ICL) is an ultra-compact version of the free-electron laser (FEL), with the undulator replaced by an ion channel. Previous studies of the ICL assumed transverse momentum amplitudes which were unrealistically small for experiments. Here we show that this restriction can be removed by correctly taking into account the dependence of the resonance between oscillations and emitted field on the betatron amplitude, which must be treated as variable. The ICL model with this essential addition is described using the well-known formalism for the FEL. Analysis of the resulting scaled equations shows a realistic prospect of building a compact ICL source for fundamental wavelengths down to UV, and harmonics potentially extending to x-rays. The gain parameter ρ can attain values as high as 0.03, which permits driving an ICL with electron bunches with realistic emittance.

Keywords: ion channel laser, compact coherent radiation source, free-electron laser, betatron oscillations



Content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](https://creativecommons.org/licenses/by/3.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

1. Introduction

The free-electron laser (FEL) [1, 2] produces highly coherent, ultra-short duration light pulses with extremely high peak brilliance, and photon energies extending to above 10 keV. FELs are very useful for ultrafast time-resolved studies of the structure of matter but require high energy electron beams and long undulators, which makes them large and expensive. In spite of the high cost, several large national and international x-ray FELs [3] have been, or are being, built because of their potential for delivering new science and applications.

FELs are based on the collective interaction of high energy electrons that are periodically deflected by an undulator. The combined undulator and radiation fields give rise to a ponderomotive force that bunches the electrons on a wavelength scale and results in intense coherent emission. The self-amplified spontaneous emission FEL [2] produces coherent radiation by amplifying incoherent synchrotron radiation spontaneously emitted by the initially uncorrelated electron beam.

However, magnetostatic undulators are not the only means of providing a periodic transverse force. Whittum *et al* [4] suggested in 1990 that an ion-channel laser (ICL) could use the ‘betatron’ motion of electrons in an ion-channel to emulate an undulator, resulting in a very compact device.

An important difference between the FEL and ICL is the spatial periodicity of the transverse oscillations. In the FEL, this is fixed by the undulator, whereas in the ICL it depends on the ion density, and both the electron energy and oscillation amplitude. Due to this latter dependence, maintaining resonance with the emitted field in an ICL requires a small amplitude spread, unless the transverse momentum is very small. Only the latter case with very small amplitudes was treated in [4] and [5]. However, this is constrained to very low emittances that are very difficult to achieve in practice.

In this paper, we consider the more general and realistic case of high transverse momentum, which requires the betatron amplitude to be treated as variable. We derive a set of equations for the ICL describing, on a slow timescale, the complex amplitude of the amplified wave, and the axial momenta, betatron amplitudes, and ponderomotive phases of the oscillating electrons. We assume ultra-relativistic axial and high transverse momenta, but non-relativistic transverse velocity; we study the steady-state regime by neglecting slippage between electrons and wave, but include space-charge effects.

The form of the equations allows one to apply the well-known scaling procedure for the FEL, with an analogous fundamental coupling parameter ρ [2]. We present analytical and numerical results showing that for small ρ the evolution of field amplitude, phase bunching, and axial momentum in the ICL is virtually identical to the FEL.

We investigate how the growth of the radiation field depends on the initial spreads of axial electron momentum and betatron amplitude. Sufficiently low betatron amplitude spreads (compared to the mean oscillation amplitude) can be achieved by injecting the electrons off-axis and/or under an angle, as shown schematically, for just two electrons, in figure 1. The admissible betatron amplitude spreads lead to a small source size for the emitted radiation, which necessitates guiding to avoid diffraction. Small overlap between the radiating electrons and the guided mode makes space-charge effects relatively much more important than in the FEL. We find that at large values of ρ electron beams with realistic amplitude spreads and emittance can be used to drive the ICL and show that by removing the restriction of small amplitude, an UV ICL with high efficiency should be feasible.

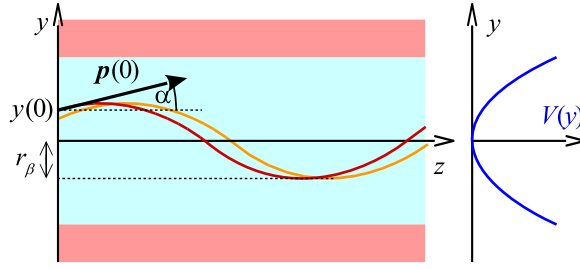


Figure 1. Schematic of electron injection into the ion channel. An offset in position ($y(0)$) and/or momentum ($p(0)$) from the channel axis (z -axis) leads to betatron oscillations (red trajectory) in the parabolic potential $V(y)$, with amplitude r_β . The orange trajectory is for an electron with slightly different initial conditions, but equal r_β . An electron bunch with suitable initial distribution in phase space can have a small betatron amplitude spread.

The next section sets out our model for the ICL. In section 3, we apply a formalism similar to that for the FEL, before presenting numerical solutions in section 4. We discuss the results in section 5, and finally draw conclusions.

2. Model

2.1. Hamiltonian

We consider test electrons in a cylindrical channel otherwise void of electrons, along the z -axis, with homogeneous stationary ion background. For motion in the y - z -plane, the electron energy is

$$H = \gamma mc^2 + V, \quad \text{with } \gamma = \left[\gamma_0^2 + (p_y + eA)^2 / (mc)^2 \right]^{1/2},$$

where

$$\gamma_0 = \left[1 + p_z^2 / (mc)^2 \right]^{1/2};$$

p_z and p_y are axial and radial components, respectively, of the canonical momentum, and A is the vector potential, taken along the y -axis. The potential energy is

$$V = m\omega_p^2 y^2 / 4,$$

where $\omega_p = [e^2 n_0 / (\epsilon_0 m)]^{1/2}$ is the plasma frequency for the background density, n_0 (with e the elementary charge, m the electron mass, and ϵ_0 the permittivity of free space).

