Analytical perturbative theories of motion in highly inhomogeneous gravitational fields

Final Report

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1   Introduction

The motion of bodies subject to inhomogeneous gravitational fields is a classical subject of research in the context of celestial mechanics. In recent years this type of research has become important to future planned missions of spacecraft to the moon and asteroids in addition to asteroid deflection missions such as ESA's "Don Quijote" concept [7]. The initial research undertaken in this area has been on the study of the effect of the Earth's inhomogeneous gravitational field on the motion of natural and artificial satellites, that is, artificial satellite theory for small and moderate eccentricities [11]. More recent studies have turned to research the effects on motion of the inhomogeneous gravitational field of other solar system bodies, including the Moon [1] and asteroids [22]. The analysis of spacecraft motion about these bodies is particularly challenging as they typically feature shapes and density distributions more irregular than those of planets. Such irregularities break symmetries and require more complicated analytical expressions for their description and increasing the complexity involved in such studies.

Numerical methods are today widely used to study the trajectories of objects orbiting specific irregular bodies [14]. Disadvantages of numerical methods in generating useful spacecraft trajectories are that they can be highly computational and require a complete re-design for different bodies. Analytical methods, by contrast, have the potential to rapidly identify useful natural motions for general bodies with inhomogeneous gravitational fields. Furthermore, analytical methods can provide a full dynamical picture of the motion around irregular bodies that can be used to search and study particular classes of useful orbits. However, current analytical methods are only used in a limited and semi-numerical way (meaning that analytical expansions constitute the first step in such studies, which are then typically carried out from a numerical standpoint [23]). The main drawback of analytical methods is that their application in the case of highly inhomogeneous bodies requires extensive symbolic computations involving algebraic manipulations. However, standard and specialist algebraic software is constantly improving in its computational ability to perform algebraic manipulations.

The potential of this software as well as the evolution of standard symbolic tools such as Maple, Mathematica or Piranha ([4], [5]) means that a fully analytical method for studying small bodies in motion subject to highly inhomogeneous gravitational fields using perturbative methods is now feasible. However, the current theory needs to be generalised so that perturbation methods can be implemented algorithmically to high-order. In this report we develop a general high-order analytical perturbative method for studying inhomogeneous gravitational fields. It is then shown that current state-of-the-art symbolic software tools such as Mathematica and Pirahna can easily cope with the extensive manipulations required to implement this high-order perturbative theory in practice.

The basic principle in analytical perturbative methods is to consider a complex physical system as the aggregate of a well-known (Hamiltonian) system and some perturbation. The unperturbed part is exactly solvable (integrable) and accounts for the dominant features of the system; the perturbed part, which is typically not solvable exactly (and renders a non-integrable Hamiltonian system), induces a deviation from the unperturbed integrable model and accounts

for the finer details of the behavior of the system. In celestial mechanics, the exactly-solvable unperturbed model is often the two-body problem, consisting of two point masses moving under reciprocal gravitational attraction. The trajectories followed by the two particles are conic sections, which can be described by the classical six orbital elements a (semi-major axis), e (eccentricity), i (inclination), w (argument of percenter), W (longitude of the ascending node) and M (mean anomaly at epoch). In the two-body problem, these orbital elements are constants of motion. When this simple model is complicated by the addition of more realistic features, such as additional and/or non-spherical bodies, the resulting system is generally not integrable.

Methods based on the variation of parameters have led to a number of important classical results in celestial mechanics, such as the existence of critical inclinations and of geosynchronous orbits, as well as the seminal works on analytical ephemerides by Charles-Eugene Delaunay and the discovery of the planet Neptune [18]. The approaches used in these classical works are not suitable for problems of higher complexity because of the number and type of algebraic manipulations involved. This reason motivated the development of methods based on Hamiltonian formalism and canonical transformations through Lie series in the 60s, which can be considered as the basis of modern Celestial Mechanics [9]. The standpoint adopted in these seminal works was that of developing a technique that could be efficiently programmed into computer languages, thus delegating to a machine the task of performing the extensive amounts of calculations involved in perturbative methods. Such a standpoint has today become essential and the increasing power of computers allows tackling increasingly complex problems. Recent results on lunar orbital motion, on the long-term propagation and stability of the Solar System [17], on the peculiar motions of Jupiter and Saturn moons have all been enabled by these approaches, which require two fundamental elements: the method and a fast and efficient computer algebra system.

In this report we show that modern computer performances and state-of-the-art algebraic manipulator software are sufficiently developed to carry out our generalised analytical perturbative theory. This report addresses three technical aspects to develop a general perturbative theory and illustrates its power by applying it to investigate the inhomogeneous gravitational fields of asteroids.

In the initial stage a truncated Hamiltonian formulation of a spacecraft in motion about an asteroid, appropriate for implementation in Piranha or Mathematica is formulated. The first phase of the study involves deriving the perturbing potential U(x) which is the gravitational potential of the inhomogeneous body uniformly rotating around the z axes. This investigation found inconsistencies in the literature when converting from the original co-ordinates to the required Nodal-polar elements. In particular there is an error in the Poisson series representation of the potential in Nodal-polar elements in [20]. To confirm the correct equations in Nodal-polar elements two independent derivations were undertaken, using two different approaches, which confirms the correct representation in Nodal-polar elements.

A general perturbative theory is then presented which considers all the terms of the gravitational potential. This generalises previous methods in the literature which have previously only considered first order terms to construct a Hamiltonian formulation of the problem. Therefore, the proposed perturbative

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theory presents a method to derive more accurate mathematical descriptions of the motions, which in turn enables the accurate identification of frozen orbits. One possible advantage of this approach is that it allows frozen orbits to be identified rapidly at the preliminary design stage. To this end once a particular frozen orbit is identified it could be employed in a numerical optimiser to fine tune the natural orbit in the full model under the effect of all relevant orbital perturbations. These frozen orbits could then be used as reference trajectories in missions that require close inspection of asteroids.An illustration of the developed general perturbative method to identify frozen orbits (including quasi-frozen orbits) for Eros 433 is presented. The method focuses on the relegation of the argument of node to arbitrary order [13] and a Delaunay normalisation [12] which reduces the complexity of the model, essentially reducing the number of degrees of freedom to a single degree of freedom. The resulting single degree of freedom model then allows the computation of frozen orbits using a simple low-dimensional root finder. Although the method is highly computational it is shown that a completely automated relegation of the argument of node to arbitrary order is feasible using standard technical computing software such as Mathematica and specialist algebraic manipulation software such as Piranha (an algebraic software tool developed by ACT).

2 The dynamical system

The initial stage of the project is dedicated to the formulation of a Hamiltonian model for the motion of a massless spacecraft in an inhomogeneous gravitational field. Such field is generated by a small body, uniformly rotating around the "z-axis" of the reference frame with constant angular velocity $\hat{\omega} = [0, 0, \omega]$.

It is therefore convenient to formulate the dynamics in a rotating frame of reference, thus describing it with the Hamiltonian:

$$H(\mathbf{x}, \mathbf{X}) = \tfrac{1}{2}(\mathbf{X} \cdot \mathbf{X}) - \hat{\omega}(\mathbf{x} \times \mathbf{X}) + U(\mathbf{x}) \tag{1}$$

where $\mathbf{x}$, $\mathbf{X} \in \mathbb{R}^3$ are respectively the position coordinates and conjugate momenta of the spacecraft, and the equations of motion are:

$$\begin{cases} \dot{\mathbf{x}} = \frac{\partial}{\partial \mathbf{X}} H(\mathbf{x}, \mathbf{X}) \\ \dot{\mathbf{X}} = -\frac{\partial}{\partial \mathbf{x}} H(\mathbf{x}, \mathbf{X}) \end{cases} \tag{2}$$

The perturbing potential $U(\mathbf{x})$ is the gravitational potential of the inhomogeneous body uniformly rotating around the z axes and with the assumption of a homogeneous internal density distribution. Due to inconsistencies in the literature in the statement of the gravitational potential it is derived here independently using two completely different approaches. This was undertaken to verify the correct representation of the gravitational potential:

The Hamiltonian is expressed in the *Nodal-Polar* variables r, θ, and ν, which, as explained in Appendix 9, are the satellite distance from the origin, the argument of latitude (i.e the angular distance of the spacecraft from the line of the ascending node on the orbital plane) and right ascension of the ascending node, that is the longitude of the ascending node respectively.

Following the change of coordinates $(\mathbf{x}, \mathbf{X}) \to (r, \theta, \nu, R, \Theta, N)$ is (fully derived in the Appendix (9)):

$$
\begin{aligned}
x &= |\mathbf{r}|(\cos\theta\cos\nu - \sin\theta\cos I\sin\nu) \\
y &= |\mathbf{r}|(\cos\theta\sin\nu + \sin\theta\cos I\cos\nu) \\
z &= |\mathbf{r}|\sin\theta\sin I \\
X &= (R\cos\theta - \tfrac{\Theta}{|\mathbf{r}|}\sin\theta)\cos\nu - (R\sin\theta + \tfrac{\Theta}{|\mathbf{r}|}\cos\theta)\cos I\sin\nu \\
Y &= (R\cos\theta - \tfrac{\Theta}{|\mathbf{r}|}\sin\theta)\sin\nu + (R\sin\theta + \tfrac{\Theta}{|\mathbf{r}|}\cos\theta)\cos I\cos\nu \\
Z &= (R\sin\theta + \tfrac{\Theta}{|\mathbf{r}|}\cos\theta)\sin I
\end{aligned}
\tag{3}
$$

with $N = \Theta_z$ and $\cos I = \frac{\Theta_z}{|\Theta|}$, yields:

$$
H(r, \theta, \nu, R, \Theta, N) = \tfrac{1}{2}(R^2 + \tfrac{\Theta^2}{r^2}) - \omega N + U(r, \theta, \nu)
\tag{4}
$$

where the perturbing potential is as constructed in the next section in two different but equivalent ways in order to clarify the results found in previous literature.

2.1   The gravitational potential (First Approach)

By Newton's Gravitational law the potential generated by a body of mass M with **spherical symmetry** whose position vector is $\mathbf{r}_1$, on a particle set in $\mathbf{r}$ is given by:

$$
U(\mathbf{r}) = -\frac{\mathcal{G}M}{|\mathbf{r} - \mathbf{r}_1|}
\tag{5}
$$

Which can be generalized for a discrete mass distribution of N masses M_i whose position vector is $\mathbf{r}_i$, as the superimposition of the single potential of each mass:

$$
U(\mathbf{r}) = -\sum_{i=1}^{N} \frac{\mathcal{G}M_i}{|\mathbf{r} - \mathbf{r}_i|}
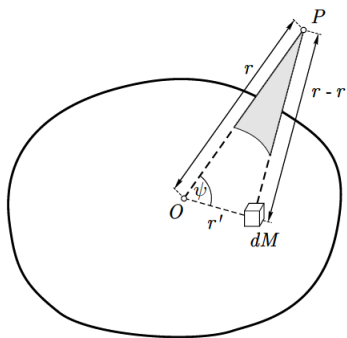\tag{6}
$$

Figure 1: *The potential generated by an arbitrarily shaped body B is the integral over the volume the infinitesimal mass elements dM*

We now consider an arbitrarily shaped body B of finite extension; denote with $\mathbf{r}' \in \mathbb{R}^3$ the position of the infinitesimal mass element dM in a cartesian

reference frame O_{xyz}. The gravity potential of such a continuous mass distribution on an external point P set in $\mathbf{r} \in \mathbb{R}^3$ can be obtained from (6) substituting the sum with an integral over the volume of the body, namely:

$$U(\mathbf{r}) = -\mathcal{G} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV \tag{7}$$

where $\rho(\mathbf{r}')$ is the density of the body and dV is the infinitesimal element of volume (i.e. $dM = \rho(\mathbf{r}')dV$) and V is the volume of the body.

Notice that, to get back to (5) it is sufficient to impose the spherical symmetry property, that is the radial distribution of density $\rho(\hat{\mathbf{r}}) = \rho(-\hat{\mathbf{r}}), \quad \forall \hat{\mathbf{r}} \in B$.

