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Optical helicity, optical spin and related quantities in electromagnetic theory

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Abstract. We examine the optical helicity, the optical spin and the ij-infra-zilches in electromagnetic theory and show that these conserved quantities can be combined to form a new description of the angular momentum associated with optical polarization: one that is analogous to the familiar description of optical energy and linear momentum. The symmetries of Maxwell's equations that underlie the conservation of our quantities are presented and discussed. We explain that a similar but distinct set of quantities, Lipkin's zilches, describe the 'angular momentum' of the curl of the electromagnetic field, rather than the angular momentum of the electromagnetic field itself.

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1. Introduction

In optics, it is well-established that a beam of light can carry both spin and orbital angular momenta parallel to the beam axis. The spin angular momentum is associated with polarization while the orbital angular momentum is associated with helical phase fronts [1, 2]. Surprisingly, a fundamental description of these mechanical properties of light within electromagnetic theory has proved controversial: it has long been known how to extract spin-like and orbital-like pieces from the total angular momentum of the electromagnetic field [3, 4], but there has been confusion as to whether these are separately meaningful as they are themselves not true angular momenta [5, 6].

In this paper we focus upon the angular momentum associated with polarization. That is, the spin angular momentum of light. The photon is massless and relativity suggests that it does not possess a well-defined spin angular momentum vector. Rather, it is only the component of the spin angular momentum in the direction of propagation, the photon's helicity, that is physically meaningful [5, 6]. Previously [7], we determined the form of the optical helicity in electromagnetic theory and demonstrated its utility. In what follows, we expand upon this work and show that the optical helicity can be combined naturally with the more familiar spin-like piece of the total electromagnetic angular momentum, as well as with six new conserved quantities, to form what is, perhaps, a more *complete* description of the angular momentum associated with polarization.

Cartesian coordinates are used exclusively throughout. Greek indices α, β etc may assume the values 0 and 1, 2, 3, corresponding to time, t, and spatial coordinates, x, y, z. Latin indices i, j etc may assume the values 1, 2, 3. When an index appears twice in a term, summation over its allowed values is to be understood. We make no distinction between raised and lowered indices, although we respect convention and use raised indices to label the components of the contravariant energy–momentum and zilch tensors.

2. Optical helicity

It was first suggested by Poynting [8], and was later confirmed experimentally by Beth [9], that a beam of light possesses an intrinsic angular momentum associated with its polarization equivalent to $\hbar\sigma$ per photon in the direction of propagation. The parameter $-1 \leqslant \sigma \leqslant 1$ takes its limiting values of ± 1 for left- or right-handed circular polarization and vanishes for linear polarization. The pseudoscalar quantity $\hbar\sigma$ is the average *helicity* per photon [10]. Recently, we asked how we might describe this property of light in electromagnetic theory. We recall below, the reasoning that led us to the desired optical helicity [7].

We work with the electric field, $\mathbf{E}$, and magnetic field, $\mathbf{B}$, in vacuum so as to describe freely propagating light. The fields obey the source-free Maxwell equations which are:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\dot{\mathbf{B}}, \\ \nabla \times \mathbf{B} &= \dot{\mathbf{E}}, \end{aligned} \tag{2.1}$$

in a natural system of units with $\epsilon_0 = \mu_0 = c = 1$.

The optical helicity we seek should be a time-even pseudoscalar with the dimensions of an angular momentum. A quantity with these properties has already been recognized in plasma physics for some time: the magnetic helicity, $\mathcal{M}$, is given by [11]:

$$\mathcal{M} = \iiint \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{r}, \tag{2.2}$$

where the vector potential, $\mathbf{A}$, is related to the magnetic field in the usual way: $\mathbf{B} = \nabla \times \mathbf{A}$ [12]. The magnetic helicity is used to quantify the twist of magnetic field lines [13], and formally resembles the vortex helicity recognized in fluid mechanics [14]. However, the magnetic helicity is *not* acceptable as a quantity that describes the helicity of freely propagating electromagnetic waves. To demonstrate this, we need only consider a symmetry, due to Heaviside [15] and Larmor [16], that is inherent in Maxwell's equations (2.1) for the free-field: the equations retain their form under a duality rotation:

$$\begin{aligned} \mathbf{E} &\to \cos\theta \, \mathbf{E} + \sin\theta \, \mathbf{B}, \\ \mathbf{B} &\to \cos\theta \, \mathbf{B} - \sin\theta \, \mathbf{E}, \end{aligned} \tag{2.3}$$

for any pseudoscalar angle θ. Accordingly, any physically meaningful property of the field must also retain its form under a duality rotation [17], a principle referred to by Berry [18] as electric–magnetic democracy. To ask whether the magnetic helicity retains its form under a duality rotation, we follow Bateman [19] and introduce a second (pseudo)vector potential, $\mathbf{C}$, such that $\mathbf{E} = -\nabla \times \mathbf{C}$. For the sake of brevity, we work with the Coulomb gauge ($\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{C} = 0$) so that our vector potentials are purely transverse ($\mathbf{A} = \mathbf{A}^{\perp}$, $\mathbf{C} = \mathbf{C}^{\perp}$) and are, therefore, gauge invariant and uniquely defined [4]. The fields can then be expressed in terms of either vector potential as:

$$\begin{aligned} \mathbf{E} &= -\nabla \times \mathbf{C} = -\dot{\mathbf{A}}, \\ \mathbf{B} &= \nabla \times \mathbf{A} = -\dot{\mathbf{C}}, \end{aligned} \tag{2.4}$$

and the duality rotation (2.3) is invoked by taking:

$$\begin{aligned} \mathbf{A} &\to \cos\theta \, \mathbf{A} + \sin\theta \, \mathbf{C}, \\ \mathbf{C} &\to \cos\theta \, \mathbf{C} - \sin\theta \, \mathbf{A}. \end{aligned} \tag{2.5}$$

The electric–magnetic democracy principle is extended to the vector potentials by requiring that physically meaningful properties of the field retain their form under the transformation (2.5). It is immediately obvious that the magnetic helicity (2.2) does not retain its form under this transformation and is therefore not an acceptable candidate for an optical helicity.