2.2. Transverse electron motion

The test electron will perform betatron oscillations

$$y(t) = r_\beta \cos(\omega_\beta t),$$

with frequency $\omega_\beta = \omega_p/(2\gamma)^{1/2}$. The associated energy,

$$W_\beta = (\gamma - \gamma_0)mc^2 + V,$$

is assumed small compared with $\gamma_0 mc^2$; approximately,

$$W_\beta \approx \gamma_0 m v_y^2 / 2 + V, \quad (1)$$

with $v_y = (p_y + eA)/(\gamma_0 m) = -v_\beta \sin(\omega_\beta t)$, and $v_\beta = \omega_\beta r_\beta \ll c$. The betatron amplitude, r_β , is defined by

$$W_\beta = m\omega_p^2 r_\beta^2 / 4.$$

$A = a_0 mc \exp(-i\phi)/(2e) + \text{c.c.}$ represents a linearly polarized propagating wave, with phase $\phi = \omega t - kz$, where $\omega \approx ck$, and slowly varying complex amplitude a_0 .

2.3. Ponderomotive force

The corresponding axial ponderomotive force is

$$\dot{p}_z \Big|_{\text{pond}} = \eta_h m \omega a_0 v_\beta \exp(i\theta)/4 + \text{c.c.},$$

where $\theta = \omega_\beta t - \phi$ is the ponderomotive phase, and the dot designates the total time derivative, $d/dt = \partial/\partial t + v_z \partial/\partial z$. The factor

$$\eta_h = J_0(Q_\beta) - J_1(Q_\beta),$$

where $J_{0,1}$ are Bessel functions, and

$$Q_\beta = kr_\beta v_\beta / (8c),$$

accounts for the modulation of the axial velocity [6]:

$$v_z = \bar{v}_z + v_m \cos(2\omega_\beta t),$$

with $v_m = v_\beta^2/(4c)$, $\bar{v}_z = v_{0z} - v_m$, and $v_{0z} = c[1 - 1/(2\gamma_0^2)]$.

2.4. Betatron amplitude evolution

From equation (1) we find $\partial W_\beta/\partial x = \partial W_\beta/\partial p_x = 0$ and $(\partial W_\beta/\partial y)\dot{y} + (\partial W_\beta/\partial p_y)\dot{p}_y = 0$; the betatron energy thus evolves as $\dot{W}_\beta = (\partial W_\beta/\partial p_z)\dot{p}_z + (\partial W_\beta/\partial z)\dot{z} + \partial W_\beta/\partial t$. As W_β depends on p_z only through γ_0 , the first term on the rhs can be written as $(\partial W_\beta/\partial \gamma_0)\dot{\gamma}_0 = -mv_y^2 \dot{\gamma}_0/2$. The minus sign is due to p_y , z , and t (and thus $p_y + eA$ rather than v_y) being kept fixed in the partial derivative. Similarly, as the only dependence on z and t is through A , the two remaining terms are equal to $(\partial W_\beta/\partial A)\dot{A} = v_y e \dot{A}$. Together, this yields

$$\dot{W}_\beta = v_y e \dot{A} - mv_y^2 \dot{\gamma}_0 / 2.$$

Hence, on a slow scale, the betatron amplitude evolves according to

$$\dot{r}_\beta = \left[i c \dot{a}_0 \exp(i\theta) + \text{c.c.} \right] / \left[2\omega_p (2\gamma_0)^{1/2} \right] - r_\beta \dot{\gamma}_0 / (4\gamma_0). \quad (2)$$

2.5. Resonance

Betatron oscillations and wave are in resonance when the ponderomotive phase is stationary. At the position $z(t) = z(0) + \bar{v}_z t$ of the electron, with time-averaged velocity \bar{v}_z , the phase evolves as

$$\dot{\theta} = \omega_\beta - \omega \left(1 - \bar{v}_z / c \right) \approx \omega_\beta - \omega / \left(2\bar{\gamma}_z^2 \right),$$

with $\bar{\gamma}_z = (1 - \bar{v}_z^2 / c^2)^{-1/2} \approx \gamma_0 / (1 + a_\beta^2 / 2)^{1/2}$; the betatron parameter

$$a_\beta = \gamma_0 v_\beta / c$$

is the normalized amplitude of transverse momentum. Resonance thus occurs for

$$\omega = \omega_p (2\gamma_0^3)^{1/2} / \left[1 + \gamma_0 \omega_p^2 r_\beta^2 / (4c^2) \right].$$

For $a_\beta \gg 1$, which for very high γ_0 is possible although $v_\beta \ll c$,

$$\dot{\theta} = \omega_p (2\gamma_0)^{-1/2} - \omega \omega_p^2 r_\beta^2 / (8\gamma_0 c^2), \quad (3)$$

and the resonance condition is $\gamma_0 = \gamma_{\text{res}}$, with

$$\gamma_{\text{res}} = \omega^2 \omega_p^2 r_\beta^4 / (32 c^4). \quad (4)$$

2.6. Space-charge

In an electron bunch, space-charge forces [6] contribute to the slow longitudinal force; thus

$$\dot{\gamma}_{0j} = \left[\frac{\eta_h \omega v_{\beta j}}{4c} a_0 - i \eta_f \frac{\omega_b^2}{\omega} \langle \exp(-i\theta) \rangle \right] \exp(i\theta_j) + \text{c.c.}, \quad (5)$$

where in the space-charge term, proportional to $\omega_b^2 = \omega_p^2 n_b / n_0$, with n_b the density in the bunch, the angled brackets denote an average over the electrons in a slice, and η_f accounts for the finite width $\sigma(y)$ of the electron bunch; $\eta_f \approx \sigma(y) / \langle r_\beta \rangle$ if this is < 1 ; else $\eta_f = 1$.