With a few algebraic manipulations it can be shown that:

$$\begin{aligned} U(|\mathbf{r}|, \psi) &= -\mathcal{G} \int_V \frac{\rho(\mathbf{r}')}{\sqrt{|\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{r}' + |\mathbf{r}'|^2}} dV \\ &= -\frac{\mathcal{G}}{|\mathbf{r}|} \int_V \frac{\rho(\mathbf{r}')}{\sqrt{1 - 2\frac{|\mathbf{r}'|}{|\mathbf{r}|}\cos(\psi) + \left(\frac{|\mathbf{r}'|}{|\mathbf{r}|}\right)^2}} dV, \end{aligned} \tag{8}$$

where ψ is the colatitude of $\mathbf{r}'$ over $\mathbf{r}$ i.e. the angle between $\mathbf{r}$ and $\mathbf{r}'$.

Indicating with $P_n(X)$ the Legendre polynomial of degree n, the expansion

$$(1 - 2XZ + Z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} Z^n P_n(X) \tag{9}$$

is now used, which can be demonstrated by the binomial theorem generalized for all exponents (other than only nonnegative integers).

Calling $r = |\mathbf{r}|$ and $r' = |\mathbf{r}'|$ and substituting $X = \cos(\psi)$ and $Z = \frac{r'}{r}$ yields, for $\frac{r'}{r} < 1$ (ray of convergence of the series):

$$\frac{1}{\sqrt{1 - 2\frac{r'}{r}\cos(\psi) + \left(\frac{r'}{r}\right)^2}} = \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos(\psi)). \tag{10}$$

Then, substituting into the potential (8), yields, for $\frac{r'}{r} < 1$:

$$U(r, \psi)) = -\frac{\mathcal{G}}{r} \int_V \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos(\psi))\rho(\mathbf{r}')dV. \tag{11}$$

The condition $\frac{r'}{r} < 1$ implies that the model is valid only outside the reference sphere that is the sphere circumscribing the asteroid.

Expressing the angle ψ in terms of the *latitude* δ and *longitude* λ we obtain that:

$$\cos \psi = \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos(\lambda - \lambda') \tag{12}$$

and the generic Legendre Polynomial $P_n(\cos \psi)$ decomposes to:

$$P_n(\cos \psi) = \sum_{m=0}^{n} \left((2 - \delta_{m,0}) \frac{(n-m)!}{(n+m)!} P_n^m(\sin \delta) P_n^m(\sin \delta') \cos(m(\lambda - \lambda')) \right) \tag{13}$$
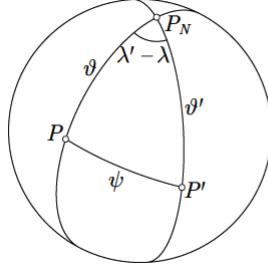
Figure 2: *The angle ψ can be expressed in terms of the* latitude δ *and* longitude λ

(see [2]), where $P_n^m(x)$ is the associated Legendre function of degree n and order m, and $\delta_{m,0}$ (different from the *longitude* δ) is the Kronecker delta that gives 1 if $m = 0$, and 0 elsewhere.

The potential becomes:

$$U(r, \delta, \lambda) = -\frac{\mathcal{G}}{r} \int_V \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n \sum_{m=0}^{n} \left((2 - \delta_{m,0}) \frac{(n-m)!}{(n+m)!} P_n^m(\sin \delta) P_n^m(\sin \delta') \cdot \right.$$

$$\left. \cdot \cos\left(m(\lambda - \lambda')\right) \right) \rho(r', \delta', \lambda') dV$$

$$(14)$$

Now, calling, $\forall 0 \leq m \leq n$:

$$C_{n,m} = \frac{(2 - \delta_{m,0})}{M} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{\alpha}\right)^n P_{n,m}(\sin \delta') \cos(m\lambda') \rho(r', \delta', \lambda') dV$$

$$S_{n,m} = \frac{(2 - \delta_{m,0})}{M} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{\alpha}\right)^n P_{n,m}(\sin \delta') \sin(m\lambda') \rho(r', \delta', \lambda') dV$$

$$(15)$$

where α is a conventionally chosen reference radius, and we have used that $P_n^m = (-1)^m P_{n,m}$, to be consistent with [16]. Note that, in particular, the equations in (15) imply that:

$$C_{0,0} = 1$$

$$C_{n,0} = \frac{1}{M} \int_V \left(\frac{r'}{\alpha}\right)^n P_n(\sin \delta') \rho(r', \delta', \lambda') dV \quad \forall n > 0$$

$$S_{n,0} = 0 \quad \forall n \geq 0$$

$$(16)$$

Moreover, centering the origin of the system of reference in the center of mass it can be demonstrated that the term $C_{1,0} = 0$.

The coefficients $C_{2,0}$ and $C_{2,2}$ express the "ellipticity" and "oblateness" of the body.

Therefore we find:

$$U(r, \delta, \lambda) = -\frac{\mathcal{G}M}{r} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n \sum_{m=0}^{n} (-1)^m \left(C_{n,m} \cos{(m\lambda)} + S_{n,m} \sin{(m\lambda)}\right)$$
$$P_n^m(\sin \delta)$$

$$(17)$$

Calling:
$$\bar{\mathcal{F}}_{n,m}^1(\lambda, \delta) := (-1)^m P_n^m(\sin \delta) \cos{(m\lambda)}$$
$$\bar{\mathcal{F}}_{n,m}^2(\lambda, \delta) := (-1)^m P_n^m(\sin \delta) \sin{(m\lambda)}$$

where, using (104) and (106):

$$\bar{\mathcal{F}}_{n,m}^1(\lambda, \delta) := (-1)^m P_n^m(\sin \delta) \cos{(m\lambda)}$$

$$= (\cos \delta)^m \frac{n!^2 m!^2}{2^n} \sum_{j=0}^{n} \sum_{\ell=\max\{0, j+m-n\}}^{\min\{j,m\}} \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} \left((-1)^{j+\ell+p} \cdot \right.$$

$$\cdot \frac{1}{(j)!(n-j)!\ell!(m-\ell)!(n-m-j+\ell)!(j-\ell)!(2p)!(m-2p)!} \cdot$$

$$\left. \cdot (1+\sin\delta)^{n-m-j+\ell}(1-\sin\delta)^{j-\ell} \cos{(\lambda)}^{m-2p} \sin{(\lambda)}^{2p} \right)$$

$$\bar{\mathcal{F}}_{n,m}^2(\lambda, \delta) := (-1)^m P_n^m(\sin \delta) \sin{(m\lambda)}$$

$$= (\cos \delta)^m \frac{n!^2 m!^2}{2^n} \sum_{j=0}^{n} \sum_{\ell=\max\{0, j+m-n\}}^{\min\{j,m\}} \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \left((-1)^{j+\ell+p} \cdot \right.$$

$$\cdot \frac{1}{(j)!(n-j)!\ell!(m-\ell)!(n-m-j+\ell)!(j-\ell)!(2p+1)!(m-2p-1)!} \cdot$$

$$\left. \cdot (1+\sin\delta)^{n-m-j+\ell}(1-\sin\delta)^{j-\ell} \cos{(\lambda)}^{m-2p-1} \sin{(\lambda)}^{2p+1} \right)$$

$$(18)$$

Therefore the potential takes the form:

$$U(r, \delta, \lambda) = -\frac{\mathcal{G}M}{r} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n \sum_{m=0}^{n} \left(C_{n,m} \bar{\mathcal{F}}_{n,m}^1(\lambda, \delta) + S_{n,m} \bar{\mathcal{F}}_{n,m}^2(\lambda, \delta)\right) \quad (19)$$

Finally the potential is expressed in the *Nodal-Polar* coordinates, by (3), which yields:

$$\mathcal{F}_{n,m}^1(I,\theta,\nu) = \tfrac{1}{2^n} m!^2 n!^2 \sum_{j=0}^{n} \sum_{\ell=\max\{0,j+m-n\}}^{\min\{j,m\}} \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} \left((-1)^{j+\ell+p} \cdot \right.$$

$$\cdot \left(\tfrac{1}{j!(n-j)!\ell!(m-\ell)!(n-m-j+\ell)!(j-\ell)!(2p)!(m-2p)!} \right) (1 + \sin(\theta)\sin(I))^{n-m-j+\ell} \cdot$$

$$\cdot (1 - \sin(\theta)\sin(I))^{j-\ell} (\cos(\theta)\cos\nu - \cos(I)\sin(\theta)\sin(\nu))^{m-2p} \cdot$$

$$\left. \cdot (\cos(\theta)\sin(\nu) + \cos(I)\sin(\theta)\cos(\nu))^{2p} \right)$$

(20)

and

$$\mathcal{F}_{k,m}^2(I,\theta,\nu) = \tfrac{1}{2^n} m!^2 n!^2 \sum_{j=0}^{n} \sum_{\ell=\max\{0,j+m-n\}}^{\min\{j,m\}} \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \left((-1)^{j+\ell+p} \cdot \right.$$

$$\cdot \left(\tfrac{1}{j!(n-j)!\ell!(m-\ell)!(n-m-j+\ell)!(j-\ell)!(2p+1)!(m-2p-1)!} \right) \cdot$$

(21)

$$\cdot (1 + \sin(\theta)\sin(I))^{n-m-j+\ell} (1 - \sin(\theta)\sin(I))^{j-\ell} \cdot$$

$$\cdot (\cos(\theta)\cos\nu - \cos(I)\sin(\theta)\sin(\nu))^{m-2p-1} \cdot$$

$$\left. \cdot (\cos(\theta)\sin(\nu) + \cos(I)\sin(\theta)\cos(\nu))^{2p+1} \right)$$

where it should be noted that, in $\mathcal{F}_{n,m}^1(\lambda,\delta)$, the term $\left(\tfrac{1}{D} \right)^m$ arising by (98) from
$\cos(\lambda)^{m-2p} \sin(\lambda)^{2p}$ in (18) cancels with the $(\cos\delta)^m = D^m$ in the same formula, and analogously in $\mathcal{F}_{n,m}^2(\lambda,\delta)$ where $\left(\tfrac{1}{D} \right)^m$ arises from

$$\cos(\lambda)^{m-2p-1} \sin(\lambda)^{2p+1}.$$

The potential becomes:

$$U(r,\theta,\nu) = -\tfrac{\mathcal{G}M}{r} \sum_{n=0}^{\infty} \left(\tfrac{\alpha}{r} \right)^n \sum_{m=0}^{n} \left(C_{n,m}\mathcal{F}_{n,m}^1(I,\theta,\nu) + S_{n,m}\mathcal{F}_{n,m}^2(I,\theta,\nu) \right).$$

(22)

2.2 The gravitational potential (Second Approach)

As there are some discrepancies between the expression of the gravitational potential found in this work and the literature (i.e. [20] or [21]), it is necessary to verify the equation by deriving the equations using a completely different approach. now want to provide a second approach for deriving the gravitational potential, and get it into the Wittaker Nodal-Polar variables, The procedure is undertaken using the non scaled spherical harmonics and Wigner's rotation theorem as follows.

Definition 1 *The non scaled spherical harmonics $Y_n^m(\delta, \lambda)$ are the angular portion of the solution to Laplace's equation in spherical coordinates where azimuthal symmetry is not present, which can be expressed as*

$$Y_n^m(\delta, \lambda) := P_n^m(\sin \delta)e^{\imath m \lambda} \tag{23}$$

Getting Back to the formulation of the potential in (11) we apply the addition formula for non scaled spherical harmonics [16]:

$$P_n(\cos \psi) = \Re\left[\sum_{m=0}^{n}(-1)^m(2-\delta_{0,m})Y_n^{-m}(\delta,\lambda)Y_n^m(\delta',\lambda')\right] \tag{24}$$

Thus obtaining:

$$U(r,\delta,\lambda)) = -\frac{\mathcal{G}}{r}\int_V \sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \Re\left[\sum_{m=0}^{n}(-1)^m(2-\delta_{0,m})Y_n^{-m}(\delta,\lambda)Y_n^m(\delta',\lambda')\right]\cdot$$

$$\cdot\rho(\mathbf{r'})dV$$

$$= -\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\Re\left[\sum_{m=0}^{n}(-1)^m\left(\frac{\alpha}{r}\right)^n Y_n^{-m}(\delta,\lambda)\left(\frac{1}{M}\int_V\left(\frac{r'}{\alpha}\right)^n(2-\delta_{0,m})\cdot\right.\right.$$

$$\left.\left.\cdot Y_n^m(\delta',\lambda')\rho(\mathbf{r'})dV)\right]\right.$$

$$= \Re\left[-\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^n\frac{(n+m)!}{(n-m)!}Y_n^{-m}(\delta,\lambda)\left(\frac{1}{M}\int_V\left(\frac{r'}{\alpha}\right)^n(2-\delta_{0,m})\cdot\right.\right.$$

$$\left.\left.\cdot\frac{(n-m)!}{(n+m)!}P_{n,m}(\sin\delta')e^{\imath m\lambda'}\rho(\mathbf{r'})dV\right)\right]$$

$$= \Re\left[-\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^n\frac{(n+m)!}{(n-m)!}Y_n^{-m}(\delta,\lambda)K_{n,m}\right],$$

$$\tag{25}$$

where $K_{n,m} = C_{n,m} + \imath S_{n,m}$ and $C_{n,m}$ and $S_{n,m}$ are as in (15) [1].

