

# THERMOVISCOUS COATING AND RIMMING FLOW

by G. A. LESLIE, S. K. WILSON<sup>‡</sup> and B. R. DUFFY

(*Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, United Kingdom*)

[Received 4th March 2012, revised 29th June and 17th July 2012]

## Summary

A comprehensive description is obtained of steady thermoviscous (i.e. with temperature-dependent viscosity) coating and rimming flow on a uniformly rotating horizontal cylinder that is uniformly hotter or colder than the surrounding atmosphere. It is found that, as in the corresponding isothermal problem, there is a critical solution with a corresponding critical load (which depends, in general, on both the Biot number  $B$  and the thermoviscosity number  $V$ ) above which no “full-film” solutions corresponding to a continuous film of fluid covering the entire outside or inside of the cylinder exist. The effect of thermoviscosity on both the critical solution and the full-film solution with a prescribed load is described. In particular, there are no full-film solutions with a prescribed load  $M$  for any value of  $B$  when  $M \geq \hat{f}^{-1/2} M_{c0}$  for positive  $V$  and when  $M > M_{c0}$  for negative  $V$ , where  $\hat{f}$  is a monotonically decreasing function of  $V$  and  $M_{c0} \simeq 4.44272$  is the critical load in the constant-viscosity case. It is also found that, for the exponential viscosity model, when the prescribed load satisfies  $M < 1.50315$  there is a narrow region of the  $B$ – $V$  parameter plane in which backflow occurs.

## 1. Introduction

The influence of thermoviscosity (i.e. temperature-dependent viscosity) effects on the two-dimensional flow of a thin film of viscous fluid on either the outside (often called “coating flow”) or the inside (often called “rimming flow”) of a uniformly rotating horizontal cylinder is studied. The corresponding isothermal problem has been studied extensively, largely building on the pioneering work of Pukhnachev<sup>†</sup> (1) and Moffatt (2). Pukhnachev (1) discussed the existence and uniqueness of solutions for coating flow of the steady two-dimensional Navier–Stokes equations and then used lubrication theory to derive an evolution equation that includes the effects of gravity, viscosity and surface tension, while Moffatt (2) considered coating flow both theoretically using a model based on lubrication theory and experimentally and, in particular, found that steady continuous, finite and non-zero solutions to his model (hereafter referred to as “full-film” solutions) are possible only below a critical load.

There have been many subsequent studies of isothermal coating flow, including those by Johnson (3), Hansen and Kelmanson (4), Thoroddsen and Mahadevan (5), Wilson and Williams (6), O’Brien and Gath (7), Hosoi and Mahadevan (8), Duffy and Wilson

---

<sup>‡</sup> Author for correspondence. Presently also a Visiting Fellow in the Oxford Centre for Collaborative Applied Mathematics (OCCAM), University of Oxford, Mathematical Institute, 24–29 St. Giles’, Oxford OX1 3LB. Email: [s.k.wilson@strath.ac.uk](mailto:s.k.wilson@strath.ac.uk)

<sup>†</sup> This author’s name has been transliterated into English as both “Pukhnachev” and “Pukhnachov”.

(9), Peterson, Jimack and Kelmanson (10), Tirumkudulu and Acrivos (11), O'Brien (12), Wilson, Hunt and Duffy (13), Ashmore, Hosoi and Stone (14), Benilov and co-workers (15, 16, 17, 18), Hinch and Kelmanson (19), Acrivos and Jin (20, 21), Evans, Schwartz and Roy (22, 23), Noakes, King and Riley (24, 25), Villegas-Díaz, Power and Riley (26), Chen, Tsai, Liu and Wu (27), Karabut (28), Hunt (29), Kelmanson (30), Tougher, Wilson and Duffy (31), and Pougatch and Frigaard (32).

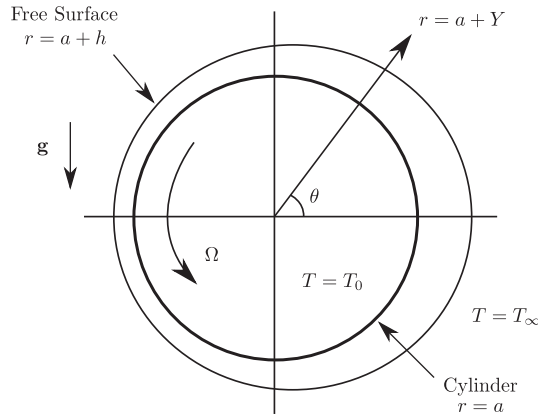
Owing to the wide range of practical situations in which thermoviscosity effects are significant there have been a number of studies of the effects of thermoviscosity on the non-isothermal flow of thin films of fluid on a variety of substrates. Thermoviscous flow of thin films and rivulets of fluid has been studied by, amongst others, Goussis and Kelly (33, 34), Hwang and Weng (35), Reisfeld and Bankoff (36), Wu and Hwang (37), Selak and Lebon (38), Kabova and Kuznetsov (39), Duffy and Wilson (40, 41, 42), and Sansom, King and Riley (43). However, relatively little work has been done on thermoviscous coating and rimming flow on a heated or cooled horizontal cylinder. Duffy and Wilson (44) used lubrication theory to study steady flow on both a stationary and a uniformly rotating cylinder in the rather special case in which the free surface is at the same uniform temperature as the surrounding atmosphere and found that the film thickness (and hence the load, but not the temperature or the velocity) can be obtained by a simple re-scaling of the isothermal solution, and recently Leslie, Wilson and Duffy (45) analysed steady thermoviscous flow with prescribed flux on a stationary cylinder. In the present work we obtain a comprehensive description of steady thermoviscous coating and rimming flow on a uniformly rotating horizontal cylinder. In particular, the effect of thermoviscosity on both the critical solution and the full-film solution with a prescribed load is described. In the Appendix a distinguished limit corresponding to strong thermoviscosity and weak heat transfer at the free surface in which thermoviscosity effects still enter the problem at leading order is discussed.

## 2. Problem Formulation

Consider the steady two-dimensional thermoviscous flow of a thin film of a Newtonian fluid with uniform density  $\rho$  and temperature-dependent viscosity  $\mu = \mu(T)$ , where  $T$  denotes the (in general) non-uniform temperature of the fluid, on either the outside (“coating flow”) or the inside (“rimming flow”) of a circular cylinder of radius  $a$  rotating in a counter-clockwise direction about its horizontal axis at uniform angular speed  $\Omega$  (so that the circumferential speed is  $a\Omega$ ), the cylinder being at the uniform temperature  $T_0$ , which may be either hotter or colder than the uniform temperature of the surrounding atmosphere, denoted by  $T_\infty$  ( $\neq T_0$ ). Where possible we will consider a general viscosity model  $\mu = \mu(T)$ , where  $\mu(T)$  is any monotonically decreasing function of  $T$  satisfying  $\mu = \mu_0$  and  $d\mu/dT = -\lambda$  ( $< 0$ ) when  $T = T_0$ , where  $\lambda$  ( $> 0$ ) is a prescribed positive constant, and so the appropriate non-dimensional measure of thermoviscosity is the *thermoviscosity number*,  $V$ , defined by

$$V = \frac{\lambda(T_0 - T_\infty)}{\mu_0}. \quad (2.1)$$

Since  $V$  has the same sign as  $T_0 - T_\infty$ , situations in which the cylinder is hotter (colder) than the atmosphere correspond to positive (negative) values of  $V$ . When it is necessary to



**Fig. 1** Geometry of the problem (drawn for coating flow): steady two-dimensional thermoviscous flow of a thin film of Newtonian fluid on a uniformly rotating horizontal cylinder that is uniformly hotter or colder than the surrounding atmosphere.

specify a particular viscosity model we will use the exponential viscosity model

$$\mu(T) = \mu_0 \exp\left(-\frac{\lambda(T - T_0)}{\mu_0}\right) \quad (2.2)$$

as used by many previous authors (for example, Goussis and Kelly (33, 34), Hwang and Weng (35), Selak and Lebon (38), Wilson and Duffy (41), and Leslie, Wilson and Duffy (45)). As, for example, Leslie, Wilson and Duffy (45) describe, physically realistic values of  $V$  vary over several orders of magnitude from arbitrarily small values (when the viscosity is effectively independent of temperature and/or when the magnitude of the heating or cooling is small) to reasonably large values (when the viscosity is strongly dependent on temperature and/or when the magnitude of the heating or cooling is large).

Referred to polar coordinates  $r = a \pm Y$  (with origin at the cylinder's axis) and  $\theta$  (measured counter-clockwise from the horizontal), as shown in Figure 1 (drawn for coating flow), we take the free surface of the fluid to be at  $r = a + h$  for coating flow and  $r = a - h$  for rimming flow, the film thickness being denoted by  $h = h(\theta)$ . The fluid velocity  $\mathbf{u} = u\mathbf{e}_\theta + v\mathbf{e}_r$  (where  $\mathbf{e}_\theta$  and  $\mathbf{e}_r$  denote unit vectors in the azimuthal and radial directions, respectively), the pressure  $p$ , and the temperature  $T$  are governed by the usual mass-conservation, Navier–Stokes and energy equations. On the cylinder  $r = a$  the velocity  $\mathbf{u}$  satisfies no-slip and no-penetration conditions, and the temperature is  $T = T_0$  (a prescribed constant). On the free surface  $r = a \pm h$  the usual normal and tangential stress balances and the kinematic condition apply, as does Newton's law of cooling  $-k_{\text{th}}\nabla T \cdot \mathbf{n} = \alpha_{\text{th}}(T - T_\infty)$ , where  $k_{\text{th}}$  denotes the thermal conductivity of the fluid (assumed constant),  $\alpha_{\text{th}} (\geq 0)$  denotes an (empirical) surface heat-transfer coefficient, and  $\mathbf{n}$  denotes the unit outward normal to the free surface. Surface tension, viscous dissipation, inertia and thermal advection are all neglected (which is valid provided that the appropriate capillary number is sufficiently large and that the appropriate Brinkman, Reynolds and Péclet numbers are sufficiently small).

We consider only thin films with small aspect ratio  $\epsilon$  defined by

$$\epsilon = \left( \frac{\mu_0 \Omega}{\rho g a} \right)^{1/2} \ll 1, \quad (2.3)$$

where  $g$  denotes the magnitude of gravitational acceleration, and non-dimensionalise and scale the system appropriately by writing

$$\begin{aligned} r &= a(1 \pm \epsilon Y^*), & h &= \epsilon a h^*, & u &= a \Omega u^*, & v &= \pm \epsilon a \Omega v^*, \\ \psi &= \pm \epsilon a^2 \Omega \psi^*, & p &= p_a \pm \epsilon a \rho g p^*, & T &= T_\infty + (T_0 - T_\infty) T^*, \\ \mu &= \mu_0 \mu^*, & Q &= \epsilon a^2 \Omega Q^*, & M &= \epsilon a^2 \rho M^*, \end{aligned} \quad (2.4)$$

in which the  $+$  in the  $\pm$  corresponds to coating flow while the  $-$  corresponds to rimming flow,  $\psi$  is the streamfunction satisfying  $u = \partial\psi/\partial Y$  and  $v = -\partial\psi/\partial\theta$  with  $\psi = 0$  on  $Y = 0$ ,  $p_a$  is the constant pressure in the surrounding atmosphere,  $Q$  is the constant azimuthal volume flux of fluid per unit axial length crossing a station  $\theta = \text{constant}$ , and  $M (> 0)$  is the constant fluid load on/in the cylinder (that is, the mass of fluid per unit length on/in the cylinder). Note that since the non-dimensionalisation of temperature given in (2.4) incorporates the factor  $T_0 - T_\infty$ , which can be either positive or negative, care is required when interpreting results for the non-dimensional temperature  $T^*$  in terms of the dimensional temperature  $T$ . For clarity the star superscripts on non-dimensional variables will be omitted henceforth. In terms of non-dimensional variables the fluid occupies  $0 \leq Y \leq h$  for  $-\pi < \theta \leq \pi$ , the general viscosity model  $\mu = \mu(T)$  satisfies  $\mu = 1$  and  $d\mu/dT = -V$  when  $T = 1$ , and the exponential viscosity model (2.2) is given by

$$\mu = \exp(-V(T - 1)). \quad (2.5)$$

For both coating and rimming flow at leading order in  $\epsilon$  the governing equations become

$$\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial Y} = 0, \quad \frac{\partial}{\partial Y} \left( \mu \frac{\partial u}{\partial Y} \right) = \cos \theta, \quad \frac{\partial p}{\partial Y} = -\sin \theta, \quad \frac{\partial^2 T}{\partial Y^2} = 0, \quad (2.6)$$

together with the boundary conditions

$$u = 1, \quad v = 0 \quad \text{and} \quad T = 1 \quad \text{on} \quad Y = 0 \quad (2.7)$$

and

$$\frac{\partial u}{\partial Y} = 0, \quad p = 0 \quad \text{and} \quad \frac{\partial T}{\partial Y} + BT = 0 \quad \text{on} \quad Y = h, \quad (2.8)$$

where the Biot number,  $B (\geq 0)$ , defined by

$$B = \frac{\epsilon a \alpha_{\text{th}}}{k_{\text{th}}}, \quad (2.9)$$

is the appropriate non-dimensional measure of heat transfer to or from the atmosphere at the free surface. The special case  $B = 0$  corresponds to a perfectly insulated free surface with no heat transfer to or from the atmosphere (i.e.  $\partial T/\partial Y = 0$  on  $Y = h$ ), while at

leading order in the limit  $B \rightarrow \infty$  the free surface is at the same uniform temperature as the atmosphere (i.e.  $T = 0$  on  $Y = h$ ).