2.7. Radiation emission

The radiation from these electrons adds to the wave amplitude:

$$(\partial/\partial t + c\partial/\partial z) a_0 = -\eta_h \eta_m \eta_f \omega_b^2 \langle v_\beta \exp(-i\theta) \rangle / (2\omega c). \quad (6)$$

Here η_h , defined above, accounts for the reduced emission at the fundamental frequency of the harmonic spectrum [6]. For a planar source, the amplitude of the ℓ th harmonic evolves as

$$(\partial/\partial t + c\partial/\partial z) a_0^{(\ell)} \approx -\eta_m \eta_f \omega_b^2 \langle v_\beta F_\ell(Q_\beta) \exp(-i\ell\theta) \rangle / (2\ell\omega c),$$

with $F_\ell(Q) = J_{(\ell-1)/2}(\ell Q) - J_{(\ell+1)/2}(\ell Q)$ (thus $\eta_h = F_1(Q_\beta)$). The resulting spectrum has a synchrotron-like envelope, with critical frequency $\omega_c \approx 3a_\beta^3\omega/8$. In resonance, $Q_\beta = a_\beta^2/(4 + 2a_\beta^2)$; $\eta_h \approx 0.7$ for $a_\beta \gg 1$. η_m accounts for the spatial overlap of current density and radiation mode, which will be discussed below.

2.8. Steady-state

In the following, we neglect slippage between electrons and wave, as in the steady-state FEL regime [2], thus $d/dt \approx \partial/\partial t + c\partial/\partial z$. Combining equations (2) and (6) then yields $\langle \gamma_0^{-1/2} d(\gamma_0^{1/2} r_\beta^2)/dt \rangle = 0$, implying that $\langle a_\beta r_\beta \rangle \propto \langle \gamma_0^{1/2} r_\beta^2 \rangle$ is conserved if correlations between electron energy and ponderomotive phase can be neglected. Furthermore, equations (5) and (6) may be combined to express energy conservation: $|a_0|^2 + 2\eta_m \eta_f \omega_b^2 \langle \gamma_0 \rangle / \omega^2 = \text{const.}$

3. FEL formalism

We express the energies $\gamma_{0,j}$ and betatron amplitudes $r_{\beta,j}$ of individual electrons in terms of their initial averages

$$\tilde{\gamma}_0 = \langle \gamma_0 \rangle(0), \quad R_\beta = \langle \gamma_0^{1/2} r_\beta^2 \rangle^{1/2} / \tilde{\gamma}_0^{1/4},$$

and relative deviations q_j and s_j , respectively:

$$\gamma_{0,j} = \tilde{\gamma}_0 (1 + q_j), \quad r_{\beta,j} = \left(\tilde{\gamma}_0 / \gamma_{0,j} \right)^{1/4} R_\beta (1 + s_j).$$

Neglecting slippage, equations (2), (3) and (5), for each electron, and (6) form a closed set of equations, which are similar to the FEL equations [2]. In analogy to the FEL-parameter, we define

$$\rho = \left(\frac{\eta \omega_b^2 R_\beta^2}{8 \tilde{\gamma}_0 c^2} \right)^{1/3} = \left(\frac{\eta n_b \tilde{\gamma}_\beta^2}{4 n_0 c^2} \right)^{1/3} \approx 0.13 \left[\frac{\eta_m \eta_f}{\tilde{\gamma}_0} \frac{n_b}{10^{18} \text{ cm}^{-3}} \left(\frac{R_\beta}{\mu\text{m}} \right)^2 \right]^{1/3}, \quad (7)$$

with $\eta = \eta_h^2 \eta_m \eta_f$, $\tilde{\gamma}_\beta = \tilde{\omega}_\beta R_\beta$ and $\tilde{\omega}_\beta = \omega_p / (2\tilde{\gamma}_0)^{1/2}$. For typical experimental parameters, $n_b = 10^{16}$ – 10^{20} cm^{-3} , $R_\beta = 1$ – $10 \mu\text{m}$, $\tilde{\gamma}_0 = 10^2$ – 10^3 , $\eta_m = 0.01$ – 0.1 , and $\eta_f = 10^{-6}$ – 0.1 , we find $\rho \approx 6 \cdot 10^{-6}$ – 0.13 . Higher values of ρ could be obtained for relativistic transverse velocities, which are beyond the scope of this study. In the following, we use $\eta_m \approx 0.01$ corresponding to propagation in a channel surrounded by underdense plasma.