To proceed, we add half of the magnetic helicity to half of the corresponding electric-helicity [7]. This gives us the desired optical helicity [20–22]:

$$\mathcal{H} = \iiint \frac{1}{2} \left(\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E} \right) \, \mathrm{d}^3\mathbf{r}. \tag{2.6}$$

The optical helicity, $\mathcal{H}$, is a time-even Lorentz pseudoscalar with the dimensions of an angular momentum. It retains its form under a duality rotation (2.5), as required. The optical helicity is a conserved quantity in that:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0. \tag{2.7}$$

By expanding the quantized field in terms of circularly polarized plane wave modes, we find that the operator form, $\hat{\mathcal{H}}$, of the optical helicity is [7]:

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \hbar \left(\hat{n}_{\mathbf{k},\mathrm{L}} - \hat{n}_{\mathbf{k},\mathrm{R}}\right), \tag{2.8}$$

where $\hat{n}_{\mathbf{k},L}$ and $\hat{n}_{\mathbf{k},R}$ are number operators for the left- and right-handed circular polarizations associated with the wavevector $\mathbf{k}$. As the helicity of a photon is $\pm\hbar$ for these polarizations, we see that (2.8) simply represents a sum over all modes of the number of photons in each mode multiplied by their helicity.

The integrand of (2.6) suggests that we take the quantity:

$$h = \tfrac{1}{2}\left(\mathbf{A}\cdot\mathbf{B} - \mathbf{C}\cdot\mathbf{E}\right), \tag{2.9}$$

to be the *helicity density* of the field. Our helicity density, h, has the dimensions of an angular momentum per unit volume, is uniquely defined and retains its form under a duality rotation (2.5).

3. Optical spin

The optical helicity, $\mathcal{H}$, is not the only quantity in electromagnetic theory that describes the angular momentum associated with polarization. We recognize, in addition, the optical spin, which we now consider.

The total angular momentum, $\boldsymbol{\mathcal{J}}$, of the field is obtained by integrating the angular momentum density, $\mathbf{j} = \mathbf{r}\times(\mathbf{E}\times\mathbf{B})$, over all space:

$$\boldsymbol{\mathcal{J}} = \iiint \mathbf{r}\times(\mathbf{E}\times\mathbf{B})\ \mathrm{d}^3\mathbf{r}. \tag{3.1}$$

Using integration by parts, it can be recast in the form [4, 23, 24]:

$$\boldsymbol{\mathcal{J}} = \iiint \frac{1}{2}\left(\mathbf{E}\times\mathbf{A} + \mathbf{B}\times\mathbf{C}\right)\ \mathrm{d}^3\mathbf{r} + \iiint \frac{1}{2}\left[E_i\left(\mathbf{r}\times\boldsymbol{\nabla}\right)A_i + B_i\left(\mathbf{r}\times\boldsymbol{\nabla}\right)C_i\right]\ \mathrm{d}^3\mathbf{r}, \tag{3.2}$$

provided the field falls off suitably as $|\mathbf{r}|\to\infty$. The first integral,

$$\boldsymbol{\mathcal{S}} = \iiint \frac{1}{2}\left(\mathbf{E}\times\mathbf{A} + \mathbf{B}\times\mathbf{C}\right)\ \mathrm{d}^3\mathbf{r}, \tag{3.3}$$

makes no explicit reference to $\mathbf{r}$ and it therefore seems natural to associate this integral with the *spin* angular momentum of light and the second integral with the *orbital* angular momentum of light. Of course, the optical spin, $\boldsymbol{\mathcal{S}}$, is a time-odd pseudovector with the dimensions of an angular momentum. Remarkably, however, it is not actually an angular momentum as its quantized form does not satisfy the required commutation relations [5, 6]. Like the optical helicity, the optical spin is conserved:

$$\frac{\mathrm{d}\boldsymbol{\mathcal{S}}}{\mathrm{d}t} = 0. \tag{3.4}$$

If we again expand the quantized field in terms of circularly polarized plane wave modes, we find the operator form, $\hat{\boldsymbol{\mathcal{S}}}$, of the optical spin to be [25]:

$$\hat{\boldsymbol{\mathcal{S}}} = \sum_{\mathbf{k}} \hbar\left(\hat{n}_{\mathbf{k},\mathrm{L}} - \hat{n}_{\mathbf{k},\mathrm{R}}\right)\frac{\mathbf{k}}{|\mathbf{k}|}. \tag{3.5}$$

This represents a sum over all modes of the number of photons in each mode multiplied by their helicity and the unit vector $\mathbf{k}/|\mathbf{k}|$ in the direction of propagation.

The integrand of (3.3) suggests that we take the quantity:

$$\mathbf{s} = \tfrac{1}{2}\left(\mathbf{E}\times\mathbf{A}+\mathbf{B}\times\mathbf{C}\right), \tag{3.6}$$

to be the *spin density* of the field [23]. Our spin density, $\mathbf{s}$, has the dimensions of an angular momentum per unit volume, is uniquely defined and retains its form under a duality rotation (2.5).