Now we apply Wigner's rotation theorem for non scaled spherical harmonics (see [27]) in order to get to the nodal polar variables.

Theorem 1 *$\forall n, m \in \mathbb{N}, n, m$ let be $Y_n^m(\delta, \lambda)$ the spherical harmonics expressed in terms of the latitude δ and longitude λ in a system of reference $O_{\hat{x},\hat{y},\hat{z}}$. Then the expression for $Y_n^m(\delta, \lambda)$ in terms of the latitude Δ and longitude Λ in another system of reference $O_{\hat{x},\hat{y},\hat{z}}$, obtained by the composition of three rotations of angles α, β and γ (around the $\hat{x}$ axes, the rotated $\hat{z}$, and the rotated $\hat{x}$ axes respectively), is given by:*

$$Y_n^m(\delta,\lambda) = \sum_{j=-n}^{n} D_{j,m}^n(-\alpha,-\beta,-\gamma)Y_n^j(\Delta,\Lambda) \tag{26}$$

_____

[1] $C_{n,m}$ and $S_{n,m}$ are the so called "Stokes coefficient" [16]

where

$$D^n_{j,m}(-\alpha,-\beta,-\gamma) = e^{\imath j(\alpha+\frac{\pi}{2})}e^{\imath m(\gamma-\frac{\pi}{2})}d^n_{j,m}(-\beta) \tag{27}$$

and

$$d^n_{j,m}(-\beta) = \sum_{t=\max\{0,j-m\}}^{\min\{n-m,n+j\}}(-1)^{m-j+3t}\frac{(n-j)!(n+m)!}{t!(n+j-t)!(n-m-t)!(m-j+t)!} \cdot$$
$$\cdot\left(\cos\left(\tfrac{\beta}{2}\right)\right)^{2n-(m-j+2t)}\left(\sin\left(\tfrac{\beta}{2}\right)\right)^{m-j+2t} \tag{28}$$

We want to apply this theorem by setting the second system of reference to be the one where the spacecraft position vector is $(0,0,r)$ therefore the three angles α, β and γ are set to be θ, I and ν, the argument of latitude, the inclination of the orbital plane and namely the right ascension of the ascending node (see Appendix 9); moreover it must be noticed that in such system of reference the new latitude Δ and longitude Λ of the spacecraft will be both equal to zero as we have set its new position vector to be $(0,0,r)$. Therefore (25) becomes:

$$U(r,\delta,\lambda)) = \Re\left[-\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^n\frac{(n+m)!}{(n-m)!}\sum_{j=-n}^{n}D^n_{j,-m}(-\alpha,-\beta,-\gamma)\right.$$
$$\left.Y^j_n(0,0)K_{n,m}\right]$$

$$= \Re\left[-\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^n\frac{(n+m)!}{(n-m)!}\sum_{j=-n}^{n}D^n_{j,-m}(-\theta,-I,-\nu)Y^j_n(0,0)K_{n,m}\right]$$

$$= \Re\left[-\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^n\frac{(n+m)!}{(n-m)!}\sum_{j=-n}^{n}e^{\imath(j\theta-m\nu)}e^{\imath\frac{\pi}{2}(k+m)}d^n_{j,-m}(-I)\cdot\right.$$
$$\left.\cdot P^j_n(0)K_{n,m}\right]$$

$$= -\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^n\frac{(n+m)!}{(n-m)!}\sum_{j=-n}^{n}\sum_{\ell=\max\{0,j+m\}}^{\min\{n+m,n+j\}}(-1)^{m-j+3\ell}\cdot$$
$$\cdot\frac{(n-j)!(n-m)!}{\ell!(n+j-\ell)!(n+m-\ell)!(-m-j+\ell)!}\left(\cos\left(\tfrac{I}{2}\right)\right)^{2n-(-m-j+2\ell)}\left(\sin\left(\tfrac{I}{2}\right)\right)^{-m-j+2\ell}\cdot$$
$$\cdot\left((-1)^j((n+j)_{\equiv 2}-1)(-1)^{\frac{n-j}{2}}\frac{(n+j-1)!!}{2^{\frac{n-j}{2}}\left(\frac{n-j}{2}\right)!}\right)\cdot$$
$$\cdot\left(C_{n,m}\left(\cos(j\theta-m\nu)\cos\left(\tfrac{\pi}{2}(j+m)\right)-\sin(j\theta-m\nu)\sin\left(\tfrac{\pi}{2}(j+m)\right)\right)\right)$$

$$+ S_{n,m} \left(-\sin\left(j\theta - m\nu\right)\cos\left(\tfrac{\pi}{2}(j+m)\right) - \cos\left(j\theta - m\nu\right)\sin\left(\tfrac{\pi}{2}(j+m)\right)\right)\right)$$

$$= -\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha}{r}\right)^{n}\sum_{j=-n}^{n}\sum_{\ell=\max\{0,j+m\}}^{\min\{n+m,n+j\}}(-1)^{m-j+3\ell}\cdot$$

$$\cdot \tfrac{(n-j)!(n+m)!}{\ell!(n+j-\ell)!(n+m-\ell)!(-m-j+\ell)!}\left(\cos\left(\tfrac{I}{2}\right)\right)^{2n-(-m-j+2\ell)}\left(\sin\left(\tfrac{I}{2}\right)\right)^{-m-j+2\ell}\cdot$$

$$\cdot (-1)^{\frac{n+j}{2}}\tfrac{1}{2^{n}}\tfrac{(n+j)!}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}((n+j)_{\equiv 2}-1)\tfrac{(n+m)!}{2^{n}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}\cdot$$

$$\cdot \left(C_{n,m}\left(\cos\left(j\theta - m\nu\right)\cos\left(\tfrac{\pi}{2}(j+m)\right) - \sin\left(j\theta - m\nu\right)\sin\left(\tfrac{\pi}{2}(j+m)\right)\right)\right.$$

$$\left. + S_{n,m}\left(-\sin\left(j\theta - m\nu\right)\cos\left(\tfrac{\pi}{2}(j+m)\right) - \cos\left(j\theta - m\nu\right)\sin\left(\tfrac{\pi}{2}(j+m)\right)\right)\right) \tag{29}$$

calling

$$\bar{\mathcal{G}}_{n,m,j}(I) = \sum_{\ell=\max\{0,j+m\}}^{\min\{n+m,n+j\}}(-1)^{m+3\ell-j}\frac{(n+m)!(n-j)!}{\ell!(n+j-\ell)!(n+m-\ell)!(\ell-m-j)!}\cdot$$

$$\cdot (-1)^{\frac{n+j}{2}}\frac{1}{2^{n}}\frac{(n+j)!}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}((n+j)_{\equiv 2}-1)\cos\left(\tfrac{I}{2}\right)^{2n+m+j-2\ell}\sin\left(\tfrac{I}{2}\right)^{2\ell-m-j} \tag{30}$$

the potential can be rearranged as:

$$U(r,\theta,\nu) = -\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^{n}\sum_{m=0}^{n}\sum_{j=-n}^{n}\bar{\mathcal{G}}_{n,m,j}(I)\cdot$$

$$\cdot \left(C_{n,m}\left(\cos\left(j\theta - m\nu\right)\cos\left(\tfrac{\pi}{2}(j+m)\right) - \sin\left(j\theta - m\nu\right)\sin\left(\tfrac{\pi}{2}(j+m)\right)\right) +\right.$$

$$\left. + S_{n,m}\left(-\sin\left(j\theta - m\nu\right)\cos\left(\tfrac{\pi}{2}(j+m)\right) - \cos\left(j\theta - m\nu\right)\sin\left(\tfrac{\pi}{2}(j+m)\right)\right)\right) \tag{31}$$

Finally we express the potential in a way that will be useful to apply the relegation algorithm described in the next section:

$$U(r,\theta,\nu) = -\frac{\mathcal{G}M}{r}\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^{n}\sum_{m=0}^{n}\sum_{j=-n}^{n}\left(\mathcal{G}_{n,m,j}^{1}(I)\cos\left(m\nu - j\theta\right) + \mathcal{G}_{n,m,j}^{2}(I)\cdot\right.$$
$$\left. \cdot \sin\left(m\nu - j\theta\right)\right), \tag{32}$$

with:

$$\mathcal{G}_{n,m,j}^{1}(I) = \bar{\mathcal{G}}_{n,m,j}(I)\left(C_{n,m}\cos\left(\tfrac{\pi}{2}(j+m)\right) - S_{n,m}\sin\left(\tfrac{\pi}{2}(j+m)\right)\right)$$

$$\mathcal{G}_{n,m,j}^{2}(I) = \bar{\mathcal{G}}_{n,m,j}(I)\left(C_{n,m}\sin\left(\tfrac{\pi}{2}(j+m)\right) + S_{n,m}\cos\left(\tfrac{\pi}{2}(j+m)\right)\right), \tag{33}$$

which is completely equivalent to the one in (22)

Remark 1 *The direct comparison with the generic formula in [20] and the restricted one in [21] (considering only the ellipticity and the oblateness terms $C_{2,0}$ and $C_{2,2}$ respectively) highlights some errors in these formulas.*

This work derived the potential using two completely different methods illustrated in the previous subsections. Both approaches lead to exactly the same result (i.e. eq. (32)) which infers that there are errors in stated expressions for the gravitational potential in polar-nodal form in the literature.

Therefore a useful representations of irregular gravitational fields has been derived and verified using two approaches.

3 The relegation of the argument of node ν

The usual technique used for implementing closed form analytical theories is the Delaunay normalization [12] that converts the principal part of the Hamiltonian into an integral of the transformed system; however it cannot be directly applied to our problem as the argument of the node is present in the Hamiltonian. The presence of the argument of node prevents the standard computation of the generator of the Lie transformation.

To overcome this problem we first perform a simplification of the Hamiltonian making use of Deprit's relegation algorithm [13]. This procedure uses repeated iterations of a transformation which decreases the importance of the perturbation in the new Hamiltonian until it can be neglected.

Taking the Hamiltonian (4) with the potential expressed as in (32) we rearrange it as a power series

$$H = \sum_{k \geq 0} \frac{\epsilon^k}{k!} H_k$$

in the generic small parameter ϵ as:

$$H(r, \theta, \nu, R, \Theta, N) = \tfrac{1}{2}(R^2 + \tfrac{\Theta^2}{r^2}) - \tfrac{M\mathcal{G}}{r} - \omega N + \epsilon U(r, \theta, \nu) \qquad (34)$$

with

$$U(r, \theta, \nu) = -\tfrac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{\infty} \left(\tfrac{\alpha}{r}\right)^n \sum_{m=0}^{n} \sum_{j=-n}^{n} \left(\mathcal{G}_{n,m,j}^1(I) \cos\left(m\nu - j\theta\right) + \right. \\ \left. + \mathcal{G}_{n,m,j}^2(I) \sin\left(m\nu - j\theta\right)\right) \qquad (35)$$

and $\mathcal{G}_{n,m,j}^1(I)$ and $\mathcal{G}_{n,m,j}^2(I)$ as in (33) and $\bar{\mathcal{G}}_{n,m,j}(I)$ as in (30).

Following the algorithm in [13] we now want to "relegate" the action of the argument of latitude ν to high orders of $\frac{1}{r}$ (i.e. to find a Lie transformation which maps the Hamiltonian into a new one where the variable ν appears only from a certain, fixed power n of $\frac{1}{r^n}$ on).

3.1 Lie transformations

In order to relegate the action of the angle ν to high orders of the Hamiltonian an to normalize the trunked Hamiltonian later on we will perform a Lie Transformation. A short description of Deprit's method for Lie Transformations, with respect to our application, is here provided, while for a full description of it and a comparison with the Von Zeipel's method which determines the Lie transformations we refer the reader to [9].