Scaling time $\tau = \rho \tilde{\omega}_\beta t$, and amplitude $\bar{a}_0 = -\eta_h a_0 / (\rho^2 \tilde{a}_\beta)$, with $\tilde{a}_\beta = \tilde{\gamma}_0 \tilde{\gamma}_\beta / c$, the evolution equations become

$$\bar{a}'_0 = (1 + \delta)b, \quad (8)$$

$$\theta'_j = \bar{P}_j \equiv \rho^{-1} \left\{ (1 + q_j)^{-1/2} - (1 + s_j)^2 / \left[(1 + \delta)(1 + q_j)^{3/2} \right] \right\}, \quad (9)$$

$$q'_j = - \left[\rho \bar{a}_0 \frac{1 + s_j}{(1 + \delta)(1 + q_j)^{3/4}} + 2i\rho^2 \frac{1 + \delta}{\eta_h^2 \eta_m} \langle \exp(-i\theta) \rangle \right] \exp(i\theta_j) + \text{c.c.}, \quad (10)$$

$$s'_j = -i \left[\rho^2 / (4\eta_h) \right] \bar{a}'_0 \left[\exp(i\theta_j) / (1 + q_j)^{1/4} \right] + \text{c.c.}, \quad (11)$$

where the prime denotes the derivative with respect to τ ,

$$\delta \equiv \left(\tilde{\gamma}_0 / \gamma_{\text{res}} \Big|_{r_\beta = R_\beta} \right)^{1/2} - 1$$

is the average detuning from resonance, and

$$b \equiv \langle (1 + s) \exp(-i\theta) / (1 + q)^{3/4} \rangle$$

is the bunching factor.

Equations (10) and (11) suggest scaling

$$\bar{q}_j = q_j / \rho, \quad \bar{s}_j = 2\eta_h s_j / \rho^2, \quad \text{and} \quad \bar{\delta} = \delta / \rho.$$

If q_j , s_j , and δ are small compared to unity, equations (8)–(11) may be linearized, and the last two merged using

$$\bar{P}_j \approx \bar{q}_j - \rho \bar{s}_j / \eta_h + \bar{\delta},$$

to obtain:

$$\bar{a}'_0 = \langle \exp(-i\theta) \rangle, \quad (12)$$

$$\theta'_j = \bar{P}_j, \quad (13)$$

$$\bar{P}'_j = -[\bar{a}_0 + i\tilde{\rho} \langle \exp(-i\bar{\theta}) \rangle] \exp(i\bar{\theta}_j) + \text{c.c.}, \quad (14)$$

with

$$\tilde{\rho} = \rho (4/\eta_m - \eta_h) / (2\eta_h^2).$$

Except for this coefficient (rather than ρ), these equations are nearly identical to those of the FEL with space-charge [6]. However, due to the small value of η_m , space-charge effects are relatively enhanced.

For small signal, assuming $\bar{a}_0 \propto \exp(i[\kappa - \bar{\delta}]\tau)$, the secular equation

$$\kappa^3 - \kappa^2 \bar{\delta} - \kappa \tilde{\rho} + 1 + \tilde{\rho} \bar{\delta} = 0$$

is similar to that for the FEL [2]. For $\bar{\delta}$ below a threshold value, there is an unstable solution with amplitude growing exponentially at a rate $\Gamma = \rho \tilde{\omega}_\beta |\text{Im}(\kappa)|$, which gives the gain of the ICL. For small $\bar{\delta}$ and $\tilde{\rho}$,