4. The ij-infra-zilches

In addition to the optical helicity, $\mathcal{H}$, and the optical spin, $\boldsymbol{\mathcal{S}}$, there exists a further six quantities in electromagnetic theory that describe the angular momentum associated with polarization. We refer to these quantities:

$$\mathcal{N}_{ij} = \iiint \frac{1}{2}\left[\delta_{ij}\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right) - A_i B_j - A_j B_i + C_i E_j + C_j E_i\right] \mathrm{d}^3\mathbf{r}, \tag{4.1}$$

as the ij-infra-zilches, in homage to Lipkin's ij-zilches (discussed in section 8) which are similar in form. That only six of the ij-infra-zilches, $\mathcal{N}_{ij}$, are independent, is a consequence of the symmetry $\mathcal{N}_{ij} = \mathcal{N}_{ji}$. The ij-infra-zilches are time-even quantities that change sign under a parity inversion and have the dimensions of an angular momentum. Like the optical helicity and the optical spin, they are conserved:

$$\frac{\mathrm{d}\mathcal{N}_{ij}}{\mathrm{d}t} = 0. \tag{4.2}$$

If we consider once more an expansion of the quantized field in terms of circularly polarized plane wave modes, we find that the operator forms, $\hat{\mathcal{N}}_{ij}$, of the ij-infra-zilches are:

$$\hat{\mathcal{N}}_{ij} = \sum_{\mathbf{k}} \hbar\left(\hat{n}_{\mathbf{k},\mathrm{L}} - \hat{n}_{\mathbf{k},\mathrm{R}}\right)\frac{k_i k_j}{|\mathbf{k}|^2}. \tag{4.3}$$

These represent a sum over all modes of the number of photons in each mode multiplied by their helicity and the i and j components of the unit vector $\mathbf{k}/|\mathbf{k}|$ in the direction of propagation. It follows that, strictly speaking, only five of the ij-infra-zilches are actually 'new' quantities: we find that $\mathcal{N}_{xx}+\mathcal{N}_{yy}+\mathcal{N}_{zz} = \mathcal{H}$, as may be confirmed by comparing (4.1) with (2.6). This is a reflection of the fact that $\left(k_x^2+k_y^2+k_z^2\right)/|\mathbf{k}|^2 = 1$.

The integrand of (4.1) suggests that we take the quantities:

$$n_{ij} = \tfrac{1}{2}\left[\delta_{ij}\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right) - A_i B_j - A_j B_i + C_i E_j + C_j E_i\right], \tag{4.4}$$

to be the *ij-infra-zilch densities* of the field. Our ij-infra-zilch densities, n_{ij}, have the dimensions of an angular momentum per unit volume, are uniquely defined and retain their forms under a duality rotation (2.5).

5. The helicity array

The similarity of (2.8), (3.5) and (4.3) suggests that the optical helicity, $\mathcal{H}$, the optical spin, $\boldsymbol{\mathcal{S}}$, and the ij-infra-zilches, $\mathcal{N}_{ij}$, are related quantities. Indeed this is the case, as we now demonstrate.

We introduce the helicity array, $N^{\alpha\beta\gamma}$: a rank-three object with components given by:

$$\begin{aligned} N^{000} &= \tfrac{1}{2}\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right), \\ N^{0i0} &= \tfrac{1}{2}\left(\mathbf{E}\times\mathbf{A}+\mathbf{B}\times\mathbf{C}\right)_i, \\ N^{ij0} &= \tfrac{1}{2}\left[\delta_{ij}\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right)-A_iB_j-A_jB_i+C_iE_j+C_jE_i\right] \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} N^{00i} &= N^{0i0}, \\ N^{0ij} &= N^{ij0}, \\ N^{ijk} &= \delta_{ij}N^{00k}+\tfrac{1}{2}\left(-A_i\partial_kC_j-A_j\partial_kC_i+C_i\partial_kA_j+C_j\partial_kA_i\right), \end{aligned} \tag{5.2}$$

where the symmetry $N^{\alpha\beta\gamma}=N^{\beta\alpha\gamma}$ is to be understood in our definitions. Our helicity array has 27 independent components. We should be clear that it is not a tensor, despite its suggestive structure. Looking at (5.1) we see, of course, that the component N^{000} is our helicity density, h (2.9); the components N^{0i0} are those of our spin density, $\mathbf{s}$ (3.6), and the components N^{ij0} are our infra-zilch densities, n_{ij} (4.4).

The significance of our helicity array lies in the fact that it obeys the continuity equation:

$$\partial_\gamma N^{\alpha\beta\gamma}=0. \tag{5.3}$$

As $N^{\alpha\beta\gamma}=N^{\beta\alpha\gamma}$, we find that (5.3) expresses ten local conservation laws. It appears that these completely describe the conservation and flow of the angular momentum associated with polarization:

(i) For $\alpha\beta=00$, (5.3) is:

$$\dot{h}+\nabla\cdot\mathbf{s}=0, \tag{5.4}$$

which describes the conservation and flow of optical *helicity*. Evidently, $\mathbf{s}$ plays a dual role in that it is simultaneously the spin density and the helicity flux density ($N^{0i0}=N^{00i}$), as we have noted previously [7].

(ii) For $\alpha\beta=0i$, (5.3) is:

$$\dot{s}_i+\partial_jn_{ij}=0, \tag{5.5}$$

which describes the conservation and flow of optical *spin*. Here we see that the n_{ij} play dual roles in that they are simultaneously the ij-infra-zilch densities and the components of the spin flux density ($N^{ij0}=N^{0ij}$).

(iii) Finally, for $\alpha\beta=ij$, (5.3) is:

$$\dot{n}_{ij}+\partial_kN^{ijk}=0, \tag{5.6}$$

which describes the conservation and flow of the *ij-infra-zilches*.

We emphasize that the optical helicity, $\mathcal{H}$, the optical spin, $\boldsymbol{\mathcal{S}}$, and the ij-infra-zilches, $\mathcal{N}_{ij}$, are distinct physical quantities. Their associated densities: h, $\mathbf{s}$ and n_{ij}, however, are related by a hierarchy of continuity equations. We see from (5.4), (5.5) and (5.6) that, loosely speaking, helicity is conserved and is transported by spin, which is itself conserved and is transported by the ij-infra-zilches, which are themselves conserved and are transported by the N^{ijk}. This three-tiered hierarchy is reminiscent of the two-tiered hierarchy found in the description of optical energy and linear momentum, where it is well known that energy is conserved and is transported by linear momentum (Poynting's theorem [26]), which is itself conserved and is transported by Maxwell's stresses [12]. We pursue this analogy in section 7.