Introducing the rescaled radial coordinate  $y = Y/h$  (so that the fluid lies within the fixed range  $0 \leq y \leq 1$ ) and solving (2.6) subject to (2.7) and (2.8) for the azimuthal velocity  $u = u(y, \theta)$ , the streamfunction  $\psi = \psi(y, \theta)^\dagger$ , the pressure  $p = p(y, \theta)$  and the temperature  $T = T(y, \theta)$  yields

$$u = 1 - h^2 \cos \theta \int_0^y \frac{1 - \tilde{y}}{\mu(T(\tilde{y}, \theta))} d\tilde{y}, \quad (2.10)$$

$$\psi = hy - h^3 \cos \theta \int_0^y \int_0^{\tilde{y}} \frac{1 - \tilde{y}}{\mu(T(\tilde{y}, \theta))} d\tilde{y} d\tilde{y} = hy - h^3 \cos \theta \int_0^y \frac{(1 - \tilde{y})(y - \tilde{y})}{\mu(T(\tilde{y}, \theta))} d\tilde{y}, \quad (2.11)$$

$$p = h(1 - y) \sin \theta \quad (2.12)$$

and

$$T = 1 - \frac{Bhy}{1 + Bh}. \quad (2.13)$$

Hence the volume flux  $Q (= \psi(1, \theta))$  is given by

$$Q = h \int_0^1 u dy = h - h^3 \cos \theta \int_0^1 \int_0^y \frac{1 - \tilde{y}}{\mu(T(\tilde{y}, \theta))} d\tilde{y} dy, \quad (2.14)$$

which may be re-written as

$$Q = h - \frac{h^3 \cos \theta}{3} f, \quad (2.15)$$

where  $f = f(\theta) (> 0)$  denotes the fluidity of the fluid film, defined by

$$f = 3 \int_0^1 \int_0^y \frac{1 - \tilde{y}}{\mu(T(\tilde{y}, \theta))} d\tilde{y} dy = 3 \int_0^1 \frac{(1 - y)^2}{\mu(T(y, \theta))} dy. \quad (2.16)$$

In the special case of constant viscosity  $\mu \equiv 1$  the fluidity is simply equal to unity, i.e.  $f \equiv 1$ .

In the present work we shall be concerned only with “full-film” solutions, i.e. solutions for which  $h$  is continuous, finite and non-zero for all  $-\pi < \theta \leq \pi$ , corresponding to a continuous film of fluid covering the entire outside or inside of the cylinder. In particular, for such solutions equations (2.13) and (2.16) show that  $f$  depends on  $\theta$  only through its dependence on  $h$ , and hence, using (2.15), only through its dependence on  $\cos \theta$ . Hence  $h$ ,  $u$  and  $T$  depend on  $\theta$  only through  $\cos \theta$ , and so the flow has top-to-bottom symmetry (but not left-to-right symmetry). Moreover, at the top and the bottom of the cylinder ( $\theta = \pm\pi/2$ ) the film thickness is simply  $h = Q$  (from which we trivially deduce that  $Q > 0$ ) and the velocity is a uniform (i.e. independent of  $y$ ) “plug flow”  $u \equiv 1$  across the film.

The fluid load in/on the cylinder,  $M (> 0)$ , is given by

$$M = \int_{-\pi}^{\pi} h d\theta = 2 \int_0^{\pi} h d\theta. \quad (2.17)$$

---

<sup>†</sup> Note that the rescaled stream function  $\psi = \psi(y, \theta)$  satisfies  $u = \frac{1}{h} \frac{\partial \psi}{\partial y}$  and  $v = -\frac{\partial \psi}{\partial \theta} + \frac{y}{h} \frac{dh}{d\theta} \frac{\partial \psi}{\partial y}$ . Leslie, Wilson and Duffy (45) erroneously omitted the second term from their corresponding expression for  $v$ . Fortunately, none of their subsequent results are affected by this oversight.

With the viscosity model  $\mu = \mu(T)$  prescribed, the film thickness  $h$  is determined in terms of  $Q$  by the algebraic equation (2.15) in which  $f$  is given by (2.16). The value of  $Q$  is determined from either an appropriate criticality condition (given in Section 5) or from the condition of prescribed load using (2.17). The properties and behaviour of the solutions in these two cases are discussed in detail in Sections 5 and 6, respectively, but in both cases, the solutions for  $u$ ,  $\psi$ ,  $p$  and  $T$  are given explicitly by (2.10)–(2.13), respectively.

### 3. The Special Case of Constant Viscosity

If either there is no heat transfer to or from the atmosphere at the free surface (i.e. in dimensional terms if  $\alpha_{\text{th}} = 0$ ) so that  $B = 0$  (in which case the fluid film is isothermal with constant temperature  $T \equiv 1$ ) or the viscosity is independent of temperature (i.e. in dimensional terms if  $\lambda = 0$ ) so that  $V = 0$  (in which case the fluid film is non-isothermal with non-constant temperature  $T \neq 1$ ), then the fluid has constant viscosity  $\mu \equiv 1$  and fluidity  $f \equiv 1$ . In either case we recover the classical constant-viscosity solution derived by Moffatt (2), denoted by  $h = h_0$ ,  $u = u_0$ ,  $\psi = \psi_0$ ,  $Q = Q_0$  and  $M = M_0$ , where

$$u_0 = 1 - \frac{h_0^2 \cos \theta}{2} (2 - y)y, \quad (3.1)$$

$$\psi_0 = h_0 y - \frac{h_0^3 \cos \theta}{6} (3 - y)y^2, \quad (3.2)$$

$$Q_0 = h_0 - \frac{h_0^3 \cos \theta}{3} \quad (3.3)$$

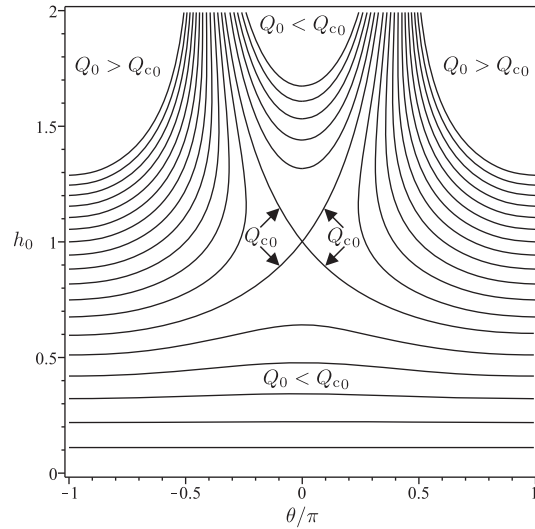
and

$$M_0 = 2 \int_0^\pi h_0 \, d\theta. \quad (3.4)$$

Figure 2 shows contours of the flux  $Q_0$  given by (3.3) in the  $\theta/\pi$ – $h_0$  plane, which, since they are by definition curves on which  $Q_0 = \text{constant}$ , represent candidate solutions for the free surface  $h_0 = h_0(\theta)$ . In particular, Figure 2 shows that full-film solutions are possible only for values of  $Q_0$  satisfying  $0 < Q_0 \leq Q_{c0}$ , where  $Q_{c0} = 2/3$  denotes the flux of the critical solution above which full-film solutions are not possible, and that these solutions have loads  $M_0$  satisfying  $0 < M_0 \leq M_{c0}$ , where  $M_{c0}$  denotes the load of the critical solution. A detailed description of the other (i.e. non-full-film) solutions of (3.3) is given by Duffy and Wilson (9). When  $0 < Q_0 \leq Q_{c0}$  the full-film solution of (3.3) for  $h_0$  may be written explicitly in terms of  $Q_0$  and  $\theta$  as

$$h_0 = \begin{cases} \frac{2}{(\cos \theta)^{1/2}} \cos \left[ \frac{2\pi}{3} - \frac{1}{3} \cos^{-1} \left( -\frac{3}{2} Q_0 [\cos \theta]^{1/2} \right) \right] & \text{if } |\theta| < \pi/2, \\ Q_0 & \text{if } |\theta| = \pi/2, \\ \frac{2}{(-\cos \theta)^{1/2}} \sinh \left[ \frac{1}{3} \sinh^{-1} \left( \frac{3}{2} Q_0 [-\cos \theta]^{1/2} \right) \right] & \text{if } \pi/2 < |\theta| \leq \pi, \end{cases} \quad (3.5)$$

and has load  $M_0$  satisfying  $0 < M_0 \leq M_{c0}$  given by (3.4). The critical solution, denoted by  $h_0 = h_{c0}$ ,  $u_0 = u_{c0}$  and  $\psi_0 = \psi_{c0}$ , is obtained by substituting  $Q_0 = Q_{c0} = 2/3$  into (3.1),



**Fig. 2** Contours of the flux in the constant-viscosity case,  $Q_0$ , given by (3.3) in the  $\theta/\pi$ - $h_0$  plane. The contours are drawn for  $Q_0/Q_{c0} = 1/6, 1/3, 1/2, \dots, 3$ , where  $Q_{c0} = 2/3$ .

(3.2) and (3.5), and from (3.4) has load  $M_0 = M_{c0}$ , where

$$M_{c0} = 2 \int_0^\pi h_{c0} d\theta \simeq 4.44272. \quad (3.6)$$

As Figure 2 shows, the critical film thickness  $h_{c0}$  has a corner at  $\theta = 0$  (with corresponding corners in all of the other streamlines  $\psi_{c0} = \text{constant}$  also at  $\theta = 0$ ) given by  $h_{c0} = 1 - H_{c0}|\theta| + O(\theta^2)$ , where the magnitude of the slope of the corner, denoted by  $H_{c0}$ , is given by  $H_{c0} = 1/\sqrt{6} \simeq 0.40825$ .<sup>†</sup>

#### 4. The General Case of Non-Constant Viscosity

In general, if there is heat transfer to or from the atmosphere at the free surface (i.e. in dimensional terms if  $\alpha_{\text{th}} > 0$ ) so that  $B > 0$  and the viscosity depends on temperature (i.e. in dimensional terms if  $\lambda > 0$ ) so that  $V \neq 0$ , then the fluid film is non-isothermal with, in general, non-constant temperature, viscosity and fluidity. In the particular case of the exponential viscosity model (2.5) we have

$$\mu = \exp(-V(T - 1)) = \exp\left(\frac{BVhy}{1 + Bh}\right) = \exp(\mathcal{V}y), \quad (4.1)$$

where, for brevity, we have introduced the notation  $\mathcal{V} = \mathcal{V}(\theta)$  defined by

$$\mathcal{V} = \frac{BVh}{1 + Bh}, \quad (4.2)$$

<sup>†</sup> Note that Moffatt (2) gave slightly inaccurate values of  $M_{c0}$  and  $H_{c0}$ , corresponding to  $M_{c0} = 4.428$  and  $H_{c0} = 0.577$  in the present notation.

so that (2.10) yields the azimuthal velocity

$$u = 1 - \frac{h^2 \cos \theta}{\mathcal{V}^2} [\mathcal{V} - 1 + (1 - \mathcal{V}(1 - y)) \exp(-\mathcal{V}y)], \quad (4.3)$$

(2.11) yields the stream function

$$\psi = hy - \frac{h^3 \cos \theta}{\mathcal{V}^3} [(\mathcal{V} - 1)(\mathcal{V}y - 1) + 1 - (2 - \mathcal{V}(1 - y)) \exp(-\mathcal{V}y)] \quad (4.4)$$

and (2.16) yields the fluidity

$$f = \frac{3}{\mathcal{V}^3} [(\mathcal{V} - 1)^2 + 1 - 2 \exp(-\mathcal{V})]. \quad (4.5)$$

Note that the azimuthal velocity profile (4.3) is, in general, very different from the simple parabolic velocity profile in the constant-viscosity case (3.1), and that the fluidity (4.5) is a monotonically decreasing function of  $\mathcal{V}$  satisfying  $f = O(\exp(-\mathcal{V})/(-\mathcal{V})^3) \rightarrow \infty$  as  $\mathcal{V} \rightarrow -\infty$ ,  $f = 1 + O(\mathcal{V})$  as  $\mathcal{V} \rightarrow 0$ , and  $f = O(1/\mathcal{V}) \rightarrow 0^+$  as  $\mathcal{V} \rightarrow \infty$ .

The contours of the flux  $Q$  given by (2.15) and (4.5) are qualitatively similar to those in the constant-viscosity case shown in Figure 2, and hence, as in the constant-viscosity case, there is a critical flux  $Q_c = Q_c(B, V)$  with a corresponding critical load  $M_c = M_c(B, V)$  (both of which, of course, now depend on both  $B$  and  $V$ ) such that full-film solutions exist only for  $0 < Q \leq Q_c$  (i.e. for  $0 < M \leq M_c$ ). For future reference, the corresponding critical solution (with  $Q = Q_c$  and  $M = M_c$ ) is denoted by  $\mu = \mu_c$ ,  $h = h_c$ ,  $u = u_c$ ,  $\psi = \psi_c$ ,  $T = T_c$  and  $f = f_c$ .

## 5. The Critical Solution

In this Section we consider the properties and behaviour of the non-constant-viscosity critical solution with load  $M = M_c$  introduced in Section 4 such that full-film solutions exist only for  $0 < M \leq M_c$ . Not only is this generalisation of the classical constant-viscosity critical solution of some interest in its own right, but, as we shall see subsequently in Section 6, it is crucial to understanding the properties and behaviour of solutions with a prescribed load.