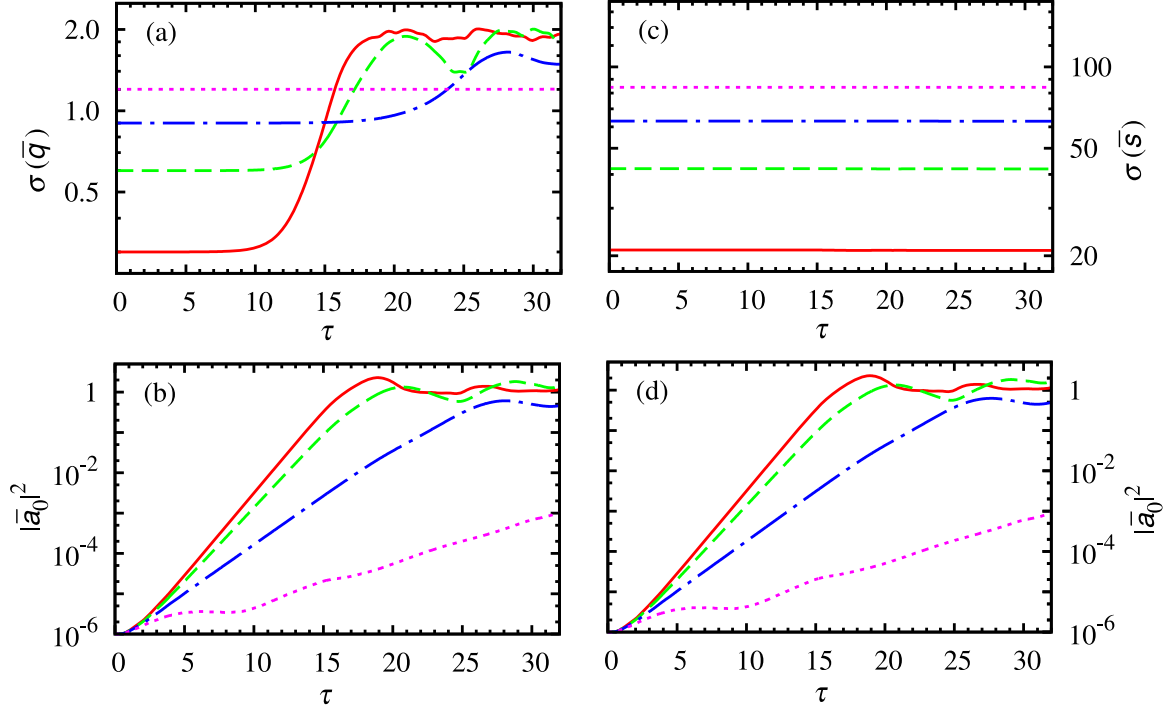


Figure 2. Evolution of spreads of momenta, $\sigma(\bar{q})$, and amplitudes, $\sigma(\bar{s})$, and corresponding field intensity $|\bar{a}_0|^2$, for $\rho = 0.01$ and varying initial conditions: (a) $\sigma(\bar{q})$ and (b) $|\bar{a}_0|^2$ for $\sigma(\bar{s}(0)) = 21$ and $\sigma(\bar{q}(0)) = 0.3$ (solid red), $\sigma(\bar{q}(0)) = 0.6$ (dashed green), $\sigma(\bar{q}(0)) = 0.9$ (dot-dashed blue), and $\sigma(\bar{q}(0)) = 1.2$ (dotted pink); (c) $\sigma(\bar{s})$ and (d) $|\bar{a}_0|^2$ for $\sigma(\bar{q}(0)) = 0.3$ and $\sigma(\bar{s}(0)) = 21$ (solid red), $\sigma(\bar{s}(0)) = 42$ (dashed green), $\sigma(\bar{s}(0)) = 63$ (dot-dashed blue), and $\sigma(\bar{s}(0)) = 84$ (dotted pink).

$$\bar{\Gamma} = \Gamma / (\rho \tilde{\omega}_\beta) \approx \sqrt{3} \left[1 - \tilde{\rho}/3 - \left(\tilde{\delta}^2 - 2\tilde{\rho}\tilde{\delta} \right) / 9 \right] / 2.$$

4. Numerical results

We have numerically solved the set of equations (8)–(11) for different values of ρ and δ , with small initial field, $|\bar{a}_0|(0) = 10^{-3}$, and vanishing initial bunching, $b(0) = 0$ (thus $\bar{a}'_0(0) = 0$). We varied the initial spreads of momenta, $\sigma(\bar{q}(0))$, and of betatron amplitudes, $\sigma(\bar{s}(0))$ (where $\sigma(f) = (\langle f^2 \rangle - \langle f \rangle^2)^{1/2}$) to explore their effect on the interaction and determine the threshold conditions for a realizable ICL. Figure 2 shows the spreads of momentum $\sigma(\bar{q})$, and betatron amplitude $\sigma(\bar{s})$ as functions of τ for varying initial values, together with the corresponding field intensities, for $\rho = 0.01$ and $\bar{\delta} = 2.0$, which is optimized for fastest growth. For small initial spreads, the evolution of intensity $|\bar{a}_0|^2$, bunching $|b|$, and average $|\langle \bar{q} \rangle|$ and spread $\sigma(\bar{q})$ of the momentum deviations is similar to the conventional FEL, with stages of lethargy, exponential growth, and saturation, where each of the scaled variables is of order unity and oscillates quasi-

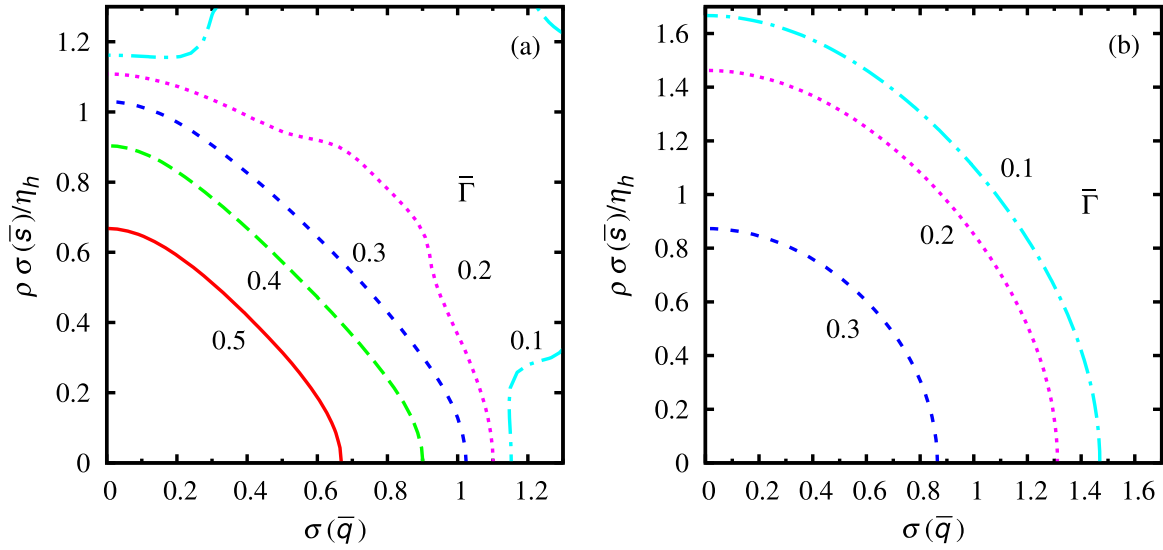


Figure 3. Contours of growth rate $\bar{\Gamma} = d \ln |\bar{a}_0| / d\tau$ for varying $\sigma(\bar{q}(0))$, and $\sigma(\bar{s}(0))$, for (a) $\rho = 0.005$, $\bar{\delta} = 1.5$, and (b) $\rho = 0.05$, $\bar{\delta} = 5.5$.