Definition 2 *A Lie transformation ϕ is a one-parameter family of mappings $\phi : (y, Y; \epsilon) \to (x, X)$, defined by the solution $x(y, Y; \epsilon)$ and $X(y, Y; \epsilon)$ of the Hamiltonian system*

$$\begin{cases} \frac{dx}{d\epsilon} = \frac{\partial W}{\partial X} \\ \frac{dX}{d\epsilon} = -\frac{\partial W}{\partial x} \end{cases}$$

With initial conditions $x(y, Y; 0) = y$ and $X(y, Y; 0) = Y$, and where the function

$$W(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} W_{s+1}(x, X)$$

is the generator of the transformation.

Due to the properties of the Hamiltonian systems, the Lie transformation ϕ is a completely canonical transformation that maps an Hamiltonian

$$H(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} H_s(x, X)$$

onto an equivalent Hamiltonian K of the form

$$K(y, Y; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} K_s(y, Y; 0).$$

We call

$$\begin{aligned} &\Delta_W^0 H := H \\ &\Delta_W^1 H := \Delta_W H = L_W(H) + \frac{\partial H}{\partial \epsilon} \\ &\Delta_W^2 H := \Delta_W(\Delta_W H) \\ &\dots \end{aligned} \tag{36}$$

where $L_W(\cdot)$ is the Lie derivative [2] with respect to W, defined as $L_W(\cdot) := [\,\cdot\,; W]$ where $[\,\cdot\,;\,\cdot\,]$ stands for the Poisson Brackets [3].

The s element $K_s(y, Y; 0)$ of the transformed Hamiltonian is given by applying s-times the Lie derivative generated by W to the Hamiltonian H and substituting $\epsilon = 0$ therein.

$$K_s(y, Y; 0) := (\Delta_W^s H)_{\epsilon=0} \tag{37}$$

that is:

$$K_1(y, Y; 0) = \left([H; W] + \frac{\partial H}{\partial \epsilon} \right)_{\epsilon=0}$$

$$K_2(y, Y; 0) = \left(\left[\left([H; W] + \frac{\partial H}{\partial \epsilon} \right); W \right] + \frac{\partial \left([H; W] + \frac{\partial H}{\partial \epsilon} \right)}{\partial \epsilon} \right)_{\epsilon=0} \tag{38}$$

$\dots$

However, considering H and W as power series of ϵ:

$$H(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} H_s(x, X)$$

---

[2] Let $W(X, x)$ be a differentiable mapping of A into C. The Lie derivative $\mathcal{L}_W$ induced by W is the operator from A to C $L_W : A \to C$ that maps any function $f(X, x)$ into its Poisson Bracket with W, namely $f(X, x) :\to [f; W]$

[3] Let A be an open subset of $C^n \times C^n$. If the mappings $f(X, x)$ and $g(X, x)$ from A to C are differentiable in A, the Poisson bracket of f and g ($[f; g]$), in that order, is the mapping from A to C ($[f; g] : A \to C$) that maps $(X, x) \to D_2 f(X, x) \cdot D_1 g(X, x) - D_1 f(X, x) \cdot D_2 g(X, x)$.

$$W(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} W_{s+1}(x, X)$$

and the properties of the Poisson brackets, the terms in (38) can be explicitly derived algorithmically, yielding:

$$
\begin{aligned}
K_1(y, Y; 0) &= [H_0; W_1] + H_1 \\
K_2(y, Y; 0) &= (H_2 + 2[H_1; W_1] + [[H_0; W_1]; W_1]) + [H_0, W_2] \\
&\dots
\end{aligned}
\tag{39}
$$

Which decomposes a Lie transformation into a series of so called homologic equations

$$[H_0; W_s] + \tilde{H}_s = K_s \quad \forall s \geq 1 \tag{40}$$

where the terms $\tilde{H}_s$ are found constructing the Lie triangle of provisional elements

$$\tilde{H}_0 := H_0$$

$$\tilde{H}_1 := H_1 \quad \tilde{H}_{1,0} := \tilde{H}_1$$

$$\tilde{H}_2 := H_2 \quad \tilde{H}_{1,1} \quad \tilde{H}_{2,0} := \tilde{H}_2 \tag{41}$$

$$\tilde{H}_3 := H_3 \quad \tilde{H}_{2,1} \quad \tilde{H}_{1,2} \quad \tilde{H}_{3,0} := \tilde{H}_3$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

by using

$$
\begin{cases}
\tilde{H}_{1,0} := \tilde{H}_1 := H_1 \\
\tilde{H}_{i,s-i} = H_s + \sum_{j=0}^{s-2} \binom{s-i}{j} [H_{s-j-i}; W_{j+1}] \quad if \quad i = 1, \ s \geq 2 \\
\tilde{H}_{i,s-i} = \tilde{H}_{i-1,s-i+1} + \sum_{j=0}^{s-i} \binom{s-i}{j} [\tilde{H}_{i-1,s-j-i} + [H_0; W_{s-j-1}]; W_{j+1}] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad if \quad i \neq 1, \ s \geq 2
\end{cases}
\tag{42}
$$

and taking the last element of each row, which, by (42), is given by

$$
\begin{aligned}
\tilde{H}_1 &:= H_1 \\
\tilde{H}_s &= \tilde{H}_{s-1,1} + [\tilde{H}_{s-1,0} + [H_0; W_{s-1}]; W_1] \quad \forall s \geq 2
\end{aligned}
\tag{43}
$$

Such series of homological equations can be both seen as
- the way to find the coefficients of the transformed Hamiltonian given a generating function of a Lie transformation
- or a way to find the generating function of the Lie transformation that maps the initial Hamiltonian into a prescribed one Hamiltonian.

The relegation and the normalization algorithms (see [13] and [12] respectively) are two different methods which will be used to solve the homological equations.
The normalization is the Lie transformation that maps a Hamiltonian

$$H = \sum_{s \geq 0} \frac{\epsilon^s}{s!} H_s$$

into an equivalent one which admits the principal term H_0 as integral of the transformed system [10].

The relegation, instead, generalizes the normalization to some exceptional situations the normalization process cannot deal with (such as secular terms or small divisors).

For the relegation algorithm, in contrast to normalization, the criteria for selecting the elements of the transformed Hamiltonian

$$K(Y, y, \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} (K_s(Y, y) + R_s(Y, y))$$

are based not on the principal term $H_0 = F + G$ but only on a part of it: the function G, which will become the integral of a truncated part of the transformed system.

3.2   The Relegation Algorithm

The relegation is applied to all those problems where:

- the principal term is a sum $H_0 = F + G$,

- the Poisson bracket $[F; G]$ is zero

- the Lie derivative $\mathcal{L}_G$ is semi-simple over a Poisson algebra of functions P.

Given the function $G(X, x)$, the relegation algorithm is a Lie transformation in which maps the hamiltonian

$$H(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} H_s(x, X)$$

into an equivalent one of the form:

$$K(Y, y, \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} (K_s(Y, y) + R_s(Y, y))$$

with $K_0 = H_0(y, Y)$ and the coefficients $K_s \in ker(\mathcal{L}_G)$.

The effect of ϕ on H is best described by saying that ϕ relegates the action of G on H to the function

$$R(Y, y, \epsilon) = \sum_{s \geq 1} \frac{\epsilon^s}{s!} R_s(Y, y)$$

since $[K; G] = [H_0; G] + [R; G]$.

In contrast with normalization, the term K_s may not belong to $ker(\mathcal{L}_G)$ due to the presence of the residual R_s. However, in practice the algorithm produces a useful approximation when the contribution of the residual R_s to K_s decreases as the iteration count p increases. Ideally, one should choose the function G so that $\lim_{p \to \infty} |R_s| = 0$ at each order s. In summary, the relegation algorithm

acting on the original Hamiltonian H produces two distinct Hamiltonians:
the first one

$$K = \sum_{s \geq 0} \frac{\epsilon^s}{s!} K_s = \sum_{s \geq 0} \frac{\epsilon^s}{s!} \left(\sum_{j=0}^{p} (K_{s,p}) + R_s \right) \tag{44}$$

which is the image of the original system by the relegating transformation ϕ,
which, in general, does not admit G as an integral,
and the second one

$$J = \sum_{s \geq 0} \frac{\epsilon^s}{s!} \sum_{j=0}^{p} K_{s,p} \tag{45}$$

which is a truncated system, close to the first one, which admits G as an integral.

For each homological equation

$$[H_0; W_s] + \tilde{H}_s = K_s \tag{46}$$

the relegation algorithm starts considering that, as $\mathcal{L}_G$ is semi-simple, there
$\exists K_{s,0}, W_{s,0} \in P$ s.t.

$$\begin{cases} \tilde{H}_s = K_{s,0} + [W_{s,0}; G] \\ K_{s,0} \in Ker(\mathcal{L}_G). \end{cases} \tag{47}$$

Therefore (46) becomes:

$$[H_0; W_s] + [W_{s,0}; G] = K_s - K_{s,0}. \tag{48}$$

Thus, setting $W_s = W_{s,0}^* + W_{s,0}$, (48) becomes:

$$[H_0; W_s^*] + [H_0 - G; W_{s,0}] = K_s - K_{s,0}. \tag{49}$$

The algorithm continues re-invoking p-times the semi-simplicity of $\mathcal{L}_G$, and
finding $\forall 1 \leq j \leq p$ $K_{s,j}, W_{s,j} \in P$ s.t.

$$\begin{cases} [H_0 - G; W_{s,j-1}] = K_{s,j} + [W_{s,j}; G] \\ K_{s,j} \in Ker(\mathcal{L}_G) \end{cases} \tag{50}$$

and setting p-times $\forall 1 \leq j \leq p$ $W_{s,j-1} = W_{s,j}^* + W_{s,j}$.

Finally the algorithm ends at a certain iteration p setting $W_{s,p}^* = 0$ and
obtaining (49) to become:

$$K_s = \sum_{j=0}^{p} (K_{s,j}) + R_s \tag{51}$$

with $R_s := [H_0 - G; W_{s,p}]$

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3.3 Application

As suggested in ([21]), we now want to apply the Relegation algorithm to the Hamiltonian (34) with the potential truncated to the terms of order $\sim \frac{1}{r^{n_{max}+1}}$ (included), therefore we set:

$$H_K := \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{M\mathcal{G}}{r}$$
$$G = -\omega N, \tag{52}$$

obtaining that:

$$H_0 = H_K + G$$
$$H_1 = U_{nmax} H_j = 0 \quad \forall j \geq 2 \tag{53}$$

with

$$U_{nmax} := -\frac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{n_{max}} \left(\frac{\alpha}{r}\right)^n \sum_{m=0}^{n} \sum_{j=-n}^{n} \left(\mathcal{G}_{n,m,j}^1(I) \cos{(m\nu - j\theta)} + \right.$$
$$\left. + \mathcal{G}_{n,m,j}^2(I) \sin{(m\nu - j\theta)}\right) \tag{54}$$

and $\mathcal{G}_{n,m,j}^1(I)$, $\mathcal{G}_{n,m,j}^2(I)$ as in (33).

It is first noticed that

$$[\,\cdot\,; G] = [\,\cdot\,; -\omega N] = -\omega \frac{\partial \cdot}{\partial \nu} \tag{55}$$

and that

$$[H_K;\,\cdot\,] = [\frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{M\mathcal{G}}{r};\,\cdot\,] = R\frac{\partial \cdot}{\partial r} + \frac{\Theta}{r^2}\frac{\partial \cdot}{\partial \theta} - (\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2})\frac{\partial \cdot}{\partial R} \tag{56}$$

$$* * * * *$$

First Homological Equation:

By (43): $\tilde{H}_1 = H_1$.
Therefore:

$$[H_0; W_1] + U_{nmax} = K_1 \tag{57}$$

First Relegation iteration:

As in (47) the first iteration consists in finding $K_{1,0}$ and $W_{1,0}$ such that

$$\begin{cases} U_{nmax} = K_{1,0} + [W_{1,0}; G] \\ K_{1,0} \in Ker(L_G). \end{cases} \tag{58}$$

That is

$$\begin{cases} U_{nmax} = K_{1,0} - \omega\frac{\partial W_{1,0}}{\partial \nu} \\ \frac{\partial K_{1,0}}{\partial \nu} = 0. \end{cases} \tag{59}$$

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Thus $K_{1,0}$ is the collection of all the terms of U_{nmax} such that their derivative with respect to ν is zero, namely the collection of all the terms of U_{nmax} which does not depend on ν, therefore:

$$K_{1,0} = -\frac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{n_{max}} \left(\frac{\alpha}{r}\right)^n \sum_{j=-n}^{n} \left(\mathcal{G}_{n,0,j}^1(I)\cos(-j\theta) + \mathcal{G}_{n,0,j}^2(I)\sin(-j\theta)\right)$$

(60)

Then, inverting (59):

$$
\begin{aligned}
W_{1,0} &= -\frac{1}{\omega}\int (U_{nmax} - K_{1,0})d\nu \\
&= -\frac{1}{\omega}\int\left(-\frac{\mathcal{G}M}{r\epsilon}\sum_{n=1}^{n_{max}}\left(\frac{\alpha}{r}\right)^n\sum_{m=1}^{n}\sum_{j=-n}^{n}\left(\mathcal{G}_{n,m,j}^1(I)\cos(m\nu - j\theta)+\right.\right. \\
&\quad \left.\left. +\mathcal{G}_{n,m,j}^2(I)\sin(m\nu - j\theta)\right)\right)d\nu
\end{aligned}
$$

(61)

which is periodic in ν.