As Figure 2 illustrates, the unique feature of the critical solution is that  $Q$  has a saddle point at  $\theta = 0$  and  $h = h_c(0)$ , at which point  $\partial Q/\partial \theta = 0$  and  $\partial Q/\partial h = 0$ , which leads to the criticality condition

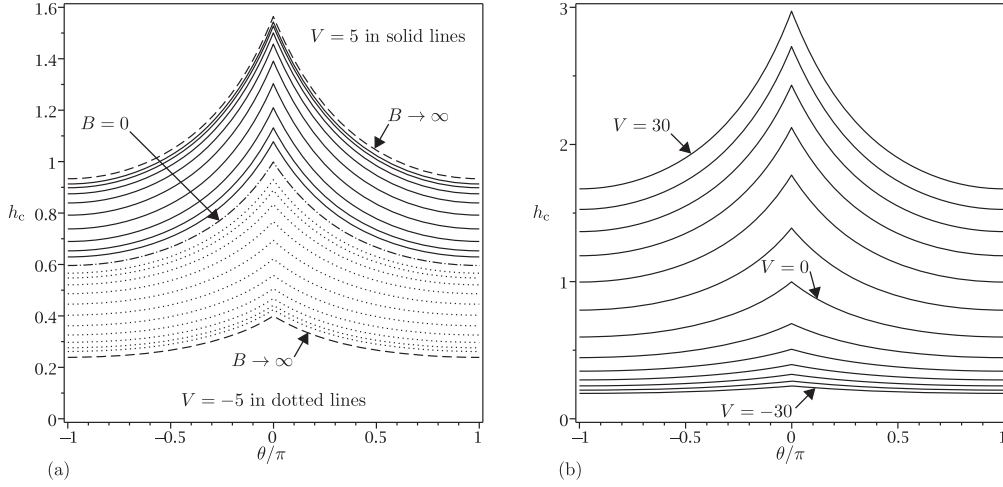
$$\frac{d}{dh} \left( \frac{fh^3}{3} \right) = 1 \quad (5.1)$$

evaluated at  $\theta = 0$ . For the exponential viscosity model (4.1) the criticality condition (5.1) yields

$$\mathcal{V}^2 + 2\mathcal{V}(V - 2) - 2V + 6 - B^2V(V - \mathcal{V})^2 - 2(\mathcal{V} - V + 3) \exp(-\mathcal{V}) = 0 \quad (5.2)$$

evaluated at  $\theta = 0$ , where  $\mathcal{V}$  is given by (4.2). Solving (5.2) yields the value of  $\mathcal{V}(0)$  and hence the value of  $h_c(0)$ , and then (2.15) with  $f_c$  given by (4.5) yields the value of  $Q_c$  and the solution for  $h_c$ . The solutions for  $u_c$ ,  $\psi_c$ ,  $T_c$  and  $M_c$  are then given by (4.3), (4.4), (2.13)





**Fig. 3** Critical film thickness  $h_c$  plotted as a function of  $\theta/\pi$  for (a)  $B = 0$  (dash-dot line) and  $B = 10^n$  ( $n = -1, -0.75, -0.5, \dots, 1.5$ ) for  $V = -5$  (dotted lines) and  $B = 10^n$  ( $n = -1, -0.75, -0.5, \dots, 1$ ) for  $V = 5$  (solid lines) and the leading-order asymptotic solution in the limit  $B \rightarrow \infty$  for  $V = -5$  and  $V = 5$  (dashed lines), and (b)  $V = -30, -25, -20, \dots, 30$  for  $B = 1$ .

and (2.17), respectively. As in the constant-viscosity case, the critical film thickness  $h_c$  has a corner at  $\theta = 0$  (with corresponding corners in all of the other streamlines  $\psi_c = \text{constant}$  also at  $\theta = 0$ ) given by  $h_c = h_c(0) - H_c|\theta| + O(\theta^2)$ , where the magnitude of the slope of the corner, denoted by  $H_c = H_c(B, V)$  (which, of course, now depends on both  $B$  and  $V$ ) is given by

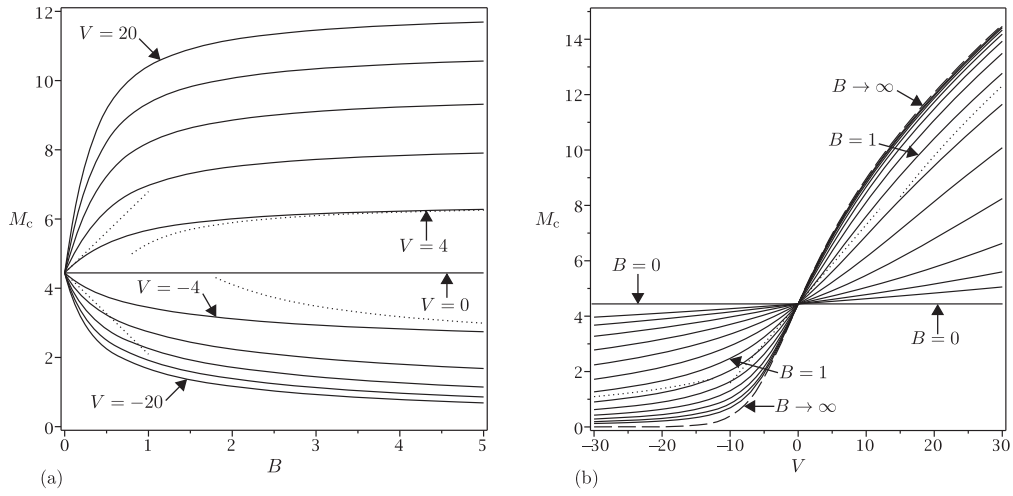
$$H_c = \left( \frac{V^2 [(\mathcal{V} - 1)^2 + 1 - 2\exp(-\mathcal{V})]}{2B^2(V - \mathcal{V})^2[V^2 + 2V(\mathcal{V} - 2) - 2\mathcal{V} + 6 - [(V - \mathcal{V} - 2)^2 + 2]\exp(-\mathcal{V})]} \right)^{1/2} \quad (5.3)$$

evaluated at  $\theta = 0$ . In fact, rather unexpectedly, a general version of (5.3) valid for any viscosity model can be also obtained and is given by

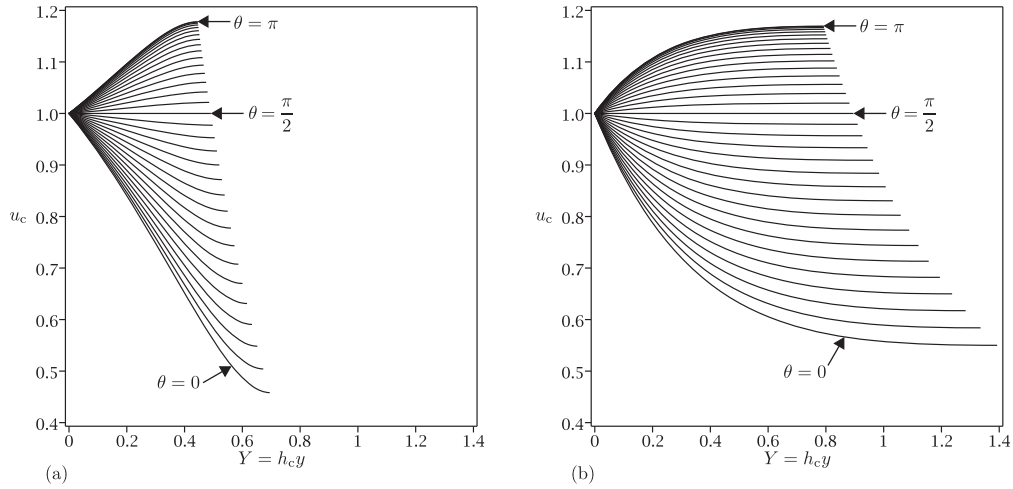
$$H_c = \left[ \frac{\int_{\bar{T}}^1 \frac{(T - \bar{T})^2}{\mu(T)} dT}{2 \int_{\bar{T}}^1 \frac{T(3T - 2\bar{T})}{\mu(T)} dT \int_{\bar{T}}^1 \frac{(T - \bar{T})(3T - \bar{T})}{\mu(T)} dT} \right]^{1/2}, \quad (5.4)$$

where  $\bar{T} = T_c(1, 0) = 1/(1 + Bh_c(0))$  is the critical free-surface temperature evaluated at  $\theta = 0$ .

Figure 3 shows the critical film thickness  $h_c$  plotted as a function of  $\theta/\pi$  for a range of values of (a)  $B$  and (b)  $V$ . In particular, Figure 3 shows that  $h_c$  is a decreasing function of  $|\theta|$ , an increasing (decreasing) function of  $B$  for positive (negative)  $V$ , and an increasing function of  $V$ . Figure 4 shows the critical load  $M_c$  plotted as a function of (a)  $B$  for a range of values of  $V$  and (b)  $V$  for a range of values of  $B$ . In particular, Figure 4 shows that  $M_c$  is



**Fig. 4** Critical load  $M_c$  plotted as a function of (a)  $B$  for  $V = -20, -16, -12, \dots, 20$  with the asymptotic solutions in the limits  $B \rightarrow 0^+$  and  $B \rightarrow \infty$  for  $V = -4$  and  $4$  (dotted lines), and (b)  $V$  for  $B = 0$  and  $B = 10^n$  ( $n = -1.5, -1.25, -1, \dots, 1.5$ ) with the asymptotic solutions in the limits  $V \rightarrow 0$ ,  $V \rightarrow \infty$  and  $V \rightarrow -\infty$  for  $B = 1$  (dotted lines) and the leading-order asymptotic solution in the limit  $B \rightarrow \infty$  (dashed line).



**Fig. 5** Critical velocity profiles  $u_c$  plotted as a function of  $Y = h_c y$  at  $\theta = 0, \pi/32, \pi/16, \dots, \pi$  when  $B = 1$  for (a)  $V = -5$  and (b)  $V = 5$ .

an increasing (decreasing) function of  $B$  for positive (negative)  $V$ , and increasing function of  $V$ .

Figure 5 shows critical velocity profiles  $u_c$  at various values of  $\theta$  in the range  $0 \leq \theta \leq \pi$  when  $B = 1$  for (a)  $V = -5$  and (b)  $V = 5$ . The corresponding profiles in the range

$-\pi < \theta < 0$  follow immediately from the top-to-bottom symmetry of the flow. In particular, Figure 5 shows that  $u_c$  is an increasing function of  $|\theta|$ , a decreasing (increasing) function of  $y$  on the right-hand side  $0 \leq |\theta| < \pi/2$  (the left-hand side  $\pi/2 < |\theta| \leq \pi$ ) of the cylinder, and a uniform plug flow  $u_c \equiv 1$  at the top and bottom of the cylinder  $|\theta| = \pi/2$ .

Leslie (46) gives plots of the critical film thickness at  $\theta = 0$ ,  $h_c(0)$ , and the magnitude of the slope of the corner in the critical film thickness at  $\theta = 0$ ,  $H_c$ , as well as the critical free-surface velocity and temperature at  $\theta = 0$ ,  $u_c(1, 0)$  and  $T_c(1, 0)$ , as functions of both  $B$  and  $V$ , all of which are omitted here for brevity.

In order to complete our understanding of the effects of varying  $B$  and  $V$  on the critical solution, in the following Subsections we analyse the behaviour of the critical solution in the asymptotic limits  $B \rightarrow 0^+$ ,  $B \rightarrow \infty$ ,  $V \rightarrow 0$ ,  $V \rightarrow \infty$  and  $V \rightarrow -\infty$ .

### 5.1 The Limit of Weak Heat Transfer $B \rightarrow 0^+$

At leading order in the limit of weak heat transfer at the free surface,  $B \rightarrow 0^+$ , the free surface is perfectly insulated (i.e.  $\partial T_c / \partial y = 0$  on  $y = 1$ ) and the fluid film is isothermal with constant temperature  $T_c \equiv 1$ , viscosity  $\mu_c \equiv 1$  and fluidity  $f_c \equiv 1$ . Hence the leading-order solutions for  $h_c$ ,  $H_c$ ,  $M_c$  and  $u_c$  are simply the constant-viscosity solutions  $h_{c0}$ ,  $H_{c0}$ ,  $M_{c0}$  and  $u_{c0}$  given in Section 3. The effect of variations in  $B$  first appear at  $O(B)$ , to which order the solutions for  $h_c$ ,  $H_c$ ,  $M_c$ ,  $u_c$  and  $T_c$  are given by

$$h_c = h_{c0} + \frac{BVh_{c0}}{24}(3h_{c0} + 1) + O(B^2), \quad (5.5)$$

$$H_c = H_{c0} + \frac{7BV}{24\sqrt{6}} + O(B^2), \quad (5.6)$$

$$M_c = M_{c0} + \frac{BV}{24}(3C_{B0} + M_{c0}) + O(B^2), \quad (5.7)$$

$$u_c = u_{c0} - \frac{BVh_{c0}^2 y \cos \theta}{24} [h_{c0}(8y^2 - 15y + 6) + 2 - y] + O(B^2) \quad (5.8)$$

and

$$T_c = 1 - Bh_{c0}y - \frac{B^2 h_{c0} y}{24} [V(3h_{c0} + 1) - 24h_{c0}] + O(B^3), \quad (5.9)$$

where the constant  $C_{B0}$  is given by

$$C_{B0} = 2 \int_0^\pi h_{c0}^2 d\theta \simeq 3.21962. \quad (5.10)$$

Note that the solutions (5.5)–(5.9) are valid for a general viscosity model satisfying  $\mu = 1$  and  $d\mu/dT = -V$  when  $T_c = 1$  to the order shown (but not to higher orders). The same is true for the asymptotic solution in the limit of weak thermoviscosity  $V \rightarrow 0$  presented subsequently in Subsection 5.3. The solution (5.9) shows that the effect of weak heat transfer at the free surface is to slightly decrease the temperature from its constant isothermal value  $T_c \equiv 1$  throughout the fluid film, and thus to slightly increase (decrease) the viscosity from its constant isothermal value  $\mu_c \equiv 1$ , causing  $h_c$ ,  $H_c$  and  $M_c$  to slightly increase (decrease) from their isothermal values when  $V > 0$  ( $V < 0$ ). Furthermore, since the sign of the first-order term in (5.8) is simply the sign of  $-V \cos \theta$ , the effect of weak heat transfer at the free surface is to slightly decrease (increase)  $u_c$  from its isothermal value  $u_{c0}$  when  $V \cos \theta > 0$  ( $V \cos \theta < 0$ ). The asymptotic result (5.7) is included in Figure 4(a).