periodically [7]. If $\sigma(\bar{q})$ initially is close to its saturation value, ~ 2.0 , it remains approximately constant, and the growth of the field is suppressed. Interestingly, the amplitude spread $\sigma(\bar{s})$, which does not play a role in the FEL but affects the resonance in the ICL, evolves in an analogous way to $\sigma(\bar{q})$. However, the threshold for $\sigma(\bar{s}(0))$ to suppress the growth of $|\bar{a}_0|^2$ is $\sim 0.7/\rho$; the contribution from $\sigma(\bar{s})$ to the relevant spread $\sigma(P)$ is scaled with ρ/η_h , cf. equation (13).

Varying ρ from 0 to 0.05, while maintaining optimized detuning, space-charge effects increase the scaled saturation intensity by about one third, and reduce the scaled gain coefficient by one half. The linearization yielding equations (12)–(14) is valid for $\rho < 0.003$.

Figure 3 shows the dependence of the growth rate on initial momentum and amplitude spreads. These plots (and similar ones for the saturation amplitude) yield an approximate condition for amplification in the ICL:

$$\left[\sigma(\gamma_0(0)) / \tilde{\gamma}_0 \right]^2 + \left[\sigma(r_\beta(0)) / R_\beta \right]^2 \leq (1 + 10\rho)^2 \rho^2, \quad (15)$$

i.e., the relative spread, between different electrons, in the variable \bar{P} , equation (9), must be less than $\sim \rho$. These admissible spreads imply optimum detuning δ . The condition $\Delta\gamma_z/\gamma_z < \rho$ in [8], referring to variations of the ‘axial energy’ within a cycle, does not apply, since these are taken into account by the emission efficiency η_h for the fundamental frequency of the harmonic spectrum.

The condition (15) suggests setting $\eta_f = \rho \bar{\eta}_f$, with $\bar{\eta}_f \approx 0.5$, resulting in a new scaling for the FEL parameter, cf equation (7):

$$\rho = \left[\eta_h^2 \eta_m \bar{\eta}_f \omega_b^2 R_\beta^2 / (8 \tilde{\gamma}_0 c^2) \right]^{1/2} \approx 3.3 \times 10^{-3} \left[(n_b / 10^{18} \text{ cm}^{-3}) / \tilde{\gamma}_0 \right]^{1/2} (R_\beta / \mu\text{m}). \quad (16)$$

For the parameters given after equation (7), $\rho \approx 10^{-5}$ – 0.03 .

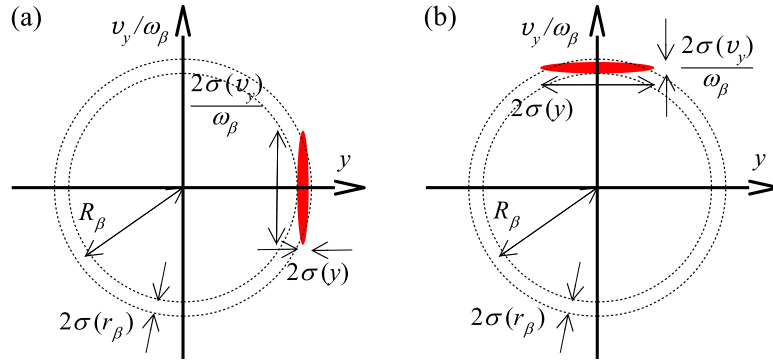


Figure 4. Schematic of initial phase-space distributions of electron bunches injected off-axis (a) and under an angle (b), respectively, with different widths $\sigma(y)$ and velocity spreads $\sigma(v_y)$, but resulting in equal betatron amplitude R_β and spread $\sigma(r_\beta) \ll R_\beta$.

5. Discussion

Whittum's original proposal for the ICL [4] would be very difficult to realize experimentally, at least for high $\tilde{\gamma}_0$, due to the restriction to very small transverse momenta, $a_\beta \ll 1$, which would also lead to very low gain and low efficiency and thus unfeasibly long devices. However, we have shown here that large transverse momenta can realistically be used, by explicitly taking into account the effect of the betatron amplitude on the resonance. This allows the coupled radiation-matter equations to be cast in a form similar to that of the conventional FEL.

An ion channel can be realized experimentally by focusing a laser pulse with relativistic amplitude $E_L > m\omega_L c/e$ (where ω_L is the laser frequency) into plasma. Its ponderomotive force displaces the electrons from its path and a 'bubble' structure is formed, which provides the required transverse field in addition to a longitudinal wakefield [9]. To minimize the variation of $\tilde{\gamma}_0$, the electron bunch should be close to dephasing, at the centre of the 'bubble'; as their velocities are different this limits the useful propagation length.