Second Relegation iteration:

By (50) we first evaluate

$$[H_0 - G; W_{1,0}] = R\frac{\partial W_{1,0}}{\partial r} + \frac{\Theta}{r^2}\frac{\partial W_{1,0}}{\partial \theta} - \left(\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2}\right)\frac{\partial W_{1,0}}{\partial R}$$

(62)

with $W_{1,0}$ is as in (61).
Notice that $[H_0 - G; W_{1,0}]$ is still ν-periodic.

Now we have to find $K_{1,1}$ and $W_{1,1}$ such that:

$$\begin{cases} [H_0 - G; W_{1,0}] = K_{1,1} + [W_{1,1}; G] \\ K_{1,1} \in Ker(\mathcal{L}_G) \end{cases}$$

(63)

That is

$$\begin{cases} [H_0 - G; W_{1,0}] = K_{1,1} - \omega\frac{\partial W_{1,1}}{\partial \nu} \\ \frac{\partial K_{1,1}}{\partial \nu} = 0 \end{cases}$$

(64)

Thus $K_{1,1}$ is the collection of all the terms of $[H_0 - G; W_{1,0}]$ such that their derivative with respect to ν is zero, namely the collection of all the terms of $[H_0 - G; W_{1,0}]$ which does not depend on ν, but $[H_0 - G; W_{1,0}]$ is periodic in ν, therefore

$$K_{1,1} = 0$$

(65)

And, inverting (64),

$$W_{1,1} = -\frac{1}{\omega}\int[H_0 - G; W_{1,0}]d\nu$$

(66)

Other Relegation iterations:

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In complete analogy with the second iteration of the relegation it is found that $\forall 2 \leq j \leq p$:

$$K_{1,j} = 0$$
$$W_{1,j} = -\frac{1}{\omega} \int [H_0 - G; W_{1,j-1}] d\nu \qquad (67)$$

Conclusion of Relegation for the first homological equation:

It must be noted that, at each step of relegation $1 \leq j \leq p$ the "remainder"

$$[H_0 - G; W_{1,j}] = R\frac{\partial W_{1,j}}{\partial r} + \frac{\Theta}{r^2}\frac{\partial W_{1,j}}{\partial \theta} - \left(\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2}\right)\frac{\partial W_{1,j}}{\partial R} \qquad (68)$$

is of order $\frac{1}{r^{j+2}}$.

As we have considered only the potential up to the terms $\sim \frac{1}{r^{n_{max}+1}}$ we will stop the relegation algorithm once the reminder is of the same order of the first neglected terms, i.e. of order $\sim \frac{1}{r^{n_{max}+2}}$, thus we set the maximum order of relegation to be $p = n_{max} - 1$.

Thus, once we have relegated $n_{max} - 1$-times, for the first homological equation we obtain:

$$W_1 = \sum_{j=0}^{p-1} W_{1,j}$$
$$K_1 = \sum_{j=0}^{p-1} K_{1,j} + R_1 = K_{1,0} \qquad (69)$$
$$R_1 = [H_0 - G; W_{1,n_{max}-1}]$$

With $K_{1,0}$ as in (60) and where in the second equation the reminder has been dropped at it made of orders higher than $\frac{1}{r^{n_{max}+1}}$ which must be truncated.

Second (and higher) Homological Equations:

By (43), considering that in (53) we have set $H_2 = 0$:
$\tilde{H}_2 = 2[H_1, W_1] + [[H_0; W_1]; W_1]$.
By direct calculations it can be easily verified that $\tilde{H}_2 \sim \frac{1}{r^8}$.
Therefore it must be noticed that we have to fix n_{max} to be greater than 8 to get any results from the second homological equations.

We set the second homological equation

$$[H_0; W_2] + \tilde{H}_2 = K_2 \qquad (70)$$

and start the relegation by setting $K_{2,0}$ equal to the terms of $\tilde{H}_2$ which does not depend on ν and $W_{2,0} = -\omega \int (\tilde{H}_2 - K_{2,0}) d\nu$ and so on.

3.4 Conclusion

As it is almost impossible to deal with so many terms "by hand", a major component of the project has been committed to building two different software scripts, written in Mathematica and Piranha, which can **deal with any arbitrary order** n_{max} of terms of the potential **as well as any arbitrary degree** s_{max} of homological equations returning as output the transformed Hamiltonian

$$K(r, \theta, R, \Theta, N) = \sum_{s=0}^{s_{max}} \frac{\epsilon^s}{s!} K_s$$

and the transformation

$$W(r, \theta, \nu, R, \Theta, N) = \sum_{s=0}^{s_{max}} \frac{\epsilon^s}{s!} W_s.$$

Remark 2 *Note that fixing the maximum order n_{max} of terms of the potential means that all the terms up to $C_{nmax,nmax}$ and $S_{nmax,nmax}$ are taken into account, which means a total of $(nmax+2)(nmax+1)$ coefficients of the potential.*

Remark 3 *It must be remarked that one of the main tasks contained in the Ariadna study call for proposals was to "determine whether modern computer performances are sufficiently developed to carry out the development of an analytical perturbative theory using the Lie series approach for an irregularly-shaped small body (e.g. asteroid)". We can conclude that both the Piranha and Mathematica software can easily deal with the extensive computations involved when computing the Lie series associated with the relegation algorithm.*

Moreover, as the transformed Hamiltonian $K(r, \theta, R, \Theta, N)$, obtained from this process, no longer depends on the argument of the node ν, the variable N becomes cyclic and therefore $-\omega N$ is a constant term, which can be dropped from the Hamiltonian.

Remark 4 *To bring the analysis to the consideration of more than just the first homological equation represents a generalisation of the usual procedure found in literature (see for example [21] and [23]). The latter usually considers only the first Homological equation due to the high number of terms which must be considered to get a coherent result for the second homological equation.*
This, together with the full consideration of all the tesseral and sectorial coefficients (i.e. not only restricted to the ellipticity and oblateness terms) provides a more refined approximation of the system than previous theories. This leads to a more accurate description of the behavior of the system.

4 Delaunay Normalization

After the elimination of the argument of node the Hamiltonian is equivalent to the one in the main problem of the artificial satellite, in which the longitude of the node is cyclic and, hence, the Coriolis term $-\omega N$ becomes constant and may be deleted. Then a Delaunay normalization can be performed, for a further reduction of the degrees of freedom, thus transforming the Hamiltonian into an integrable one. The Delaunay normalization is classically implemented in closed form, i.e. without using series expansion in the eccentricity, only to the first order of tesseral/sectorial coefficients and first degree in the homologic equations. An arbitrary order application for the first homologic equation is thus developed, enabled by changing the independent variable to be the true anomaly. It has been demonstrated that such procedure can be carried on in an automated way up to an arbitrary order of tesseral/sectorial coefficients for the first Homologic equation, but can only be developed in a semi-automated way for the higher homologic equations.

In order to perform the Delaunay normalisation the Hamiltonian must be changed from the Wittaker to the so called Delaunay coordinates.

4.1 Delaunay Coordinates

The Delaunay coordinates are symplectic action-angle variables (L, G, H, ℓ, g, h), where the angles ℓ, g and h are conjugated to the actions L, G and H respectively, where

- ℓ is the *mean anomaly* measured from the pericenter;

- g is the argument of the pericenter;

- h is the argument of the node;

- L is related to the major semi-axis, a, by $L = \sqrt{\mathcal{G}Ma}$;

- G is the *total angular momentum* of the spacecraft with respect to the Asteroid (in the inertial frame), related to the eccentricity and the variable L by $e = \sqrt{1 - \frac{G^2}{L^2}}$;

- H is the z-component of the total angular momentum, i.e. $H = G \cos I$.

Moreover the relation between the *True anomaly* and the *Eccentric anomaly u* is defined as:

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{u}{2}\right), \tag{71}$$

which, in particular, implies

$$r = a(1 - e \cos u) = a \frac{1 - e^2}{1 + e \cos f}. \tag{72}$$

A quick derivation of such coordinates is here provided, while a full derivation can be found in [3] and [8] (also see [15], [25]).

The change of coordinates which brings from the Wittaker variables to the Delaunay ones is generated by the function

$$S = \int_{r_-(-\frac{(\mathcal{G}M)^2}{L^2})}^{r} \sqrt{-\frac{G^2}{r^2} + 2\frac{M\mathcal{G}}{r} - \frac{(\mathcal{G}M)^2}{L^2}} \, d\rho + G(\theta - f) + H\nu \tag{73}$$

It is completely canonical as

$$d\ell \wedge dL + dg \wedge dG + dh \wedge dH = dr \wedge dR + d\theta \wedge d\Theta + d\nu \wedge dN \tag{74}$$

that is:

$$\begin{aligned}
R &= \frac{\partial S}{\partial r} = \sqrt{-\frac{\Theta^2}{r^2} + 2\frac{M\mathcal{G}}{r} - \frac{(\mathcal{G}M)^2}{L^2}} \\
\Theta &= \frac{\partial S}{\partial \theta} = G \\
N &= \frac{\partial S}{\partial \nu} = H \\
\ell &= \frac{\partial S}{\partial L} = \dots = u - e \sin u \\
g &= \frac{\partial S}{\partial G} = (\theta - f) \\
h &= \frac{\partial S}{\partial H} = \nu.
\end{aligned} \tag{75}$$

Plus, by Section 6.1.2 we know that

$$N = G \cos I \quad \Rightarrow \quad H = G \cos I. \tag{76}$$

The Hamiltonian obtained by the Relegation algorithm

$$\begin{aligned}
K_0 &:= H_0 = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{M\mathcal{G}}{r} \\
K_1 &= -\frac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{n_{max}} \left(\frac{\alpha}{r}\right)^n \dots.
\end{aligned} \tag{77}$$

where we have dropped the Coriolis terms from H_0.
In the Delaunay variables the new Hamiltonian

$$K' = \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} K'_s$$

becomes:

$$\begin{aligned}
K'_0 &= -\frac{(\mathcal{G}M)^2}{2L^2} \\
K'_1 &= -\frac{\mathcal{G}M}{\epsilon} \sum_{n=1}^{n_{max}} \alpha^n \left(\frac{(a(1-e^2))}{(1+e\cos f)}\right)^{n+1} \sum_{j=-n}^{n} \left(\mathcal{G}^1_{n,0,j}(G,H)\cos\left(-j(f+g)\right)+ \right. \\
&\quad \left. +\mathcal{G}^2_{n,0,j}(G,H)\sin\left(-j(f+g)\right)\right) \\
&\dots
\end{aligned} \tag{78}$$

4.2 The Normalization algorithm

The full, arbitrary order theory is here illustrated, which, instead of using the expansions of r and f in powers of the eccentricity, computes the integrals with respect to ℓ changing the independent variable to be the true anomaly f.

Definition 3

A formal series $J(y, Y, \epsilon) = \sum\limits_{s=0}^{\infty} \dfrac{\epsilon^s}{s!} J_s(y, Y)$ is said to be in Delaunay normal form if the Lie derivative $L_{J_0} J$ is zero, that is $[J_s, J_0] = 0 \ \ \forall s \geq 0$.