For a plane wave of angular frequency ω, wavevector $\mathbf{k}$ and polarization σ, explicit calculation reveals that the cycle-averaged components, $\overline{N^{\alpha\beta\gamma}}$, of our helicity array, normalized by the cycle-averaged energy density, $\overline{w}$, are:

$$\frac{\overline{N^{\alpha\beta\gamma}}}{\overline{w}} = \frac{\hbar\sigma}{\hbar\omega}\frac{k_\alpha k_\beta k_\gamma}{|\mathbf{k}|^3}, \tag{5.7}$$

where, because of our choice of units, $k_0 = \omega = |\mathbf{k}|$. We recognize $\hbar\omega$ as the energy of a photon which suggests, in turn, that the $\overline{N^{\alpha\beta\gamma}}$ are proportional to $\hbar\sigma$ per photon. Our classical result (5.7) reflects (2.8), (3.5) and (4.3), and should be contrasted with the angular momentum density, $\mathbf{j} = \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$, for which the component in the direction of propagation of a plane wave vanishes [7].

We noted earlier that the optical spin has been a source of controversy because it is not a true angular momentum [5, 6]. Our present findings suggest, perhaps, that it is more meaningfully thought of as one piece of a larger description of optical *helicity*, as embodied in our helicity array: $N^{\alpha\beta\gamma}$.

6. Underlying symmetries

The optical helicity, $\mathcal{H}$; the optical spin, $\boldsymbol{\mathcal{S}}$, and the ij-infra-zilches, $\mathcal{N}_{ij}$, are conserved quantities, as indicated by (2.7), (3.4) and (4.2). It was shown by Noether [27] that such global conservation laws are associated with symmetries and it is instructive to ask what the symmetries underlying our conservation laws are. This question may be answered most elegantly by considering the transformations *generated* by our quantities.

We begin with the optical helicity. Considering the quantized electric and magnetic fields, $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$, we find that:

$$\begin{aligned} \exp(\mathrm{i}\hat{\mathcal{H}}\theta/\hbar)\hat{\mathbf{E}}\exp(-\mathrm{i}\hat{\mathcal{H}}\theta/\hbar) &= \hat{\mathbf{E}} + \theta\,\hat{\mathbf{B}}, \\ \exp(\mathrm{i}\hat{\mathcal{H}}\theta/\hbar)\hat{\mathbf{B}}\exp(-\mathrm{i}\hat{\mathcal{H}}\theta/\hbar) &= \hat{\mathbf{B}} - \theta\,\hat{\mathbf{E}}, \end{aligned} \tag{6.1}$$

where θ is an infinitesimal pseudoscalar angle. We recognize the transformation (6.1), generated by the optical helicity, as an infinitesimal duality rotation (2.3). This is, of course, a *symmetry* of Maxwell's equations (2.1) in that the equations retain their form under the transformation:

$$\begin{aligned} \mathbf{E} &\to \mathbf{E} + \theta\,\mathbf{B}, \\ \mathbf{B} &\to \mathbf{B} - \theta\,\mathbf{E}. \end{aligned} \tag{6.2}$$

It follows that this, the Heaviside–Larmor symmetry, underlies the conservation of optical helicity, as we have noted previously [7]. For a plane wave, it can be shown that the transformation (6.2) invokes an infinitesimal rotation of the field vectors about the wavevector, $\mathbf{k}$, as depicted in figure 1. It seems reasonable, therefore, that the Heaviside–Larmor symmetry is associated with optical helicity.

Let us now consider the optical spin. It has been shown previously [23] that:

$$\begin{aligned} \exp(\mathrm{i}\hat{\mathbf{S}}\cdot\boldsymbol{\theta}/\hbar)\hat{\mathbf{E}}\exp(-\mathrm{i}\hat{\mathbf{S}}\cdot\boldsymbol{\theta}/\hbar) &= \hat{\mathbf{E}} + (\boldsymbol{\theta}\times\hat{\mathbf{E}})^{\perp}, \\ \exp(\mathrm{i}\hat{\mathbf{S}}\cdot\boldsymbol{\theta}/\hbar)\hat{\mathbf{B}}\exp(-\mathrm{i}\hat{\mathbf{S}}\cdot\boldsymbol{\theta}/\hbar) &= \hat{\mathbf{B}} + (\boldsymbol{\theta}\times\hat{\mathbf{B}})^{\perp}, \end{aligned} \tag{6.3}$$

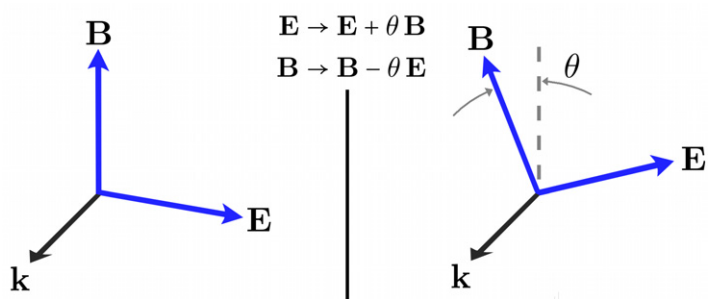

Figure 1. The optical helicity generates an infinitesimal duality rotation. For a plane wave, the field vectors are rotated about the wavevector, as depicted. The magnitude of this rotation has been exaggerated for the purpose of illustration.