### 5.2 The Limit of Strong Heat Transfer $B \rightarrow \infty$

At leading order in the limit of strong heat transfer at the free surface,  $B \rightarrow \infty$ , the free surface is at the same uniform temperature as the atmosphere (i.e.  $T_c = 0$  on  $y = 1$ ) and the fluid film has non-constant temperature  $T_c = \hat{T} = 1 - y$  and viscosity  $\mu_c = \hat{\mu} = \mu(\hat{T})$ . As Duffy and Wilson (44) showed, from (2.10) and (2.16) the leading-order solutions for  $u_c$  and  $f_c$ , denoted by  $\hat{u}$  and  $\hat{f}$ , are given by

$$\hat{u} = 1 - \hat{h}^2 \cos \theta \int_{\hat{T}}^1 \frac{T}{\mu(T)} dT \quad (5.11)$$

and

$$\hat{f} = 3 \int_0^1 \frac{T^2}{\mu(T)} dT, \quad (5.12)$$

where  $\hat{h}$  denotes the leading-order solution for  $h_c$ . Closed-form expressions for  $\hat{f}$  for the linear, exponential and Eyring viscosity models are given by Wilson and Duffy (41). Moreover, since, for any viscosity model,  $\hat{f} = \hat{f}(V)$  is a constant (and not a function of  $\theta$  as, in general,  $f$  is) the leading-order solutions for  $h_c$ ,  $H_c$  and  $M_c$ , denoted by  $\hat{h}$ ,  $\hat{H}$  and  $\hat{M}$ , are given simply by rescaling the corresponding constant-viscosity solutions  $h_{c0}$ ,  $H_{c0}$ ,  $Q_{c0}$  and  $M_{c0}$  according to  $\hat{h} = \hat{f}^{-1/2} h_{c0}$ ,  $\hat{H} = \hat{f}^{-1/2} H_{c0} = (6\hat{f})^{-1/2}$ ,  $\hat{Q} = \hat{f}^{-1/2} Q_{c0}$  and  $\hat{M} = \hat{f}^{-1/2} M_{c0}$ . For the exponential viscosity model  $\mathcal{V} \sim V$  in the limit  $B \rightarrow \infty$ , and so the leading-order solutions for  $\mu_c$ ,  $u_c$ ,  $\psi_c$  and  $f_c$  are given simply by (4.1), (4.3)–(4.5) with  $\mathcal{V}$  replaced by  $V$ , respectively.

As might have been anticipated, this simple re-scaling property does not extend to higher orders. Specifically, to higher order the solutions for  $h_c$ ,  $H_c$ ,  $M_c$ ,  $u_c$  and  $T_c$  are given by

$$h_c = \hat{h} + \frac{3 - (V + 3)\hat{f}}{3B\hat{f}} + O\left(\frac{1}{B^2}\right), \quad (5.13)$$

$$H_c = \hat{H} + \frac{V^2 \hat{f}^2 + 3(V + 6)\hat{f} - 18}{3B^2(6\hat{f})^{3/2}} + O\left(\frac{1}{B^3}\right), \quad (5.14)$$

$$M_c = \hat{M} + \frac{2\pi[3 - (V + 3)\hat{f}]}{3B\hat{f}} + O\left(\frac{1}{B^2}\right), \quad (5.15)$$

$$\begin{aligned} u_c = \hat{u} + \frac{h_{c0} \cos \theta}{3V^2 B \hat{f}^{3/2}} & \left[ \left\{ V(5 - 3y)\hat{f} - 6 \right\} Vy \exp(-Vy) \right. \\ & \left. + \left\{ V(2V + 1)\hat{f} - 6(V - 1) \right\} (1 - \exp(-Vy)) \right] + O\left(\frac{1}{B^2}\right) \end{aligned} \quad (5.16)$$

and

$$T_c = \hat{T} + \frac{\hat{f}^{1/2} y}{B h_{c0}} - \frac{(3 - V\hat{f})y}{3B^2 h_{c0}^2} + O\left(\frac{1}{B^3}\right). \quad (5.17)$$

The solution (5.17) shows that the effect of large-but-finite heat transfer at the free surface is to slightly increase the temperature from its leading-order value  $T_c = \hat{T} = 1 - y$  throughout

the fluid film, and thus to slightly decrease (increase) the viscosity from its leading-order value  $\mu_c = \hat{\mu} = \exp(Vy)$ , causing  $h_c$ ,  $H_c$  and  $M_c$  to slightly decrease (increase) from their leading-order values when  $V > 0$  ( $V < 0$ ). The asymptotic result (5.15) is included in Figure 4(a).

### 5.3 The Limit of Weak Thermoviscosity $V \rightarrow 0$

At leading order in the limit of weak thermoviscosity,  $V \rightarrow 0$ , the fluid film has non-constant temperature  $T_c \neq 1$  but constant viscosity  $\mu_c \equiv 1$  and fluidity  $f_c \equiv 1$ . Hence, like in the limit  $B \rightarrow 0^+$ , the leading-order solutions for  $h_c$ ,  $H_c$ ,  $M_c$  and  $u_c$  are simply the constant-viscosity solutions  $h_{c0}$ ,  $H_{c0}$ ,  $M_{c0}$  and  $u_{c0}$  given in Section 3. The effect of variations in  $V$  first appear at  $O(V)$ , to which order the solutions for  $h_c$ ,  $H_c$ ,  $M_c$ ,  $u_c$  and  $T_c$  are given by

$$h_c = h_{c0} + \frac{BVh_{c0}[3(1+B)h_{c0}+1]}{24(1+B)(1+Bh_{c0})} + O(V^2), \quad (5.18)$$

$$H_c = H_{c0} + \frac{BV(3B^2+9B+7)}{24\sqrt{6}(1+B)^3} + O(V^2), \quad (5.19)$$

$$M_c = M_{c0} + \frac{BV}{12(1+B)} \int_0^\pi \frac{h_{c0}[3(1+B)h_{c0}+1]}{1+Bh_{c0}} d\theta + O(V^2), \quad (5.20)$$

$$u_c = u_{c0} - \frac{BVh_{c0}^2 y \cos \theta [(1+B)h_{c0}(8y^2-15y+6)+2-y]}{24(1+B)(1+Bh_{c0})} + O(V^2) \quad (5.21)$$

and

$$T_c = 1 - \frac{Bh_{c0}y}{1+Bh_{c0}} - \frac{B^2Vh_{c0}y[3(1+B)h_{c0}+1]}{24(1+B)(1+Bh_{c0})^3} + O(V^2). \quad (5.22)$$

The solutions in this limit are somewhat similar to those in the limit  $B \rightarrow 0^+$  described previously in Subsection 5.1 and have the corresponding physical interpretation. There is, however, one noteworthy difference. Whereas in the limit  $B \rightarrow 0^+$  the sign of the first-order term in (5.8) is simply the sign of  $-V \cos \theta$ , in the present limit it is the sign of the somewhat more complicated expression  $-V \cos \theta [(1+B)h_{c0}(8y^2-15y+6)+2-y]$ . Thus for  $0 < h_{c0} \leq 1/(1+B)$  and for  $0 < y < y_{c0}$  when  $1/(1+B) < h_{c0} \leq 1$  the effect of weak thermoviscosity is to slightly decrease (increase)  $u_c$  from its isothermal value  $u_{c0}$  when  $V \cos \theta > 0$  ( $V \cos \theta < 0$ ), whereas for  $y_{c0} < y \leq 1$  when  $1/(1+B) < h_{c0} \leq 1$  the effect is the opposite, where  $y = y_{c0}$  ( $0 < y_{c0} \leq 1$ ) satisfies  $(1+B)h_{c0}(8y^2-15y+6)+2-y = 0$ . This behaviour is the first hint of the occurrence of the backflow described subsequently in Section 7. The asymptotic result (5.20) is included in Figure 4(b).

### 5.4 The Limit of Strong Positive Thermoviscosity $V \rightarrow \infty$

In the limit of strong positive thermoviscosity,  $V \rightarrow \infty$ , the solutions for  $h_c$ ,  $H_c$ ,  $M_c$ ,  $u_c$  and  $T_c$  are given by

$$h_c = \frac{V^{1/2}h_{c0}}{\sqrt{3}} - \frac{1}{3B} + O\left(\frac{1}{V^{1/2}}\right), \quad (5.23)$$

$$H_c = \frac{V^{1/2}}{3\sqrt{2}} + \frac{\sqrt{2}(3B^2-1)}{18B^2V^{1/2}} + O\left(\frac{1}{V}\right), \quad (5.24)$$

$$M_c = \frac{V^{1/2}M_{c0}}{\sqrt{3}} - \frac{2\pi}{3B} + O\left(\frac{1}{V^{1/2}}\right), \quad (5.25)$$

$$u_c = 1 - \frac{h_{c0}^2 \cos \theta}{3} [1 - \exp(-Vy)] - \frac{\sqrt{3}h_{c0} \cos \theta}{9BV^{1/2}} [1 - (3Vy + 1) \exp(-Vy)] + O\left(\frac{1}{V}\right) \quad (5.26)$$

and

$$T_c = 1 - y + \frac{\sqrt{3}y}{BV^{1/2}h_{c0}} - \frac{2y}{B^2Vh_{c0}^2} + O\left(\frac{1}{V^{3/2}}\right). \quad (5.27)$$

These solutions show that at leading order in the limit of strong positive thermoviscosity the temperature is given by  $T_c = 1 - y$  and the viscosity  $\mu_c = O(\exp(Vy))$  is exponentially large outside a narrow boundary layer of width  $O(1/V) \ll 1$  near the cylinder  $y = 0$ , resulting in a uniform plug flow  $u_c \equiv 1 - h_{c0}^2 \cos \theta / 3$  outside the boundary layer and a large film thickness and load of  $O(V^{1/2}) \gg 1$ . The asymptotic result (5.25) is included in Figure 4(b).

### 5.5 The Limit of Strong Negative Thermoviscosity $V \rightarrow -\infty$

In the limit of strong negative thermoviscosity,  $V \rightarrow -\infty$ , the solutions for  $h_c$ ,  $H_c$ ,  $M_c$ ,  $u_c$  and  $T_c$  are given by

$$h_c = \frac{1}{B(-V)} \left[ \log\left(\frac{B^2V^2}{2}\right) - W_{V\infty} - 1 \right] + O\left(\frac{\log(-V)^2}{V^2}\right), \quad (5.28)$$

$$H_c = \frac{1}{B(-V)} + O\left(\frac{\log(-V)}{V^2}\right), \quad (5.29)$$

$$M_c = \frac{1}{B(-V)} \left[ 2\pi \log\left(\frac{B^2V^2}{2}\right) - C_{V\infty} - 2\pi \right] + O\left(\frac{\log(-V)^2}{V^2}\right), \quad (5.30)$$

$$u_c = 1 - \frac{\cos \theta}{B^2V^2} \left[ \left(-\frac{B^2V^2W_{V\infty}}{2 \cos \theta}\right)^y \left\{ \left[ \log\left(\frac{B^2V^2}{2}\right) - W_{V\infty} \right] (1 - y) + y \right\} - \log\left(\frac{B^2V^2}{2}\right) + W_{V\infty} \right] + O\left(\frac{\log(-V)^2}{-V}\right) \quad (5.31)$$

and

$$T_c = 1 - \frac{y}{(-V)} \left[ \log\left(\frac{B^2V^2}{2}\right) - W_{V\infty} - 1 \right] + O\left(\frac{\log(-V)}{V^2}\right), \quad (5.32)$$

where the function  $W_{V\infty} = W_{V\infty}(\theta)$  is given by

$$W_{V\infty} = W_0 \left(-\frac{\cos \theta}{e}\right) \quad (5.33)$$

and the constant  $C_{V\infty}$  is given by

$$C_{V\infty} = 2 \int_0^\pi W_{V\infty} d\theta \simeq -0.73144, \quad (5.34)$$

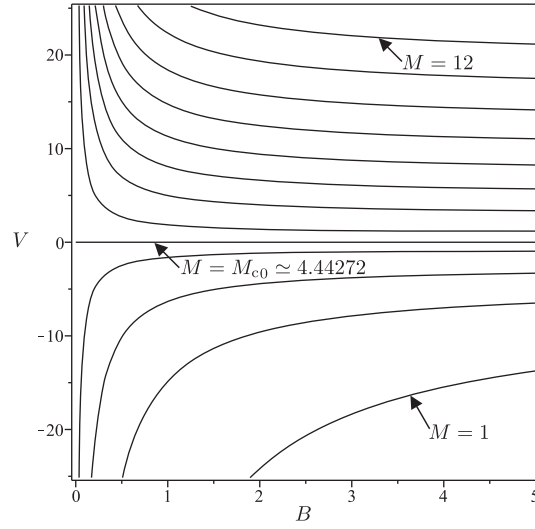
in which  $W_0 = W_0(x)$  denotes the principal real branch of the Lambert W function<sup>†</sup>, and it can be shown that the sign of  $W_{V_\infty}$  is the same as the sign of  $-\cos\theta$ . The solutions (5.28)–(5.32) show that at leading order in the limit of strong negative thermoviscosity the temperature is given by  $T_c \equiv 1$  and the viscosity  $\mu = O(V^{-2y})$  decreases from  $O(1)$  at the cylinder  $y = 0$  to  $O((-V)^{-2}) \ll 1$  at the free surface  $y = 1$ , resulting in a velocity that increases (decreases) from  $u_c = 1$  at the cylinder [where there is a narrow boundary layer of width  $O(1/\log(-V)) \ll 1$ ] to  $u_c = 1 + W_{V_\infty}/2$  at the free surface [where there is another narrow boundary layer also of width  $O(1/\log(-V)) \ll 1$ ] when  $\cos\theta < 0$  ( $\cos\theta > 0$ ), and a small film thickness and load of  $O(\log(-V)/(-V)) \ll 1$ . The asymptotic result (5.30) is included in Figure 4(b).