In our case, as $J_0 = K_0' = -\dfrac{(\mathcal{G}M)^2}{2L^2}$, the Lie derivative

$$L_{J_0}(\cdot) = \frac{(\mathcal{G}M)^2}{L^3} \frac{\partial(\cdot)}{\partial \ell}$$

therefore the new Hamiltonian $J(y, Y, \epsilon) = \sum\limits_{s=0}^{\infty} \dfrac{\epsilon^s}{s!} J_s(y, Y)$ will be in normal form if and only if

$$\frac{\partial J_s}{\partial \ell} = 0 \ \ \forall s \geq 1$$

As in (46) for every homologic equation:

$$
\begin{aligned}
&[J_0; \bar{W}_s] + \tilde{K}_s' = J_s \\
\Leftrightarrow\ &-\frac{(\mathcal{G}M)^2}{L^3} \frac{\partial \bar{W}_s}{\partial \ell} + \tilde{K}_s' = J_s
\end{aligned}
\tag{79}
$$

with $\tilde{K}_s'$ found as in (43).

Now, as we want J_s to be in Delaunay normal form ($\Leftrightarrow \frac{\partial J_s}{\partial \ell} = 0 \forall s \in \mathbb{N}$), we set

$$J_s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{K}_s' d\ell \tag{80}$$

This integral is solved by changing the independent variable from ℓ to be the true anomaly f by the relation

$$\frac{df}{d\ell} = \frac{df}{du} \frac{du}{d\ell} = \left(\frac{1 + e \cos f}{\sqrt{1 - e^2}} \right) \left(\frac{1}{1 - e \cos u} \right) = \frac{r^2}{a^2 \sqrt{1 - e^2}} \tag{81}$$

Thus, inverting (79), yields the first order of the generating function that is:

$$\bar{W}_s = \int \frac{L^3}{(\mathcal{G}M)^2} \left(\tilde{K}_s' - J_s \right) d\ell = \int \frac{L^3}{(\mathcal{G}M)^2} \left(\tilde{K}_s' - \frac{1}{2\pi} \int_0^{2\pi} \tilde{K}_s' d\ell \right) d\ell \tag{82}$$

Remark 5 *The script to execute this algorithm in an automated way has been developed, which normalizes the first homologic equation considering an arbitrary number of Stokes coefficients. However, for higher homologic equations, the process cannot be iterated in an automated way as there appear some terms with both the mean anomaly ℓ and the true anomaly f which must be solved considering each singular case alone (i.e. can only be processed with a semi-automated procedure).*

Remark 6 *This leads to an integrable, two degrees of freedom, Hamiltonian which is an arbitrary order in tesseral/sectorial coefficients approximation of*

the initial Hamiltonian for the first homologic equation. Which can now be applied (using the computer algebra system built) to every specific asteroid in order to determine possible orbits useful for scientific observational missions such as frozen orbits.

Moreover, restricting the result to the case where all the $S_{n,m}$ and $C_{n,m}$ coefficients to be zero except for the ellipticity and oblateness ones, the correct versions of the integrable Normalized Hamiltonian contained in [21] and [23] are found.

Finally it must be noted that considering arbitrary order of tesseral/sectorial coefficients the resulting Hamiltonian, will, in general, contain both the relegated variables G and g, thus containing one variable more than the final Hamiltonian in [21]. This leads to a system still integrable but where the solution cannot be explicitly solved, i.e.it is no longer "trivially integrable" as in [21].

5 Frozen Orbits

The Hamiltonian obtained is of the form: $J(L, G, H, \_, g, \_)$ thus the equations of motion are:

$$
\begin{aligned}
\ell'(t) &= \frac{\partial J}{\partial L} \\
g'(t) &= \frac{\partial J}{\partial G} \\
h'(t) &= \frac{\partial J}{\partial H} \\
L'(t) &= 0 \\
G'(t) &= -\frac{\partial J}{\partial g} \\
H'(t) &= 0,
\end{aligned}
\tag{83}
$$

which means that L and H are constants and all the other motions will only depend on $G(t)$ and $g(t)$.

Definition 4 *(Frozen orbit)*
A frozen orbit is an orbit in which the Inclination, the Eccentricity and the Argument of perigee remains constant during the motion.

This in particular implies that such an orbit is then perfectly periodic except for the orbital plane precession.
Thus, in order to get a frozen orbit we must set to zero

$$
\begin{aligned}
\dot{e} &= \frac{d}{dt} \frac{\sqrt{L^2 - G^2}}{L} = 0 \\
\dot{I} &= \frac{d}{dt} arcsin \frac{\sqrt{H}}{G} = 0 \\
\dot{g} &= 0.
\end{aligned}
\tag{84}
$$

For the relegated system it is equivalent to solve

$$
\begin{aligned}
\dot{G} &= 0 \\
\dot{g} &= 0.
\end{aligned}
\tag{85}
$$

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Thus fixing eccentricity e and inclination I for the desired frozen orbit the solutions (L_0, G_0, H_0, g_0) of the system

$$
\begin{aligned}
\dot{G} &= 0 \\
\dot{g} &= 0 \\
e &= \frac{\sqrt{L^2 - G^2}}{L} \\
I &= arcsin \frac{\sqrt{H}}{G},
\end{aligned}
\tag{86}
$$

are the initial conditions that lead to a frozen orbit .

Some examples of frozen orbits have been generated for the asteroid 433 Eros of the main belt which are contained in the next section.

6 Application

The script developed and provided with the report is divided into three main parts:
- the relegation of the argument of node
- the change to Delaunay variables and normalization (i.e. averaging over the mean anomaly)
- the integration of the reduced system.

Fixing the maximum order of coefficients of the potential to be taken into account $nmax$ (i.e. $(nmax+2)(nmax+1)$ coefficients in total) the system yields the resulting relegated and normalized Hamiltonians and the functions that generate the two changes of coordinates that lead to them. Moreover, choosing a reference asteroid and fixing the initial conditions $\mathbf{x_0} = (x_0, y_0, z_0)$ and $\mathbf{X_0} = (X_0, Y_0, Z_0)$ in the cartesian system of reference (or in any other system of coordinates) it integrates and displays the orbit obtained in the normalized Delaunay system taken into account all the $(nmax + 2)(nmax + 1)$ terms of the potential; the accuracy of the orbit obtained will increase with the order of coefficients taken into account as well as with the degree of homologic equations normalized and relegated. Finally, fixing a desired eccentricity and inclination, it gives in output the initial conditions that yield to the frozen orbit required (if it exist) and display the orbit obtained. Again the accuracy of the result increases with the number of tesseral/sectorial coefficients taken into account.

6.1 The relegation script

INPUT REQUIRED:

· smax= max order of homologic equations to be considered for the relegation;

· nmax= max degree of tesseral/sectorial coefficients to be included;

OUTPUT:

· Seconds used for the computation

· Hamiltonian in the relegated polar-nodal variables $K(r^*, \theta^*, \_, R^*, \Theta^*, N^*)$ where the $*$ has been dropped to simplify notation.

· The generating function $W(r, \theta, \nu, R, \Theta, N)$ of the transformation of coordinates which is conserved by the relegation (i.e. $W(r^*, \theta^*, \nu^*, R^*, \Theta^*, N^*) = W(r, \theta, \nu, R, \Theta, N)$).

6.2 the normalization script

INPUT REQUIRED:
· it automatically takes the relegated Hamiltonian as input.
OUTPUT:
· Seconds used for the computation
· Hamiltonian in the normalized Delaunay variables $J(L, G, H, \_, g, \_)$ where the * has been dropped to simplify notation.
· The generating function $U(L, G, H, f, l, g, \_)$ of the transformation of coordinates which is conserved by the normalization.

6.3 Integration of the system

INPUT REQUIRED:
· it automatically takes the Notmalized Hamiltonian as input.
· tesseral/sectorial coefficients of the asteroid.
· total mass of the asteroid taken in Kg.
· angular velocity ω of rotation of the asteroid around the z-axes in s^{-1}.
· reference radius in m the asteroid .
OUTPUT:
· Equations of motion of $\dot{G}^* = -\frac{\partial J}{\partial g}$ and $\dot{g}^* = \frac{\partial J}{\partial G}$

6.4 Analysis of the computational complexity of the algorithms

The table (6.4) below collects the time used to process the relegation algorithm using the software Mathematica 6.0, depending on the maximum order n_{max} of coefficients to be taken into account (i.e. including all the coefficients up to $C_{nmax,nmax}$ and $S_{nmax,nmax}$) and the number of homologic equations s_{max} considered. The coefficients marked with * have been estimated by extrapolation of previous data.

$\downarrow n_{max}$ \\ $s_{max} \rightarrow$	1	2	3	4	5
1 (6 coefficients)	0.015	0.062	0.078	0.109	0.125
2 (12 coefficients)	0.312	0.437	0.578	1.233	1.966
3 (20 coefficients)	2.356	2.886	3.962	13.073	79.779
4 (30 coefficients)	11.216	13.182	21.326	151.789	2092.49
5 (42 coefficients)	42.183	49.624	114.068	1768.13	39329.1
6 (56 coefficients)	79.482	108.436	505.257	18560.1	145079*
7 (72 coefficients)	115.051	217.466	2316.47	64225.7*	352630*
8 (90 coefficients)	266.886	906.257	6669.28*	152463*	695272*
9 (110 coefficients)	707.667	2704.35*	14685.3*	296970*	1.20629×10^6*
10 (132 coefficients)	1610.07*	6141.29*	27486*	511444*	1.9189×10^6*

Table 1: Relegation Algorithm:Seconds per coefficients/homologic equations considered

The data in the table are summarized in Figure (4), each line has been

plotted considering $Smax$ homologic equations. The Figure shows the time (in hours) used for processing the coefficients up to $C_{nmax,nmax}$ and $S_{nmax,nmax}$ (n_{max} displayed on the horizontal axes). Notice that, given n_{max}, the coefficients considered are $(n_{max} + 1)(n_{max} + 2)$.
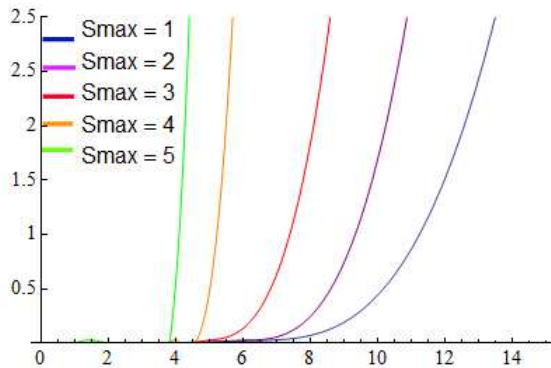

Figure 3: *Relegation Algorithm: hours per maximum order of coefficients/homologic equations considered*

The Table (6.4) below collects the time used from to process the normalization algorithm using the software Mathematica 6.0, depending on the maximum order n_{max} of coefficients to be taken into account (i.e. including all the coefficients up to $C_{nmax,nmax}$ and $S_{nmax,nmax}$) considering the first homologic equation.

n_{max}	s_{max}
1 (6 coefficients)	0.0031
2 (12 coefficients)	0.016
3 (20 coefficients)	0.093
4 (30 coefficients)	0.92
5 (42 coefficients)	2.761
6 (56 coefficients)	7.675
7 (72 coefficients)	16.536
8 (90 coefficients)	33.852
9 (110 coefficients)	56.27
10 (132 coefficients)	84.88

Table 2: Normalization Algorithm: minutes per coefficients considered

Notice that the actual time used may vary depending on the computer on which are evaluated. However the behavior would remain the same.

Remark 7 *Notice that, to include the coefficients up to $C_{20,20}$, $S_{20,20}$ (i.e. 462 coefficients) the relegation algorithm would take around 20 hours while he normalization algorithm will take around 15 minutes.*
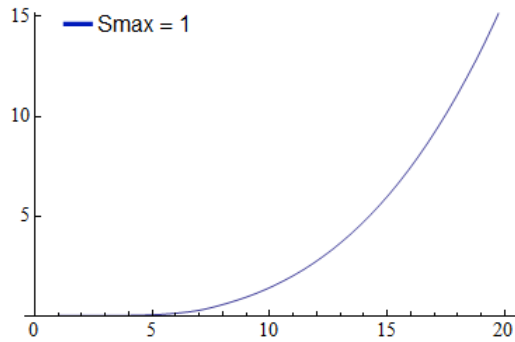
Figure 4: *Normalization Algorithm: seconds per maximum order of coefficients for the first homologic equation*

7 Results

Taking the tesseral/sectorial coefficients of $433 - Eros$ from [24] (up to the coefficients $C_{4,4}$ and $S_{4,4}$) some frozen orbits around the asteroid have been plotted, for some (fixed) values of the eccentricity e_0 of the orbit, the inclination I_0.