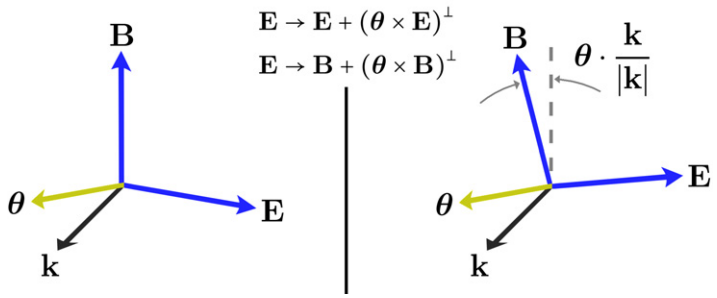

Figure 2. The optical spin generates the closest approximation to an infinitesimal rotation of the field vectors about $\boldsymbol{\theta}$ that is consistent with the requirement that the field remains transverse. For a plane wave, the field vectors are rotated in the transverse plane through the angle $\boldsymbol{\theta} \cdot \mathbf{k}/|\mathbf{k}|$, as depicted. The magnitude of this rotation has been exaggerated for the purpose of illustration.

where $\boldsymbol{\theta}$ is an infinitesimal pseudovector whose magnitude and direction define an axis and an angle of rotation about that axis, respectively. The transformation (6.3) generated by the optical spin is the closest approximation to an infinitesimal rotation of the field vectors, leaving the spatial distribution of the field unchanged, that is consistent with the requirement that the field remains transverse [23]. In fact, Maxwell's equations retain their form under the transformation:

$$\begin{aligned} \mathbf{E} &\to \mathbf{E} + (\boldsymbol{\theta} \times \mathbf{E})^{\perp}\,, \\ \mathbf{E} &\to \mathbf{B} + (\boldsymbol{\theta} \times \mathbf{B})^{\perp}\,, \end{aligned} \tag{6.4}$$

and we associate this symmetry with the conservation of optical spin. The approximate nature of the rotation (6.4) is a reflection of the fact that the optical spin is not a true angular momentum [23]. See figure 2.

The considerations above serve to demonstrate that the optical helicity and the optical spin are indeed distinct physical quantities. For a plane wave, the optical helicity generates an exact rotation of the field vectors about the wavevector, $\mathbf{k}$, whose orientation in space is, of course, arbitrary. In contrast, the optical spin generates an approximate rotation of the field vectors about an axis, defined by the direction of $\boldsymbol{\theta}$, that is fixed in space. When looking towards the incoming wave, the sense of the rotation generated by the optical helicity is always seen to be

anti-clockwise (for $\theta > 0$), whereas the magnitude and sense of the rotation generated by the optical spin is seen to depend upon the orientation of $\mathbf{k}$ relative to $\boldsymbol{\theta}$.

Finally, let us consider the ij-infra-zilches. We find that:

$$\begin{aligned}\exp(\mathrm{i}\hat{\mathcal{N}}_{ij}\theta_{ij}/\hbar)\hat{E}_l\exp(-\mathrm{i}\hat{\mathcal{N}}_{ij}\theta_{ij}/\hbar) &= \hat{E}_l + \tfrac{1}{2}[(\tilde{\theta}_{li}\hat{B}_i)^{\perp} + \epsilon_{lij}\partial_i(\tilde{\theta}_{jk}\hat{A}_k)^{\perp}],\\ \exp(\mathrm{i}\hat{\mathcal{N}}_{ij}\theta_{ij}/\hbar)\hat{B}_l\exp(-\mathrm{i}\hat{\mathcal{N}}_{ij}\theta_{ij}/\hbar) &= \hat{B}_l - \tfrac{1}{2}[(\tilde{\theta}_{li}\hat{E}_i)^{\perp} - \epsilon_{lij}\partial_i(\tilde{\theta}_{jk}\hat{C}_k)^{\perp}],\end{aligned} \tag{6.5}$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ are the quantized vector potentials, $\epsilon_{123} = +1$ is the Levi-Civita symbol and the symmetric arrays of infinitesimal pseudoscalar angles $\theta_{ij} = \theta_{ji}$ and $\tilde{\theta}_{ij} = \tilde{\theta}_{ji}$ are related by:

$$\tilde{\theta}_{ij} = \delta_{ij}\theta_{kk} - 2\theta_{ij}. \tag{6.6}$$

The transformation (6.5) generated by the ij-infra-zilches resembles the infinitesimal duality rotation (6.1) generated by the optical helicity, but with a sense of directionality mixed in through the various θ_{ij}. Indeed, taking $\theta_{xx} = \theta_{yy} = \theta_{zz} = \theta$ with $\theta_{xy} = \theta_{xz} = \theta_{yz} = 0$ reduces (6.5) to (6.1), as it must. We may interpret (6.5) loosely as follows: The terms $(\tilde{\theta}_{li}B_i)^{\perp}$ and $-(\tilde{\theta}_{li}E_i)^{\perp}$ give the closest approximation to a directional duality transformation that is consistent with the requirement of transversality. However, on its own, this transformation 'breaks the rules' and the additional terms $\epsilon_{lij}\partial_i(\tilde{\theta}_{jk}\hat{A}_k)^{\perp}$ and $\epsilon_{lij}\partial_i(\tilde{\theta}_{jk}\hat{C}_k)^{\perp}$ involving the vector potentials are apparently required to ensure that the *complete* transformation, which is the average of the two contributions, is physically acceptable. Maxwell's equations do indeed retain their form under the transformation:

$$\begin{aligned}E_l &\to E_l + \tfrac{1}{2}[(\tilde{\theta}_{li}B_i)^{\perp} + \epsilon_{lij}\partial_i(\tilde{\theta}_{jk}A_k)^{\perp}],\\ B_l &\to B_l - \tfrac{1}{2}[(\tilde{\theta}_{li}E_i)^{\perp} - \epsilon_{lij}\partial_i(\tilde{\theta}_{jk}C_k)^{\perp}],\end{aligned} \tag{6.7}$$

and we associate this symmetry with the conservation of the ij-infra-zilches. The physical significance of the conservation of the ij-infra-zilches is considered below.

The formal derivation of the results above using Noether's theorem sheds light on the appearance of the transverse pieces of the vector potentials in our physical quantities. The fields are simply related to these pieces by time derivatives ($\mathbf{E} = -\dot{\mathbf{A}}^{\perp}$, $\mathbf{B} = -\dot{\mathbf{C}}^{\perp}$). Consequently, when we rotate the vector potentials in the Coulomb gauge, the fields rotate in step with them. We shall return to this point elsewhere.