## 6. Full-Film Solution with a Prescribed Load

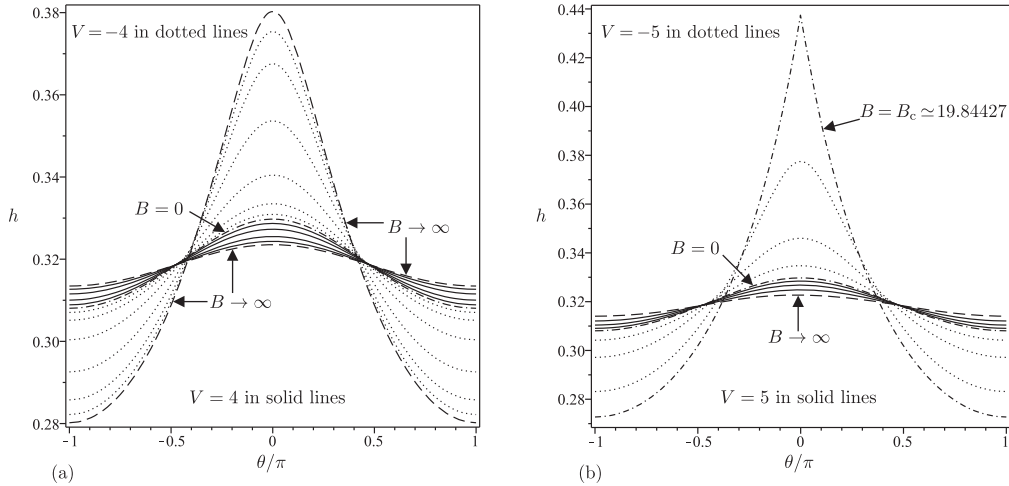
In this Section we consider the properties and behaviour of the non-constant-viscosity full-film solution with a prescribed load  $M$ . As described in Section 4, such solutions exist only when  $0 < M \leq M_c$ , where  $M_c$  is the load of the critical solution described in Section 5 and shown in Figure 4. Thus for a prescribed value of  $M$  such solutions exist, in general, only for restricted ranges of values of  $B$  and  $V$ , and the critical values of  $B$  and  $V$  for such solutions to exist, denoted by  $B_c$  and  $V_c$ , are precisely the values of  $B$  and  $V$  for which the critical solution has load  $M_c = M$ . The ranges of values of  $B$  and  $V$  depend on the relative size of the prescribed load  $M$  compared to the load of the critical solution in the limit  $B \rightarrow 0^+$  (i.e. in the constant-viscosity case), namely  $M_{c0} \simeq 4.44272$ , and the load of the critical solution in the limit  $B \rightarrow \infty$ , namely  $\hat{M} = \hat{f}^{-1/2}M_{c0}$ , where  $\hat{f} = \hat{f}(V)$  is given by (5.12), described previously. Specifically, as Figure 4(a) shows, for positive  $V$  there are no full-film solutions for any value of  $B$  when  $M \geq \hat{M}$ , full-film solutions for  $B \geq B_c$  when  $M_{c0} < M < \hat{M}$ , and full-film solutions for all values of  $B$  when  $M \leq M_{c0}$ , while for negative  $V$  there are no full-film solutions for any value of  $B$  when  $M > M_{c0}$ , full-film solutions for  $B \leq B_c$  when  $\hat{M} < M \leq M_{c0}$ , and full-film solutions for all values of  $B$  when  $M \leq \hat{M}$ . In particular, if we denote the value of  $V$  satisfying  $M = \hat{M}$  by  $V = V_\infty$ , then there are full-film solutions in the limit  $B \rightarrow \infty$  for  $V \geq V_\infty$ , but not for  $V < V_\infty$ . Similarly, as Figure 4(b) shows, for all values of  $B$  there are full-film solutions for  $V \geq V_c$ , but not for  $V < V_c$ . This behaviour is summarised in Figure 6, which shows the critical curves  $(B_c, V_c)$  for a range of values of  $M$ . For each value of  $M$  the corresponding critical curve  $(B_c, V_c)$  divides the  $B$ - $V$  parameter plane into the region above the curve in which full-film solutions exist and the region below the curve in which they do not exist.

Figure 7 shows the film thickness  $h$  when  $M = 2 (< M_{c0})$  plotted as a function of  $\theta/\pi$  for a range of values of  $B$  for (a)  $V = \pm 4$  and (b)  $V = \pm 5$ . Figure 7 shows that when  $M = 2$  for  $V = 5$ ,  $V = 4$  and  $V = -4$  (all of which satisfy  $V > V_\infty \simeq -4.51567$ ) there are full-film solutions for all values of  $B$ , but for  $V = -5 < V_\infty$  there are full-film solutions only for  $B \leq B_c \simeq 19.84427$ . Figure 8(a) shows the film thickness  $h$  plotted as a function of  $\theta/\pi$  for a range of values of  $B$  when  $M = 6 (> M_{c0})$  and  $V = 5$ , in which case there are full-film solutions for  $B \geq B_c \simeq 0.99344$ . Note that there is no corresponding plot for  $M = 6 (> M_{c0})$  and  $V = -5$  because there are no full-film solutions when  $M > M_{c0}$  and

<sup>†</sup> The Lambert W function, which is defined to be the solution for  $W = W(x)$  of  $W \exp(W) = x$ , has two real branches, namely the principal real branch  $W_0(x)$  with domain  $[-1/e, \infty)$  and range  $[-1, \infty)$  and the lower real branch  $W_{-1}(x)$  with domain  $[-1/e, 0)$  and range  $(-\infty, -1]$ .

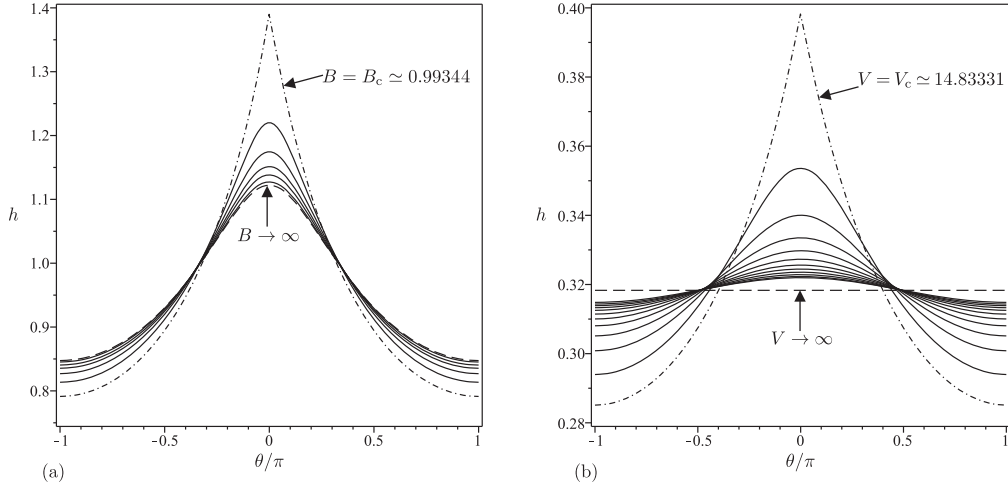


**Fig. 6** The critical curves  $(B_c, V_c)$  for  $M = 1, 2, 3, \dots, 12$  and  $M = M_{c0} \simeq 4.44272$ . For each value of  $M$  the corresponding curve  $(B_c, V_c)$  divides the  $B$ - $V$  parameter plane into the region above the curve in which full-film solutions exist and the region below the curve in which they do not exist.



**Fig. 7** Film thickness  $h$  when  $M = 2 (< M_{c0})$  plotted as a function of  $\theta/\pi$  for (a)  $B = 0$  (dash-dash-dotted line),  $B = 10^n$  ( $n = -0.5, 0, 0.5, 1, 1.5, 2$ ) for  $V = -4$  (dotted lines) and  $B = 10^n$  ( $n = -0.5, 0, 0.5, 1$ ) for  $V = 4$  (solid lines) together with the leading-order asymptotic solution in the limit  $B \rightarrow \infty$  for  $V = -4$  and  $V = 4$  (dashed lines), and (b)  $B = 0$  (dash-dash-dotted line),  $B = 10^n$  ( $n = 0, 0.5, 1$ ) for  $V = -5$  (dotted lines) and  $B = 10^n$  ( $n = -0.5, 0, 0.5$ ) for  $V = 5$  (solid lines) together with the leading-order asymptotic solution in the limit  $B \rightarrow \infty$  for  $V = 5$  (dashed line) and the critical solution for  $V = -5$  with  $B = B_c \simeq 19.84427$  (dash-dotted line).





**Fig. 8** Film thickness  $h$  plotted as a function of  $\theta/\pi$  for (a)  $B = 10^n$  ( $n = 0.25, 0.5, 0.75, 1, 1.25$ ) (solid lines) together with the leading-order asymptotic solution in the limit  $B \rightarrow \infty$  (dashed line) and the critical solution with  $B = B_c \simeq 0.99344$  (dash-dotted line) when  $M = 6 (> M_{c0})$  and  $V = 5$ , and (b)  $V = -12, -8, -4, \dots, 20$  (solid lines) together with the leading-order asymptotic solution in the limit  $V \rightarrow \infty$  (i.e.  $h \simeq 0.31831$ ) (dashed line) and the critical solution with  $V = V_c \simeq -14.83331$  (dash-dotted line) when  $M = 2 (< M_{c0})$  and  $B = 1$ .

$V < 0$ . Figure 8(b) shows the film thickness  $h$  plotted as a function of  $\theta/\pi$  for a range of values of  $V$  when  $M = 2 (< M_{c0})$  and  $B = 1$ , in which case there are full-film solutions for  $V \geq V_c \simeq -14.83331 (< 0)$ . Note that the corresponding plot for  $M = 6 (> M_{c0})$  (which is omitted for brevity) is qualitatively similar to Figure 8(b) except that  $V_c > 0$ . Figures 7 and 8 also show that  $h$  is a decreasing function of  $|\theta|$ , but, in contrast to the corresponding results for  $h_c$  shown in Figure 3, show that near  $\theta = 0$  the film thickness  $h$  is a decreasing (increasing) function of  $B$  for positive (negative)  $V$  and a decreasing function of  $V$ , but that near  $\theta = \pi$  it behaves in the opposite way in order to satisfy the condition of prescribed load.

The velocity profiles  $u$  are qualitatively similar to the critical velocity profiles shown in Figure 5.

Leslie (46) gives plots of the film thickness, free-surface velocity and free-surface temperature at  $\theta = 0$  and  $\theta = \pi$  as functions of both  $B$  and  $V$ , all of which are again omitted here for brevity.

In the following Subsections we analyse the behaviour of the full-film solution with a prescribed load in the asymptotic limits  $B \rightarrow 0^+$ ,  $B \rightarrow \infty$ ,  $V \rightarrow 0$ ,  $V \rightarrow \infty$  and  $M \rightarrow 0$ . Note that (unlike in the corresponding analysis of the critical solution described in Subsection 5.5) there is no solution with prescribed load in the limit  $V \rightarrow -\infty$ , and, of course, there is no solution with prescribed load in the limit  $M \rightarrow \infty$ . Moreover, there are solutions with a prescribed load in the limits  $B \rightarrow 0^+$  and  $V \rightarrow 0$  only when  $M \leq M_{c0}$ .

### 6.1 *The Limit of Weak Heat Transfer $B \rightarrow 0^+$*

In the limit of weak heat transfer at the free surface,  $B \rightarrow 0^+$ , the solutions for  $h$ ,  $u$  and  $T$  are given by

$$h = h_0 + \frac{BV(I_{B0} - h_0^4 \cos \theta)}{12(1 - h_0^2 \cos \theta)} + O(B^2), \quad (6.1)$$

$$u = u_0 - \frac{BVh_0y \cos \theta}{12(1 - h_0^2 \cos \theta)} [h_0^2(1 - h_0^2 \cos \theta)(4y^2 - 7y + 2) + (I_{B0} - h_0^2)(2 - y)] + O(B^2) \quad (6.2)$$

and

$$T = 1 - Bh_0y - \frac{B^2y [V(I_{B0} - h_0^4 \cos \theta) - 12h_0^2(1 - h_0^2 \cos \theta)]}{12(1 - h_0^2 \cos \theta)} + O(B^3), \quad (6.3)$$

where the constant  $I_{B0} = I_{B0}(M_0)$  ( $0 < I_{B0} < 1$ ) is given by

$$I_{B0} = \frac{\int_0^\pi \frac{h_0^4 \cos \theta}{1 - h_0^2 \cos \theta} d\theta}{\int_0^\pi \frac{1}{1 - h_0^2 \cos \theta} d\theta}, \quad (6.4)$$

and  $h_0$  and  $u_0$  are the constant-viscosity solutions with load  $M_0 = M (\leq M_{c0})$  given in Section 3. Note that the solutions (6.1)–(6.3) are valid for a general viscosity model satisfying  $\mu = 1$  and  $d\mu/dT = -V$  when  $T = 1$  to the order shown (but not to higher orders). The same is true for the asymptotic solution in the limit of weak thermoviscosity  $V \rightarrow 0$  presented subsequently in Subsection 6.3. Like the corresponding behaviour of the critical solution described in Subsection 5.1, the solution (6.3) shows that the effect of weak heat transfer at the free surface is to slightly decrease the temperature from its constant isothermal value  $T \equiv 1$  throughout the fluid film, and thus to slightly increase (decrease) the viscosity from its constant isothermal value  $\mu \equiv 1$  when  $V > 0$  ( $V < 0$ ). However, unlike the corresponding behaviour of the critical solution, the effect on  $h$  depends on the sign of  $V(I_{B0} - h_0^4 \cos \theta)$  (rather than just  $V$ ). Thus when  $V > 0$  ( $V < 0$ ) the effect of weak heat transfer at the free surface is to slightly decrease (increase)  $h$  from its isothermal value when  $|\theta| < \bar{\theta}$  with the opposite behaviour when  $|\theta| > \bar{\theta}$ , where  $\theta = \bar{\theta}$  ( $0 < \bar{\theta} \leq \pi/2$ ) satisfies  $I_{B0} - h_0^4 \cos \theta = 0$ .