Figure 4 shows possible initial phase-space distributions resulting in low amplitude spread. Small $\sigma(r_\beta)/R_\beta$, $\sim 0.7\rho$, can be achieved by injecting electrons either off-axis at a distance R_β , with $\sigma(y) \approx 0.5\rho R_\beta$ (figure 4(a)) and a range of betatron phases $\sigma(\varphi_\beta) \approx \sigma(v_y)/\tilde{v}_\beta \leq 0.8\sqrt{\rho}$, or at an angle $\arctan(\tilde{v}_\beta/c)$, with $\sigma(v_y) \approx 0.5\rho\tilde{v}_\beta$ and $\sigma(y) \approx 0.8\sqrt{\rho}R_\beta$ (figure 4(b)). An electron bunch trapped in the 'bubble' could be offset from the axis by perturbing the propagation direction of the laser and thus of the 'bubble' [10].

The normalized emittance,

$$\epsilon_{yn} \approx \pi\tilde{\gamma}_0\sigma(y)\sigma(v_y)/c \leq 0.4\pi\rho^{3/2}\tilde{a}_\beta R_\beta,$$

sets a lower limit for the emitted fundamental wavelength: using equations (4) and (16)

$$\lambda = \pi\tilde{\omega}_\beta R_\beta^2 / (2c) \geq \epsilon_{yn} / (0.8\rho^{3/2}\tilde{\gamma}_0) = 6.6 \times 10^3 \epsilon_{yn} / \left[(n_b/10^{18} \text{ cm}^{-3})^{3/4} (R_\beta/\mu\text{m})^{3/2} \tilde{\gamma}_0^{1/4} \right].$$

In the x -direction, perpendicular to the polarization, the matched bunch width for $\epsilon_{xn} = \epsilon_{yn}$ is $\sigma(x) \leq 0.6\rho^{3/4}R_\beta$. The source size is thus $\pi\sigma(x)R_\beta/4 \leq 0.5\rho^{3/4}R_\beta^2$, and the Rayleigh length $0.5\rho^{3/4}R_\beta^2/\lambda = 0.05\rho^{3/4}\lambda_\beta$. This is shorter than the gain length, $l_g = \lambda_\beta/(\sqrt{3}\pi\rho)$ by a factor of

$\sim 0.3\rho^{7/4}$, making some form of guiding necessary. Optical guiding is naturally provided by the plasma channel, since the refractive index in the channel (≈ 1) is higher than in the surrounding plasma ($[1 - \omega_p^2/\omega^2]^{1/2}$). An analysis similar to [11] shows that the plasma channel can guide high-frequency modes if its radius exceeds c/ω_p . The overlap factor of the lowest order mode with an electron bunch, oscillating with amplitude matched to the channel radius, is $\eta_m \approx 0.01$ for $\omega \gg \omega_p$.

Electron bunches can be accelerated in laser wakefields to $\tilde{\gamma}_0 = 200\text{--}300$ with relative energy spread as small as $\sigma(\gamma_0(0))/\tilde{\gamma}_0 \sim 0.01$ [12], and normalized emittance $\epsilon_{yn} \sim 10^{-6} \pi \text{ m}$ [13]. This emittance together with $n_b = 10^{20} \text{ cm}^{-3}$, $R_\beta = 10 \mu\text{m}$, and $\tilde{\gamma}_0 = 300$, thus $\rho = 0.02$, yields $\lambda \geq 5 \mu\text{m}$; for the shortest wavelength, $\tilde{\omega}_\beta = 10^{13} \text{ s}^{-1}$, $\omega_p = 2.4 \times 10^{14} \text{ s}^{-1}$, $n_0 = 1.8 \times 10^{19} \text{ cm}^{-3}$, $\tilde{v}_\beta = 10^8 \text{ ms}^{-1} = 0.33c$, $\tilde{a}_\beta = 100$, and gain length $l_g = 1.7 \text{ mm}$; the critical harmonic number of the synchrotron-like spectrum is $h_c = 3\tilde{a}_\beta^3/8 = 3.8 \times 10^3$, corresponding to a critical wavelength $\lambda_c = 1.3 \times 10^{-11} \text{ m}$. These latter values are for guidance only; the high oscillation amplitude, larger than the bunch width, leads to a modified emission spectrum. Improving the emittance to $\epsilon_{yn} \sim 4 \times 10^{-8} \pi \text{ m}$ [14] allows $\lambda \geq 2 \times 10^{-7} \text{ m}$; in this case, $\tilde{\omega}_\beta = 4 \times 10^{11} \text{ s}^{-1}$, $\omega_p = 9.6 \times 10^{12} \text{ s}^{-1}$, $n_0 = 2.7 \times 10^{16} \text{ cm}^{-3}$, $\tilde{v}_\beta = 4 \times 10^6 \text{ ms}^{-1} = 0.013c$, $\tilde{a}_\beta = 4$, $l_g = 4.5 \text{ cm}$, $h_c = 24$, and $\lambda_c = 8.3 \times 10^{-9} \text{ m}$. The current in these cases is $I \approx 600 \text{ A}$, and the efficiency of converting kinetic electron energy into radiated energy, $\rho|\tilde{a}_0|^2 \approx 2\%$ at saturation, giving a peak power, at the fundamental wavelength, of 2 GW and respective photon rates of $5 \times 10^{28} \text{ s}^{-1}$ and $2 \times 10^{27} \text{ s}^{-1}$. Increasing n_b to $1.5 \times 10^{20} \text{ cm}^{-3}$ and R_β to $15 \mu\text{m}$, with $\tilde{\gamma}_0 = 200$, results in $\rho = 0.049$. For emittance $\epsilon_{xn} = \epsilon_{yn} = 10^{-6} \pi \text{ m}$ and $\omega_p = 3.4 \times 10^{13} \text{ s}^{-1}$, the fundamental wavelength is $\lambda = 2 \mu\text{m}$ and the gain length 4 mm . In this case, $\tilde{a}_\beta = 17$ and $\lambda_c = 1.1 \times 10^{-9} \text{ m}$; the current is 10 kA , the peak power emitted at λ , 50 GW , and the photon rate $5 \times 10^{29} \text{ s}^{-1}$.