7. An interesting analogy

The $N^{\alpha\beta 0}$ components (5.1) of our helicity array are remarkably similar in form to the components of the contravariant symmetric energy–momentum tensor, $T^{\alpha\beta} = T^{\beta\alpha}$, given by:

$$\begin{aligned}T^{00} &= \tfrac{1}{2}\,(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\,,\\ T^{0i} &= (\mathbf{E}\times\mathbf{B})_i\,,\\ T^{ij} &= \tfrac{1}{2}\delta_{ij}\,(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}) - E_iE_j - B_iB_j.\end{aligned} \tag{7.1}$$

The component T^{00} is the energy density of the field, the components T^{0i} are those of the energy flux density or linear momentum density and the components T^{ij} are those of the linear momentum flux density. The continuity equation $\partial_\beta T^{\alpha\beta} = 0$ expresses energy conservation for $\alpha = 0$ and linear momentum conservation for $\alpha = 1, 2, 3$ [12].

We note that the density components of our helicity array are mapped onto the components of the energy–momentum tensor, $N^{\alpha\beta 0} \to T^{\alpha\beta}$, when we make the superficial transformation of the vector potentials:

$$\begin{aligned} \mathbf{A} &\to \mathbf{B}, \\ \mathbf{C} &\to -\mathbf{E}, \end{aligned} \tag{7.2}$$

whilst leaving the fields unchanged ($\mathbf{E} \to \mathbf{E}$, $\mathbf{B} \to \mathbf{B}$), as may be confirmed by comparing (5.1) with (7.1). The explanation of this follows from the observation that the vector potentials of a left- ($\sigma = +1$) or right-($\sigma = -1$) handed circularly polarized plane wave of angular frequency ω are related to its fields by:

$$\begin{aligned} \mathbf{A} &= \frac{\sigma}{\omega}\mathbf{B}, \\ \mathbf{C} &= -\frac{\sigma}{\omega}\mathbf{E}. \end{aligned} \tag{7.3}$$

Of course, an arbitrary optical field can be expressed as a superposition of many circularly polarized plane waves. From (7.3) we see that the transformation (7.2) is equivalent to letting $\hbar\sigma \to \hbar\omega$, which is simply a mapping of photon helicity to photon energy. Accordingly, we find that $N^{\alpha\beta 0} \to T^{\alpha\beta}$.

Our analogy is incomplete in the sense that our helicity array, $N^{\alpha\beta\gamma}$, has three indices whereas the energy–momentum tensor, $T^{\alpha\beta}$, only has two. We may enquire as to the physical significance of the third index. In particular, we ask whether the conservation of the ij-infra-zilches:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint N^{ij0}\,\mathrm{d}^3\mathbf{r} = 0, \tag{7.4}$$

which has no analog in the description of energy and linear momentum,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint T^{ij}\,\mathrm{d}^3\mathbf{r} \neq 0, \tag{7.5}$$

is in any sense trivial or superfluous. That it is not, can be demonstrated as follows. Consider the situation depicted in figure 3(a). Here we have two plane waves with equal amplitudes and frequencies, but with opposite helicities ($\sigma = +1$ and $\sigma = -1$), propagating in perpendicular directions. Suppose we were to 'close our eyes' at time $t = t_1$ and open them later, at $t = t_2$, to find that both waves had changed the signs of their wavevectors *and* the signs of their helicities, as depicted in figure 3(b). That is, suppose our two-wave configuration were to undergo a parity inversion, spontaneously, during this time interval. Such an evolution is clearly unnatural and yet is *not* forbidden by either helicity or spin conservation, as:

$$\mathcal{H}(t_2) = \mathcal{H}(t_1) = 0 \quad \text{and} \quad \boldsymbol{\mathcal{S}}(t_2) = \boldsymbol{\mathcal{S}}(t_1). \tag{7.6}$$

In fact, our hypothetical evolution *is* forbidden by the conservation of the ij-infra-zilches, as:

$$\mathcal{N}_{ij}(t_2) = -\mathcal{N}_{ij}(t_1), \tag{7.7}$$

which violates the conservation laws (7.4) because some or all of the $\mathcal{N}_{ij}$ are non-zero, depending on the coordinate system used. Conservation laws are important because they constrain the evolution of a system. Our simple argument demonstrates that the conservation of the ij-infra-zilches is neither trivial nor superfluous: It provides constraints that the conservation of optical helicity and optical spin do not.

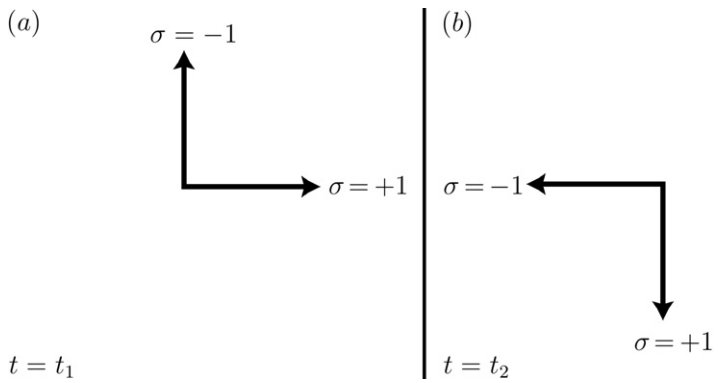

Figure 3. (a) A configuration of two waves with opposite helicities and perpendicular wavevectors. (b) The configuration obtained by changing the signs of the helicities and the signs of the wavevectors of both waves.