### 6.2 *The Limit of Strong Heat Transfer $B \rightarrow \infty$*

In the limit of strong heat transfer at the free surface,  $B \rightarrow \infty$ , the solutions for  $h$ ,  $u$  and  $T$  are given by

$$h = \hat{h} + \frac{h_1}{B} + O\left(\frac{1}{B^2}\right), \quad (6.5)$$

$$u = \hat{u} + \frac{h_0 \cos \theta}{V^2 B \hat{f}^{1/2}} [\{V(1 - y) - 2(1 + h_1)\} Vy \exp(-Vy) - \{V(1 + 2h_1) - 2(1 + h_1)\} (1 - \exp(-Vy))] + O\left(\frac{1}{B^2}\right) \quad (6.6)$$

and

$$T = \hat{T} + \frac{\hat{f}^{1/2}y}{Bh_0} - \frac{\hat{f}(1+h_1)y}{B^2h_0^2} + O\left(\frac{1}{B^3}\right), \quad (6.7)$$

where  $h_1 = h_1(\theta)$  is given by

$$h_1 = \frac{(I_{B\infty} - h_0^2 \cos \theta)[3 - (V+3)\hat{f}]}{3\hat{f}(1 - h_0^2 \cos \theta)} \quad (6.8)$$

and the constant  $I_{B\infty} = I_{B\infty}(M_0)$  ( $0 < I_{B\infty} < 1$ ) is given by

$$I_{B\infty} = \frac{\int_0^\pi \frac{h_0^2 \cos \theta}{1 - h_0^2 \cos \theta} d\theta}{\int_0^\pi \frac{1}{1 - h_0^2 \cos \theta} d\theta}. \quad (6.9)$$

As in the corresponding analysis of the critical solution described in Subsection 5.2, the leading-order solutions for  $h$  and  $Q$ , denoted by  $\hat{h}$  and  $\hat{Q}$ , are given simply by rescaling the corresponding constant-viscosity solutions  $h_0$  and  $Q_0$  according to  $\hat{h} = \hat{f}^{-1/2}h_0$  and  $\hat{Q} = \hat{f}^{-1/2}Q_0$ , where  $M = \hat{f}^{-1/2}M_0$ , and the leading-order solutions for  $u$  and  $f$ , denoted by  $\hat{u}$  and  $\hat{f}$ , are again given by (5.11) and (5.12), respectively. Like the corresponding behaviour of the critical solution described in Subsection 5.2, the solution (6.7) shows that the effect of large-but-finite heat transfer at the free surface is to slightly increase the temperature from its leading-order value  $T = \hat{T} = 1 - y$  throughout the fluid film, and thus to slightly decrease (increase) the viscosity from its leading-order value  $\mu = \hat{\mu} = \exp(Vy)$  when  $V > 0$  ( $V < 0$ ). However, unlike the corresponding behaviour of the critical solution, the effect on  $h$  depends on the sign of  $(I_{B\infty} - h_0^2 \cos \theta)[3 - (V+3)\hat{f}]$  (rather than just  $3 - (V+3)\hat{f}$ ). Thus when  $V > 0$  ( $V < 0$ ), the effect of large-but-finite heat transfer at the free surface is to slightly increase (decrease)  $h$  from its isothermal value when  $|\theta| < \bar{\theta}$  with the opposite behaviour when  $|\theta| > \bar{\theta}$ , where  $\theta = \bar{\theta}$  ( $0 < \bar{\theta} \leq \pi/2$ ) satisfies  $I_{B\infty} - h_0^2 \cos \theta = 0$ .

### 6.3 The Limit of Weak Thermoviscosity $V \rightarrow 0$

In the limit of weak thermoviscosity,  $V \rightarrow 0$ , the solutions for  $h$ ,  $u$  and  $T$  are given by

$$h = h_0 + \frac{BV[I_{V0}(1+Bh_0) - h_0^4 \cos \theta]}{12(1 - h_0^2 \cos \theta)(1+Bh_0)} + O(V^2), \quad (6.10)$$

$$u = u_0 - \frac{BVh_0y \cos \theta}{12(1 - h_0^2 \cos \theta)(1+Bh_0)} \left\{ h_0^2(1 - h_0^2 \cos \theta)(4y^2 - 7y + 2) + [I_{V0}(1+Bh_0) - h_0^2] (2 - y) \right\} + O(V^2) \quad (6.11)$$

and

$$T = 1 - \frac{Bh_0y}{1+Bh_0} - \frac{B^2Vy [I_{V0}(1+Bh_0) - h_0^4 \cos \theta]}{12(1 - h_0^2 \cos \theta)(1+Bh_0)^3} + O(V^2), \quad (6.12)$$

where the constant  $I_{V_0} = I_{V_0}(M_0, B)$  ( $0 < I_{V_0} < 1$ ) is given by

$$I_{V_0} = \frac{\int_0^\pi \frac{h_0^4 \cos \theta}{(1 - h_0^2 \cos \theta)(1 + Bh_0)} d\theta}{\int_0^\pi \frac{1}{1 - h_0^2 \cos \theta} d\theta}, \quad (6.13)$$

and  $h_0$  and  $u_0$  are the constant-viscosity solutions with load  $M_0 = M (\leq M_{c0})$  given in Section 3. Like the corresponding behaviour of the critical solution described in Subsection 5.3, the solutions in this limit are somewhat similar to those in the limit  $B \rightarrow 0^+$  described previously in Subsection 6.1 and have the corresponding physical interpretation.

#### 6.4 The Limit of Strong Positive Thermoviscosity $V \rightarrow \infty$

In the limit of strong positive thermoviscosity,  $V \rightarrow \infty$ , the solutions for  $h$ ,  $u$  and  $T$  are given by

$$h = \frac{M}{2\pi} + \frac{M^2 \cos \theta (2\pi + BM)}{8\pi^3 BV} + O\left(\frac{1}{V^2}\right), \quad (6.14)$$

$$u = 1 - \frac{M \cos \theta (2\pi + BM)}{4\pi^2 BV} \left[ 1 - \exp\left(-\frac{BVM y}{2\pi + BM}\right) \right] + O\left(\frac{1}{V^2}\right) \quad (6.15)$$

and

$$T = 1 - \frac{BM y}{2\pi + BM} - \frac{M^2 y \cos \theta}{2\pi(2\pi + BM)V} + O\left(\frac{1}{V^2}\right). \quad (6.16)$$

These solutions show that, like the corresponding behaviour of the critical solution described in Subsection 5.4, at leading order in the limit of strong positive thermoviscosity the temperature is given by  $T = 1 - BM y / (2\pi + BM)$  and the viscosity

$$\mu = O\left[\exp\left(\frac{BVM y}{2\pi + BM}\right)\right] \quad (6.17)$$

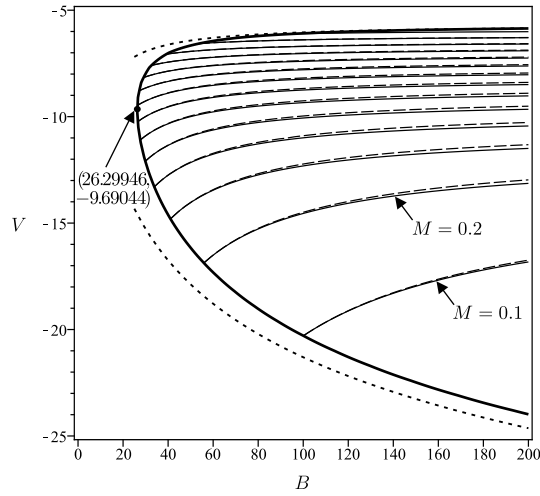
is exponentially large outside a narrow boundary layer of width  $O(1/V) \ll 1$  near the cylinder  $y = 0$ , resulting in a uniform plug flow  $u \equiv 1$  outside the boundary layer. However, unlike the corresponding behaviour of the critical solution, the leading-order film thickness is an  $O(1)$  constant, and the effect of large-but-finite positive thermoviscosity is to slightly decrease  $T$  from its leading-order value throughout the fluid film, and thus to slightly decrease (increase)  $u$  outside the boundary layer and slightly increase (decrease)  $h$ , respectively, from their leading-order values when  $\cos \theta > 0$  ( $\cos \theta < 0$ ).

#### 6.5 The Limit of Small Load $M \rightarrow 0$

In the limit of small load,  $M \rightarrow 0$ , the solutions for  $h$ ,  $u$  and  $T$  are given by

$$h = \frac{M}{2\pi} + \frac{M^3 \cos \theta}{24\pi^3} + O(M^4), \quad (6.18)$$

$$u = 1 - \frac{M^2 \cos \theta (2 - y)y}{8\pi^2} + \frac{M^3 BV \cos \theta (3 - 2y)y^2}{48\pi^3} + O(M^4) \quad (6.19)$$



**Fig. 9** The regions of the  $B$ - $V$  parameter plane in which backflow occurs for  $M = 0.1, 0.2, 0.3, \dots, 1.4$ . For each value of  $M$  the region in which backflow occurs is bounded below by the critical curve  $(B_c, V_c)$  (solid lines) and above by the curve on which  $u(1,0) = 0$  (dashed lines), which meet on the bounding curve on which  $u_c(1,0) = 0$  (thick solid line). Also shown are the asymptotic solutions for the upper and lower branches of the bounding curve given by (7.1) and (7.2), respectively, (dashed lines).

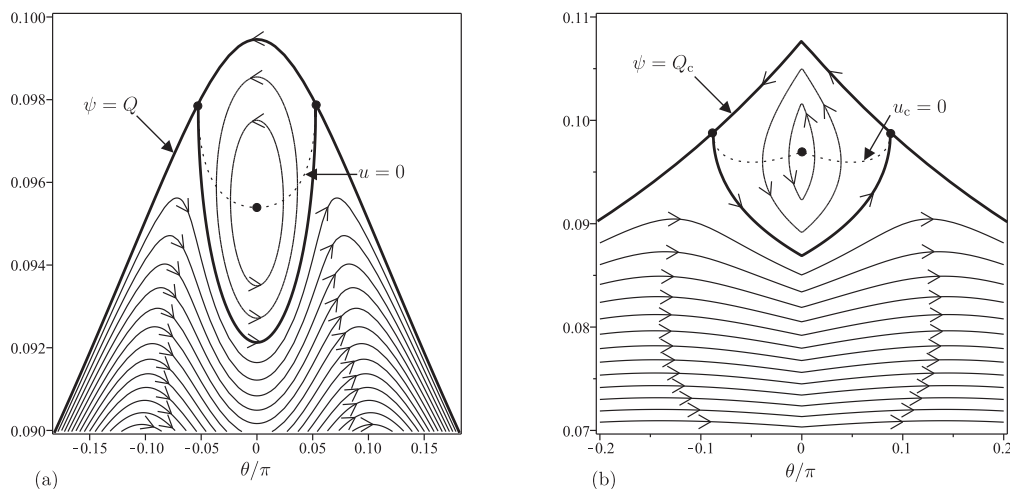
and

$$T = 1 - \frac{MB y}{2\pi} + \frac{M^2 B^2 y}{4\pi^2} + O(M^3). \quad (6.20)$$

These solutions show that, as might have been anticipated, at leading order in the limit of small load the film is isothermal,  $T \equiv 1$ , the viscosity is constant,  $\mu \equiv 1$ , the film thickness is a small  $O(M) \ll 1$  constant, and there is a uniform plug flow  $u \equiv 1$ .