The results obtained have been compared with the ones found for other four asteroids:

- 1989 UQ
- 1999 Ju3_2
- Phobos
- Deimos,

considering the coefficients up to order 10 (i.e. 132 coefficients).

For each fixed eccentricity/inclination the resulting type of orbit has been plotted for 433-Eros while the results for all the asteroids have been summarized in the corresponding table.
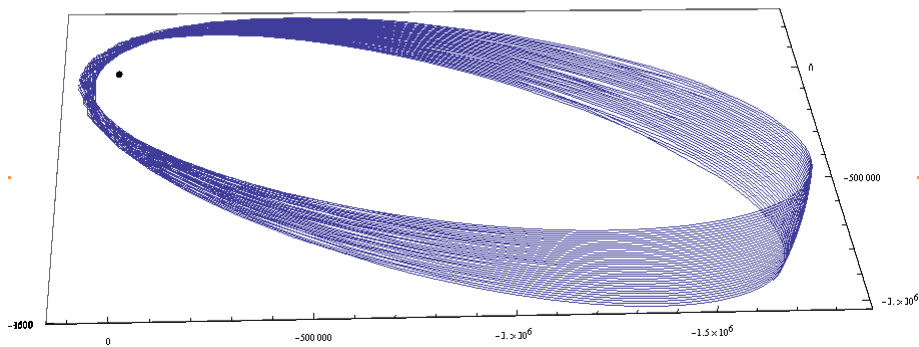

Figure 5: *433-Eros Frozen Orbit: e=0.9; I=0.001, 10 years*

	Mass (Kg)	Rotation vel. (m/s)	Reference rad. (Km)	a0 (m)	T0 (s)	g0 (rad)
EROS	6.6904×10^{15}	3.31182×10^{-4}	16	1.0013×10^{6}	9.42134×10^{6}	$\pi/2$
1989 UQ	3×10^{11}	3.59211×10^{-5}	0.38	24828.4	5.49358×10^{6}	$\pi/2$
1999 UJ3_02	5×10^{11}	3.6075×10^{-5}	0.45	1.01249×10^{10}	110813×10^{15}	$\pi/2$
Phobos	1.07×10^{16}	3.63108×10^{-5}	11.1	61973.1	114711	$\pi/2$
Deimos	2.244×10^{15}	9.13743×10^{-6}	6.2	2.25599×10^{6}	5.50157×10^{7}	$-\pi/2$

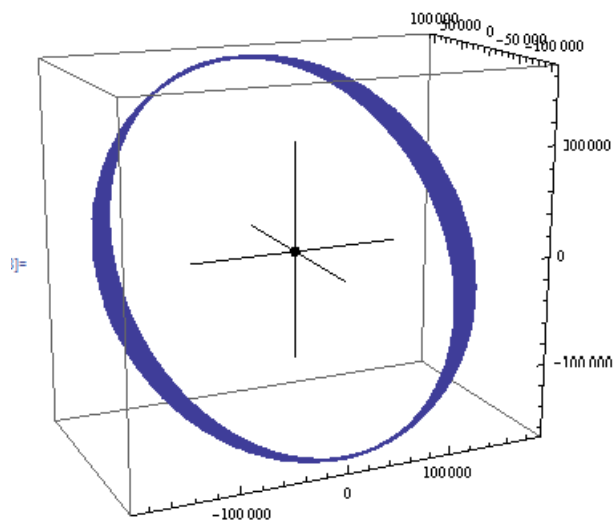Figure 6: *Frozen Orbit: e=0.9; I=0.001*

Figure 7: *433-Eros Frozen Orbit: e=0.001; I=1.0472, 3 years*

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	Mass (Kg)	Rotation vel. (m/s)	Reference rad. (Km)	a0 (m)	T0 (s)	g0 (rad)
EROS	6.6904×10^{15}	3.31182×10^{-4}	16	187765	765048	$-\pi/2$
1989 UQ	3×10^{11}	3.59211×10^{-5}	0.38	4511.03	425477	$-\pi/2$
1999 UJ3_02	5×10^{11}	3.6075×10^{-5}	0.45	1.85069×10^9	8.65977×10^{13}	$-\pi/2$
Phobos	1.07×10^{16}	3.63108×10^{-5}	11.1	117634	299986	$-\pi/2$
Deimos	2.244×10^{15}	9.13743×10^{-6}	6.2	412363	4.29933×10^6	$\pi/2$

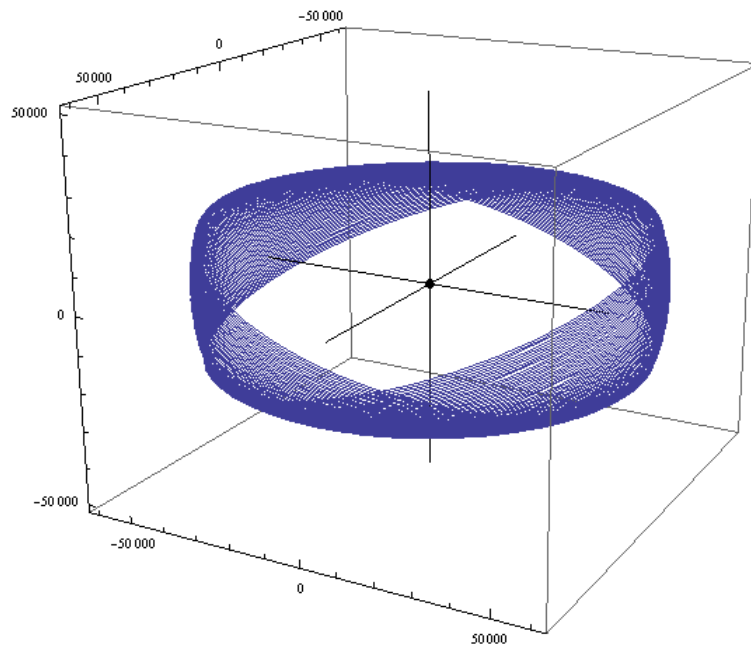Figure 8: *Frozen Orbit:e=0.001; I=1.0472*

Figure 9: *433-Eros Frozen Orbit: e=0.001; I=0.3, 4 months*

	Mass (Kg)	Rotation vel. (m/s)	Reference rad. (Km)	a0 (m)	T0 (s)	g0 (rad)
EROS	6.6904×10^{15}	3.31182×10^{-4}	16	67146.6	163607	$\pi/2$
1989 UQ	3×10^{11}	3.59211×10^{-5}	0.38	1766.61	104266	$\pi/2$
1999 UJ3_02	5×10^{11}	3.6075×10^{-5}	0.45	7.05059×10^{8}	2.03632×10^{13}	$\pi/2$
Phobos	1.07×10^{16}	3.63108×10^{-5}	11.1	9467.23	6849.11	$\pi/2$
Deimos	2.244×10^{15}	9.13743×10^{-6}	6.2	157099	1.01097×10^{6}	$-\pi/2$

Figure 10: *Frozen Orbit:e=0.001; I=0.3*

8 Conclusions

A general perturbative theory is developed that considers all the terms of the gravitational potential. This generalises previous methods developed into the literature and enables an algorithmic procedure to identify frozen orbits. The relegation method presents a completely autonomous method that can be implemented in algebraic software such as Mathematica and Piranha.

A mathematical representation of an inhomogeneous gravitational field has been developed in polar-nodal coordinates. This expression is shown to be different from those stated in the literature [21], [20]. This is an important result as much of the current work in this field is using the incorrect equations in their analysis. The equations that were derived in this paper were verified by using two completely different approaches.

An automated derivation of an analytical perturbative theory has been developed. This method was implemented in modern computer algebra tools both in Mathematica and Piranha. Both Mathematic and Pirahna dealt with the extensive computations involved in the high dimensional series expansions.

The relegation algorithm has been completely automated in algebraic software.
The project has developed two different software scripts, written in Mathematica and Piranha, which can **deal with any arbitrary order** n_{max} of terms of the potential **as well as any arbitrary degree** s_{max} of homological equations returning as output the transformed Hamiltonian

$$K(r, \theta, R, \Theta, N) = \sum_{s=0}^{s_{max}} \frac{\epsilon^s}{s!} K_s$$

and the transformation

$$W(r, \theta, \nu, R, \Theta, N) = \sum_{s=0}^{s_{max}} \frac{\epsilon^s}{s!} W_s.$$

The second homological equation is considered representing a generalisation of the usual procedure found in the literature (see for example [21] and [23]) which consider only the first Homological equation. Previous work only considers the first equation due to the high number of terms which must be considered. With the availability of powerful algebraic software it has been shown that the second homological equation providing more accurate approximations can be exploited.

Both Mathematica and Piranha easily deal with the extensive algebraic manipulations required in the perturbative method. The order of relegation obtainable using this software is way beyond the data available for the coefficients of the celestial bodies gravitational field.

The analytical perturbative theory is based on combining Deprit's method [13] to relegate the argument of node and the classical Delaunay Normalization

[12]. The derived method is used to reduce the number of degrees of freedom to a (integrable) single degree of freedom model.

The analytical perturbative theory that is presented here and implemented in Piranha and Mathematica is able to derive high order approximations of the Hamiltonian. As this method generalises previous methods and can be used to obtain higher orders than previously obtained we are able to find more accurate analytical descriptions of frozen orbits.

The general method is illustrated through an application to the asteroid 433-Eros of the main asteroid belt. This asteroid represents the "classical" example on which most of the literature referenced in this paper is focused. This enables a comparison to previous method for finding frozen orbits to be made. The work here is to higher order and therefore a more accurate description of the spacecraft's motion about this asteroid is derived. Furthermore, we are able to identify new classes of frozen orbits. The generality of the method would allow it to be applied to any celestial body where the relevant coefficients are available.

This method enables a rapid preliminary mission analysis of frozen orbits, whereby the user can simply state the required eccentricty and inclination and a frozen orbit will be generated.

The results obtained have been compared with the ones found for other four asteroids:
- 1989 UQ
- 1999 Ju3_2
- Phobos
- Deimos,
considering the coefficients up to order 10 (i.e. 132 coefficients).

9 Appendix A: Mathematical Preview

9.1 Changes of coordinates

As we want to transform the perturbing potential from the polar spherical coordinates (longitude, latitude and ray) to the so called nodal polar variables (see [26]), in this section we will explicit the relations between three sets of coordinates:
- the polar spherical coordinates - the cartesian - the polar-nodal (or Wittaker variables)

9.1.1 from polar spherical to cartesian and back

Given a cartesian system of reference $O_{\hat{x},\hat{y},\hat{z}}$ and a point P whose position and velocity vectors in cartesian coordinates are $\mathbf{r} = [x, y, z]$, $\mathbf{X} = [X, Y, Z]$, we want to express in terms of the cartesian coordinates its distance from the origin r, longitude λ and latitude δ; recall that $\lambda \in [0, 2\pi)$ is the angle expressing the horizontal displacement of the spacecraft from a reference semiplane that we fix to be the $\hat{x}/\hat{z}$ semiplane of the $x > 0$, and $\delta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the vertical angular displacement from the $\hat{x}/\hat{y}$ plane. By such definition it is easy to see that :

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\cos\delta &= \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \\
\sin\delta &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\
\cos\lambda &= \frac{x}{\sqrt{x^2+y^2}} \\
\sin\lambda &= \frac{y}{\sqrt{x^2+y^2}}
\end{aligned}
\tag{87}
$$

And inverting these relations, calling $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$:

$$
\begin{aligned}
\sqrt{x^2 + y^2} &= r\cos\delta \\
z &= r\sin\delta \\
x &= r\cos\delta\cos\lambda \\
y &= r\sin\lambda\cos\delta
\end{aligned}
\tag{88}
$$

9.1.2 from polar nodal to cartesian and back

The three coordinates (r, θ, ν) of the nodal polar variables are respectively the satellite distance from the origin, the argument of latitude (i.e the angular distance of the spacecraft from the line of the ascending node on the orbital plane) and right ascension of the ascending node, that is the longitude of the ascending node; their conjugate momenta (R, Θ, N) are radial velocity, angular momentum, polar component of angular momentum. The angular momentum $\Theta\hat{\mathbf{h}}$, (where $\hat{\mathbf{h}}$ is the unity vector indicating the direction and Θ the magnitude) is defined as:

$$
\Theta\hat{\mathbf{h}} = \mathbf{r} \times \mathbf{X} = \left\| \begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ X & Y & Z \end{matrix} \right\| = [yZ - zY, zX - xZ, xY - yX]
\tag{89}
$$

where $\times$ denotes the vectorial product and $\|.\|$ the determinant.