The helicity of a plane wave, which is proportional to σ, can be positive or negative. Consequently, helicities can both add and subtract in general. Consider, for example, a linearly polarized wave ($\sigma = 0$), which can be thought of as a superposition of circularly polarized waves whose opposite helicities ($\sigma = \pm 1$) cancel. Similarly, $\mathcal{H} = 0$ for the configurations shown in figure 3. The simple argument above suggests that the conservation of the ij-infra-zilches is necessary to account for the degree of freedom that is the sign of σ. This degree of freedom is intimately related to the handedness, or chirality, of the field and indeed the ij-infra-zilches change sign under a parity inversion. In contrast, the energy of a wave is strictly positive, which perhaps explains why the conservation laws (7.4) have no analog in the description of optical energy and linear momentum: (7.5).

8. Lipkin's zilches

The appearance of the vector potentials, **A** and **C**, in our description of the angular momentum associated with polarization is surprising and raises the question of gauge dependence. We stress here that we are working with the transverse pieces of the vector potentials ($\mathbf{A} = \mathbf{A}^{\perp}$, $\mathbf{C} = \mathbf{C}^{\perp}$), which are gauge invariant [4]. Nevertheless, it is natural to ask whether a description exists which only makes explicit reference to the fields, **E** and **B**. We now address this question.

In 1964, Lipkin [28] introduced his rank-three zilch[2] tensor, $Z^{\alpha\beta\gamma} = Z^{\beta\alpha\gamma}$, with contravariant components given by:

$$\begin{aligned} Z^{000} &= \tfrac{1}{2}\left[\mathbf{E}\cdot(\nabla\times\mathbf{E}) + \mathbf{B}\cdot(\nabla\times\mathbf{B})\right], \\ Z^{0i0} &= \tfrac{1}{2}\left[\mathbf{E}\times(\nabla\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{E})\right]_i, \\ Z^{ij0} &= \tfrac{1}{2}\left\{\delta_{ij}\left[\mathbf{E}\cdot(\nabla\times\mathbf{E}) + \mathbf{B}\cdot(\nabla\times\mathbf{B})\right] - E_i\,(\nabla\times\mathbf{E})_j - E_j\,(\nabla\times\mathbf{E})_i \right. \\ &\qquad \left. - B_i\,(\nabla\times\mathbf{B})_j - B_j\,(\nabla\times\mathbf{B})_i\right\} \end{aligned} \tag{8.1}$$

[2] In his paper, Lipkin does not elaborate upon his choice of the name 'zilch'.

and

$$\begin{aligned} Z^{00i} &= Z^{0i0}, \\ Z^{0ij} &= Z^{ij0}, \\ Z^{ijk} &= \delta_{ij} Z^{00k} + \tfrac{1}{2}\left(-E_i \partial_k B_j - E_j \partial_k B_i + B_i \partial_k E_j + B_j \partial_k E_i\right), \end{aligned} \tag{8.2}$$

where we have omitted physically insignificant transverse contributions to the $Z^{\alpha\beta k}$ components. As may be confirmed,

$$\partial_\gamma Z^{\alpha\beta\gamma} = 0, \tag{8.3}$$

which describes the conservation and flow of the quantities:

$$\mathcal{Z}_{\alpha\beta} = \iiint Z^{\alpha\beta 0} \mathrm{d}^3\mathbf{r}. \tag{8.4}$$

Lipkin referred to these quantities, $\mathcal{Z}_{\alpha\beta}$, as the $\alpha\beta$-zilches of the field. Nine of the zilches are independent.

For a plane wave of angular frequency ω, wavevector $\mathbf{k}$ and polarization σ, explicit calculation reveals that:

$$\frac{\overline{Z^{\alpha\beta\gamma}}}{\overline{w}} = \omega^2 \frac{\hbar\sigma}{\hbar\omega} \frac{k_\alpha k_\beta k_\gamma}{|\mathbf{k}|^3}. \tag{8.5}$$

Evidently, for a plane wave, the cycle-averaged components, $\overline{Z^{\alpha\beta\gamma}}$, of Lipkin's zilch tensor are proportional to the cycle-averaged components, $\overline{N^{\alpha\beta\gamma}}$, of our helicity array, as may be confirmed by comparing (8.5) with (5.7). In fact, it can be shown that this relationship:

$$\overline{Z^{\alpha\beta\gamma}} = \omega^2\, \overline{N^{\alpha\beta\gamma}}, \tag{8.6}$$

holds for *any* monochromatic field. This suggests that the zilches might provide a description of the angular momentum associated with polarization that avoids the issue of gauge dependence, as (8.1) and (8.2) make no explicit reference to the vector potentials. Indeed, in his original paper [28], Lipkin noted that the zilches might be related to the intrinsic spin of the field, although he recognized that their frequency dependence is unusual and that they do not have the dimensions of an angular momentum. Shortly after the appearance of Lipkin's paper, it was conjectured by Candlin [20] that the zilches are members of an infinite hierarchy of conserved quantities related to the helicity of the field (which he referred to as its screw action) and that they possess no physical significance. A similar conclusion was reached by Kibble [29]. We note that transforming the vector potentials (and the fields) into their curls:

$$\begin{aligned} \mathbf{A} &\to \nabla \times \mathbf{A}, \\ \mathbf{C} &\to \nabla \times \mathbf{C}, \end{aligned} \tag{8.7}$$

maps our helicity array onto Lipkin's zilch tensor, $N^{\alpha\beta\gamma} \to Z^{\alpha\beta\gamma}$, as may be confirmed by comparing (5.1) and (5.2) with (8.1) and (8.2), making use of (2.4). We may explain this relationship as follows. The equations:

$$\begin{aligned} \nabla \cdot \mathbf{A} &= 0, \\ \nabla \cdot \mathbf{C} &= 0, \\ \nabla \times \mathbf{A} &= -\dot{\mathbf{C}}, \\ \nabla \times \mathbf{C} &= \dot{\mathbf{A}}, \end{aligned} \tag{8.8}$$

which govern the vector potentials, **A** and **C**, are identical *in form* to Maxwell's equations (2.1), which govern the fields, **E** and **B**. The fields are given by the curls of the vector potentials (2.4), suggesting the existence of a pattern. Indeed, we find that the curls of the fields, which we denote $\mathbf{G} = \nabla \times \mathbf{E}$ and $\mathbf{M} = \nabla \times \mathbf{B}$, satisfy the equations:

$$\begin{aligned} &\nabla \cdot \mathbf{G} = 0, \\ &\nabla \cdot \mathbf{M} = 0, \\ &\nabla \times \mathbf{G} = -\dot{\mathbf{M}}, \\ &\nabla \times \mathbf{M} = \dot{\mathbf{G}}, \end{aligned} \tag{8.9}$$

which are again identical in form to Maxwell's equations. Furthermore, we find that the curls of the curls of the fields obey a set of Maxwell-like equations and so on, ad infinitum. As a consequence of this infinite hierarchy of Maxwell-like equations, we can take *any* conserved quantity and replace, superficially, the fields with their curls, obtaining a 'new' quantity, of different dimensions, that is also conserved. For example, it is well-known that the energy, W, of the field:

$$W = \iiint \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B})\ \mathrm{d}^3\mathbf{r}, \tag{8.10}$$

is conserved ($\mathrm{d}W/\mathrm{d}t = 0$). The proof of this follows from Maxwell's equations (2.1). Therefore, replacing **E** and **B** with $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{B}$ in (8.10) yields a new quantity:

$$X = \iiint \frac{1}{2} (\mathbf{G} \cdot \mathbf{G} + \mathbf{M} \cdot \mathbf{M})\ \mathrm{d}^3\mathbf{r}, \tag{8.11}$$

which is itself conserved ($\mathrm{d}X/\mathrm{d}t = 0$) by virtue of the Maxwell-like equations (8.9). This obscure quantity, X, is the 'energy' of the curl of the field and, as such, possesses no obvious significance. We can repeat this procedure indefinitely, obtaining an infinite hierarchy of quantities, all of which are conserved and have, for a monochromatic field, cycle-averaged values that are proportional to the cycle-averaged energy of the field. Of course, only one of these conserved quantities, namely (8.10), actually has the dimensions of an energy and couples to matter in a physically meaningful way.

We recognize the optical helicity, $\mathcal{H}$; the optical spin, $\boldsymbol{\mathcal{S}}$, and the ij-infra-zilches, $\mathcal{N}_{ij}$, as being physically meaningful because they have the dimensions of an angular momentum. The zilches, however, do not. In fact, we see from (8.7) and the considerations above that the zilches describe the 'angular momentum' of the curl of the field, rather than the angular momentum of the field itself. Consequently, the zilches are no more meaningful as a description of the angular momentum associated with polarization, than, for example, the quantity X above is meaningful as a description of optical energy.

Recently, the zilches have been reintroduced into the literature by Tang, Cohen and Yang [30, 31]. In particular, they refer to the 00-zilch density, Z^{000}, as the optical chirality and have used it to successfully predict and describe the results of experiments [32–34]. We believe that the quantity of interest in these experiments is in fact the helicity density, N^{000}, of the field, rather than the 00-zilch density, Z^{000}, which we now recognize as being the 'helicity' density of the *curl* of the field. The use of Z^{000} to describe these experiments has, we suggest, succeeded because of the existence of the proportionality (8.6), which *only* holds for the special case of a monochromatic field. For example, the dissymmetry factor, g, derived by Tang and Cohen

for a chiral absorbing molecule irradiated by a monochromatic field of angular frequency ω is (equation 6 of [30]):

$$g = -\left(\frac{G''}{\alpha''}\right)\left(\frac{2\overline{Z^{000}}}{\omega\overline{w_e}}\right), \tag{8.12}$$

where G'' and α'' describe the response of the molecule to the field and $w_e = E^2/2$ is the electric energy density. Tang and Cohen advocate Z^{000} as a measure of the chirality of the field on the grounds that it appears in (8.12). However, making use of (8.6), their dissymmetry factor can be written in the form:

$$g = -\left(\frac{G''}{\alpha''}\right)\left(\frac{2\omega\overline{N^{000}}}{\overline{w_e}}\right), \tag{8.13}$$

which now reflects the fact that the helicity density, N^{000}, not the 00-zilch density, Z^{000}, provides a physically meaningful measure of the twist and chirality associated with the angular momentum of the field.

A connection between the zilches and the helicity of the field was proposed recently by Bliokh and Nori [35]. However, they restricted their attention to monochromatic fields and were in fact observing the existence of the proportionality (8.6). This proportionality was also observed recently by Coles and Andrews [36, 37]. We emphasize that this proportionality only holds for the special case of a monochromatic field. In general, no proportionality like (8.6) holds and, for this reason, we surmise that it is our helicity array, $N^{\alpha\beta\gamma}$, rather than Lipkin's zilch tensor, $Z^{\alpha\beta\gamma}$, that embodies the phenomenon of chirality.

9. Discussion

We have examined the optical helicity, the optical spin and the ij-infra-zilches in electromagnetic theory and shown that these conserved quantities can be combined to form a new description of the angular momentum associated with polarization: one that is analogous to the familiar description of optical energy and linear momentum. The existence of this description strengthens the idea that the division of optical angular momentum into parts separately associated with polarization and the spatial distribution of the field is meaningful.

Our description is exact. Furthermore, it is gauge invariant, as we have only made reference to the transverse pieces of the vector potentials ($\mathbf{A} = \mathbf{A}^{\perp}$, $\mathbf{C} = \mathbf{C}^{\perp}$). However, the isolation of these pieces raises questions regarding locality and relativity. Ultimately, it is perhaps the total helicity, $\mathcal{H}$; spin, $\boldsymbol{\mathcal{S}}$, and ij-infra-zilches, $\mathcal{N}_{ij}$, that are physically meaningful, rather than their associated densities: h, $\mathbf{s}$ and n_{ij}.

We have been concerned exclusively with the free-field. It remains for us to discuss the significance possessed by our quantities in the presence of matter.

We shall return to these ideas in future publications.

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