## 7. Backflow

In the special case of constant viscosity the azimuthal velocity  $u$  is always in the same direction as the rotation of the cylinder (i.e.  $u \geq 0$  for all  $0 \leq y \leq 1$ ) and so backflow (i.e.  $u < 0$  somewhere in  $0 < y \leq 1$ ) never occurs. It is therefore interesting to discover that in the general case of non-constant viscosity there is a region of the  $B$ - $V$  parameter plane in which backflow occurs in a region on the right-hand side of the cylinder containing the point on the free surface at  $\theta = 0$  (i.e. the point furthest from the cylinder at the azimuthal location at which the local effect of gravity is strongest). Figure 9 shows the regions of the  $B$ - $V$  parameter plane in which backflow occurs for a range of values of  $M$ . The “bounding” curve corresponding to  $u_c(1,0) = 0$  divides the  $B$ - $V$  parameter plane into the region to the left of the curve in which  $u_c(1,0) > 0$  in which backflow is impossible from the region to the right of the curve in which  $u_c(1,0) < 0$  in which backflow is possible (but may or may not actually occur). As we have already seen, for each prescribed value of  $M$  full-film solutions exist only above the corresponding critical curve  $(B_c, V_c)$  (described in Section 6 and shown in Figure 4). Hence, for sufficiently small prescribed values of  $M$  backflow occurs



**Fig. 10** Details of the streamlines of the flow near  $y = 1$  and  $\theta = 0$  when  $B = 100$  and  $V = -10$  for (a) the solution with prescribed load  $M = 1/2$  ( $Q \simeq 0.07642$ ) in which the streamlines are drawn for  $\psi/Q = 1487/1500$  (the lowest visible streamline),  $1488/1500$ ,  $1489/1500$ ,  $\dots$ ,  $1$  (the stagnation streamline),  $1501/1500$  and  $1502/1500$  (inside the recirculation region), and (b) the critical solution with load  $M = M_c \simeq 0.50800$  ( $Q = Q_c \simeq 0.07716$ ) in which the streamlines are drawn for  $\psi/Q = 87/100$  (the lowest visible streamline),  $88/100$ ,  $89/100$ ,  $\dots$ ,  $1$  (the stagnation streamline),  $101/100$  and  $102/100$  (inside the recirculation region). In both cases the curve on which  $u = 0$  is shown with a dashed line and the stagnation points are shown with large dots.

in the narrow region bounded below by the critical curve  $(B_c, V_c)$  and above by the curve on which  $u(1, 0) = 0$ , and these curves meet on the bounding curve on which  $u_c(1, 0) = 0$ . In the limit  $B \rightarrow \infty$  the upper and lower branches of the bounding curve satisfy

$$V = V_{\max} + \frac{2\sqrt{3}V_{\max}(V_{\max} + 3)}{B(2V_{\max} + 9)(V_{\max} + 6)^{1/2}} + O\left(\frac{1}{B^2}\right) \quad (7.1)$$

where  $V_{\max} \simeq -5.65658$  satisfies  $2V_{\max}^2 - 5V_{\max} + 6 - (V_{\max} + 6)\exp(-V_{\max}) = 0$ , and

$$V = -4 \log B - 4 \log(\log B) + 6 - 4 \log 2 + O\left(\frac{\log(\log B)}{\log B}\right), \quad (7.2)$$

respectively. In particular, (7.1) and (7.2) show that backflow never occurs for any value of  $B$  when  $V \geq V_{\max}$ , or, equivalently, when  $M \geq M_{\max} = \hat{f}(V_{\max})^{-1/2} M_{c0} \simeq 1.50315$ . Furthermore, since the bounding curve lies to the right of the point  $(26.29946, -9.69044)$ , backflow never occurs for any value of  $V$  when  $B < B_{\min} \simeq 26.29946$ .

In order to illustrate backflow, Figure 10 shows details of the streamlines of the flow near  $y = 1$  and  $\theta = 0$  when  $B = 100$  and  $V = -10$  in two situations in which backflow occurs. Specifically, Figure 10(a) shows streamlines of the solution with prescribed load  $M = 1/2$ , while Figure 10(b) shows streamlines of the critical solution with load  $M = M_c \simeq 0.50800$ .

## 8. Conclusions

In the present work we obtained a comprehensive description of steady thermoviscous coating and rimming flow on a uniformly rotating horizontal cylinder. We found that, as in the corresponding isothermal problem studied by Moffatt (2), there is a critical solution with a corresponding critical load (which depends, in general, on both the Biot number  $B$  and the thermoviscosity number  $V$ ) above which no full-film solutions exist.

In Section 5 we showed that the critical film thickness,  $h_c$ , the magnitude of the slope of the corner in the critical film thickness at  $\theta = 0$ ,  $H_c$ , and the critical load,  $M_c$ , are increasing (decreasing) functions of  $B$  for positive (negative)  $V$ , and increasing functions of  $V$ . For a positive (negative) fixed value of  $V$  the maximum (minimum) possible critical load is attained in the limit of strong heat transfer at the free surface  $B \rightarrow \infty$  and is given by  $\hat{M} = \hat{f}^{-1/2} M_{c0}$ , where  $\hat{f} = \hat{f}(V)$  is given by (5.12) and  $M_{c0} \simeq 4.44272$  is the critical load in the constant-viscosity case. For a fixed value of  $B$  the critical film thickness, and hence the critical load, become small like  $O(\log(-V)/(-V)) \ll 1$  in the limit of strong negative thermoviscosity  $V \rightarrow -\infty$  and become large like  $O(V^{1/2}) \gg 1$  in the limit of strong positive thermoviscosity  $V \rightarrow \infty$ .

Full-film solutions with a prescribed load  $M$  exist only when  $0 < M \leq M_c$ , and Figure 6 shows how for each prescribed value of  $M$  the corresponding critical curve  $(B_c, V_c)$  divides the  $B$ - $V$  parameter plane into the region above the curve in which full-film solutions exist from the region below the curve in which they do not exist. In particular, there are no full-film solutions with a prescribed load for any value of  $B$  when  $M \geq \hat{M}$  for positive  $V$  and when  $M > M_{c0}$  for negative  $V$ . In Section 6 we showed that near  $\theta = 0$  the film thickness  $h$  is a decreasing (increasing) function of  $B$  for positive (negative)  $V$ , and a decreasing function of  $B$ , but that near  $\theta = \pi$  it behaves in the opposite way. For a fixed value of  $V$  there are full-film solutions in the limit  $B \rightarrow \infty$  for  $V \geq V_\infty$ , but not for  $V < V_\infty$ , where  $V = V_\infty$  satisfies  $M = \hat{M}$ . For a fixed value of  $B$  there is no full-film solution in the limit  $V \rightarrow -\infty$ , while the leading-order film thickness is the  $O(1)$  constant  $h = M/2\pi$  in the limit  $V \rightarrow \infty$ .

In Section 7 we showed that, while by far the most common behaviour of the azimuthal velocity is that it is always in the same direction as the rotation of the cylinder, for sufficiently small prescribed values of  $M$  satisfying  $M < M_{\max} \simeq 1.50315$  there is a narrow region of the  $B$ - $V$  parameter plane shown in Figure 9 in which backflow occurs in a region on the right-hand side of the cylinder containing the point of the free surface at  $\theta = 0$ . In particular, backflow never occurs for any value of  $B$  when  $V \geq V_{\max} \simeq -5.65658$  or for any value of  $V$  when  $B < B_{\min} \simeq 26.29946$ .

Lastly, in Appendix A we considered the distinguished limit of strong thermoviscosity  $|V| \rightarrow \infty$  and weak heat transfer  $B \rightarrow 0^+$  with  $\hat{V} = BV = O(1)$  in which the variation of temperature across the film is small but the variation of viscosity across the film is still  $O(1)$ , and found qualitative agreement with but some quantitative differences from the previous results. In particular, unlike in the limit  $V \rightarrow \infty$  described above, in the limit  $\hat{V} \rightarrow \infty$  the critical film thickness, and hence the critical load, become large like  $O(\hat{V}) \gg 1$ .

## Acknowledgements

The first author (GAL) gratefully acknowledges the financial support of the United Kingdom Engineering and Physical Sciences Research Council (EPSRC) via a Doctoral

Training Account (DTA) research studentship. Part of this work was undertaken while the corresponding author (SKW) was a Visiting Fellow in the Department of Mechanical and Aerospace Engineering, School of Engineering and Applied Science, Princeton University, USA, and it was completed while he was a Visiting Fellow in the Oxford Centre for Collaborative Applied Mathematics (OCCAM), Mathematical Institute, University of Oxford, United Kingdom. This publication was based on work supported in part by Award No KUK-C1-013-04, made by King Abdullah University of Science and Technology (KAUST).

## References

1. V. V. Pukhnachev, Motion of a liquid film on the surface of a rotating cylinder in a gravitational field, *J. Appl. Mech. Tech. Phys.* (1977) **18** 344–351.
2. H. K. Moffatt, Behaviour of a viscous film on the outer surface of a rotating cylinder, *J. Méc.* (1977) **16** 651–673.
3. R. E. Johnson, Steady-state coating flows inside a rotating horizontal cylinder, *J. Fluid Mech.* (1988) **190** 321–342.
4. E. B. Hansen and M. A. Kelmanson, Steady, viscous, free-surface flow on a rotating cylinder, *J. Fluid Mech.* (1994) **272** 91–107.
5. S. T. Thoroddsen and L. Mahadevan, Experimental study of coating flows in a partially-filled horizontally rotating cylinder, *Exp. Fluids* (1997) **23** 1–13.
6. S. D. R. Wilson and J. Williams, The flow of a liquid film on the inside of a rotating cylinder, and some related problems, *Phys. Fluids* (1997) **9** 2184–2190.
7. S. B. G. O’Brien and E. G. Gath, The location of a shock in rimming flow, *Phys. Fluids* (1998) **10** 1040–1042.
8. A. E. Hosoi and L. Mahadevan, Axial instability of a free-surface front in a partially filled horizontal rotating cylinder, *Phys. Fluids* (1999) **11** 97–106.
9. B. R. Duffy and S. K. Wilson, Thin-film and curtain flows on the outside of a rotating horizontal cylinder, *J. Fluid Mech.* (1999) **394** 29–49.
10. R. C. Peterson, P. K. Jimack and M. A. Kelmanson, On the stability of viscous free-surface flow supported by a rotating cylinder, *Proc. R. Soc. Lond. A* (2001) **457** 1427–1445.
11. M. Tirumkudulu and A. Acrivos, Coating flows within a rotating horizontal cylinder: Lubrication analysis, numerical computations, and experimental measurements, *Phys. Fluids* (2001) **13** 14–19.
12. S. B. G. O’Brien, A mechanism for linear instability in two-dimensional rimming flow, *Q. Appl. Math.* (2002) **60** 283–299.
13. S. K. Wilson and R. Hunt and B. R. Duffy, On the critical solutions in coating and rimming flow on a uniformly rotating horizontal cylinder, *Q. J. Mech. Appl. Math.* (2002) **55** 357–383.
14. J. Ashmore, A. E. Hosoi and H. A. Stone, The effect of surface tension on rimming flows in a partially filled rotating cylinder, *J. Fluid Mech.* (2003) **479** 65–98.
15. E. S. Benilov, S. B. G. O’Brien and I. A. Sazonov, A new type of instability: explosive disturbances in a liquid film inside a rotating horizontal cylinder, *J. Fluid Mech.* (2003) **497** 201–224.



16. E. S. Benilov, N. Kopteva and S. B. G. O'Brien, Does surface tension stabilize the liquid films inside a rotating horizontal cylinder?, *Q. J. Mech. Appl. Math.* (2005) **58** 185–200.
17. E. S. Benilov and S. B. G. O'Brien, Inertial instability of a liquid film inside a rotating horizontal cylinder, *Phys. Fluids* (2005) **17** 052106.
18. E. S. Benilov, M. S. Benilov and N. Kopteva, Steady rimming flows with surface tension, *J. Fluid Mech.* (2008) **597** 91–118.
19. E. J. Hinch and M. A. Kelmanson, On the decay and drift of free-surface perturbations in viscous thin-film flow exterior to a rotating cylinder, *Proc. R. Soc. Lond. A* (2003) **459** 1193–1213.
20. A. Acrivos and B. Jin, Rimming flows within a rotating horizontal cylinder: asymptotic analysis of the thin-film lubrication equations and stability of their solutions, *J. Eng. Math.* (2004) **50** 99–120.
21. B. Jin and A. Acrivos, Rimming flows with an axially varying viscosity, *Phys. Fluids* (2004) **16** 633–640.
22. P. L. Evans, L. W. Schwartz and R. V. Roy, Steady and unsteady solutions for coating flow on a rotating horizontal cylinder: Two-dimensional theoretical and numerical modeling, *Phys. Fluids* (2004) **16** 2742–2756.
23. P. L. Evans, L. W. Schwartz and R. V. Roy, Three-dimensional solutions for coating flow on a rotating horizontal cylinder: Theory and experiment, *Phys. Fluids* (2005) **17** 072102.
24. C. J. Noakes, J. R. King and D. S. Riley, On three-dimensional stability of a uniform, rigidly rotating film on a rotating cylinder, *Q. J. Mech. Appl. Math.* (2005) **58** 229–256.
25. C. J. Noakes, J. R. King and D. S. Riley, On the development of rational approximations incorporating inertial effects in coating and rimming flows: a multiple-scales approach, *Q. J. Mech. Appl. Math.*, (2006) **59** 163–190.
26. M. Villegas-Díaz, H. Power and D. S. Riley, Analytical and numerical studies of the stability of thin-film rimming flow subject to surface shear, *J. Fluid Mech.* (2005) **541** 317–344.
27. P.-J. Chen, Y.-T. Tsai, T.-J. Liu and P.-Y. Wu, Low volume fraction rimming flow in a rotating horizontal cylinder, *Phys. Fluids* (2007) **19** 128107.
28. E. A. Karabut, Two regimes of liquid film flow on a rotating cylinder, *J. Appl. Mech. Tech. Phys.* (2007) **48** 55–64.
29. R. Hunt, Numerical solution of the free-surface viscous flow on a horizontal rotating elliptical cylinder, *Num. Meth. Part. Diff. Eqns* (2008) **24** 1094–1114.
30. M. A. Kelmanson, On inertial effects in the Moffatt-Pukhnachov coating-flow problem, *J. Fluid Mech.* (2009) **633** 327–353.
31. C. H. Tougher, S. K. Wilson and B. R. Duffy, On the approach to the critical solution in leading order thin-film coating and rimming flow, *Appl. Math. Lett.* (2009) **22** 882–886.
32. K. Pougatch and I. Frigaard, Thin film flow on the inside surface of a horizontally rotating cylinder: Steady state solutions and their stability, *Phys. Fluids* (2011) **23** 022102.
33. D. Goussis and R. E. Kelly, Effects of viscosity variation on the stability of film flow down heated or cooled inclined surfaces: Long-wavelength analysis, *Phys. Fluids* (1985) **28**