Calling $\mathbf{n}$ the vector identifying the position of the ascending node we find that

$$\mathbf{n} = [0,0,1] \times \Theta \hat{\mathbf{h}} = \left\| \begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \Theta_x & \Theta_y & \Theta_z \end{matrix} \right\| = [-\Theta_y, \Theta_x, 0] = [xZ - zX, yZ - zY, 0] \tag{90}$$

as the line of the ascending node is defined at the interception of the orbital plane and the horizontal plane $\hat{x}/\hat{y}$ and therefore it has to be perpendicular to both the angular momentum and the $\hat{z}$ axe. The longitude of the ascending node ν is therefore found by:

$$\begin{aligned} \cos \nu &= \frac{n_x}{|\mathbf{n}|} = \frac{xZ - zX}{\sqrt{(xZ-zX)^2+(yZ-zY)^2}} \\ \sin \nu &= \frac{n_y}{|\mathbf{n}|} = \frac{yZ-zY}{\sqrt{(xZ-zX)^2+(yZ-zY)^2}} \end{aligned} \tag{91}$$

In order to find the argument of latitude θ the definition of the inclination of the orbit I is firstly needed, which is the latitude of a unity vector $\hat{\mathbf{i}}$ contained in the orbital plane and perpendicular to the nodal line, namely:

$$\hat{\mathbf{i}} = \frac{\hat{\mathbf{h}} \times \mathbf{n}}{|\hat{\mathbf{h}} \times \mathbf{n}|} = \frac{1}{|\mathbf{i}|} \left[(Yx - Xy)(-Zy + Yz), (Yx - Xy)(Zx - Xz), Z^2(x^2 + y^2) \right.$$

$$\left. -2Z(Xx + Yy)z + (X^2 + Y^2)z^2 \right] \tag{92}$$

where

$$|\mathbf{i}| = \left(((Yx - Xy)(-Zy + Yz))^2 + ((Yx - Xy)(Zx - Xz))^2 + \left(Z^2(x^2 + y^2) \right.\right.$$

$$\left.\left. -2Z(Xx + Yy)z + (X^2 + Y^2)z^2 \right)^2 \right)^{\frac{1}{2}} \tag{93}$$

Therefore

$$\begin{aligned} \sin I &= \hat{\mathbf{i}}_z = \frac{Z^2(x^2+y^2) - 2Z(Xx+Yy)z + (X^2+Y^2)z^2}{|\mathbf{i}|} \\ \cos I &= \frac{\Theta_z}{|\Theta|} = \frac{Yx-Xy}{\sqrt{(Yx-Xy)^2+(Zx-Xz)^2+(Zy-Yz)^2}} \end{aligned} \tag{94}$$

Finally the argument of latitude θ is the angle between the normalized vector $\frac{\mathbf{r}}{|\mathbf{r}|}$ expressing the position of the spacecraft and normalized vector identifying the position of the of the ascending node, $\frac{n}{|\mathbf{n}|}$, therefore:

$$\begin{aligned} \cos \theta &= \frac{r}{|\mathbf{r}|} . \hat{\mathbf{h}} = \frac{Z(x^2+y^2) - (Xx+Yy)z}{\sqrt{x^2+y^2+z^2}\sqrt{(Zx-Xz)^2+(Zy-Yz)^2}} \\ \sin \theta &= \frac{r}{|\mathbf{r}|} . \hat{\mathbf{i}} = \frac{z}{\sqrt{x^2+y^2+z^2}} \sqrt{\frac{(Z^2(x^2+y^2) - 2XZxz - 2Yy(Xx+Zz) + Y^2(x^2+z^2) + X^2(y^2+z^2))}{(Z^2(x^2+y^2) - 2Z(Xx+Yy)z + (X^2+Y^2)z^2)}} \end{aligned} \tag{95}$$

Finally $N = |\Theta| \cos I$ and given the momenta vector $\mathbf{X} = [X, Y, Z]$ R will be defined as $R = \mathbf{X}.\frac{\mathbf{r}}{|\mathbf{r}|}$. The inverse of this transformation is given in [19] and is easily verifiable:

$$\begin{aligned} x &= |\mathbf{r}|(\cos \theta \cos \nu - \sin \theta \cos I \sin \nu) \\ y &= |\mathbf{r}|(\cos \theta \sin \nu + \sin \theta \cos I \cos \nu) \\ z &= |\mathbf{r}| \sin \theta \sin I \\ X &= \left(R \cos \theta - \frac{\Theta}{|\mathbf{r}|} \sin \theta \right) \cos \nu - \left(R \sin \theta + \frac{\Theta}{|\mathbf{r}|} \cos \theta \right) \cos I \sin \nu \\ Y &= \left(R \cos \theta - \frac{\Theta}{|\mathbf{r}|} \sin \theta \right) \sin \nu + \left(R \sin \theta + \frac{\Theta}{|\mathbf{r}|} \cos \theta \right) \cos I \cos \nu \\ Z &= \left(R \sin \theta + \frac{\Theta}{|\mathbf{r}|} \cos \theta \right) \sin I \end{aligned} \tag{96}$$

9.1.3 from polar spherical to nodal polar

Composing the transformations above it is easy to demonstrate that:

$$
\begin{aligned}
\sin \delta &= \sin \theta \sin I \\
\cos \delta &= \sqrt{\cos \theta^2 + \cos I^2 \sin \theta^2} \\
\sin \lambda &= \frac{\cos \theta \cos \nu - \sin \theta \cos I \sin \nu}{\sqrt{\cos \theta^2 + \cos I^2 \sin \theta^2}} \\
\cos \lambda &= \frac{\cos \theta \sin \nu + \sin \theta \cos I \cos \nu}{\sqrt{\cos \theta^2 + \cos I^2 \sin \theta^2}}
\end{aligned}
\tag{97}
$$

which, calling

$$
D = \sqrt{\cos \theta^2 + \cos I^2 \sin \theta^2}
\tag{98}
$$

can be rearranged as:

$$
\begin{aligned}
\sin \delta &= \sin \theta \sin I \\
\cos \delta &= D \\
\sin \lambda &= \frac{\cos \theta \cos \nu - \sin \theta \cos I \sin \nu}{D} \\
\cos \lambda &= \frac{\cos \theta \sin \nu + \sin \theta \cos I \cos \nu}{D}
\end{aligned}
\tag{99}
$$

9.2 Legendre Polynomial and some properties

Definition 5 *(Legendre Polynomial)*
$\forall n \in \mathbb{N}$ *the Legendre Polynomial $P_n(x)$ is:*

$$
P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n
\tag{100}
$$

Formula 1 *(Explicit formulation of Legendre Polynomials)*
$\forall n \in \mathbb{N}$ *it is true that:*

$$
P_n(x) = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j}^2 (-1)^j (1+x)^{n-j} (1-x)^j
\tag{101}
$$

Proof:

$$
\begin{aligned}
P_n(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \\
&= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x+1)^n (x-1)^n \right) \\
&= \frac{1}{2^n n!} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{d^j}{dx^j} (x+1)^n \right) \left(\frac{d^{n-j}}{dx^{n-j}} (x-1)^n \right) \\
&= \frac{1}{2^n n!} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{n!}{(n-j)!} (x+1)^{n-j} \right) \left(\frac{n!}{j!} (x-1)^j \right) \\
&= \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j}^2 (-1)^j (1+x)^{n-j} (1-x)^j
\end{aligned}
\tag{102}
$$

Definition 6 *(Associated Legendre Polynomials)* $\forall n, m \in \mathbb{N}; \quad n < k$ *the Associated Legendre Polynomials are defined as:*

$$
P_n^m(x) := (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} (P_n(x))
\tag{103}
$$

Formula 2 *(Explicit formulation of Associated Legendre Polynomials)*
$\forall n, m \in \mathbb{N}; \quad m < n$ *it is true that:*

$$P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{n!^2 m!^2}{2^n} \sum_{j=0}^{n} \sum_{\ell=\max\{0,j+m-n\}}^{\min\{j,m\}} \left((-1)^{j+\ell+p} \cdot \right.$$

$$\left. \cdot \frac{1}{(j)!(n-j)!\ell!(m-\ell)!(n-m-j+\ell)!(j-\ell)!} (1+x)^{n-m-j+\ell}(1-x)^{j-\ell} \right) \tag{104}$$

Proof:

Using Formula 1 yields:

$$P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m}(P_n(x))$$
$$= (-1)^m (1-x^2)^{\frac{m}{2}} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j}^2 (-1)^j \frac{d^m}{dx^m}(1+x)^{n-j}(1-x)^j$$

$$= (-1)^m (1-x^2)^{\frac{m}{2}} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j}^2 (-1)^j \sum_{\ell=0}^{m} \binom{m}{\ell} \left(\frac{d^{m-\ell}}{dx^{m-\ell}}(1+x)^{n-j} \right)$$

$$\left(\frac{d^{\ell}}{dx^{\ell}}(1-x)^j \right)$$

$$= (-1)^m (1-x^2)^{\frac{m}{2}} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j}^2 (-1)^j \sum_{\ell=\max\{0,j+m-n\}}^{\min\{j,m\}} \left(\frac{(n-j)!}{(n-j-m+\ell)!} \right.$$

$$(1+x)^{n-j-m+\ell} \left(\frac{j!}{(j-\ell)!}(-1)^{\ell}(1-x)^{j-\ell} \right)$$

$$= (-1)^m (1-x^2)^{\frac{m}{2}} \frac{n!^2 m!^2}{2^n} \sum_{j=0}^{n} \sum_{\ell=\max\{0,j+m-n\}}^{\min\{j,m\}} \left((-1)^{j+\ell+p} \cdot \right.$$

$$\left. \cdot \frac{1}{(j)!(n-j)!\ell!(m-\ell)!(n-m-j+\ell)!(j-\ell)!} (1+x)^{n-m-j+\ell}(1-x)^{j-\ell} \right) \tag{105}$$

9.3 Useful trigonometric formulas

Formula 3 $\forall m \in \mathbb{N}$ *it is true that:*

$$\cos(mx) = \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2p}(-1)^p \cos(x)^{m-2p} \sin(x)^{2p}$$

$$\sin(mx) = \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2p+1}(-1)^p \cos(x)^{m-2p-1} \sin(x)^{2p+1} \tag{106}$$

Proof: Using Euler's Formula yields:

$$(\cos(mx) + \imath \sin(mx)) = \sum_{s=0}^{m} \binom{m}{s} \imath^s \cos(x)^{m-s} \sin(x)^s \tag{107}$$

as

$$(\cos(mx) + \imath\sin(mx)) = e^{\imath mx}$$
$$= (e^{\imath x})^m$$
$$= (\cos(x) + \imath\sin(x))^m \tag{108}$$
$$= \sum_{s=0}^{m} \binom{m}{s} \imath^s \cos(x)^{m-s} \sin(x)^s$$

Therefore, equalling the Real part of both sides of (107):

$$\cos(mx) = \Re\left(\sum_{s=0}^{m} \binom{m}{s} \imath^s \cos(x)^{m-s} \sin(x)^s\right)$$
$$= \sum_{s=0,\ s\ even}^{m} \binom{m}{s} \imath^n \cos(x)^{m-s} \sin(x)^s$$
$$= \sum_{p=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m}{2p} \imath^{2p} \cos(x)^{m-2p} \sin(x)^{2p} \tag{109}$$
$$= \sum_{p=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m}{2p} (-1)^p \cos(x)^{m-2p} \sin(x)^{2p}$$

Analogously, equalling the Imaginary part of both sides of (107):

$$\sin(mx) = \Im\left(\sum_{s=0}^{m} \binom{m}{s} \imath^s \cos(x)^{m-s} \sin(x)^s\right)$$
$$= \sum_{s=0,\ s\ odd}^{m} \binom{m}{s} \imath^s \cos(x)^{m-s} \sin(x)^s$$
$$= \sum_{p=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2p+1} \imath^{2p+1} \cos(x)^{m-2p-1} \sin(x)^{2p+1} \tag{110}$$
$$= \sum_{p=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2p+1} (-1)^p \cos(x)^{m-2p-1} \sin(x)^{2p+1}$$

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