While in the present study we consider planar electron motion and linear polarization of the radiation field, the theory can be extended to describe betatron oscillations in both transverse directions, leading to elliptical polarization. The simplest case, equal amplitudes in both directions, with a phase difference of $\pi/2$, results in circular polarization. In this case, the distance from the axis and the magnitude of the transverse momentum are slowly-varying; as a consequence, there will be no synchrotron-like spectrum. Experimentally, these cases can be realized by combining off-axis and oblique injection out of plane.

6. Conclusions

In conclusion, we have developed a comprehensive model for the ICL by studying the collective interaction of electrons moving in an ion channel with a propagating wave. Including the effect of variable betatron amplitude on the resonance between field and electrons is essential to correctly describe experimentally accessible regimes. We show that the ICL can be described in a similar way to the FEL and define an analogous ρ -parameter, which can realistically reach 0.03 , which is high for an FEL. However, space charge effects are relatively much more important here, with a corresponding new coupling parameter $\tilde{\rho} \sim 400\rho$.

Numerical solutions with varying initial spreads in axial momentum and betatron amplitude confirm the condition, known from the FEL, that the relative momentum spread must be less than $\sim \rho$, and give an analogous condition for the amplitude spread. A numerical

example indicates the requirements for operating an ICL down to UV fundamental wavelengths, with harmonics potentially extending to x-rays. A high value of ρ is essential for reconciling the requirement of low betatron amplitude for emission at short wavelength with experimentally accessible amplitude spread. While the present study covers the steady-state case and neglects longitudinal plasma fields, the model can be readily extended to the time-dependent regime where superradiant pulses will evolve [15–19]. The resonance with a propagating wave also plays an important role in laser-driven betatron oscillations [20, 21].

Acknowledgments

We acknowledge fruitful discussions with G R M Robb and B W J McNeil and support by the United Kingdom Engineering and Physical Sciences Research Council (Grant EP/J018171/1), the Extreme Light Infrastructure Project, and the European Commission FP7 projects Laserlab-Europe (Grant 284464) and EuCARD-2 (Grant 312453).

References

- [1] Madey J 1971 *J. Appl. Phys.* **42** 1906
Deacon D A G, Elias L R, Madey J M J, Ramian G J, Schwettman H A and Smith T I 1977 *Phys. Rev. Lett.* **38** 892
- [2] Bonifacio R, Pellegrini C and Narducci L M 1984 *Opt. Commun.* **50** 373
- [3] McNeil B W J and Thompson N R 2010 *Nature Photon.* **4** 814
- [4] Whittum D H, Sessler A M and Dawson J M 1990 *Phys. Rev. Lett.* **64** 2511
- [5] Chen K R, Katsouleas T C and Dawson J M 1990 *IEEE Trans. Plasma Sci.* **18** 837
Chen K R, Dawson J M, Lin A T and Katsouleas T C 1991 *Phys. Fluids B* **3** 1270
- [6] Murphy J B, Pellegrini C and Bonifacio R 1985 *Opt. Commun.* **53** 197
- [7] Jaroszynski D A, Bakker R J, van der Meer A F G, Oepts D and van Amersfoort P W 1993 *Phys. Rev. Lett.* **70** 3412
- [8] Esarey E, Shadwick B A, Catravas P and Leemans W P 2002 *Phys. Rev. E* **65** 056505
- [9] Pukhov A, Gordienko S, Kiselev S and Kostyukov I 2004 *Plasma Phys. Control. Fusion* **46** B 179
- [10] Reitsma A J W and Jaroszynski D A 2007 *Phys. Plasmas* **14** 053104
- [11] Snitzer E 1961 *J. Opt. Soc. Am.* **51** 491
- [12] Wiggins S M *et al* 2010 *Plasma Phys. Control. Fusion* **52** 124032
- [13] Brunetti E *et al* 2010 *Phys. Rev. Lett.* **105** 215007
- [14] Hidding B, Pretzler G, Rosenzweig J B, Königstein T, Schiller D and Bruhwiler D L 2012 *Phys. Rev. Lett.* **108** 035001
- [15] Bonifacio R and Casagrande F 1985 *Nucl. Instrum. Methods Phys. Res. A* **239** 36
- [16] Bonifacio R, Maroli C and Piovela N 1988 *Opt. Commun.* **68** 369
- [17] Bonifacio R, De Salvo Souza L, Pierini P and Piovela N 1990 *Nucl. Instrum. Methods Phys. Res. A* **296** 358
- [18] Piovela N, Chaix P, Shvets G and Jaroszynski D A 1995 *Phys. Rev. E* **52** 5470
- [19] Jaroszynski D A, Chaix P, Piovela N, Oepts D, Knippels G M H, van der Meer A F G and Weits H H 1997 *Phys. Rev. Lett.* **78** 1699
- [20] Németh K, Shen B, Li Y, Shang H, Crowell R, Harkay K C and Cary J R 2008 *Phys. Rev. Lett.* **100** 095002
- [21] Cipiccia S *et al* 2011 *Nature Phys.* **7** 867