3207–3214

34. D. A. Goussis and R. E. Kelly, Effects of viscosity variation on the stability of a liquid film flow down heated or cooled inclined surfaces: Finite wavelength analysis, *Phys. Fluids* (1987) **30** 974–982.
35. C.-C. Hwang and C.-I. Weng, Non-linear stability analysis of film flow down a heated or cooled inclined plane with viscosity variation, *Int. J. Heat Mass Transfer* (1988) **31** 1775–1784.
36. B. Reisfeld and S. G. Bankoff, Nonlinear stability of a heated thin liquid film with variable viscosity, *Phys. Fluids* (1990) **2** 2066–2067.
37. M.-C. Wu and C.-C. Hwang, Nonlinear theory of film rupture with viscosity variation, *Int. Comm. Heat Mass Transfer* (1991) **18** 705–713.
38. R. Selak and G. Lebon, Bénard-Marangoni thermoconvective instability in [the] presence of a temperature-dependent viscosity, *J. Phys. II France* (1993) **3** 1185–1199.
39. Y. O. Kabova and V. V. Kuznetsov, Downward flow of a nonisothermal thin liquid film with variable viscosity, *J. Appl. Mech. Tech. Phys.* (2002) **43** 895–901.
40. B. R. Duffy and S. K. Wilson, A rivulet of perfectly wetting fluid with temperature-dependent viscosity draining down a uniformly heated or cooled slowly varying substrate, *Phys. Fluids* (2003) **15** 3236–3239.
41. S. K. Wilson and B. R. Duffy, On the gravity-driven draining of a rivulet of fluid with temperature-dependent viscosity down a uniformly heated or cooled substrate, *J. Eng. Math.* (2002) **42** 359–372.
42. S. K. Wilson and B. R. Duffy, Strong temperature-dependent-viscosity effects on a rivulet draining down a uniformly heated or cooled slowly varying substrate, *Phys. Fluids* (2003) **15** 827–840.
43. A. Sanson and J. R. King and D. S. Riley, Degenerate-diffusion models for the spreading of thin non-isothermal gravity currents, *J. Eng. Math.* (2004) **48** 43–68.
44. B. R. Duffy and S. K. Wilson, Large-Biot-number non-isothermal flow of a thin film on a stationary or rotating cylinder, *Eur. Phys. J. Special Topics* (2009) **166** 147–150.
45. G. A. Leslie, S. K. Wilson and B. R. Duffy, Non-isothermal flow of a thin film of fluid with temperature-dependent viscosity on a stationary horizontal cylinder, *Phys. Fluids* (2011) **23** 062101.
46. G. A. Leslie, Thin-film flows on cylinders, Ph. D. Thesis, University of Strathclyde, Glasgow, United Kingdom (2012).

## APPENDIX A

*The Distinguished Limit of Strong Thermoviscosity and Weak Heat Transfer*  $|V| \rightarrow \infty$  and  $B \rightarrow 0^+$  with  $\hat{V} = BV = O(1)$

In this Appendix we consider the distinguished limit discussed by Wilson and Duffy (42) of strong thermoviscosity,  $|V| \rightarrow \infty$ , and weak heat transfer at the free surface,  $B \rightarrow 0^+$ , such that  $\hat{V} = BV = O(1)$ , in which, although the variation in temperature across the fluid film is small, specifically  $T = 1 - Bhy + O(B^2)$ , thermoviscosity effects still enter the problem at leading order, i.e. the variation in viscosity across the fluid film is still  $O(1)$ . Note that in this limit the *effective thermoviscosity number*,  $\hat{V} = BV$ , defined in terms of dimensional quantities by

$$\hat{V} = \frac{\lambda(T_0 - T_\infty)\epsilon a \alpha_{\text{th}}}{\mu_0 k_{\text{th}}}, \quad (\text{A.1})$$

and not the previously defined thermoviscosity number,  $V$ , is the appropriate non-dimensional measure of thermoviscosity effects. In the particular case of the exponential viscosity model (4.1) in this limit  $\mathcal{V} \sim \hat{V}h$  and so the leading-order expressions for  $\mu$ ,  $u$ ,  $\psi$  and  $f$  are given simply by (4.1), (4.3)–(4.5) with  $\mathcal{V}$  replaced by  $\hat{V}h$ , respectively.

### A.1 The Critical Solution

Unlike in the general case considered in Section 5, in the present distinguished limit explicit expressions can be obtained for the critical film thickness at  $\theta = 0$ ,  $h_c(0)$ , the magnitude of the slope of the corner in  $h_c$  at  $\theta = 0$ ,  $H_c$ , and the critical flux,  $Q_c$ , namely

$$h_c(0) = \frac{\hat{V}^2 + 2(W_{\hat{V}} + 1)}{2\hat{V}}, \quad (\text{A.2})$$

$$H_c = \left( \frac{\hat{V}^4 + 4\hat{V}^2 W_{\hat{V}} + 4(W_{\hat{V}} + 1)^2}{8\hat{V}(W_{\hat{V}} + 1)} \right)^{1/2} \quad (\text{A.3})$$

and

$$Q_c = \frac{\hat{V}^2(\hat{V}^2 + 4) - 4(W_{\hat{V}} + 1)^2}{4\hat{V}^3}, \quad (\text{A.4})$$

where  $W_{\hat{V}} = W_{\hat{V}}(\hat{V})$  is given by

$$W_{\hat{V}} = \begin{cases} W_0 \left( -\exp \left[ -\frac{\hat{V}^2}{2} - 1 \right] \right) & \text{if } \hat{V} \geq 0, \\ W_{-1} \left( -\exp \left[ -\frac{\hat{V}^2}{2} - 1 \right] \right) & \text{if } \hat{V} < 0, \end{cases} \quad (\text{A.5})$$

where  $W_0(x)$  and  $W_{-1}(x)$  are again the principal and lower real branches of the Lambert W function, respectively. Figure A shows the quantities  $h_c(0)$ ,  $H_c$ ,  $Q_c$  and  $M_c$  plotted as functions of  $\hat{V}$ .

In the limit of weak thermoviscosity,  $\hat{V} \rightarrow 0$ , the solutions for  $h_c$ ,  $H_c$ ,  $M_c$  and  $u_c$  are given by the corresponding results in the limit  $B \rightarrow 0$  given in Subsection 5.1, namely (5.5)–(5.8), with  $BV$  and  $O(B^2)$  replaced by  $\hat{V}$  and  $O(\hat{V}^2)$ , respectively, and hence have the same physical interpretation.

In the limit of strong positive thermoviscosity,  $\hat{V} \rightarrow \infty$ , the solutions for  $h_c$ ,  $H_c$ ,  $M_c$  and  $u_c$  are given by

$$h_c = \frac{\hat{V} \left[ 1 - (1 - \cos \theta)^{1/2} \right]}{2 \cos \theta} + \frac{1}{\hat{V}} + O \left( \frac{1}{\hat{V}^3} \right), \quad (\text{A.6})$$

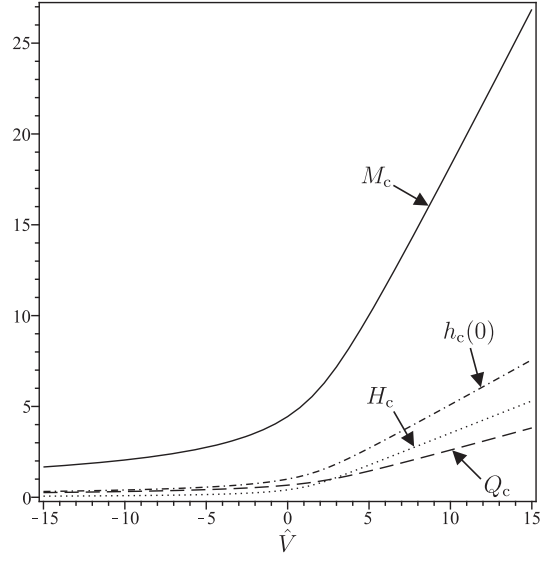
$$H_c = \frac{\hat{V}}{2\sqrt{2}} + O \left( \frac{1}{\hat{V}^3} \right), \quad (\text{A.7})$$

$$M_c = 2\hat{V} \log(1 + \sqrt{2}) + \frac{2\pi}{\hat{V}} + O \left( \frac{1}{\hat{V}^3} \right) \quad (\text{A.8})$$

and

$$u_c = 1 - \frac{1}{2} \left[ 1 - (1 - \cos \theta)^{1/2} \right] \left[ 1 - \exp \left( -\frac{\hat{V}^2 \left[ 1 - (1 - \cos \theta)^{1/2} \right] y}{2 \cos \theta} \right) \right] + O \left( \frac{1}{\hat{V}^2} \right). \quad (\text{A.9})$$

These solutions differ from the corresponding results in the limit  $V \rightarrow \infty$  given in Subsection 5.4,



**Fig. A** The quantities  $h_c(0)$  given by (A.2) (dash-dotted line),  $H_c$  given by (A.3) (dotted line),  $Q_c$  given by (A.4) (dashed line), and  $M_c$  (solid line) plotted as functions of  $\hat{V}$ .

namely (5.23)–(5.26), but have a qualitatively similar physical interpretation. In particular, these solutions show that at leading order in the limit of strong positive thermoviscosity the viscosity

$$\mu = O \left[ \exp \left( \frac{\hat{V}^2 [1 - (1 - \cos \theta)^{1/2}] y}{2 \cos \theta} \right) \right] \quad (\text{A.10})$$

is exponentially large outside a narrow boundary layer of width  $O(1/\hat{V}^2) \ll 1$  near the cylinder  $y = 0$ , resulting in a uniform plug flow  $u_c \equiv [1 + (1 - \cos \theta)^{1/2}]/2$  outside the boundary layer and a large film thickness and load of  $O(\hat{V}) \gg 1$ .

In the limit of strong negative thermoviscosity,  $\hat{V} \rightarrow -\infty$ , the solutions for  $h_c$ ,  $H_c$ ,  $M_c$ , and  $u_c$  are given by

$$h_c = \frac{1}{(-\hat{V})} \left[ \log \left( \frac{\hat{V}^2}{2} \right) - W_{V\infty} - 1 \right] + O \left( \frac{\log(-\hat{V})^2}{\hat{V}^3} \right), \quad (\text{A.11})$$

$$H_c = \frac{1}{(-\hat{V})} + O \left( \frac{\log(-\hat{V})^2}{\hat{V}^3} \right), \quad (\text{A.12})$$

$$M_c = \frac{1}{(-\hat{V})} \left[ 2\pi \log \left( \frac{\hat{V}^2}{2} \right) - C_{V\infty} - 2\pi \right] + O \left( \frac{\log(-\hat{V})^2}{(-\hat{V})^3} \right) \quad (\text{A.13})$$

and

$$u_c = 1 - \frac{\cos \theta}{\hat{V}^2} \left[ \left( -\frac{\hat{V}^2 W_{V\infty}}{2 \cos \theta} \right)^y \left\{ \left[ \log \left( \frac{\hat{V}^2}{2} \right) - W_{V\infty} \right] (1 - y) + y \right\} - \log \left( \frac{\hat{V}^2}{2} \right) + W_{V\infty} \right] + O \left( \frac{\log(-\hat{V})^2}{\hat{V}^2} \right), \quad (\text{A.14})$$

where the function  $W_{V\infty} = W_{V\infty}(\theta)$  and the constant  $C_{V\infty}$  are again given by (5.33) and (5.34), respectively. At leading (but not higher) order these solutions coincide with the corresponding results in the limit  $V \rightarrow -\infty$  given in Subsection 5.5, namely (5.28)–(5.31), and hence have the same physical interpretation.

#### A.2 Full-Film Solution with a Prescribed Load

In the limit of weak thermoviscosity,  $\hat{V} \rightarrow 0$ , the solutions for  $h$  and  $u$  are given by the corresponding results in the limit  $B \rightarrow 0$  given in Subsection 6.1, namely (6.1) and (6.2), with  $BV$  and  $O(B^2)$  replaced by  $\hat{V}$  and  $O(\hat{V}^2)$ , respectively, and hence have the same physical interpretation.

In the limit of strong positive thermoviscosity,  $\hat{V} \rightarrow \infty$ , the solutions for  $h$  and  $u$  are given by

$$h = \frac{M}{2\pi} + \frac{M^2 \cos \theta}{4\pi^2 \hat{V}} + O\left(\frac{1}{\hat{V}^2}\right) \quad (\text{A.15})$$

and

$$u = 1 - \frac{M \cos \theta}{2\pi \hat{V}} \left[ 1 - \exp\left(-\frac{M \hat{V} y}{2\pi}\right) \right] + O\left(\frac{1}{\hat{V}^2}\right). \quad (\text{A.16})$$

These solutions are similar to the corresponding results in the limit  $V \rightarrow \infty$  given in Subsection 6.4, namely (6.14) and (6.15), from which they can be obtained by writing  $V = \hat{V}/B$  and taking the limit  $B \rightarrow 0$ , and hence have the same physical interpretation.

In the limit of small load,  $M \rightarrow 0$ , the solutions for  $h$  and  $u$  are given by the corresponding results in the limit  $M \rightarrow 0$  given in Subsection 6.5, namely (6.18) and (6.19), with  $BV$  replaced by  $\hat{V}$ , and hence have the same physical interpretation.